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1. Determine the values $\int_1^2 e^x \sin(4x)dx$ with $h = 0.1$ by

a. Use the composite trapezoidal rule

b. Use the composite Simpsons' method

c. Use the composite midpoint rule

$x = x_1 + sh$

$= x_1 + 0.1s$

$\frac{2-1}{0.1} = 10$

1. a. $\int_a^b f(x)\,dx = \frac{h}{2}[f(a) + 2\sum_{i=1}^{2n-1} f(x_i) + f(b)]$

$= \frac{0.1}{2}[e^1\sin(4) + 2\sum_{i=1}^{9}\left(e^{1+0.1i}\sin(4+0.4i)\right) + e^2\sin(8)] = 0.3961425922$ #

b. $\int_a^b f(x)\,dx = \int_{x_0}^{x_{2n}} f(x)\,dx = \sum_{i=1}^{n}\int_{x_{2(i-1)}}^{x_{2i}} f(x)\,dx = \sum_{i=1}^{n}\frac{h}{3}[f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})]$

$= \frac{h}{3}\{[f(x_0) + 4f(x_1) + \boxed{f(x_2)] + [f(x_2)} + 4f(x_3) + \boxed{f(x_4)] + [f(x_4)} + f(x_5) + f(x_6)] + \cdots + [f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]\}$

$= \frac{h}{3}[f(x_0) + 2\sum_{i=2}^{n} f(x_{2n-2}) + 4\sum_{i=1}^{n} f(x_{2n-1}) + f(x_{2n})]$

$x_{2i-2} \Rightarrow s = 2i-2$ $\quad s = 2i-1$ $\quad s = 2i$

$= \frac{0.1}{3}[-2.05720247] + 2\times(-0.4886126538) + 4\times(1.823478018) + 7.310423586]$

$= 0.385663596$ #

c. $\int_a^b f(x)\,dx = 2h[f(x_1) + f(x_3) + \cdots + f(x_{2n-1})]$

$x_3 = x_1 + 0.1\times 2$
$= 1 + 0.2$

$x_5 = x_1 + 0.1\times 4$
$= 1 + 0.4$

$= 2\times 0.1[e^{1+0.1}\sin(4.4) + e^{1+0.3}\sin(5.2) + \cdots + e^{1+0.9}\sin(7.6)]$

$= 0.2[\sum_{i=1}^{9} e^{1+(2i-1)0.1}\sin(4+4(2i-1)0.1)]$

$2i-1 = 9 \Rightarrow i = 5$

$4(2i-1)0.1$: $4 - 0.4 + 0.8i = 3.6 + 0.8i$

$= 0.2[1.823478018] = 0.3646956036$ #

Actual value $= 0.3859357293$

2. Approximate $\int_1^{1.5} x^2 \ln x dx$ using Gaussian Quadrature with $n=3$ and $n=4$. Then compare the result to the exact value of the integral.

$n=3 \Rightarrow x_1=-0.775,\ x_2=0,\ x_3=0.775$

$C_1=0.556=C_3,\ C_2=0.889$

$$\int_1^{1.5} x^2 \ln x\, dx$$

$$= \frac{1.5-1}{2}\sum_{i=1}^{3} C_i x_i^2 \ln x_i, \text{ where } x_i = \frac{1.5-1}{2}\eta_i + \frac{1.5+1}{2} = 0.25\eta_i + 1.25$$

$\times \ln(0.25\times 0+1.25)$ $\times \ln(0.25\times 0.775+1.25)$

$$= 0.25\left\{(0.556)[(0.25)(-0.775)+1.25]^2 \ln(0.25\times(-0.775)+1.25) + (0.889)[(0.25)(0)+1.25]^2 + (0.556)[(0.25)(0.775)+1.25]^2\right\}$$

$$= 0.25\left\{\underbrace{\boxed{0.6203092188 \times 0.05472489969}}_{=0.03394635978} + \underbrace{\boxed{1.3890625 \times 0.2231435513}}_{=0.3099603392} + \underbrace{\boxed{1.158934219 \times 0.3672438953}}_{=0.425611517}\right\} = 0.192379554$$

\#

$n=4 \Rightarrow$ $n=4$, $x_1=-x_4=-0.861$, $x_2=-x_3=0.340$, $c_1=c_4=0.348$, $c_2=c_3=0.652$

$x_1=(0.25)(-0.861)+1.25=1.03475$

$x_2=(0.25)(0.340)+1.25=1.335$

$x_3=(0.25)(-0.340)+1.25=1.165$

$x_4=(0.25)(0.861)+1.25=1.46525$

$$\int_1^{1.5} x^2 \ln x\, dx$$

$$= \frac{1.5-1}{2}\sum_{i=1}^{4} C_i x_i^2 \ln x_i, \text{ where } x_i = \frac{1.5-1}{2}\eta_i + \frac{1.5+1}{2} = 0.25\eta_i + 1.25$$

$$= (0.25)\{(0.348)(1.03475)^2\ln(1.03475) + (0.652)(1.335)^2\ln(1.335)$$

$$+ (0.652)(1.165)^2\ln(1.165) + (0.348)(1.46525)^2\ln(1.46525)\}$$

$$= 0.192260309$$

\#

Actual value $= 0.1922593577$

$n=3 \Rightarrow$ Absolute error

$\Delta y = y - fl(y)$

$= 0.1922593577 - 0.192379554$

$= -1.19245\times 10^{-4}$ \#

Relative error

$\varepsilon = \left|\frac{\Delta y}{y}\right|$

$= 6.2022989\times 10^{-4}$

\#

$n=4 \Rightarrow$ Absolute error

$\Delta y = y - fl(y)$

$= 0.1922593577 - 0.192260309$

$= -9.513\times 10^{-7}$ \#

Relative error

$\varepsilon = \left|\frac{\Delta y}{y}\right|$

$= 4.94800363\times 10^{-6}$

\#

3. Approximate $\int_0^{\pi/4}\int_{\sin x}^{\cos x}(2y\sin x+\cos^2 x)dydx$ using

a. Simpson's rule for $n=4$ and $m=4$

b. Gaussian Quadrature, $n=3$ and $m=3$

c. Compare these results with the exact value.

3.

a. Simpson's $\overset{n=4}{\underset{m=4}{}}$ $\frac{h}{3}\left[f(x_0)+4\sum_{j=1}^{m}f(x_{2j-1})+2\sum_{j=2}^{m}f(x_{2j-2})+f(x_{2m})\right]$

$y_0=\sin(x)$ $\quad m=4$

$y_{2m}=\cos(x)$ $\quad \Rightarrow y_0\sim y_8$

$y_j=y_0+j\cdot\frac{[\cos(x)-\sin(x)]}{8}$, $\quad k(x)=\frac{\cos(x)-\sin(x)}{2m}=\frac{\cos x-\sin x}{8}$

$\int_{y_0}^{y_{2m}}f(x,y)\,dy$

$=\frac{k(x)}{3}\left[f(x,y_0)+2\sum_{j=2}^{m}f(x,y_{2j-2})+4\sum_{j=1}^{m}f(x,y_{2j-1})+f(x,y_{2m})\right]$

$=\frac{\cos x-\sin x}{24}\left\{\left[\sin x+2\left(3\sin x+\frac{12}{8}\cos x-\frac{12}{8}\sin x\right)+4\left(3\sin x+\frac{16}{8}\cos x-\frac{16}{8}\sin x\right)+\cos x\right]\cdot 2\sin x+24\cos^2 x\right\}$

$=\frac{\cos x-\sin x}{24}\left[(8\sin x+12\cos x)\cdot 2\sin x+24\cos^2 x\right]=\frac{(\cos x-\sin x)(16\sin^2 x+24\sin x\cos x+24\cos^2 x)}{24}$

$=g(x)$

$=\frac{(\cos x-\sin x)(16+12\sin 2x+8\cos^2 x)}{24}$

$=\frac{(\cos x-\sin x)(4+3\sin 2x+2\cos^2 x)}{6}$

$x_0=0$ $\quad n=4\Rightarrow x_0\sim x_8$

$x_{2n}=\frac{\pi}{4}$ $\quad x_i=x_0+i\cdot\frac{\frac{\pi}{4}-0}{8}=\frac{\pi}{32}i$, $\quad h=\frac{\frac{\pi}{4}-0}{8}=\frac{\pi}{32}$

$\int_0^{\frac{\pi}{4}}g(x)\,dx$

$=\frac{h}{3}\left[g(x_0)+2\sum_{i=2}^{n}g(x_{2i-2})+4\sum_{i=1}^{n}g(x_{2i-1})+g(x_{2n})\right]$

$=\frac{\pi}{96}\left\{g(0)+2\left[g\left(\frac{2\pi}{32}\right)+g\left(\frac{4\pi}{32}\right)+g\left(\frac{6\pi}{32}\right)\right]+4\left[g\left(\frac{\pi}{32}\right)+g\left(\frac{3}{32}\pi\right)+g\left(\frac{5}{32}\pi\right)+g\left(\frac{7}{32}\pi\right)\right]+g\left(\frac{\pi}{4}\right)\right\}$

$=\frac{\pi}{96}[1+2(0.926063269+0.706119038+0.374962064)+4(0.981808806+0.833117389+0.550785642$
$+0.187998292)+0]$

$=\frac{\pi}{96}(15.2291293)=0.498372091$ #

Actual value $=0.5118446353$11

$\Rightarrow$ Absolute error $\delta y=y-y^*$

$=0.5118446353$11 -0.498372091

$=1.34725443\times10^{-2}$ #

Relative error $\varepsilon=\left|\frac{\delta y}{y}\right|$

$=\left|\frac{1.34725443\times10^{-2}}{0.5118446353{11}}\right|=2.63215503\times10^{-2}$ #

b. Gaussian $\Rightarrow n=3, m=3 \Rightarrow$ η : $0.7746,\ 0,\ 0.7746$

c : $0.5556,\ 0.8889,\ 0.5556$

$$\Rightarrow \int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} C_i f(x_i),\quad x_i = \eta_i \cdot \frac{b-a}{2} + \frac{b+a}{2}$$

$$\int_{\sin x}^{\cos x} (2y\sin x + \cos^2 x)\,dy$$

$$= \frac{\cos x - \sin x}{2} \sum_{j=1}^{m} C_j (2y_j \sin x + \cos^2 x),\quad y_j = \eta_j \cdot \frac{\cos x - \sin x}{2} + \frac{\cos x + \sin x}{2}$$

$$= \frac{\cos x - \sin x}{2} \Big\{ (0.5556)\Big[2\Big(-0.7746 \cdot \frac{\cos x - \sin x}{2} + \frac{\cos x + \sin x}{2}\Big)\sin x + \cos^2 x\Big]$$
$$+ (0.8889)\Big[2\Big(0 \cdot \frac{\cos x - \sin x}{2} + \frac{\cos x + \sin x}{2}\Big)\sin x + \cos^2 x\Big]$$
$$+ (0.5556)\Big[2\Big(0.7746 \cdot \frac{\cos x - \sin x}{2} + \frac{\cos x + \sin x}{2}\Big)\sin x + \cos^2 x\Big] \Big\}$$

$$= \frac{\cos x - \sin x}{2} \Big[(0.12523224 \sin x\cos x + 0.98596776 \sin^2 x + 0.5556\cos^2 x)$$
$$+ (0.8889 \sin x\cos x + 0.8889\sin^2 x + 0.8889\cos^2 x)$$
$$+ (0.98596776 \sin x\cos x + 0.12523224 \sin^2 x + 0.5556\cos^2 x) \Big]$$

$$= \frac{\cos x - \sin x}{2} \left(2\sin x\cos x + 2\sin^2 x + 2\cos^2 x\right) = g(x)$$

$$\int_0^{\frac{\pi}{4}} g(x)\,dx$$

$$= \frac{\frac{\pi}{4} - 0}{2} \sum_{i=1}^{n} C_i\, g(x_i),\quad x_i = \xi_i \cdot \frac{\frac{\pi}{4} - 0}{2} + \frac{\frac{\pi}{4} + 0}{2} = \xi_i \cdot \frac{\pi}{8} + \frac{\pi}{8}$$

$$x_1 = -0.7746 \cdot \frac{\pi}{8} + \frac{\pi}{8} = 0.088514373$$
$$x_2 = 0 \cdot \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{8} = 0.392699082$$
$$x_3 = 0.7746 \cdot \frac{\pi}{8} + \frac{\pi}{8} = 0.221825\pi = 0.69688379$$

$$= \frac{\pi}{8}\Big[(0.5556)\,g(x_1) + (0.8889)\,g(x_2) + (0.5556)\,g(x_3)\Big]$$

$$= \frac{\pi}{8}\Big[(0.5556)(0.98761062) + (0.8889)(0.73253781(5) + (0.5556)(0.18654534\mathrm{l})$$

$$= \frac{\pi}{8} \times 1.30351392 = 0.511888719$$

#

Actual value $= 0.5118446353\mathrm{l}\mathrm{l}$

$\Rightarrow$ Absolute error $\delta y = y - y^*$

$$= 0.511844635311 - 0.511888719$$

$$= -4.4083689 \times 10^{-5}$$

#

Relative error $\varepsilon = \left|\frac{\delta y}{y}\right|$

$$= \left|\frac{-4.4083689\times 10^{-5}}{0.511844635311}\right| = 8.61270901 \times 10^{-5}$$

#

4. Use the composite Simpson's rule and $n=4$ to approximate the improper integral a) $\int_0^1 x^{-1/4}\sin x\,dx$, b) $\int_1^\infty x^{-4}\sin x\,dx$ by use the transform $t=x^{-1}$

(a) $\int_0^1 x^{-\frac{1}{4}}\sin x\,dx$

$\rightarrow f(x)=\dfrac{g(x)}{(x-a)^p}$

$P_4(x)=g(a)+g'(a)(x-a)+\dfrac{g''(a)}{2!}(x-a)^2+\dfrac{g'''(a)}{3!}(x-a)^3+\dfrac{g^{(4)}(a)}{4!}(x-a)^4$

$\int_a^b f(x)\,dx$

$=\int_a^b \dfrac{g(x)-P_4(x)}{(x-a)^p}\,dx+\int_a^b \dfrac{P_4(x)}{(x-a)^p}\,dx$

$G(x)=\begin{cases}\dfrac{g(x)-P_4(x)}{(x-a)^p}\,dx & \text{if } a<x\le b\\ 0 & \text{if } x=a\end{cases}$, $h=\dfrac{b-a}{n}$

$\Rightarrow \int_a^b G(x)\,dx \approx \dfrac{h}{3}[G(x_0)+4G(x_1)+2G(x_2)+4G(x_3)+\cdots+4G(x_{n-1})+G(x_n)]$.

$\Rightarrow P_4(x)=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}$ (Taylor expansion for $\sin x$ about $x=0$)

$\int_0^1 \dfrac{P_4(x)}{x^{\frac{1}{4}}}\,dx$

$=\int_0^1 \dfrac{x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}}{x^{\frac{1}{4}}}\,dx$

$=\int_0^1 \left(x^{\frac{3}{4}}-\frac{1}{6}x^{\frac{11}{4}}+\frac{1}{120}x^{\frac{19}{4}}-\frac{1}{5040}x^{\frac{27}{4}}\right)dx$

$=\lim\limits_{M\to 0^+}\left[\frac{4}{7}x^{\frac{7}{4}}-\frac{1}{6}\cdot\frac{4}{15}x^{\frac{15}{4}}+\frac{1}{120}\cdot\frac{4}{23}x^{\frac{23}{4}}-\frac{1}{5040}\cdot\frac{4}{31}x^{\frac{31}{4}}\right]\Big|_M^1=\frac{4}{7}-\frac{4}{90}+\frac{4}{2760}-\frac{4}{156240}\approx \underline{0.5284078007}$

$G(x)=\begin{cases}\frac{1}{x^{\frac{1}{4}}}(\sin x-P_4(x)) & \text{if } 0<x\le 1\\ 0 & \text{if } x=0.\end{cases}$

$h=\frac{1-0}{4}=0.25 \Rightarrow x=0\rightarrow G(0)=0$

$x=0.25\rightarrow G(0.25)=1.48568456\times10^{-11}$

$x=0.5\rightarrow G(0.5)=6.38613198\times10^{-9}$

$x=0.75\rightarrow G(0.75)=2.212098824\times10^{-7}$

$x=1\rightarrow G(1)=2.730839641\times10^{-6}$

$\int_0^1 G(x)\,dx\approx\frac{0.25}{3}[0+4(1.48568456\times10^{-11}+2.212098824\times10^{-7})+2(6.386131398\times10^{-9})+2.730839641\times10^{-6}]$

$=\underline{3.023759051\times10^{-7}}$

$\int_0^1 x^{-\frac{1}{4}}\sin x\,dx\approx 0.5284078007+0.0000003023759051$

$=0.5284081031$ #

(b) $\int_1^\infty x^{-4}\sin x\,dx$

$\Rightarrow t = x^{-1}$, $\quad dt = -x^{-2}dx \Rightarrow dx = -x^2 dt = -t^{-2}dt$

$\Rightarrow \int_1^\infty x^{-4}\sin x\,dx$

$= \int_{t=1}^{t=0} t^4 \sin\left(\frac{1}{t}\right)\cdot\left(-\frac{1}{t^2}\right)dt$

$= \int_0^1 \underbrace{t^2\sin\left(\frac{1}{t}\right)}_{f(t)} dt$

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

$t=0 \rightarrow f(0)=0$

$t=0.25 \rightarrow f(0.25) = -0.0473001596$

$t=0.5 \rightarrow f(0.5) = 0.2273243567$

$t=0.75 \rightarrow f(0.75) = 0.5467150695$

$t=1 \rightarrow f(1) = 0.8414709848$

$\int_0^1 f(t)\,dt$

$\approx \frac{0.25}{3}\left[0 + 4(-0.0473001596 + 0.5467150695) + 2(0.2273243567) + 0.8414709848\right]$

$= 0.2744816127$ #