$$w_{i+1} = w_i + \frac{h}{12}[5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})] + O(h^4)$$

Adams- Moulton three-step explicit method

$w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2$

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})] + O(h^5)$$

Adams- Moulton four-step explicit method

$w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2$, $w_3 = \alpha_3$

$$w_{i+1} = w_i + \frac{h}{720}[251f(t_{i+1}, w_{i+1}) + 646f(t_i, w_i) - 246f(t_{i-1}, w_{i-1}) + 106f(t_{i-2}, w_{i-2})$$

$$-19f(t_{i-3}, w_{i-3})] + O(h^6)$$

Milne's method

$$w_{i+1} = w_{i-3} + \frac{4h}{3}[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})] + O(h^5)$$

Simpson's method

$$w_{i+1} = w_{i-1} + \frac{h}{3}[f(t_{i+1}, w_{i+1}) + 4f(t_i, w_i) + f(t_{i-1}, w_{i-1})] + O(h^5)$$

Runge-Kutta-Fehlberg method (order of 5)

$$w_{i+1} = w_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6$$

$$k_1 = hf(t_i, w_i), \quad k_2 = hf(t_i + \frac{h}{4}, w_i + \frac{k_1}{4}), \quad k_3 = hf(t_i + \frac{3h}{8}, w_i + \frac{3k_1}{32} + \frac{9k_2}{32}),$$

$$k_4 = hf(t_i + \frac{12h}{13}, w_i + \frac{1932k_1}{2197} - \frac{7200k_2}{2197} + \frac{7296k_3}{2197}),$$

$$k_5 = hf(t_i + h, w_i + \frac{439k_1}{216} - 8k_2 + \frac{3680k_3}{513} - \frac{845k_4}{4104})$$

$$k_6 = hf(t_i + \frac{1}{2}h, w_i - \frac{8k_1}{27} + 2k_2 - \frac{3544k_3}{2565} + \frac{1859k_4}{4104} - \frac{11k_5}{40})$$

HW5

1. The initial-value problem

   $y' = 1 + (y/t) + (y/t)^2$, $1 \le t \le 2$, $y(1) = 0$ has the exact

   solution $y(t) = t\tan(\ln t)$.

a. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

b. Use Taylor's method of order 2 with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

```
=== Problem 1: Euler & Taylor Method Comparison ===

Euler Method:
      t  Euler_Approx  Exact_Solution
0   1.0      0.000000        0.000000
1   1.1      0.100000        0.105160
2   1.2      0.209917        0.221243
3   1.3      0.330471        0.349121
4   1.4      0.462354        0.489682
5   1.5      0.606285        0.643875
6   1.6      0.763041        0.812753
7   1.7      0.933475        0.997494
8   1.8      1.118537        1.199439
9   1.9      1.319293        1.420116
10  2.0      1.536943        1.661282

Taylor Order 2 Method:
      t  Taylor_Approx  Exact_Solution
0   1.0       0.000000        0.000000
1   1.1       0.105000        0.105160
2   1.2       0.220919        0.221243
3   1.3       0.348612        0.349121
4   1.4       0.488954        0.489682
5   1.5       0.642883        0.643875
6   1.6       0.811438        0.812753
7   1.7       0.995787        0.997494
8   1.8       1.197252        1.199439
9   1.9       1.417344        1.420116
10  2.0       1.657795        1.661282
```

2. The system of initial-value problems

$$u_1' = 9u_1 + 24u_2 + 5\cos t - \frac{1}{3}\sin t, \quad u_1(0) = \frac{4}{3},$$

$$u_2' = -24u_1 - 52u_2 - 9\cos t + \frac{1}{3}\sin t, \quad u_2(0) = \frac{2}{3},$$

has the unique solution

$$u_1 = 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t, \quad u_2 = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t.$$

Try $h = 0.05$ and $h = 0.1$ in Runge-Kutta method, and compare their results with the exact value.

=== Problem 2: Runge-Kutta 4th Order Comparison ===

Runge-Kutta 4th Order (h=0.1):

	t	RK4_u1_Approx	RK4_u2_Approx	Exact_u1	Exact_u2
0	0.0	1.333333×10^+00	6.666667×10^-01	1.333333	0.666667
1	0.1	-3.052437×10^+00	8.989305×10^+00	1.793063	-1.032002
2	0.2	-2.384779×10^+01	5.119270×10^+01	1.423902	-0.874681
3	0.3	-1.301652×10^+02	2.692692×10^+02	1.131577	-0.724999
4	0.4	-6.802315×10^+02	1.399369×10^+03	0.909409	-0.608214
5	0.5	-3.531300×10^+03	7.258242×10^+03	0.738788	-0.515658
6	0.6	-1.831280×10^+04	3.763496×10^+04	0.605710	-0.440411
7	0.7	-9.495133×10^+04	1.951319×10^+05	0.499860	-0.377404
8	0.8	-4.923065×10^+05	1.011722×10^+06	0.413671	-0.322954
9	0.9	-2.552514×10^+06	5.245579×10^+06	0.341614	-0.274409
10	1.0	-1.323428×10^+07	2.719729×10^+07	0.279675	-0.229888

Runge-Kutta 4th Order (h=0.05):

	t	RK4_u1_Approx	RK4_u2_Approx	Exact_u1	Exact_u2
0	0.00	1.333333	0.666667	1.333333	0.666667
1	0.05	1.721880	-0.499599	1.912059	-0.909077
2	0.10	1.726915	-0.832598	1.793063	-1.032002
3	0.15	1.617161	-0.890373	1.601967	-0.961459
4	0.20	1.481687	-0.861042	1.423902	-0.874681
5	0.25	1.348945	-0.807505	1.267646	-0.795221
6	0.30	1.227063	-0.750341	1.131577	-0.724999
7	0.35	1.117478	-0.695886	1.012999	-0.663060
8	0.40	1.019525	-0.645732	0.909409	-0.608214
9	0.45	0.931977	-0.599934	0.818630	-0.559389
10	0.50	0.853541	-0.558092	0.738788	-0.515658
11	0.55	0.783017	-0.519706	0.668275	-0.476225
12	0.60	0.719337	-0.484290	0.605710	-0.440411
13	0.65	0.661560	-0.451407	0.549909	-0.407635
14	0.70	0.608868	-0.420673	0.499860	-0.377404
15	0.75	0.560547	-0.391754	0.454695	-0.349296
16	0.80	0.515980	-0.364365	0.413671	-0.322954
17	0.85	0.474633	-0.338259	0.376158	-0.298076
18	0.90	0.436043	-0.313226	0.341614	-0.274409
19	0.95	0.399812	-0.289089	0.309583	-0.251739
20	1.00	0.365600	-0.265698	0.279675	-0.229888