數值方法 HW4 E94116198 劉羿伶

1. Determine the values $\int_1^2 e^x \sin(4x)dx$ with $h = 0.1$ by

a. Use the composite trapezoidal rule

b. Use the composite Simpsons' method

c. Use the composite midpoint rule

$f(x) = e^x \sin(4x)$, $f(\overset{x_0}{1}) = -2.057$, $f(\overset{x_{10}}{2}) = 7.310$

$f(\overset{x_1}{1.1}) = -2.859$, $f(\overset{x_2}{1.2}) = -3.307$, $f(\overset{x_3}{1.3}) = -3.242$, $f(\overset{x_4}{1.4}) = -2.6$, $f(\overset{x_5}{1.5}) = -1.252$

$f(\overset{x_6}{1.6}) = 0.577$, $f(\overset{x_7}{1.7}) = 2.705$, $f(\overset{x_8}{1.8}) = 4.801$, $f(\overset{x_9}{1.9}) = 6.471$

$f'(x) = e^x \sin(4x) + 4e^x \cos(4x)$

$f''(x) = e^x \sin(4x) + 4e^x \cos(4x) + 4e^x \cos(4x) - 16e^x \sin(4x)$ $\qquad f''(\xi) = \max_{x\in[1,2]} |f''(x)| \approx 43.5$

$\Rightarrow f''(x) = 8e^x \cos(4x) - 15e^x \sin(4x)$

$f^{iv}(x) = 161e^x \sin(4x) - 240e^x \cos(4x)$ $\qquad f^{iv}(\xi) \approx 2963.19$

a. Composite trapezoidal rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{2n-1} f(x_i) + f(b)\right] - \frac{b-a}{12}h^2 f''(\xi)$$

$$\int_1^2 e^x \sin(4x)dx = \frac{0.1}{2}\left[e^1\sin(4) + 2\sum_{i=1}^{9} f(x_i) + e^2\sin(8)\right] - \frac{1}{12}\times(0.1)^2\times f''(\xi)$$

$$= 0.05[-2.057 + 2\times1.294 + 7.310] - \frac{1}{12}\times(0.1)^2\times 43.5 = 0.3961476 - \overset{誤差}{\underline{0.03625}} = 0.3598976 \ \#$$

b. Composite Simpson's method

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 2\sum_{i=2}^{n} f(x_{2i-2}) + 4\sum_{i=1}^{n} f(x_{2i-1}) + f(x_{2n})\right] - \frac{b-a}{180}h^4 f^{iv}(\xi)$$

$$\Rightarrow \int_1^2 e^x \sin(4x)dx = \frac{0.1}{3}\left[f(1) + 2\sum_{i=2}^{5} f(x_{2i-2}) + 4\sum_{i=1}^{5} f(x_{2i-1}) + f(x_{10})\right] - \frac{1}{180}(0.1)^4 f^{iv}(\xi)$$

$$= 0.3856636 - \overset{誤差}{\underline{0.01646}} = 0.38401 \ \#$$

c. Composite midpoint rule

$$\int_a^b f(x)dx = 2h[f(x_1) + f(x_3) + \ldots + f(x_{2n-1})] + \frac{b-a}{6}h^2 f''(\xi)$$

$$\Rightarrow \int_1^2 e^x \sin(4x)dx = 0.2[f(x_1) + f(x_3) + f(x_5) + f(x_7) + f(x_9)] + \frac{1}{6}\times(0.1)^2\times f''(\xi)$$

$$= 0.3646956 - \overset{誤差}{\underline{0.0725}} = 0.2921956 \ \#$$

2. Approximate $\int_{1}^{1.5} x^2 \ln x dx$ using Gaussian Quadrature with $n=3$ and $n=4$. Then compare the result to the exact value of the integral.

$x = \frac{1.5-1}{2} t + \frac{1.5+1}{2} = 0.25t + 1.25$, $dx = 0.25 dt$

$\int_{-1}^{1} \underbrace{((0.25t+1.25)^2 \ln(0.25t+1.25)) \cdot 0.25}_{f(t)} dt = \sum_{i=1}^{n} c_i f(x_i)$

$n=3$: $\int_{1}^{1.5} x^2 \ln x \, dx = 0.25\left[0.556\left(f(-0.775)+f(0.775)\right)+0.889 \times f(0)\right] = \underline{0.192229}$ #

$n=4$: $\int_{1}^{1.5} x^2 \ln x \, dx = 0.25\left[0.348\left(f(-0.861)+f(0.861)\right)+0.652\left(f(0.34)+f(-0.34)\right)\right] = \underline{0.1922603}$ #

Exact value: $\int_{1}^{1.5} x^2 \ln x \, dx = \int_{1}^{1.5} \ln x \, d\left(\frac{x^3}{3}\right) = \left[\frac{x^3}{3} \ln x \Big|_{1}^{1.5}\right] - \int_{1}^{1.5} \frac{x^3}{3} d(\ln x)$

$= 0.456148 - \int_{1}^{1.5} \frac{x^2}{3} dx = 0.456148 - \left[\frac{x^3}{9}\Big|_{1}^{1.5}\right] = 0.192259$

Compare:

Method	Value	error
$n=3$	0.192229	-0.00003
$n=4$	0.1922603	0.0000013
Exact	0.192259	

#

n=4的方法比n=3的方法更準確

3. Approximate $\int_{0}^{\pi/4} \int_{\sin x}^{\cos x} (2y \sin x + \cos^2 x) dy dx$ using

a. Simpson's rule for $n=4$ and $m=4$

b. Gaussian Quadrature, $n=3$ and $m=3$

c. Compare these results with the exact value.

a. Simpson's rule, $h = \frac{\frac{\pi}{4}-0}{2\times4} = \frac{\pi}{32}$, $k(x) = \frac{\cos x - \sin x}{8}$, $y_j = \sin x + j\frac{\cos x - \sin x}{8}$

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0	$\frac{\pi}{32}$	$\frac{\pi}{16}$	$\frac{3\pi}{32}$	$\frac{\pi}{8}$	$\frac{5\pi}{32}$	$\frac{3\pi}{16}$	$\frac{7\pi}{32}$	$\frac{\pi}{4}$
y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
0	0.210	0.392	0.540	0.653	0.728	0.762	0.756	0.707

$\int_{0}^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} (2y\sin x + \cos^2 x) dy dx$

$= \frac{\pi}{32} \times \frac{1}{9}\left[k_0 f(x_0,y_0) + 2\sum_{i=2}^{4} k_{2i-2} f(x_{2i-2},y_0) + 4\sum_{i=1}^{4} k_{2i-1} f(x_{2i-1},y_0) + k_8 f(x_8,y_0)\right]$

$+ \frac{\pi}{32} \times \frac{2}{9}\left[k_0 \sum_{j=2}^{4} f(x_0,y_{2j-2}) + 2\sum_{i=2}^{4} k_{2i-2}\left(\sum_{j=2}^{4} f(x_{2i-2},y_{2j-2})\right) + 4\sum_{i=1}^{4} k_{2i-1}\left(\sum_{j=2}^{4} f(x_{2i-1},y_{2j-2})\right) + k_8\left(\sum_{j=2}^{4} f(x_8,y_{2j-2})\right)\right]$

$+ \frac{\pi}{32} \times \frac{4}{9}\left[k_0 \sum_{j=1}^{4} f(x_0,y_{2j-1}) + 2\sum_{i=2}^{4} k_{2i-2}\left(\sum_{j=1}^{4} f(x_{2i-2},y_{2j-2})\right) + 4\sum_{i=1}^{4} k_{2i-1}\left(\sum_{j=1}^{4} f(x_{2i-1},y_{2j-1})\right) + k_8\left(\sum_{j=1}^{4} f(x_8,y_{2j-1})\right)\right]$

$+ \frac{\pi}{32} \times \frac{1}{9}\left[k_0 f(x_0,y_8) + 2\sum_{i=2}^{n} k_{2i-2} f(x_{2i-2},y_8) + 4\sum_{i=1}^{4} k_{2i-1} f(x_{2i-1},y_8) + k_8 f(x_8,y_8)\right]$

$= \underline{0.5119875}$ #

b. Gaussian Quadrature

$$x_i = \frac{\pi}{8}\xi_i + \frac{\pi}{8}$$

$$\int_0^{\frac{\pi}{4}}\int_{\sin x}^{\cos x}(2y\sin x+\cos^2 x)\,dy\,dx$$

$$= \frac{\pi}{8}\sum_{i=1}^{3} c_i \sum_{j=1}^{3} c_j G_1(\xi_i) f\left[\frac{\pi}{8}\xi_i+\frac{\pi}{8},\ G_1(\xi_i)\eta_j + G_2(\xi_i)\right] = \underline{0.5118655} \ \#$$

c. Exact Value

$$\int_0^{\frac{\pi}{4}}\int_{\sin x}^{\cos x}(2y\sin x+\cos^2 x)\,dy\,dx = \int_0^{\frac{\pi}{4}}\left[y^2\sin x + y\cos^2 x\Big|_{\sin x}^{\cos x}\right]dx$$

$$= \int_0^{\frac{\pi}{4}}(\cos^2 x\sin x + \cos^3 x - \sin^3 x - \sin x\cos^2 x)\,dx = 0.5118446$$

Compare:

Method	Value	Error
Simpson's Rule	0.5119875	0.0001429
Guassian Quadrature	0.5118655	0.0000209
Exact Value	0.5118446	

- (a) Simpson’s Rule 結果 = 0.5119875440121252
- (b) Gaussian Quadrature 結果 = 0.511865539945296
- (c) 精確值 = 0.511844635310913
- ➡ Simpson 誤差 = 0.000142908701212696
- ➡ Gaussian 誤差 = 2.09046343835340e-5

4. Use the composite Simpson’s rule and $n=4$ to approximate the improper integral a) $\int_0^1 x^{-1/4}\sin x\,dx$, b) $\int_1^{\infty} x^{-4}\sin x\,dx$ by use the transform $\underline{t=x^{-1}}$ $\Rightarrow x=\frac{1}{t}$, $dx=-\frac{1}{t^2}dt$

(a) $\int_0^1 x^{-\frac{1}{4}}\sin x\,dx = \int_{\infty}^{1}\left(\left(\frac{1}{t}\right)^{-\frac{1}{4}}\sin\left(\frac{1}{t}\right)\cdot\left(-\frac{1}{t^2}\right)\right)dt = \int_1^{\infty} t^{\frac{1}{4}}\sin\left(\frac{1}{t}\right)\left(\frac{1}{t^2}\right)dt = \int_1^{\infty} t^{-\frac{7}{4}}\sin\left(\frac{1}{t}\right)dt$

代100為∞ $\Rightarrow \int_1^{100} \underbrace{t^{-\frac{7}{4}}\sin\left(\frac{1}{t}\right)}_{f(t)}dt$ $\quad n=4,\ h=\frac{100-1}{4}=24.75$

i	t_i	$f(t_i)$
0	1	0.841471
1	25.75	0.003294
2	50.5	0.001110
3	75.25	0.000561
4	100	0.000316

$$\int_0^1 x^{-\frac{1}{4}}\sin x\,dx$$

$$= \frac{24.75}{3}\left[f(t_0)+4f(t_1)+2f(t_2)+4f(t_3)+f(t_4)\right]$$

$$= \underline{7.0903}\ \#$$

(b) $\int_1^{\infty} x^{-4}\sin x\,dx = \int_0^1 t^2\sin\left(\frac{1}{t}\right)dt$

$\Rightarrow \int_{0.01}^{1} \underbrace{t^2\sin\left(\frac{1}{t}\right)}_{f(t)}dt$ $\quad n=4,\ h=\frac{1-0.01}{4}=0.2475$

i	t_i	$f(t_i)$
0	0.01	-0.0000506
1	0.2575	0.0660
2	0.505	0.2475
3	0.7525	0.4871
4	1	0.8415

$$\int_1^{\infty} x^{-4}\sin x\,dx$$

$$= \frac{0.2475}{3}\left[f(t_0)+4f(t_1)+2f(t_2)+4f(t_3)+f(t_4)\right]$$

$$= \underline{0.2928}\ \#$$

Approximation for (a): 6.947084101379209
Approximation for (b): 0.27465825002428546