ELEMENTARY NUMBER THEORY FINAL EXAM

Question 1. The following is a table of indices for the prime 13 relative to the primitive root 2:

a	1	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{ind}_2(a)$	12	1	4	2	9	5	11	3	8	10	7	6

With the aid of this table, solve the following congruences:

(1)(6%) $11x^5 \equiv 4 \bmod 13$; (2)(6%) $6x^8 \equiv 5 \bmod 13$; (3)(6%) $2x^2 + 6x + 5 \equiv 0 \bmod 13$.

Question 2. Let p be a prime number.

(1) (10%) Show that for each $n \in \mathbb{N}$, we have

$$\sum_{m=1}^{p-1} m^n \equiv \begin{cases} -1 \bmod p, & \text{if } (p-1) \mid n, \\ 0 \bmod p, & \text{otherwise.} \end{cases}$$

(2) (6%) Given a polynomial $f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k$ with $a_0, ..., a_k \in \mathbb{Z}$ and $k < p-1$. Show that

$$\sum_{m=0}^{p-1} f(m) \equiv 0 \bmod p.$$

Question 3. Determine whether the following congruence equations are solvable:

(1)(6%) $x^2 \equiv 38 \bmod 97$; (2)(6%) $x^2 + 4x \equiv 29 \bmod 157$; (3)(6%) $x^2 + 377x + 154 \equiv 0 \bmod 383$.

Question 4.

(1) (6%) Let p be a prime number. Verify that $\left(\frac{-6}{p}\right) = 1$ if and only if $p \equiv 1, 5, 7, 11 \mod 24$.

(2) (10%) Show that there are infinitely many prime numbers of the form either $24k + 5$ or $24k + 11$. (Hint: Given prime numbers $p_1, ..., p_n$ of the form either $24k + 5$ or $24k + 11$, consider the positive integer $N := 9(p_1 \cdots p_n)^2 + 6$.)

Question 5. (12%) Show that the only solutions in positive integers of the equation $x^2 + y^2 = z^4$ with $x \equiv 1 \bmod 4$ and $\gcd(x, y, z) = 1$ are given by

$$x = s^4 - 6s^2t^2 + t^4, \quad y = 4s^3t - 4st^3, \quad z = s^2 + t^2,$$

where $s, t \in \mathbb{N}$, $s > t$, $\gcd(s, t) = 1$, and $s \equiv t \bmod 2$.

Question 6.

(1) (8%) Express $53/75$ and $117/41$ as finite simple continued fractions.

(2) (8%) Determine the rational numbers represented by the finite simple continued fractions $[2; 3, 1, 4, 3, 5]$ and $[1; 3, 5, 7, 9, 2]$.

Question 7.

(1) (10%) Express $\sqrt{34}$ as a periodic simple continued fraction.

(2) (10%) Find the fundamental positive solution of the Pell equation $x^2 - 34y^2 = 1$.

Date: June 23, 2021.