Fraleigh Excerpts

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of ambiguity, $*$ is **not well defined**. If Condition 2 is violated, then S is **not closed under** $*$.

Following are several illustrations of attempts to define binary operations on sets. Some of them are worthless. The symbol $*$ is used for the attempted operation in all these examples.

2.19 Example On $\mathbb{Q}$, let $a * b = a/b$. Here $*$ is *not everywhere defined* on $\mathbb{Q}$, for no rational number is assigned by this rule to the pair (2, 0). ▲

2.20 Example On $\mathbb{Q}^+$, let $a * b = a/b$. Here both Conditions 1 and 2 are satisfied, and $*$ is a binary operation on $\mathbb{Q}^+$. ▲

2.21 Example On $\mathbb{Z}^+$, let $a * b = a/b$. Here Condition 2 fails, for $1 * 3$ is not in $\mathbb{Z}^+$. Thus $*$ is not a binary operation on $\mathbb{Z}^+$, since $\mathbb{Z}^+$ is *not closed under* $*$. ▲

2.22 Example Let F be the set of all real-valued functions with domain $\mathbb{R}$ as in Example 2.7. Suppose we "define" $*$ to give the usual quotient of f by g, that is, $f * g = h$, where $h(x) = f(x)/g(x)$. Here Condition 2 is violated, for the functions in F were to be defined for *all* real numbers, and for some $g \in F$, $g(x)$ will be zero for some values of x in $\mathbb{R}$ and $h(x)$ would not be defined at those numbers in $\mathbb{R}$. For example, if $f(x) = \cos x$ and $g(x) = x^2$, then $h(0)$ is undefined, so $h \notin F$. ▲

2.23 Example Let F be as in Example 2.22 and let $f * g = h$, where h is the function greater than both f and g. This "definition" is completely worthless. In the first place, we have not defined what it means for one function to be greater than another. Even if we had, any sensible definition would result in there being many functions greater than both f and g, and $*$ would still be *not well defined.* ▲

2.24 Example Let S be a set consisting of 20 people, no two of whom are of the same height. Define $*$ by $a * b = c$, where c is the tallest person among the 20 in S. This is a perfectly good binary operation on the set, although not a particularly interesting one. ▲

2.25 Example Let S be as in Example 2.24 and let $a * b = c$, where c is the shortest person in S who is taller than both a and b. This $*$ is *not everywhere defined,* since if either a or b is the tallest person in the set, $a * b$ is not determined. ▲

■ EXERCISES 2

Computations

Exercises 1 through 4 concern the binary operation $*$ defined on $S = \{a, b, c, d, e\}$ by means of Table 2.26.

1. Compute $b * d$, $c * c$, and $[(a * c) * e] * a$.
2. Compute $(a * b) * c$ and $a * (b * c)$. Can you say on the basis of this computations whether $*$ is associative?
3. Compute $(b * d) * c$ and $b * (d * c)$. Can you say on the basis of this computation whether $*$ is associative?

2.26 Table

$*$	a	b	c	d	e
a	a	b	c	b	d
b	b	c	a	e	c
c	c	a	b	b	a
d	b	e	b	e	d
e	d	b	a	d	c

2.27 Table

$*$	a	b	c	d
a	a	b	c	
b	b	d		c
c	c	a	d	b
d	d			a

2.28 Table

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d				

4. Is $*$ commutative? Why?

5. Complete Table 2.27 so as to define a commutative binary operation $*$ on $S = \{a, b, c, d\}$.

6. Table 2.28 can be completed to define an associative binary operation $*$ on $S = \{a, b, c, d\}$. Assume this is possible and compute the missing entries.

In Exercises 7 through 11, determine whether the binary operation $*$ defined is commutative and whether $*$ is associative.

7. $*$ defined on $\mathbb{Z}$ by letting $a * b = a - b$

8. $*$ defined on $\mathbb{Q}$ by letting $a * b = ab + 1$

9. $*$ defined on $\mathbb{Q}$ by letting $a * b = ab/2$

10. $*$ defined on $\mathbb{Z}^+$ by letting $a * b = 2^{ab}$

11. $*$ defined on $\mathbb{Z}^+$ by letting $a * b = a^b$

12. Let S be a set having exactly one element. How many different binary operations can be defined on S? Answer the question if S has exactly 2 elements; exactly 3 elements; exactly n elements.

13. How many different commutative binary operations can be defined on a set of 2 elements? on a set of 3 elements? on a set of n elements?

Concepts

In Exercises 14 through 16, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

14. A binary operation $*$ is *commutative* if and only if $a * b = b * a$.

15. A binary operation $*$ on a set S is *associative* if and only if, for all $a, b, c \in S$, we have $(b * c) * a = b * (c * a)$.

16. A subset H of a set S is *closed* under a binary operation $*$ on S if and only if $(a * b) \in H$ for all $a, b \in S$.

In Exercises 17 through 22, determine whether the definition of $*$ does give a binary operation on the set. In the event that $*$ is not a binary operation, state whether Condition 1, Condition 2, or both of these conditions on page 24 are violated.

17. On $\mathbb{Z}^+$, define $*$ by letting $a * b = a - b$.

18. On $\mathbb{Z}^+$, define $*$ by letting $a * b = a^b$.

19. On $\mathbb{R}$, define $*$ by letting $a * b = a - b$.

20. On $\mathbb{Z}^+$, define $*$ by letting $a * b = c$, where c is the smallest integer greater than both a and b.

Now suppose that G' is any other group of three elements and imagine a table for G' with identity element appearing first. Since our filling out of the table for $G = \{e, a, b\}$ could be done in only one way, we see that if we take the table for G' and rename the identity e, the next element listed a, and the last element b, the resulting table for G' must be the same as the one we had for G. As explained in Section 3, this renaming gives an isomorphism of the group G' with the group G. Definition 3.7 defined the notion of *isomorphism* and of *isomorphic binary structures*. Groups are just certain types of binary structures, so the same definition pertains to them. Thus our work above can be summarized by saying that all groups with a single element are isomorphic, all groups with just two elements are isomorphic, and all groups with just three elements are isomorphic. We use the phrase *up to isomorphism* to express this identification using the equivalence relation $\simeq$. Thus we may say, "There is only one group of three elements, up to isomorphism."

4.19 Table

$*$	e	a
e	e	a
a	a	e

4.20 Table

$*$	e	a	b
e	e	a	b
a	a		
b	b		

4.21 Table

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

EXERCISES 4

Computations

In Exercises 1 through 6, determine whether the binary operation $*$ gives a group structure on the given set. If no group results, give the first axiom in the order $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$ from Definition 4.1 that does not hold.

1. Let $*$ be defined on $\mathbb{Z}$ by letting $a * b = ab$.

2. Let $*$ be defined on $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ by letting $a * b = a + b$.

3. Let $*$ be defined on $\mathbb{R}^+$ by letting $a * b = \sqrt{ab}$.

4. Let $*$ be defined on $\mathbb{Q}$ by letting $a * b = ab$.

5. Let $*$ be defined on the set $\mathbb{R}^*$ of nonzero real numbers by letting $a * b = a/b$.

6. Let $*$ be defined on $\mathbb{C}$ by letting $a * b = |ab|$.

7. Give an example of an abelian group G where G has exactly 1000 elements.

8. We can also consider multiplication $\cdot_n$ modulo n in $\mathbb{Z}_n$. For example, $5 \cdot_7 6 = 2$ in $\mathbb{Z}_7$ because $5 \cdot 6 = 30 = 4(7) + 2$. The set $\{1, 3, 5, 7\}$ with multiplication $\cdot_8$ modulo 8 is a group. Give the table for this group.

9. Show that the group $\langle U, \cdot \rangle$ is not isomorphic to either $\langle \mathbb{R}, + \rangle$ or $\langle \mathbb{R}^*, \cdot \rangle$. (All three groups have cardinality $|\mathbb{R}|$.)

10. Let n be a positive integer and let $n\mathbb{Z} = \{nm \mid m \in \mathbb{Z}\}$.

a. Show that $\langle n\mathbb{Z}, + \rangle$ is a group.

b. Show that $\langle n\mathbb{Z}, + \rangle \simeq \langle \mathbb{Z}, + \rangle$.

In Exercises 11 through 18, determine whether the given set of matrices under the specified operation, matrix addition or multiplication, is a group. Recall that a **diagonal matrix** is a square matrix whose only nonzero entries lie on the **main diagonal,** from the upper left to the lower right corner. An **upper-triangular matrix** is a square matrix with only zero entries below the main diagonal. Associated with each $n \times n$ matrix A is a number called the determinant of A, denoted by $\det(A)$. If A and B are both $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$. Also, $\det(I_n) = 1$ and A is invertible if and only if $\det(A) \neq 0$.

11. All $n \times n$ diagonal matrices under matrix addition.

12. All $n \times n$ diagonal matrices under matrix multiplication.

13. All $n \times n$ diagonal matrices with no zero diagonal entry under matrix multiplication.

14. All $n \times n$ diagonal matrices with all diagonal entries 1 or -1 under matrix multiplication.

15. All $n \times n$ upper-triangular matrices under matrix multiplication.

16. All $n \times n$ upper-triangular matrices under matrix addition.

17. All $n \times n$ upper-triangular matrices with determinant 1 under matrix multiplication.

18. All $n \times n$ matrices with determinant either 1 or -1 under matrix multiplication.

19. Let S be the set of all real numbers except -1. Define $*$ on S by

$$a * b = a + b + ab.$$

- **a.** Show that $*$ gives a binary operation on S.
- **b.** Show that $\langle S, * \rangle$ is a group.
- **c.** Find the solution of the equation $2 * x * 3 = 7$ in S.

20. This exercise shows that there are two nonisomorphic group structures on a set of 4 elements.

Let the set be $\{e, a, b, c\}$, with e the identity element for the group operation. A group table would then have to start in the manner shown in Table 4.22. The square indicated by the question mark cannot be filled in with a. It must be filled in either with the identity element e or with an element different from both e and a. In this latter case, it is no loss of generality to assume that this element is b. If this square is filled in with e, the table can then be completed in two ways to give a group. Find these two tables. (You need not check the associative law.) If this square is filled in with b, then the table can only be completed in one way to give a group. Find this table. (Again, you need not check the associative law.) Of the three tables you now have, two give isomorphic groups. Determine which two tables these are, and give the one-to-one onto renaming function which is an isomorphism.

- **a.** Are all groups of 4 elements commutative?
- **b.** Which table gives a group isomorphic to the group U_4, so that we know the binary operation defined by the table is associative?
- **c.** Show that the group given by one of the other tables is structurally the same as the group in Exercise 14 for one particular value of n, so that we know that the operation defined by that table is associative also.

21. According to Exercise 12 of Section 2, there are 16 possible binary operations on a set of 2 elements. How many of these give a structure of a group? How many of the 19,683 possible binary operations on a set of 3 elements give a group structure?

Concepts

22. Consider our axioms $\mathscr{G}_1$, $\mathscr{G}_2$, and $\mathscr{G}_3$ for a group. We gave them in the order $\mathscr{G}_1\mathscr{G}_2\mathscr{G}_3$. Conceivable other orders to state the axioms are $\mathscr{G}_1\mathscr{G}_3\mathscr{G}_2$, $\mathscr{G}_2\mathscr{G}_1\mathscr{G}_3$, $\mathscr{G}_2\mathscr{G}_3\mathscr{G}_1$, $\mathscr{G}_3\mathscr{G}_1\mathscr{G}_2$, and $\mathscr{G}_3\mathscr{G}_2\mathscr{G}_1$. Of these six possible