1.

113.2091509　142.0410482　195.1341156　256.7968991

x	4.0	4.2	4.5	4.7	5.1	5.5	5.9	6.3
y	102.6	113.2	130.1	142.1	167.5	195.1	224.9	256.8

f(x)　102.6103548　130.1060182　169.5169914　224.8924129

$y = ax^2 + bx + c$

a

$$\begin{bmatrix} \sum x_i^2 & \sum x_i & \sum 1 \\ \sum x_i^3 & \sum x_i^2 & \sum x_i \\ \sum x_i^4 & \sum x_i^3 & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

$\sum_{i=1}^{8} 1 = 8$　$\sum_{i=1}^{8} x_i = 40.2$　$\sum_{i=1}^{8} x_i^2 = 206.74$

$\sum_{i=1}^{8} x_i^3 = 1087.488$　$\sum_{i=1}^{8} x_i^4 = 5843.815$

$\sum_{i=1}^{8} y_i = 1332.3$　$\sum_{i=1}^{8} x_i y_i = 7011.21$　$\sum_{i=1}^{8} x_i^2 y_i = 37691.573$

$\Rightarrow a = 6.691184389$　$b = -1.883746442$　$c = 3.086393299$

$E = \sum_{i=1}^{8} [y_i - f(x_i)]^2 = 5.245681488 \times 10^{-3}$

b.

$f(x)$	114.33908552	139.5466122	191.9423408	264.0111545
$\ln y$	4.729156166	4.936531015	5.253512248	5.548297572

x	4.0	4.2	4.5	4.7	5.1	5.5	5.9	6.3
y	102.6	113.2	130.1	142.1	167.5	195.1	224.9	256.8

$\ln y$ 4.630583933 4.886830386 5.120983351 5.415655859

$f(x)$ 105.5781997 128.8565297 163.6609403 225.1109037

$y = be^{ax}$

$\ln y = ax + \ln b$

$\ln y \to Y$, $\ln b \to B$

$$\begin{cases} \sum_{i=1}^{8} 1 = 8 \\ \sum_{i=1}^{8} x_i = 40.2 \\ \sum_{i=1}^{8} x_i^2 = 206.74 \end{cases}$$

$\sum Y_i = 40.54327755$

$\sum x_i Y_i = 205.61684455$

$$\Rightarrow \begin{bmatrix} 40.2 & 8 \\ 206.74 & 40.2 \end{bmatrix} \begin{bmatrix} a \\ \ln b \end{bmatrix} = \begin{bmatrix} 40.54327755 \\ 205.61684455 \end{bmatrix}$$

$\Rightarrow a = 0.398495419\,5$

$\ln b = 3.065470211 \Rightarrow b = 21.44454308$

$E = \sum_{i=1}^{8} [y_i - f(x_i)]^2 = 94.98305701$

C

$f(x_i)$	113.2002251	142.0703764	195.1522325	256.7362719
$\ln x$	1.435084525	1.547562509	1.704748092	1.840549633
$\ln y$	4.729156166	4.956531015	5.273512248	5.548297572

x	4.0	4.2	4.5	4.7	5.1	5.5	5.9	6.3
y	102.6	113.2	130.1	142.1	167.5	195.1	224.9	256.8

$\ln y$	4.630839233	4.868830386	5.120983351	5.415655859
$\ln x$	1.386294361	1.504077397	1.629240541	1.774952351
$f(x_i)$	102.5776289	130.1253896	167.5500802	224.8799849

$$y = bx^n = be^{n\ln x} \Rightarrow \underbrace{\ln y}_{Y} = n\underbrace{\ln x}_{X} + \underbrace{\ln b}_{B}$$

$$\begin{cases} \sum 1 = 8 \\ \sum X_i = 12.82250941 \\ \sum X_i^2 = 20.73714978 \end{cases} \qquad \begin{cases} \sum Y_i = 40.54329755 \\ \sum X_i Y_i = 65.35706181 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 12.82250941 & 8 \\ 20.73714978 & 12.82250941 \end{bmatrix} \begin{bmatrix} n \\ B \end{bmatrix} = \begin{bmatrix} 40.54329755 \\ 65.35706181 \end{bmatrix}$$

$$n = 2.019634262 \qquad B = 1.830812298 \Rightarrow b = 6.238952365$$

$$E = \sum_{i=1}^{8} [y_i - f(x_i)]^2 = 0.011720841$$

2. $f(x)=\frac{1}{2}\cos x+\frac{1}{4}\sin 2x \qquad x\in[-1,1] \qquad P_2(x)=a_2x^2+a_1x+a_0$

$$\begin{bmatrix} \int_{-1}^{1}1\,dx & \int_{-1}^{1}x\,dx & \int_{-1}^{1}x^2dx \\ \int_{-1}^{1}x\,dx & \int_{-1}^{1}x^2dx & \int_{-1}^{1}x^3dx \\ \int_{-1}^{1}x^2dx & \int_{-1}^{1}x^3dx & \int_{-1}^{1}x^4dx \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}=\begin{bmatrix} \int_{-1}^{1}f(x)\,dx \\ \int_{-1}^{1}xf(x)\,dx \\ \int_{-1}^{1}x^2f(x)\,dx \end{bmatrix}$$

$\int_{-1}^{1}1\,dx=x\Big|_{-1}^{1}=2$

$\int_{-1}^{1}x\,dx=\frac{1}{2}x^2\Big|_{-1}^{1}=0$

$\int_{-1}^{1}x^2dx=\frac{x^3}{3}\Big|_{-1}^{1}=\frac{2}{3}$

$\int_{-1}^{1}x^3dx=\frac{x^4}{4}\Big|_{-1}^{1}=0$

$\int_{-1}^{1}x^4dx=\frac{x^5}{5}\Big|_{-1}^{1}=\frac{2}{5}$

$\Rightarrow$ $a_0=0.498279307\,5$

$a_1=0.326548331\,3$

$a_2=-0.232681445\,2$

$\int_{-1}^{1}f(x)\,dx=0.841470984\,8$

$\int_{-1}^{1}xf(x)\,dx=0.217698887\,5$

$\int_{-1}^{1}x^2f(x)\,dx=0.239133626\,9$

$\Rightarrow -0.232681445\,2x^2+0.326548331\,3x+0.498279307\,5$

3. $m=16$ S_4 of $f(x)=x^2\sin x$ $x\in[0,1]$ $\Rightarrow x_i=\frac{i}{16}$

$$S_4=a_0+\sum_{k=1}^{4} a_k\cos(2\pi k x_i)+b_k\sin(2\pi k x_i)$$

$$a_0=\frac{1}{m}\sum_{i=0}^{m-1} f(x_i)=\frac{1}{16}\sum_{i=0}^{15}\left(\frac{i}{16}\right)^2\sin\left(\frac{i}{16}\right)=0.1976\,721965$$

$$a_k=\frac{2}{m}\sum_{i=0}^{m-1} f(x_i)\cos(2\pi k x_i) \qquad b_k=\frac{2}{m}\sum_{i=0}^{m-1} f(x_i)\sin(2\pi k x_i)$$

$k=1\Rightarrow a_1=0.07282665799$ $\quad b_1=-0.23724918\,26$

$k=2\Rightarrow a_2=-0.02226182\,64$ $\quad b_2=-0.12385900\,24$

$k=3\Rightarrow a_3=-0.03839016\,979$ $\quad b_3=-0.07780\,8\,9\,9068$

$k=4\Rightarrow a_4=-0.04386478933$ $\quad b_4=-0.05222268084$

b. $\int_0^1 S_4\,dx=\int_0^1 a_0\,dx+\sum_{k=1}^{4}\int_0^1 a_k\cos(2\pi k x)+b_k\sin(2k\pi x)\,dx$

$\int_0^1\cos(2\pi k x)\,dx=\frac{1}{2\pi k}\sin(2\pi k x)\Big|_0^1=0$

$\int_0^1\sin(2\pi k x)\,dx=\frac{-1}{2\pi k}\cos(2\pi k x)\Big|_0^1=0$

$\Rightarrow\int_0^1 S_4\,dx=a_0 x\Big|_0^1=a_0=0.1976\,721965$

c. $\int_0^1 x^2\sin x\,dx=0.2232\,442955$ $\quad$ error $=\left|\int_0^1 x^2\sin x\,dx-\int_0^1 S_4\,dx\right|=0.0255\,709858$

relative error $=\dfrac{\left|\int_0^1 x^2\sin x\,dx-\int_0^1 S_4\,dx\right|}{\int_0^1 x^2\sin x\,dx}=11.4547\,779\%$

d. $E(S_4)=\sum_{i=0}^{15}\left(f(x_i)-S_4(x_i)\right)^2$ $\xrightarrow{\text{wolfram }\alpha}$ 0.0774016