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1. $f[t_i, y(t_i)] = 1 + \frac{y(t_i)}{t_i} + \left(\frac{y(t_i)}{t_i}\right)^2 \qquad y(t_0) = 0$

a. $y(t_{i+1}) = y(t_i) + f[t_i, y(t_i)]\,\overset{0.1}{h}$ (Euler's method)

Actual value - Euler's method value

t_i		Euler's method $y(t_i)$	Exact $y(t_i) = t \times \tan(\ln t)$	Error
t_0	1	0	$1 \times \tan(\ln 1) = 0$	0
t_1	1.1	$0 + 0.1 \times (1 + \frac{0}{1} + (\frac{0}{1})^2) = 0.1$	$1.1 \times \tan(\ln 1.1) = 0.105$	0.005
t_2	1.2	$0.1 + 0.1 \times (1 + \frac{0.1}{1.1} + (\frac{0.1}{1.1})^2) = 0.21$	$1.2 \times \tan(\ln 1.2) = 0.221$	0.011
t_3	1.3	$0.21 + 0.1 \times (1 + \frac{0.21}{1.2} + (\frac{0.21}{1.2})^2) = 0.33$	$1.3 \times \tan(\ln 1.3) = 0.349$	0.019
t_4	1.4	$0.33 + 0.1 \times (1 + \frac{0.33}{1.3} + (\frac{0.33}{1.3})^2) = 0.462$	$1.4 \times \tan(\ln 1.4) = 0.489$	0.027
t_5	1.5	$0.462 + 0.1 \times (1 + \frac{0.462}{1.4} + (\frac{0.462}{1.4})^2) = 0.606$	$1.5 \times \tan(\ln 1.5) = 0.644$	0.038
t_6	1.6	$0.606 + 0.1 \times (1 + \frac{0.606}{1.5} + (\frac{0.606}{1.5})^2) = 0.763$	$1.6 \times \tan(\ln 1.6) = 0.813$	0.05
t_7	1.7	$0.763 + 0.1 \times (1 + \frac{0.763}{1.6} + (\frac{0.763}{1.6})^2) = 0.933$	$1.7 \times \tan(\ln 1.7) = 0.997$	0.064
t_8	1.8	$0.933 + 0.1 \times (1 + \frac{0.933}{1.7} + (\frac{0.933}{1.7})^2) = 1.119$	$1.8 \times \tan(\ln 1.8) = 1.199$	0.08
t_9	1.9	$1.119 + 0.1 \times (1 + \frac{1.119}{1.8} + (\frac{1.119}{1.8})^2) = 1.319$	$1.9 \times \tan(\ln 1.9) = 1.420$	0.101
t_{10}	2.0	$1.319 + 0.1 \times (1 + \frac{1.319}{1.9} + (\frac{1.319}{1.9})^2) = 1.537$	$2.0 \times \tan(\ln 2.0) = 1.661$	0.124

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b. $y(t_{i+1}) = y(t_i) + \overset{0.1}{h}\, T^{(2)}[t_i, y(t_i)]$ (Taylor's method of order 2)

$$T^{(2)} = f + \frac{h}{2}\left[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right] = f[t_i, y(t_i)] + \frac{h}{2}\left(-\frac{y(t_i)}{t_i^2} - 2\frac{y(t_i)^2}{t_i^3} + f[t_i, y(t_i)]\left(\frac{1}{t_i} + \frac{2y(t_i)}{t_i^2}\right)\right)$$

Actual - Taylor

t_i		Taylor method of order 2 $y(t_i)$	Exact	Error
t_0	1	0	0	0
t_1	1.1	$0 + 0.1\left[1 + \frac{0}{1.1} + \frac{0^2}{1.1^2} + \frac{0.1}{2}\left(-\frac{0}{1.1^2} - 2\frac{0^2}{1.1^3} + \left(1 + \frac{0}{1.1} + \frac{0^2}{1.1^2}\right)\left(\frac{1}{1.1} + \frac{2\times 0}{1.1^2}\right)\right)\right] = 0.105$	0.1051	0.0001
t_2	1.2	$0.105 + 0.1\left[1 + \frac{0.105}{1.2} + \frac{0.105^2}{1.2^2} + \frac{0.1}{2}\left(-\frac{0.105}{1.2^2} - 2\frac{0.105^2}{1.2^3} + \left(1 + \frac{0.105}{1.2} + \frac{0.105^2}{1.2^2}\right)\left(\frac{1}{1.2} + \frac{2\times 0.105}{1.2^2}\right)\right)\right] = 0.2209$	0.2212	0.0003
t_3	1.3	$0.2209 + 0.1\left[1 + \frac{0.2209}{1.3} + \frac{0.2209^2}{1.3^2} + \frac{0.1}{2}\left(-\frac{0.2209}{1.3^2} - 2\frac{0.2209^2}{1.3^3} + \left(1 + \frac{0.2209}{1.3} + \frac{0.2209^2}{1.3^2}\right)\left(\frac{1}{1.3} + \frac{2\times 0.2209}{1.3^2}\right)\right)\right] = 0.3486$	0.3491	0.0005
t_4	1.4	$0.3486 + 0.1\left[1 + \frac{0.3486}{1.4} + \frac{0.3486^2}{1.4^2} + \frac{0.1}{2}\left(-\frac{0.3486}{1.4^2} - 2\frac{0.3486^2}{1.4^3} + \left(1 + \frac{0.3486}{1.4} + \frac{0.3486^2}{1.4^2}\right)\left(\frac{1}{1.4} + \frac{2\times 0.3486}{1.4^2}\right)\right)\right] = 0.489$	0.4897	0.0007
t_5	1.5	$0.489 + 0.1\left[1 + \frac{0.489}{1.5} + \frac{0.489^2}{1.5^2} + \frac{0.1}{2}\left(-\frac{0.489}{1.5^2} - 2\frac{0.489^2}{1.5^3} + \left(1 + \frac{0.489}{1.5} + \frac{0.489^2}{1.5^2}\right)\left(\frac{1}{1.5} + \frac{2\times 0.489}{1.5^2}\right)\right)\right] = 0.6429$	0.6439	0.001
t_6	1.6	$0.6429 + 0.1\left[1 + \frac{0.6429}{1.6} + \frac{0.6429^2}{1.6^2} + \frac{0.1}{2}\left(-\frac{0.6429}{1.6^2} - 2\frac{0.6429^2}{1.6^3} + \left(1 + \frac{0.6429}{1.6} + \frac{0.6429^2}{1.6^2}\right)\left(\frac{1}{1.6} + \frac{2\times 0.6429}{1.6^2}\right)\right)\right] = 0.8114$	0.8127	0.0013
t_7	1.7	$0.8114 + 0.1\left[1 + \frac{0.8114}{1.7} + \frac{0.8114^2}{1.7^2} + \frac{0.1}{2}\left(-\frac{0.8114}{1.7^2} - 2\frac{0.8114^2}{1.7^3} + \left(1 + \frac{0.8114}{1.7} + \frac{0.8114^2}{1.7^2}\right)\left(\frac{1}{1.7} + \frac{2\times 0.8114}{1.7^2}\right)\right)\right] = 0.9958$	0.9975	0.0017
t_8	1.8	$0.9958 + 0.1\left[1 + \frac{0.9958}{1.8} + \frac{0.9958^2}{1.8^2} + \frac{0.1}{2}\left(-\frac{0.9958}{1.8^2} - 2\frac{0.9958^2}{1.8^3} + \left(1 + \frac{0.9958}{1.8} + \frac{0.9958^2}{1.8^2}\right)\left(\frac{1}{1.8} + \frac{2\times 0.9958}{1.8^2}\right)\right)\right] = 1.1972$	1.1994	0.0022
t_9	1.9	$1.1972 + 0.1\left[1 + \frac{1.1972}{1.9} + \frac{1.1972^2}{1.9^2} + \frac{0.1}{2}\left(-\frac{1.1972}{1.9^2} - 2\frac{1.1972^2}{1.9^3} + \left(1 + \frac{1.1972}{1.9} + \frac{1.1972^2}{1.9^2}\right)\left(\frac{1}{1.9} + \frac{2\times 1.1972}{1.9^2}\right)\right)\right] = 1.4173$	1.4201	0.0028
t_{10}	2	$1.4173 + 0.1\left[1 + \frac{1.4173}{2} + \frac{1.4173^2}{2^2} + \frac{0.1}{2}\left(-\frac{1.4173}{2^2} - 2\frac{1.4173^2}{2^3} + \left(1 + \frac{1.4173}{2} + \frac{1.4173^2}{2^2}\right)\left(\frac{1}{2} + \frac{2\times 1.4173}{2}\right)\right)\right] = 1.6578$	1.6613	0.0035

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=== (a) Euler's Method ===

	t	Euler	Exact	Error_Euler
0	1.000000	0.000000	0.000000	0.000000
1	1.100000	0.100000	0.105160	0.005160
2	1.200000	0.209917	0.221243	0.011325
3	1.300000	0.330471	0.349121	0.018651
4	1.400000	0.462354	0.489682	0.027328
5	1.500000	0.606285	0.643875	0.037590
6	1.600000	0.763041	0.812753	0.049711
7	1.700000	0.933475	0.997494	0.064019
8	1.800000	1.118537	1.199439	0.080902
9	1.900000	1.319293	1.420116	0.100823
10	2.000000	1.536943	1.661282	0.124338

=== (b) Taylor's Method (Order 2) ===

	t	Taylor	Exact	Error_Taylor
0	1.000000	0.000000	0.000000	0.000000
1	1.100000	0.105000	0.105160	0.000160
2	1.200000	0.220919	0.221243	0.000324
3	1.300000	0.348612	0.349121	0.000509
4	1.400000	0.488954	0.489682	0.000728
5	1.500000	0.642883	0.643875	0.000993
6	1.600000	0.811438	0.812753	0.001315
7	1.700000	0.995787	0.997494	0.001707
8	1.800000	1.197252	1.199439	0.002187
9	1.900000	1.417344	1.420116	0.002772
10	2.000000	1.657795	1.661282	0.003487

2.

$h = 0.05$

$k_{11} = hf_1(0, \frac{4}{3}, \frac{2}{3}) = (9\times\frac{4}{3} + 24\times\frac{2}{3} + 5\cos(0) - \frac{1}{3}\sin(0))\times 0.05 = 1.65$

$k_{12} = hf_2(0, \frac{4}{3}, \frac{2}{3}) = (-24\times\frac{4}{3} - 52\times\frac{2}{3} - 9\cos(0) + \frac{1}{3}\sin(0))\times 0.05 = -\frac{227}{60}$

$k_{21} = hf_1(0+\frac{0.05}{2}, \frac{4}{3}+\frac{1.65}{2}, \frac{2}{3}-\frac{227}{120}) = 0.05 f(\frac{1}{40}, \frac{259}{120}, -\frac{49}{40}) = -0.2507$

$k_{22} = hf_2(\frac{1}{40}, \frac{259}{120}, -\frac{49}{40}) = 0.1455$

$k_{31} = hf_1(\frac{1}{40}, \frac{4}{3}+\frac{k_{21}}{2}, \frac{2}{3}+\frac{k_{22}}{2}) = 1.6804$

$k_{32} = hf_2(\frac{1}{40}, 1.2079, 0.739) = -3.8215$

$k_{41} = hf_1(0.05, \frac{4}{3}+k_{31}, \frac{2}{3}+k_{32}) = -2.1808$

$k_{42} = hf_2(0.05, 3.0139, -3.155) = 4.1375$

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=== Results for h = 0.05 ===
     t        u1_num      u1_exact      u1_error        u2_num      u2_exact      u2_error
==========================================================================
  0.00      1.333333      1.333333      0.00e+00      0.666667      0.666667      1.11e-16
  0.05      1.721880      1.912059      1.90e-01     -0.499599     -0.909077      4.09e-01
  0.10      1.726915      1.793063      6.61e-02     -0.832598     -1.032002      1.99e-01
  0.15      1.617161      1.601967      1.52e-02     -0.890373     -0.961459      7.11e-02
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$u_1(0.05) = u_1(0) + \frac{1}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}) = 1.7214$

$u_2(0.05) = u_2(0) + \frac{1}{6}(k_{12} + 2k_{22} + 2k_{32} + k_{42}) = -0.4996$

Runge-Kutta method

$u_1(0.05) = 2e^{-3\times 0.05} - e^{-39\times 0.05} + \frac{1}{3}\cos(0.05) = 1.9125$

$u_2(0.05) = -e^{-3\times 0.05} + 2e^{-39\times 0.05} - \frac{1}{3}\cos(0.05) = -0.9095$

Exact Value

	Runge-Kutta	Exact Value	Error
$u_1(0.05)$	1.7214	1.9125	0.1911
$u_2(0.05)$	-0.4996	-0.9095	-0.4099

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$h = 0.1$

$k_{11} = hf_1(0, \frac{4}{3}, \frac{2}{3}) = 3.3$, $k_{12} = hf_2(0, \frac{4}{3}, \frac{2}{3}) = -\frac{227}{30}$

$k_{21} = hf_1(0+\frac{h}{2}, u_1+\frac{k_{11}}{2}, u_2+\frac{k_{12}}{2}) = 0.1 f(0+\frac{0.1}{2}, \frac{4}{3}+\frac{1}{2}\times 3.3, \frac{2}{3}-\frac{1}{2}\times\frac{227}{30}) = -4.297$

$k_{22} = hf_2(0.05, 2.9833, -3.1166) = 8.189$

$k_{31} = hf_1(0+\frac{h}{2}, u_1+\frac{k_{21}}{2}, u_2+\frac{k_{22}}{2}) = 11.191$

$k_{32} = hf_2(0.05, -0.8153, 4.7613) = -23.6997$

$k_{41} = hf_1(0+h, u_1+k_{31}, u_2+k_{32}) = -43.513$

$k_{42} = hf_2(0.1, 12.524, -23.033) = 88.821$

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=== Results for h = 0.1 ===
     t        u1_num      u1_exact      u1_error        u2_num      u2_exact      u2_error
==========================================================================
  0.00      1.333333      1.333333      0.00e+00      0.666667      0.666667      1.11e-16
  0.10     -3.052437      1.793063      4.85e+00      8.989305     -1.032002      1.00e+01
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$u_1(0.1) = u_1(0) + \frac{1}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}) = -3.071$

$u_2(0.1) = u_2(0) + \frac{1}{6}(k_{21} + 2k_{22} + k_{32} + k_{42}) = 9.0389$

Runge-Kutta method

$u_1(0.1) = 2e^{-3\times 0.1} - e^{-39\times 0.1} + \frac{1}{3}\cos(0.1) = 1.8352$

$u_2(0.1) = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos(0.1) = -1.0336$

Exact Value

	Runge-Kutta	Exact	Error
$u_1(0.1)$	-3.071	1.8352	4.9062
$u_2(0.1)$	9.0389	-1.0336	-10.0725

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