1. Use the Lagrange interpolating polynomials of degree one, two, three and four to approximate $\cos(0.750) = 0.7317$ if $\cos(0.698) = 0.7661$, $\cos(0.733) = 0.7432$, $\cos(0.768) = 0.7193$, $\cos(0.803) = 0.6946$.

Find the error bound.

⇒ ∵只有4個值
⇒ 算不出 Degree four #

① Degree 3

⇒ 共4個點找 $P_3(0.750)$

$\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \{x_2, f(x_2)\}, \{x_3, f(x_3)\}$

$\Rightarrow P_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$ where $L_i(x) = L_{4,i}(x) = \prod_{\substack{j \neq i \\ j=0}}^{4} \frac{x - x_j}{x_i - x_j}$

(1) Let $x_0 = 0.698$, $x_1 = 0.733$, $x_2 = 0.768$, $x_3 = 0.803$

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-0.733)(x-0.768)(x-0.803)}{(0.698-0.733)(0.698-0.768)(0.698-0.803)} = \frac{(x-0.733)(x-0.768)(x-0.803)}{-0.000257725}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0.698)(x-0.768)(x-0.803)}{(0.733-0.698)(0.733-0.768)(0.733-0.803)} = \frac{(x-0.698)(x-0.768)(x-0.803)}{0.00008575}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0.698)(x-0.733)(x-0.803)}{(0.768-0.698)(0.768-0.733)(0.768-0.803)} = \frac{(x-0.698)(x-0.733)(x-0.803)}{-0.00008575}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0.698)(x-0.733)(x-0.768)}{(0.803-0.698)(0.803-0.733)(0.803-0.768)} = \frac{(x-0.698)(x-0.733)(x-0.768)}{0.000257725}$$

(2) Approximate 0.750

$$\Rightarrow L_0(0.750) = \frac{(0.75-0.733)(0.75-0.768)(0.75-0.803)}{-0.000257725} = \frac{0.000016218}{-0.000257725} = -0.0630437378$$

$$L_1(0.750) = \frac{(0.75-0.698)(0.75-0.768)(0.75-0.803)}{0.00008575} = \frac{0.000049608}{0.00008575} = 0.578518954$$

$$L_2(0.750) = \frac{(0.75-0.698)(0.75-0.733)(0.75-0.803)}{-0.00008575} = \frac{-0.000046852}{-0.00008575} = 0.546379008$$

$$L_3(0.750) = \frac{(0.75-0.698)(0.75-0.733)(0.75-0.768)}{0.000257725} = \frac{-0.000015912}{0.000257725} = -0.0618542741$$

$$P_3(0.750) = (-0.06304373)(0.7661) + (0.5785189)(0.7432) + (0.546379)(0.7193) + (-0.06185423)(0.6946)$$
$$= 0.7317039556 \ \#$$

(3) Error bound:

$$|f(x) - P_n(x)| = \left| \frac{f^{(n+1)}(\varepsilon(x))}{(n+1)!} \prod_{k=0}^{n} (x-x_k) \right| \Rightarrow |f(x) - P_3(x)| = \left| \frac{f^{(4)}(\varepsilon(x))}{4!} \prod_{k=0}^{3} (x-x_k) \right|$$

$$\prod_{k=0}^{3} (x-x_k) = (0.75-0.698)(0.75-0.733)(0.75-0.768)(0.75-0.803) = 8.43336 \times 10^{-7}$$

$f(x) = \cos(x)$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$

∵ 在 $[0.698, 0.803]$ 区間内, $f^{(4)}(x)$ 嚴格遞減 ⇒ max 在 $f(0.698)$

∴ $f^{(4)}(\varepsilon(x)) = f(0.698) = 0.76612909$.

$$\varepsilon = |f(x) - P_3(x)| = \left| \frac{f^{(4)}(\varepsilon(x))}{4!} \prod_{k=0}^{3} (x-x_k) \right| \leq \frac{0.76612909}{24} \cdot 8.43336 \cdot 10^{-7} = 2.6921 \times 10^{-8} \ \#$$

② Degree 2

$\Rightarrow$ 用3個点找 $P_2(0.750)$ $\Rightarrow$ 有2種算法

$\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \{x_2, f(x_2)\}$

$\Rightarrow P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$ where $L_i(x) = L_{3,i}(x) = \prod_{\substack{j\neq i \\ j=0}}^{3} \frac{x-x_j}{x_i-x_j}$

a.

(1) 取 $x_0 = 0.698$, $x_1 = 0.733$, $x_2 = 0.768$

(2) $L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0.733)(x-0.768)}{0.00245} \Rightarrow L_0(0.75) = \frac{-0.000306}{0.00245} = -0.1248979592$

$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0.698)(x-0.768)}{-0.001225} \Rightarrow L_1(0.75) = \frac{-0.000936}{-0.001225} = 0.7640816327$

$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0.698)(x-0.733)}{0.00245} \Rightarrow L_2(0.75) = \frac{0.000884}{0.00245} = 0.3608163265$

$P_2(0.75) = (-0.1248979592)(0.7661) + (0.7640816327)(0.7432) + (0.3608163265)(0.7193)$

$= 0.7317163265$ #

(3) Error bound:

$|f(x) - P_2(x)| = \left| \frac{f'''(\xi(x))}{3!} \prod_{k=0}^{2} (x-x_k) \right|$

$\prod_{k=0}^{2}(x-x_k) = (0.75-0.698)(0.75-0.733)(0.75-0.768) = -1.5912\times10^{-5}$

From above, $f'''(x) = \sin x$ $\Rightarrow$ 嚴格遞增, 区間: $[0.698, 0.768]$ $\Rightarrow$ max在 $f'''(0.768) = 0.694698026$

$\varepsilon = |f(x) - P_2(x)| = \left| \frac{f'''(\xi(x))}{3!} \prod_{k=0}^{2}(x-x_k) \right|$

$\leq \frac{0.694698026}{6} \times 1.5912\times10^{-5} = 1.842339\times10^{-6}$ #

b.

(1) 取另3点 $x_0 = 0.733$, $x_1 = 0.768$, $x_2 = 0.803$

(2) $L_0(x) = \frac{(x-0.768)(x-0.803)}{(0.733-0.768)(0.733-0.803)} = \frac{(x-0.768)(x-0.803)}{0.00245} \Rightarrow L_0(0.75) = \frac{0.000954}{0.00245} = 0.3893877551$

$L_1(x) = \frac{(x-0.733)(x-0.803)}{(0.768-0.733)(0.768-0.803)} = \frac{(x-0.733)(x-0.803)}{-0.001225} \Rightarrow L_1(0.75) = \frac{-0.000901}{-0.001225} = 0.7355102041$

$L_2(x) = \frac{(x-0.733)(x-0.768)}{(0.803-0.733)(0.803-0.768)} = \frac{(x-0.733)(x-0.768)}{0.00245} \Rightarrow L_2(0.75) = \frac{-0.000306}{0.00245} = -0.1248979592$

$\Rightarrow P_2'(0.75) = (0.3893877551)(0.7432) + (0.7355102041)(0.7193) + (-0.1248979592)(0.6946) = 0.731691347$ #

(3) Error bound:

$\left|\prod_{k=0}^{2}(x-x_k)\right| = |(0.75-0.733)(0.75-0.768)(0.75-0.803)| = 1.6218\times10^{-5}$

区間 $[0.733, 0.803]$ $\Rightarrow$ max在 $f'''(0.803) = 0.7194429798$

$\varepsilon \leq \frac{0.7194429798}{6} \times 1.6218\times10^{-5} = 1.944654374\times10^{-6}$ #

③ Degree 1.

$\Rightarrow$ 2個点 $\{x_0, f(x_0)\}, \{x_1, f(x_1)\}$ 找 $P_1(0.750)$

$\Rightarrow P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$ where $L_i(x) = L_{2,i}(x) = \prod_{\substack{j\neq i \\ j=0}}^{2} \frac{x-x_j}{x_i-x_j}$

(1) 取 $x_0 = 0.733,\ x_1 = 0.768$

(2) $L_0(x) = \frac{(x-x_1)}{(x_0-x_1)} = \frac{(x-0.768)}{(0.733-0.768)} = \frac{(x-0.768)}{-0.035} \Rightarrow L_0(0.75) = \frac{(0.75-0.768)}{-0.035} = 0.5142857143$

$L_1(x) = \frac{(x-x_0)}{(x_1-x_0)} = \frac{(x-0.733)}{(0.768-0.733)} = \frac{(x-0.733)}{0.035} \Rightarrow L_1(0.75) = \frac{(0.75-0.733)}{0.035} = 0.4857142857$

$\Rightarrow P_1(0.75) = (0.5142857143)(0.7432) + (0.4857142857)(0.7193) = 0.7315914286$ #

(3) Error bound:

$|f(x) - P_1(x)| = \left|\frac{f''(\varepsilon(x))}{2!} \prod_{k=0}^{1} (x-x_k)\right| \Rightarrow \left|\prod_{k=0}^{1} (x-x_k)\right| = |(0.75-0.733)(0.75-0.768)| = 3.06\times10^{-4}$

From above, $f''(x) = -\cos x \Rightarrow$ 嚴格遞增的負数, 区間: $[0.733, 0.768] \Rightarrow \max|-\cos x|$ 在 0.733

$|f''(0.733)| = 0.7431704432$

$\therefore \varepsilon = |f(x) - P_1(x)| = \left|\frac{f''(\varepsilon(x))}{2!} \prod_{k=0}^{1} (x-x_k)\right|$

$\leq \frac{0.7431704432}{2} \times 3.06\times10^{-4} = 1.137050\times10^{-4}$ #

2. Use iterated inverse interpolation to find an approximation to the solution $x - e^{-x} = 0$ using the data $e^{-0.3} = 0.740818$, $e^{-0.4} = 0.670320$, $e^{-0.5} = 0.606531$, $e^{-0.6} = 0.548812$.

$x_0 = 0.3$, $x_1 = 0.4$, $x_2 = 0.5$, $x_3 = 0.6$

$f[x_0] = 0.3 - e^{-0.3} = -0.440818$

$f[x_1] = 0.4 - e^{-0.4} = -0.27032$

$f[x_2] = 0.5 - e^{-0.5} = -0.106531$

$f[x_3] = 0.6 - e^{-0.6} = 0.051188$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{-0.27032 - (-0.440818)}{0.4 - 0.3} = 1.70498$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{-0.106531 - (-0.27032)}{0.5 - 0.4} = 1.63789$$

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = 1.57719$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1.63789 - 1.70498}{0.5 - 0.3} = -0.33545$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{1.57719 - 1.63789}{0.6 - 0.4} = -0.3035$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-0.3035 - (-0.33545)}{0.6 - 0.3} = 0.1065$$

$$P_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$= -0.440818 + (1.70498)(x - 0.3) + (-0.33545)(x - 0.3)(x - 0.4) + (0.1065)(x - 0.3)(x - 0.4)(x - 0.5)$$

$$= -0.440818 + 1.70498x - 0.511494 - 0.33545x^2 + 0.234815x - 0.040254 + 0.1065x^3 - 0.1278x^2 + 0.050055x - 0.00639$$

$$= 0.1065x^3 - 0.46325x^2 + 1.98985x - 0.998956$$

Newton's method ⇒ 取 0.6 開始.

$P_3'(x) = 0.3195x^2 - 0.9265x + 1.98985$

$$x_1^* = 0.6 - \frac{0.051188}{1.54897} = 0.5669535$$

$$x_2^* = 0.5669535 - \frac{-3.003949781 \times 10^{-4}}{1.990128346} = 0.567104425$$

$$x_3^* = x_2^* - \frac{P_3(x_2^*)}{P_3'(x_2^*)} = 0.5671451984$$

$$x_4^* = x_3^* - \frac{P_3(x_3^*)}{P_3'(x_3^*)} = 0.5671451987$$ #

解

$P_3(x) = 0 \Rightarrow x = 0.567145$, $1.89131 \pm 3.6i$ #

3. A car travelling along a straight road is clocked at a number of points. The data from the observations are given in the following table, where the time ***T*** is in seconds, the distance ***D*** is in feet, and the speed ***V*** is in feet per second.

		x_0	x_1	x_2	x_3	x_4
	T	0	3	5	8	13
$f(x)$	D	0	200	375	620	990
$f'(x)$	V	75	77	80	74	72

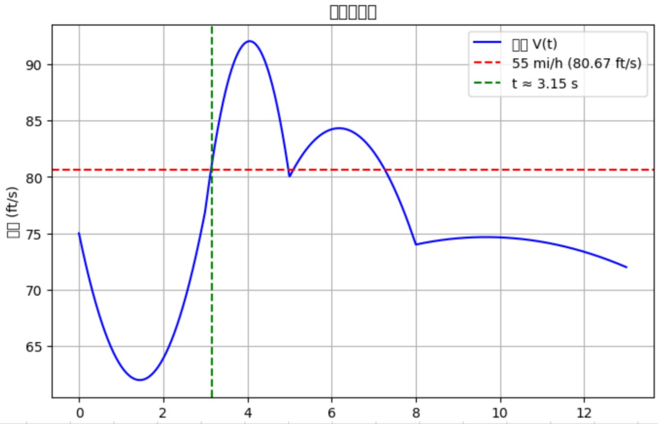
a. Use a Hermite polynomial to predict the position of the car and its speed when $t = 10$ s.

b. Use the derivative of the Hermite polynomial to determine whether the car ever exceeds a 55 mi/h speed limit on the road. If so, what is the first time the car exceeds this speed?

c. What is the predicted maximum speed for the car ?

程式碼執行

```
(a) D(10) = 768.96 ft
(a) V(10) = 74.64 ft/s
(b) 車速首次超過 55 mi/h 時間: t ≈ 3.15 秒
(c) 預測最大車速: 92.04 ft/s
(c) 最大車速發生在 t ≈ 4.06 秒
```



a. distance → $f(x)$.

x	z	$f(z)$	1st divided difference	2nd	3rd	4th	5th	6th	7th	8th
x_0	$z_0=0$	$f(x_0)=0$	$f'(x_0)=75$	$f[z_0,z_1,z_2]=\frac{66.\bar{6}-75}{3-0}=-\frac{25}{9}$	$f[z_0\sim z_3]=\frac{56}{27}$	$f[z_0,\dots z_4]=-\frac{37}{106}$	$f[z_0\sim z_5]=-\frac{1189}{9540}$	$f[z_0\sim z_6]=\frac{403}{8480}$	$f[z_0\sim z_7]=-0.01447054071$	$f[z_0\sim z_8]=0.001721223107$
x_0	$z_1=0$	$f(x_0)=0$	$f[x_0,x_1]=\frac{200-0}{3-0}=66.\bar{6}$	$f[z_1,z_2,z_3]=\frac{77-66.\bar{6}}{3-0}=\frac{31}{9}$	$f[z_1\sim z_4]=\frac{13}{36}$	$f[z_1,\dots z_5]=-\frac{35}{36}$	$f[z_1\sim z_6]=\frac{23}{90}$	$f[z_1\sim z_7]=-\frac{737}{10800}$	$f[z_1\sim z_8]=7.905359687\times10^{-3}$	$f[z_1\sim z_9]=-9.370620344\times10^{-4}$
x_1	$z_2=3$	$f(x_1)=200$	$f'(x_1)=77$	$f[z_2,z_3,z_4]=\frac{87.5-77}{5-3}=5.25$	$f[z_2\sim z_5]=-4.5$	$f[z_2,\dots z_6]=\frac{193}{180}$	$f[z_2\sim z_7]=-\frac{196}{675}$	$f[z_2\sim z_8]=0.03452893519$	$f[z_2\sim z_9]=0.004276444676$	
x_1	$z_3=3$	$f(x_1)=200$	$f[x_1,x_2]=\frac{375-200}{5-3}=87.5$	$f[z_3,z_4,z_5]=\frac{80-87.5}{5-3}=-3.75$	$f[z_3\sim z_6]=\frac{31}{36}$	$f[z_3,\dots,z_7]=-\frac{41}{108}$	$f[z_3\sim z_8]=\frac{949}{17280}$	$f[z_3\sim z_9]=-8.235532407\times10^{-3}$		
x_2	$z_4=5$	$f(x_2)=375$	$f'(x_2)=80$	$f[z_4,z_5,z_6]=\frac{5}{9}$	$f[z_4\sim z_7]=-\frac{28}{27}$	$f[z_4,\dots,z_8]=\frac{293}{1728}$	$f[z_4\sim z_9]=-0.02743634259$			
x_2	$z_5=5$	$f(x_2)=375$	$f[x_2,x_3]=\frac{620-375}{8-5}=81.\bar{6}$ $\frac{245}{3}$	$f[z_5,z_6,z_7]=-\frac{23}{9}$	$f[z_5\sim z_8]=\frac{23}{72}$	$f[z_5,\dots,z_9]=-\frac{719}{14400}$				
x_3	$z_6=8$	$f(x_3)=620$	$f'(x_3)=74$	$f[z_6,z_7,z_8]=0$	$f[z_6\sim z_9]=-0.08$					
x_3	$z_7=8$	$f(x_3)=620$	$f[x_3,x_4]=\frac{990-620}{13-8}=74$	$f[z_7,z_8,z_9]=-0.4$						
x_4	$z_8=13$	$f(x_4)=990$	$f'(x_4)=72$							
x_4	$z_9=13$	$f(x_4)=990$								

9th

$\Rightarrow f[z_0\sim z_9] = -2.044834724\times10^{-4}$

$$H_9(x)=f[z_0]+\sum_{k=1}^{9} f[z_0,\cdots,z_k](x-z_0)\cdots(x-z_{k-1})$$

$$=0+75(x-0)-\frac{25}{9}(x-0)^2+\frac{56}{27}(x-0)^2(x-3)-\frac{37}{106}(x-0)^2(x-3)^2-\frac{1189}{9540}(x-0)^2(x-3)^2(x-5)+\frac{403}{8480}(x-0)^2(x-3)^2(x-5)^2$$

$$-0.01447054071(x-0)^2(x-3)^2(x-5)^2(x-8)+0.001721223107(x-0)^2(x-3)^2(x-5)^2(x-8)^2-0.0469349824 5(x-0)^2(x-3)^2(x-5)^2(x-8)^2(x-13)$$

$H_9(10)=580.5318891$ (手算答案)

用程式碼估算 ⇒ 768.96 feet #

$V(10)=H_9'(10)=74.64$ ft/s #

b. 已知 $V(x)$，且 $55\,mi/h = 55\,\frac{mi}{h}\times\frac{5280\,ft}{1\,mi}\times\frac{1\,h}{3600\,s}=80.67\,ft/s$

⇒ 找 $V(x)=80.67$ 的点 ⇒ 約 $t\approx3.15$ s. #

c. Maximum ⇒ 92.04 ft/s #

($t\approx4.06$ s)