1. Use the Lagrange interpolating polynomials of degree one, two, three and four to approximate $\cos(0.750) = 0.7317$ if $\cos(0.698) = 0.7661$, $\cos(0.733) = 0.7432$, $\cos(0.768) = 0.7193$, $\cos(0.803) = 0.6946$. Find the error bound.

原. $\cos x$ ① $-\sin x$ ② $-\cos x$ ③ $\sin x$ ④ $\cos x$ ⑤ $-\sin x$

x	$y(x)$
0.698	0.7661
0.733	0.7432
0.768	0.7193
0.803	0.6946

$$\text{Lagrange polynomial}: \sum_{J=1}^{n} y_j \prod_{\substack{k=1\\k\neq j}}^{n} \frac{x-x_k}{x_j-x_k}$$

絕對值最大可能值

$$\text{Lagrange error bound}: E_n(x) = \frac{\text{MAX}[f^{(n+1)}(z)]}{(n+1)!} \boxed{\prod_{i=0}^{n}(x-x_i)}$$ 求最大值

1階

$$f(x) = y_1 \cdot \frac{x-x_2}{x_1-x_2} + y_2 \cdot \frac{x-x_1}{x_2-x_1} = 0.7661 \cdot \frac{x-0.733}{0.698-0.733} + 0.7432 \cdot \frac{x-0.698}{0.733-0.698}$$

$$= -21.8885714\,3(x-0.733) + 21.2342857\,1(x-0.698)$$

$$= -0.6543x + 1.2228 \quad , \quad f(0.750) = 0.7321 \qquad (x-0.698)(x-0.733)$$

$$E_1(x) = \frac{\text{MAX}[f''(z)]}{2!}(x-x_0)(x-x_1) \qquad |-\cos(0.733)| = |-0.7432| = 0.7432$$

$$E_1(x) = \frac{1}{2}\underline{(x-0.698)(x-0.7432)}$$ ⊖ 最大值在端點 ⊖ $x = 0.7206$

$$E_1(x) = -0.0002553\,8$$

2階

$$f(x) = y_1 \cdot \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \cdot \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \cdot \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

$$= 0.7661 \cdot \frac{(x-0.733)(x-0.768)}{(0.698-0.733)(0.698-0.768)} + 0.7432 \cdot \frac{(x-0.698)(x-0.768)}{(0.733-0.698)(0.733-0.768)}$$

$$+ 0.7193 \cdot \frac{(x-0.698)(x-0.733)}{(0.768-0.698)(0.768-0.733)}$$

$$= 312.6938776 \cdot (x^2 - 1.501x + 0.562944) - 606.6938776 \cdot (x^2 - 1.466x + 0.536064)$$

$$+ 293.5918367 (x^2 - 1.431x + 0.511634)$$

$$= -0.4081633x^2 - 0.07024053 7x + 1.013961208 , \quad f(0.75) = 0.7317$$

$$E_2(x) = \left| \frac{MAX[f'''(x)]}{3!} (x-x_0)(x-x_1)(x-x_2) \right| \qquad \underline{\sin(0.768) = 0.0134}$$

$$= \left| \frac{0.0134}{6} (x-0.698)(x-0.733)(x-0.768) \right|$$

$$= \left| (x^2 - 1.431x + 0.511634)(x-0.768) \cdot \frac{0.0134}{6} \right|$$

$$= \left| (x^3 - 2.199x^2 + 1.610642x - 0.392934912) \cdot \frac{0.0134}{6} \right|$$

$$E_2(0.75) = 1.3427 \times 10^{-7}$$

$\boxed{3階}$

$$f(x) = 0.7661 \cdot \frac{(x-0.733)(x-0.768)(x-0.803)}{(-0.035)(-0.07)(-0.105)} + 0.7432 \times \frac{(x-0.698)(x-0.768)(x-0.803)}{0.035 \cdot (-0.035)(-0.07)}$$

$$+ 0.7193 \cdot \frac{(x-0.698)(x-0.733)(x-0.803)}{0.07 \cdot 0.035 \cdot (-0.035)} + 0.6946 \cdot \frac{(x-0.698)(x-0.733)(x-0.768)}{0.105 \cdot 0.07 \cdot 0.035}$$

$$f(0.75) = 0.7317$$

$$E(x) = \frac{0.7661}{4!} \cdot (x-0.698)(x-0.733)(x-0.768)(x-0.803)$$

$$E(0.75) = 2.6920 \times 10^{-8} \ \#$$

2. Use iterated inverse interpolation to find an approximation to the solution $x-e^{-x}=0$ using the data $e^{-0.3}=0.740818$, $e^{-0.4}=0.670320$, $e^{-0.5}=0.606531$, $e^{-0.6}=0.548812$.

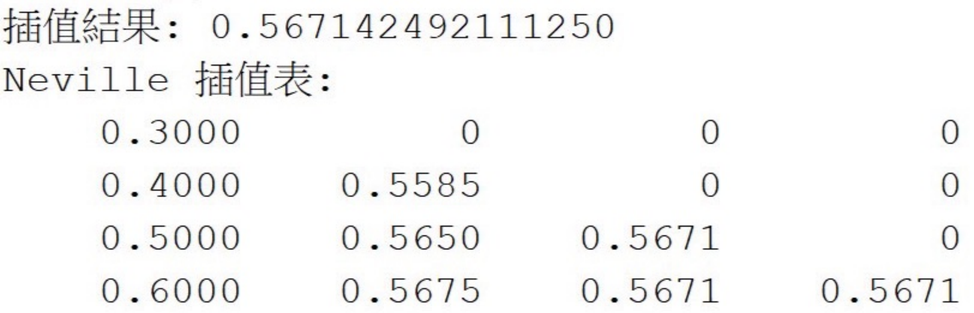
逆插+改用e^{x}做线性插值

用 iterated inverse interpolation 需使用 naille's method,
但老師沒教 ∴ 用 matlab

x	0.3	0.4	0.5	0.6
e^{-x}	0.740818	0.67032	0.606531	0.548812

⇓

$y=x-e^{-x}$	-0.440818	-0.270320	-0.106531	0.0511884	→ x
x	0.3	0.4	0.5	0.6	→ f



```
插值結果: 0.567142492111250
Neville 插值表:
    0.3000         0         0         0
    0.4000    0.5585         0         0
    0.5000    0.5650    0.5671         0
    0.6000    0.5675    0.5671    0.5671
```

$\therefore x=0.5671$

試用手算

$x=e^{-x} \to y=e^{-x}$, $x_1=0.5$, $y_1=0.606531$, $x_2=0.6$, $y_2=0.548812$

$$x = x_1 + \frac{y-y_1}{y_2-y_1}(x_2-x_1) = 0.5 + \frac{x-0.606531}{0.548812-0.606531}(0.6-0.5)$$

$$\Rightarrow -0.057718x = -0.028859 + 0.1x - 0.060653$$

$$\Rightarrow x = 0.5675446, \underline{e^{x} = 0.566916}$$ → 替換掉 $x_2=0.6$ ∵較遠

$$x = 0.5 + \frac{x-0.606531}{0.566916-0.606531}(0.566916-0.5)$$

$$\Rightarrow -0.039615x = -0.0198075 + 0.066916x - 0.0405866284 \quad \Rightarrow x = 0.5669$$

一般線性插值 $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow y = y_1 + \frac{x-x_1}{x_2-x_1}(y_2-y_1)$

逆插法 $x = x_1 + \frac{y-y_1}{y_2-y_1}(x_2-x_1)$

3. A car travelling along a straight road is clocked at a number of points. The data from the observations are given in the following table, where the time ***T*** is in seconds, the distance ***D*** is in feet, and the speed ***V*** is in feet per second.

T	0	3	5	8	13
D	0	200	375	620	990
V	75	77	80	74	72

a. Use a Hermite polynomial to predict the position of the car and its speed when $t = 10$ s.

b. Use the derivative of the Hermite polynomial to determine whether the car ever exceeds a 55 mi/h speed limit on the road. If so, what is the first time the car exceeds this speed?

c. What is the predicted maximum speed for the car ?

$f(x_2) = 990$

$f(x_1) = 620$

$$f[z_1, z_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= 74$$

$f[z_2, z_3] = f'(x_2) = f'(13)$

$= 72$

$z_0 = z_1 = x_1 = 8, \quad z_2 = z_3 = x_2 = 13$

a. $H_3(x) = f[z_0] + f[z_0, z_1](x - z_0) + f'[z_0, z_1, z_2](x - z_0)(x - z_1) + f[z_0, z_1, z_2, z_3](x - z_0)(x - z_1)(x - z_2)$

$$= f(8) + f'(8)(x-8) + \frac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0}(x-8)^2 + \frac{f[z_1, z_2, z_3] - f[z_0, z_1, z_2]}{z_3 - z_0}(x-8)^2(x-13)$$

$$= 620 + 74(x-8) + \frac{1}{x_2 - x_1}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - f'(x_1)\right](x-8)^2$$

$$+ \frac{1}{x_2 - x_1}\left\{\frac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1} - f[z_0, \overset{0}{\cancel{z_1}}, z_2]\right\}(x-8)^2(x-13)$$

$$= 74x + 28 + \frac{1}{5}(x^2 - 16x + 64)(\underset{0}{\cancel{74 - 74}}) + \frac{1}{5} \cdot \frac{1}{5} \cdot \left(f'(x_2) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)(x-8)^2(x-13)$$

$$= 74x + 28 + \frac{1}{25}(\overset{-2}{\cancel{72 - 74}})(x^2 - 16x + 64)(x - 13)$$

$$= 74x + 28 - \frac{2}{25}(x^3 - 13x^2 - 16x^2 + 208x + 64x - 832)$$

$$= \frac{-2}{25}x^3 + \frac{58}{25}x^2 + 52.24x + 94.56 \qquad f(10) = 768.96 \; \#$$

(b) $55\,mi/h = 55 \times \frac{5280}{3600} = 80.67\ ft/s$

位置: $H_3(x) = \frac{-2}{25}x^3 + \frac{58}{25}x^2 + 52.24x + 94.56$

速度: $H_3'(x) = \frac{-6}{25}x^2 + \frac{116}{25}x + 52.24$

$H_3''(x) = \frac{-12}{25}x + \frac{116}{25} = 0$ → 最大速度 $\Rightarrow x = \frac{29}{3}$ 代回 $H_3'(x)$

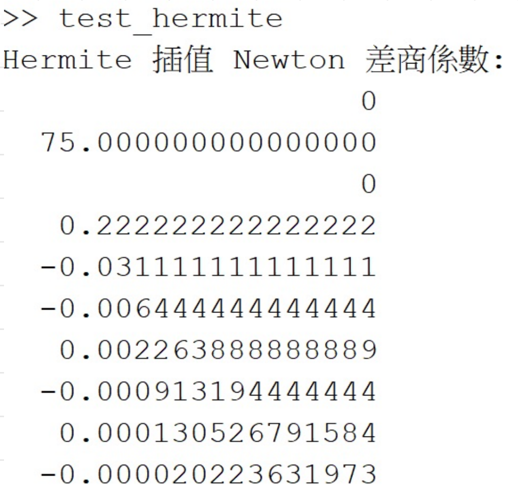
(c) $H_3'(\frac{29}{3}) = 74.67$ → (b)不可能超過 $80.67\ ft/s$！

but 其實這題在4~5秒時會超過 $80.67\ ft/s$，但為了用手算捨棄不看 $t<8$ 的所有值 → 用電腦重算！

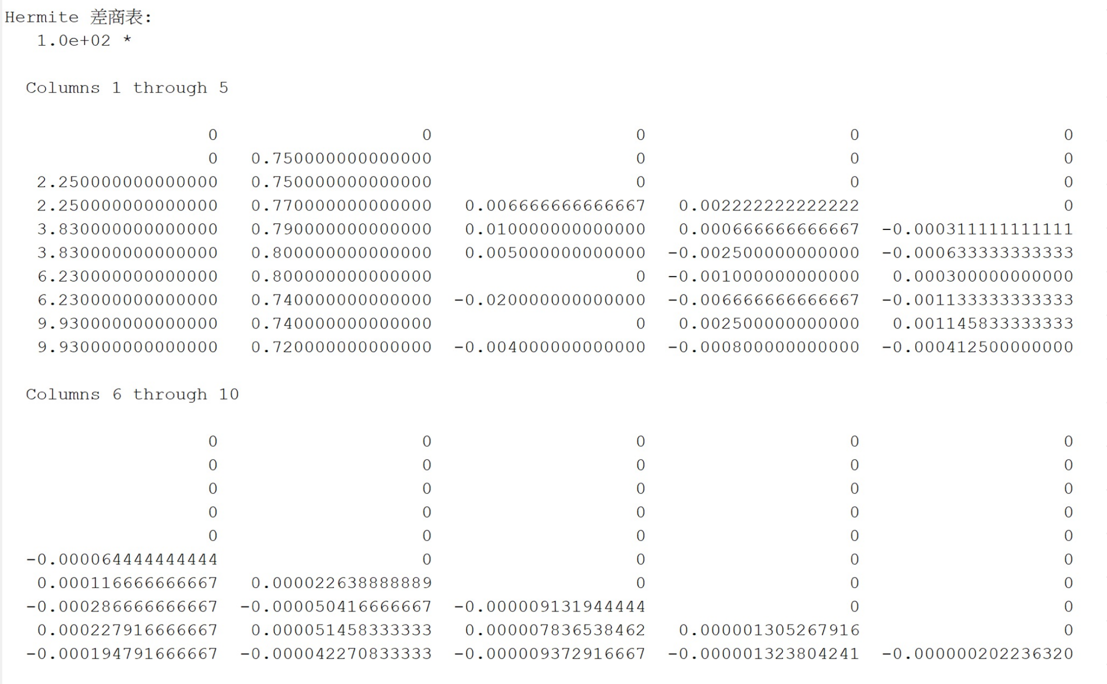
$x = [0,3,5,8,13]$

$f = [0,260,375,620,990]$

$f_p = [75,77,80,74,72]$



```
>> test_hermite
Hermite 插值 Newton 差商係數:
                 0
  75.000000000000000
                 0
   0.222222222222222
  -0.031111111111111
  -0.006444444444444
   0.002263888888889
  -0.000913194444444
   0.000130526791584
  -0.000020223631973
```



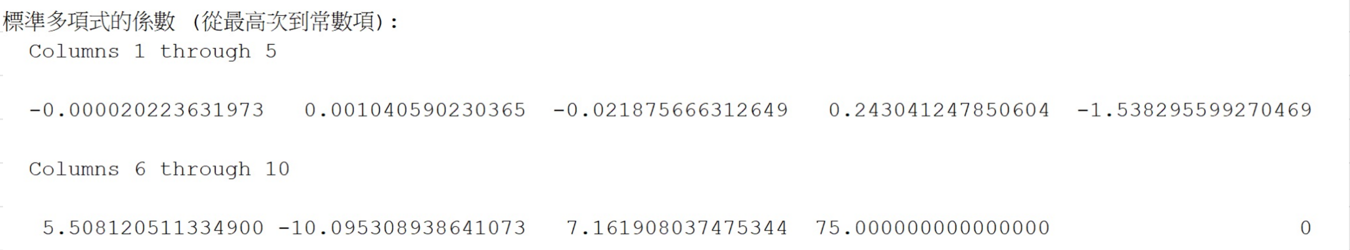
```
Hermite 差商表:
   1.0e+02 *

  Columns 1 through 5

                 0                 0                 0                 0                 0
                 0   0.750000000000000                 0                 0                 0
   2.250000000000000   0.750000000000000                 0                 0                 0
   2.250000000000000   0.770000000000000   0.006666666666667   0.002222222222222                 0
   3.830000000000000   0.790000000000000   0.010000000000000   0.000666666666667  -0.000311111111111
   3.830000000000000   0.800000000000000   0.005000000000000  -0.002500000000000  -0.000633333333333
   6.230000000000000   0.800000000000000                 0  -0.001000000000000   0.000300000000000
   6.230000000000000   0.740000000000000  -0.020000000000000  -0.006666666666667  -0.001133333333333
   9.930000000000000   0.740000000000000                 0   0.002500000000000   0.001145833333333
   9.930000000000000   0.720000000000000  -0.004000000000000  -0.000800000000000  -0.000412500000000

  Columns 6 through 10

                 0                 0                 0                 0                 0
                 0                 0                 0                 0                 0
                 0                 0                 0                 0                 0
                 0                 0                 0                 0                 0
                 0                 0                 0                 0                 0
  -0.000064444444444                 0                 0                 0                 0
   0.000116666666667   0.000022638888889                 0                 0                 0
  -0.000286666666667  -0.000050416666667  -0.000009131944444                 0                 0
   0.000227916666667   0.000051458333333   0.000007836538462   0.000001305267916                 0
  -0.000194791666667  -0.000042270833333  -0.000009372916667  -0.000001323804241  -0.000000202236320
```



```
標準多項式的係數 (從最高次到常數項):
  Columns 1 through 5

  -0.000020223631973   0.001040590230365  -0.021875666312649   0.243041247850604  -1.538295599270469

  Columns 6 through 10

   5.508120511334900 -10.095308938641073   7.161908037475344  75.000000000000000                   0
```

可得知 $Hq(x) = -0.000020223632x^9 + 0.001040590230365x^8 - 0.0218756663x^7$

$+ 0.24304x^6 - 1.5382956x^5 + 5.50812x^4 - 10.095309x^3$

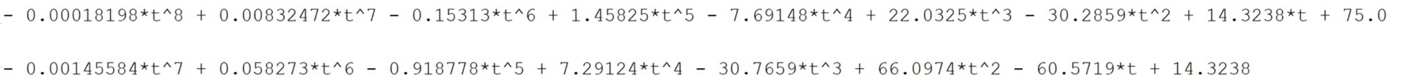
$+ 7.1619x^2 + 75x$

$Hq'(x) = -0.00018198x^8 + 0.00832472x^7 - 0.15313x^6 + 1.4582495x^5$

$-7.691478x^4 + 22.03248x^3 - 30.2859219x^2 + 14.323816x + 75$

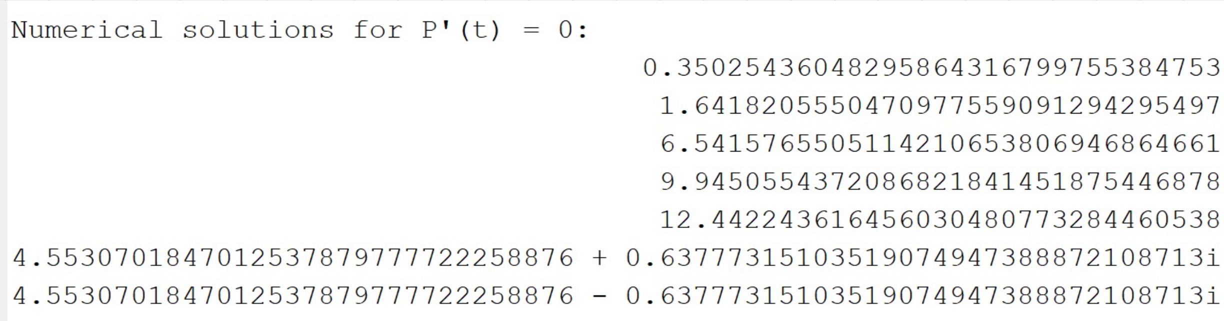
$Hq''(x) = -0.00145584x^7 + 0.0582730x^6 - 0.918778x^5 + 7.291235x^4$

$-30.765912x^3 + 66.097446x^2 - 60.571854x + 14.323816 = 0$



```
- 0.00018198*t^8 + 0.00832472*t^7 - 0.15313*t^6 + 1.45825*t^5 - 7.69148*t^4 + 22.0325*t^3 - 30.2859*t^2 + 14.3238*t + 75.0

- 0.00145584*t^7 + 0.058273*t^6 - 0.918778*t^5 + 7.29124*t^4 - 30.7659*t^3 + 66.0974*t^2 - 60.5719*t + 14.3238
```



```
Numerical solutions for P'(t) = 0:
                                   0.3502543604829586431679975538475З
                                    1.6418205550470977559091294295497
                                    6.5415765505114210653806946864661
                                    9.9450554372086821841451875446878
                                    12.442243616456030480773284460538
4.5530701847012537879777722258876 + 0.6377731510351907494738887210871Зi
4.5530701847012537879777722258876 - 0.6377731510351907494738887210871Зi
```

虛數解不看，可知在 $x = 0.3503, 1.642, 6.54, 9.945, 12.44$ $Hq'(x)$ 有最大值

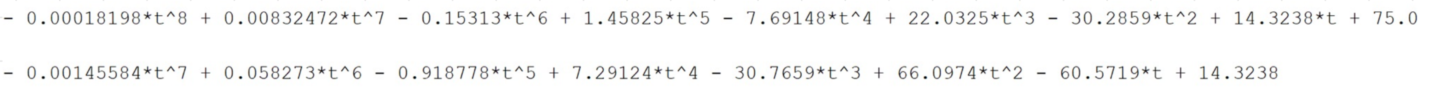
代回 $Hq'(x)$ 求這些時間的速度

$V(0.55) = 77.1399$ $U(1.642) = 73.1552$, $V(6.54) = 81.5644$

$V(9.945) = 51.56288$, $U(12.44) = 137.6032$

(b) ∴ 在 $x = 6.54$ 時 時會第1次超過 81.5644

(c) 在 $x = 12.44$ 時 會存最大值 137.6032



```
- 0.00018198*t^8 + 0.00832472*t^7 - 0.15313*t^6 + 1.45825*t^5 - 7.69148*t^4 + 22.0325*t^3 - 30.2859*t^2 + 14.3238*t + 75.0

- 0.00145584*t^7 + 0.058273*t^6 - 0.918778*t^5 + 7.29124*t^4 - 30.7659*t^3 + 66.0974*t^2 - 60.5719*t + 14.3238
```