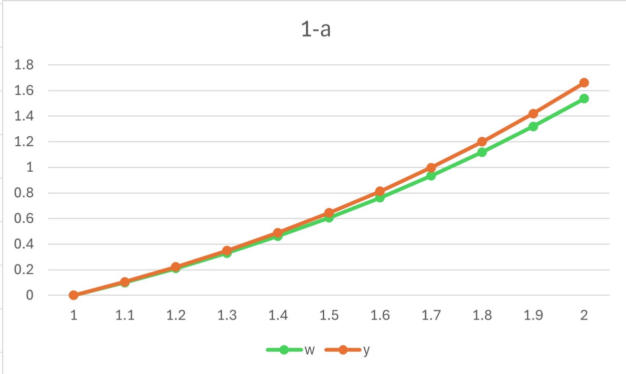
1. The initial-value problem

$y' = 1 + (y/t) + (y/t)^2$, $1 \le t \le 2$, $y(1) = 0$ has the exact solution $y(t) = t\tan(\ln t)$.

a. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

b. Use Taylor's method of order 2 with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.



1\. a. $h=0.1 = t_{i+1} - t_i$, $y(t_{i+1}) = y(t_i) + f[t_i, y(t_i)]h = y(t_i) + \frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i} h$ —(1)

$\Rightarrow t_{i+1} = t_i + 0.1$ $\qquad \Rightarrow y(t_0) = y(1) = 0$

$t_i = 1 + 0.1i$

$f(t,y) = 1 + (\frac{y}{t}) + (\frac{y}{t})^2 \Rightarrow f(t_i, y_i) = 1 + \frac{y_i}{t_i} + (\frac{y_i}{t_i})^2 \Rightarrow f(t_i, w_i) = 1 + \frac{w_i}{1+0.1i} + (\frac{w_i}{1+0.1i})^2$ —(2)

(1)&(2)

$w_{i+1} = w_i + (0.1)[1 + \frac{w_i}{1+0.1i} + (\frac{w_i}{1+0.1i})^2]$, $i = 1, 2, 3, \cdots, 9, 10$

$\therefore$ By $\begin{cases} t_i = t_0 + ih = 1 + 0.2i, \quad i = 0, 1, 2, \cdots, 10 \\ w_{i+1} = w_i + (0.1)[1 + \frac{w_i}{1+0.1i} + (\frac{w_i}{1+0.1i})^2] \end{cases}$

i	t_i	w_i (approximate)	y_i (exact)	Δy absolute error	ε relative error
0	1.0	0	0	0	0
1	1.1	0.1	0.1052	0.00520	0.0494
2	1.2	$\frac{301}{1440}$=0.2099	0.2212	0.0113	0.0511
3	1.3	0.3305	0.3491	0.0186	0.0533
4	1.4	0.4624	0.4897	0.0273	0.0557
5	1.5	0.6063	0.6439	0.0376	0.0584
6	1.6	0.7630	0.8128	0.0498	0.0613
7	1.7	0.9335	0.9975	0.0640	0.0642
8	1.8	1.1185	1.1994	0.0809	0.0675
9	1.9	1.3193	1.4201	0.101	0.0710
10	2.0	1.5369	1.6613	0.124	0.0749

b. Taylor's $\rightarrow$ $y(t_{i+1}) = y(t_i) + hT^{(n)}[t_i, y(t_i)]$, $y' = 1 + (\frac{y}{t}) + (\frac{y}{t})^2$

Order 2 $\rightarrow$ $T^{(2)} = f + \frac{h}{2!}\frac{df}{dt} = f + \frac{h}{2}[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}]$

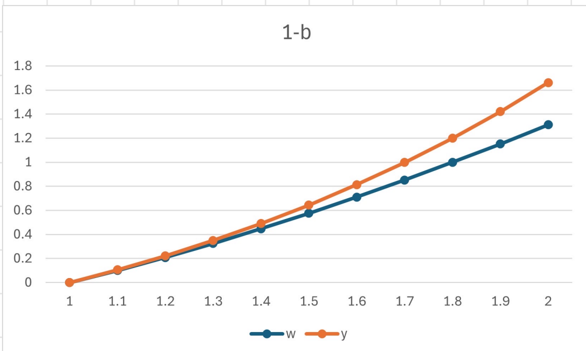
$h = 0.1$

令 $w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$

Let $f(t,y) = 1 + \frac{y}{t} + (\frac{y}{t})^2$

$\frac{df}{dt} = \frac{\partial f}{\partial t} + f\frac{df}{dy}$

$= -\frac{y}{t^2} - \frac{2y^2}{t^3} + [1 + \frac{y}{t} + (\frac{y}{t})^2][\frac{1}{t} + \frac{2y}{t^2}]$



$= -\cancel{\frac{y}{t^2}} - \cancel{\frac{2y^2}{t^3}} + \frac{1}{t} + \frac{2y}{t^2} + \cancel{\frac{y}{t^2}} + \cancel{\frac{2y^2}{t^3}} + \frac{y^2}{t^3} + \frac{2y^3}{t^4} = \frac{1}{t} + \frac{2y}{t^2} + \frac{y^2}{t^3} + \frac{2y^3}{t^4}$

$\to \begin{cases} w_{i+1} = w_i + h\left[\frac{1}{t_i} + \frac{2w_i}{t_i^2} + \frac{w_i^2}{t_i^3} + \frac{2w_i^3}{t_i^4}\right] = w_i + (0.1)\left[\frac{1}{1+0.1i} + \frac{2w_i}{(1+0.1i)^2} + \frac{w_i^2}{(1+0.1i)^3} + \frac{2w_i^3}{(1+0.1i)^4}\right] \\ t_i = 1+0.1i \end{cases}$

i	t_i	w_i (approximate)	y_i (exact)	Δy absolute error	ε relative error
0	1.0	0	0	0	0
1	1.1	0.1	0.1052	0.00520	0.0494
2	1.2	0.2083	0.2212	0.0129	0.0582
3	1.3	0.3239	0.3491	0.0251	0.0720
4	1.4	0.4463	0.4897	0.0433	0.0884
5	1.5	0.5752	0.6439	0.0686	0.1066
6	1.6	0.7104	0.8128	0.1024	0.1260
7	1.7	0.8517	0.9975	0.1458	0.1462
8	1.8	0.9990	1.1994	0.2004	0.1671
9	1.9	1.1523	1.4201	0.2678	0.1886
10	2.0	1.3116	1.6613	0.3497	0.2105

2. The system of initial-value problems

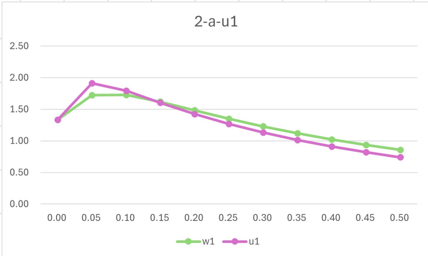
$u_1' = 9u_1 + 24u_2 + 5\cos t - \frac{1}{3}\sin t$, $\underline{u_1(0) = \frac{4}{3}}$, $u_1' = f_1$

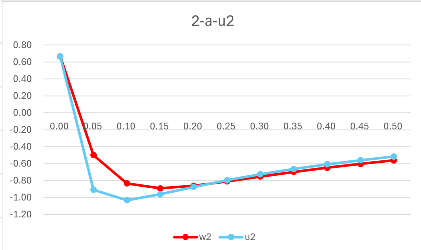
$u_2' = -24u_1 - 52u_2 - 9\cos t + \frac{1}{3}\sin t$, $\underline{u_2(0) = \frac{2}{3}}$, $u_2' = f_2$

has the unique solution

$u_1 = 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t$, $u_2 = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t$.

Try $h = 0.05$ and $h = 0.1$ in Runge-Kutta method, and compare their results with the exact value.





Runge-Kutta method $\to$

u_j: $f(t, u_1, u_2)$

$k_{1,j} = hf_1(t_i, w_{i,1}, w_{i,2})$

$k_{2,j} = hf_1(t_i + \frac{h}{2}, w_{i,1} + \frac{1}{2}k_{1,1}, w_{i,2} + \frac{1}{2}k_{1,2})$

$k_{3,j} = hf_1(t_i + \frac{h}{2}, w_{i,1} + \frac{1}{2}k_{2,1}, w_{i,2} + \frac{1}{2}k_{2,2})$

$k_{4,j} = hf_1(\overset{t_i+h}{t_{i+1}}, w_{i,1} + k_{3,1}, w_{i,2} + k_{3,2})$

$w_{i+1,j} = w_{i,j} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$

Let, $0 \le t \le 1$

$u_1'(t) = f_1(t, u_1, u_2, \cdots, u_m)$

$u_2'(t) = f_2(t, u_1, u_2, \cdots, u_m)$

$\vdots$

$u_m'(t) = f_m(t, u_1, u_2, \cdots, u_m)$

$h = 0.05$

u_1: $k_{1,1} = h[9w_{i,1} + 24w_{i,2} + 5\cos t_i - \frac{1}{3}\sin t_i]$

$= (0.05)[9w_{i,1} + 24w_{i,2} + 5\cos(0.05i) - \frac{1}{3}\sin(0.05i)]$

$k_{2,1} = h[9(w_{i,1} + \frac{1}{2}k_{1,1}) + 24(w_{i,2} + \frac{1}{2}k_{1,2}) + 5\cos(t_i + \frac{h}{2}) - \frac{1}{3}\sin(t_i + \frac{h}{2})]$

$= (0.05)[9(w_{i,1} + \frac{1}{2}k_{1,1}) + 24(w_{i,2} + \frac{1}{2}k_{1,2}) + 5\cos(0.05i + 0.025) - \frac{1}{3}\sin(0.05i + 0.025)]$

$k_{3,1} = h[9(w_{i,1} + \frac{1}{2}k_{2,1}) + 24(w_{i,2} + \frac{1}{2}k_{2,2}) + 5\cos(t_i + \frac{h}{2}) - \frac{1}{3}\sin(t_i + \frac{h}{2})]$

$= (0.05)[9(w_{i,1} + \frac{1}{2}k_{2,1}) + 24(w_{i,2} + \frac{1}{2}k_{2,2}) + 5\cos(0.05i + 0.025) - \frac{1}{3}\sin(0.05i + 0.025)]$

$k_{4,1} = h[9(w_{i,1} + k_{3,1}) + 24(w_{i,2} + k_{3,2}) + 5\cos(t_{i+1}) - \frac{1}{3}\sin(t_{i+1})]$

$= (0.05)[9(w_{i,1} + k_{3,1}) + 24(w_{i,2} + k_{3,2}) + 5\cos(0.05i + 0.05) - \frac{1}{3}\sin(0.05i + 0.05)]$

$w_{i+1,1} = w_{i,1} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$

u_2: $k_{1,2} = h[-24w_{i,1} - 52w_{i,2} - 9\cos t_i + \frac{1}{3}\sin t_i]$

$= (0.05)[-24w_{i,1} - 52w_{i,2} - 9\cos(0.05i) + \frac{1}{3}\sin(0.05i)]$

$k_{2,2} = (0.05)[-24(w_{i,1} + \frac{1}{2}k_{1,1}) - 52(w_{i,2} + \frac{1}{2}k_{1,2}) - 9\cos(0.05i + 0.025) + \frac{1}{3}\sin(0.05i + 0.025)]$

$k_{3,2} = (0.05)[-24(w_{i,1} + \frac{1}{2}k_{2,1}) - 52(w_{i,2} + \frac{1}{2}k_{2,2}) - 9\cos(0.05i + 0.025) + \frac{1}{3}\sin(0.05i + 0.025)]$

$k_{4,2} = (0.05)[-24(w_{i,1} + k_{3,1}) - 52(w_{i,2} + k_{3,2}) - 9\cos(0.05i + 0.05) + \frac{1}{3}\sin(0.05i + 0.05)]$

$w_{i+1,2} = w_{i,2} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$

t_i	j	$k_{1,j}$	$k_{2,j}$	$k_{3,j}$	$k_{4,j}$	$w_{i,j}$
0	1	1.65	-0.2492	1.6808	-2.1817	1.3333
	2	-3.7833	0.1456	-3.8225	4.1395	0.6667
0.05	1	0.4242	-0.2107	0.4428	-0.8581	1.7219
	2	-1.2159	0.1114	-1.2331	1.4614	-0.4996
0.10	1	0.0251	-0.1826	0.0385	-0.3955	1.7269
	2	-0.3536	0.0927	-0.3629	0.5474	-0.8326
0.15	1	-0.0960	-0.1599	-0.0853	-0.2265	1.6172
	2	-0.0681	0.0803	-0.0743	0.2320	-0.8904
0.20	1	-0.1248	-0.1408	-0.1157	-0.1586	1.4817
	2	0.0230	0.0708	0.0182	0.1203	-0.8610
0.25	1	-0.1239	-0.1244	-0.1161	-0.1263	1.3489
	2	0.0489	0.0630	0.0450	0.0781	-0.8075
0.30	1	-0.1143	-0.1103	-0.1096	-0.1074	1.2271
	2	0.0534	0.0564	0.0501	0.0602	-0.7503
0.35	1	-0.1031	-0.0981	-0.0972	-0.0940	1.1175
	2	0.0513	0.0508	0.0485	0.0510	-0.6959
0.40	1	-0.0923	-0.0875	-0.0873	-0.0835	1.0195
	2	0.0475	0.0460	0.0450	0.0452	-0.6457
0.45	1	-0.0827	-0.0783	-0.0783	-0.0747	0.9320
	2	0.0435	0.0419	0.0414	0.0409	-0.5999
0.50	1	-0.0742	-0.0704	-0.0704	-0.0673	0.8535
	2	0.0399	0.0384	0.0380	0.0375	-0.5581

Compare with exact value:

t_i	$w_{i,1}$	$u_1(t_i)$	δy absolute error	ε relative error	$w_{i,2}$	$u_2(t_i)$	δy absolute error	ε relative error
0	1.3333	1.3333	0	0	0.6667	0.6667	0	0
0.05	1.7219	1.9121	0.1902	0.0995	-0.4996	-0.9091	-0.4095	0.4504
0.10	1.7269	1.7931	0.06615	0.0369	-0.8326	-1.0320	-0.1994	0.1932
0.15	1.6172	1.6020	-0.0152	-0.00948	-0.8904	-0.9615	-0.0711	0.0739
0.20	1.4817	1.4239	-0.0578	-0.04058	-0.8610	-0.8747	-0.01364	0.0156
0.25	1.3489	1.2676	-0.0813	-0.06413	-0.8075	-0.7952	0.01228	-0.0155
0.30	1.2271	1.1316	-0.0955	-0.08438	-0.7503	-0.7250	0.0253	-0.0350
0.35	1.1175	1.0130	-0.1045	-0.1031	-0.6959	-0.6631	0.0328	-0.0495
0.40	1.0195	0.9094	-0.1101	-0.1211	-0.6457	-0.6082	0.0375	-0.0617
0.45	0.9320	0.8186	-0.1134	-0.1385	-0.5999	-0.5594	0.0405	-0.0725
0.50	0.8535	0.7388	-0.1148	-0.1553	-0.5581	-0.5157	0.0424	-0.0823

$h=0.1$:

u_1: $k_{1,1}=(0.1)[9w_{i,1}+24w_{i,2}+5\cos(0.1i)-\frac{1}{3}\sin(0.1i)]$

$k_{2,1}=(0.1)[9(w_{i,1}+\frac{1}{2}k_{1,1})+24(w_{i,2}+\frac{1}{2}k_{1,2})+5\cos(0.1i+0.05)-\frac{1}{3}\sin(0.1i+0.05)]$

$k_{3,1}=(0.1)[9(w_{i,1}+\frac{1}{2}k_{2,1})+24(w_{i,2}+\frac{1}{2}k_{2,2})+5\cos(0.1i+0.05)-\frac{1}{3}\sin(0.1i+0.05)]$

$k_{4,1}=(0.1)[9(w_{i,1}+k_{3,1})+24(w_{i,2}+k_{3,2})+5\cos(0.1i+0.1)-\frac{1}{3}\sin(0.1i+0.1)]$

$w_{i+1,1}=w_{i,1}+\frac{1}{6}(k_{1,1}+2k_{2,1}+2k_{3,1}+k_{4,1})$

u_2: $k_{1,2}=(0.1)[-24w_{i,1}-52w_{i,2}-9\cos(0.1i)+\frac{1}{3}\sin(0.1i)]$

$k_{2,2}=(0.1)[-24(w_{i,1}+\frac{1}{2}k_{1,1})-52(w_{i,2}+\frac{1}{2}k_{1,2})-9\cos(0.1i+0.05)+\frac{1}{3}\sin(0.1i+0.05)]$

$k_{3,2}=(0.1)[-24(w_{i,1}+\frac{1}{2}k_{2,1})-52(w_{i,2}+\frac{1}{2}k_{2,2})-9\cos(0.1i+0.05)+\frac{1}{3}\sin(0.1i+0.05)]$

$k_{4,2}=(0.1)[-24(w_{i,1}+k_{3,1})-52(w_{i,2}+k_{3,2})-9\cos(0.1i+0.1)+\frac{1}{3}\sin(0.1i+0.1)]$

$w_{i+1,2}=w_{i,2}+\frac{1}{6}(k_{1,1}+2k_{2,1}+2k_{3,1}+k_{4,1})$

(divergent)

t_i	j	$k_{1,j}$	$k_{2,j}$	$k_{3,j}$	$k_{4,j}$	$w_{i,j}$
0	1	3.3	-4.2973	11.1433	-43.307	1.3333
	2	-7.5667	8.1495	-23.5957	88.395	0.6667
0.1	1	19.321	-20.362	59.736	-222.84	-3.0524
	2	-40.311	41.319	-123.30	457.49	8.989
0.2	1	101.88	-104.09	311.24	-1154.1	-23.848
	2	-209.84	213.5	-640.03	2371.4	51.193
0.3	1	529.57	-538.52	1614.9	-5982.7	-130.17
	2	-1088.7	1106.4	-3319.0	12294	269.27
0.4	1	2746.7	-2791.2	8373.8	-31018	-680.23
	2	-5645.0	5735.9	-17209	63744	1399.4
0.5	1	14242	-14471	43417	-160822	-3531.3
	2	-29269	29739	-89225	330500	7258.2

t_i	$w_{i,1}$	$u_1(t_i)$	δy absolute error	ε relative error	$w_{i,2}$	$u_2(t_i)$	δy absolute error	ε relative error
0	1.3333	1.3333	0	0	0.6667	0.6667	0	0
0.1	-3.0524	1.7931	4.8455	2.7024	8.9893	-1.0320	-10.021	9.7105
0.2	-23.848	1.4239	25.272	17.748	51.193	-0.8745	-52.067	59.527
0.3	-130.165	1.1316	131.30	116.03	269.27	-0.7250	-269.99	372.41
0.4	-680.231	0.9094	681.14	748.99	1399.4	-0.6082	-1399.98	2301.78
0.5	-3531.30	0.7388	3532.0	4780.9	7258.2	-0.5157	-7258.8	14076.7

$\Rightarrow$ $\because$ h too big $\therefore$ w_i is divergent.