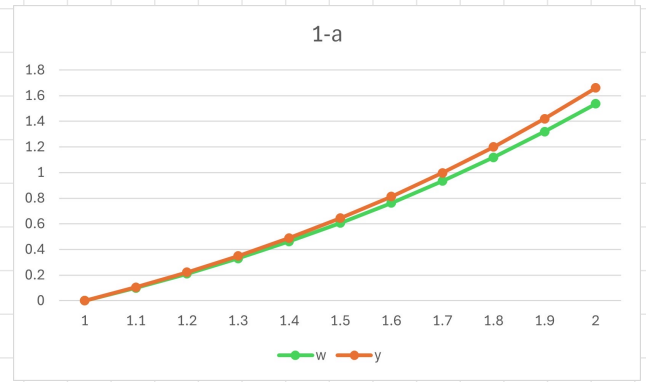1. The initial-value problem

$y' = 1 + (y/t) + (y/t)^2$, $1 \le t \le 2$, $y(1) = 0$ has the exact

solution $y(t) = t \tan(\ln t)$.

a. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

b. Use Taylor's method of order 2 with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

1-a

1.

a. $h = 0.1 = t_{i+1} - t_i$, $y(t_{i+1}) = y(t_i) + f[t_i, y(t_i)]h = y(t_i) + \dfrac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i} h$ —— (1)

$\Rightarrow t_{i+1} = t_i + 0.1$ $\Rightarrow y(t_0) = y(1) = 0$

$t_i = 1 + 0.1i$

$f(t, y) = 1 + \left(\dfrac{y}{t}\right) + \left(\dfrac{y}{t}\right)^2 \Rightarrow f(t_i, y_i) = 1 + \dfrac{y_i}{t_i} + \left(\dfrac{y_i}{t_i}\right)^2 \Rightarrow f(t_i, w_i) = 1 + \dfrac{w_i}{1 + 0.1i} + \left(\dfrac{w_i}{1 + 0.1i}\right)^2$ —— (2)

(1) & (2)

$w_{i+1} = w_i + (0.1)\left[1 + \dfrac{w_i}{1 + 0.1i} + \left(\dfrac{w_i}{1 + 0.1i}\right)^2\right]$, $i = 1, 2, 3, \cdots, 9, 10$

$\therefore$ By $\begin{cases} t_i = t_0 + ih = 1 + 0.2i, & i = 0, 1, 2, \cdots, 10 \\ w_{i+1} = w_i + (0.1)\left[1 + \dfrac{w_i}{1 + 0.1i} + \left(\dfrac{w_i}{1 + 0.1i}\right)^2\right] \end{cases}$

i	t_i	w_i (approximate)	y_i (exact)	Δy absolute error	ε relative error
0	1.0	0	0	0	0
1	1.1	0.1	0.1052	0.00520	0.0494
2	1.2	$\dfrac{301}{1440} = 0.2099$	0.2212	0.0113	0.0511
3	1.3	0.3305	0.3491	0.0186	0.0533
4	1.4	0.4624	0.4897	0.0273	0.0557
5	1.5	0.6063	0.6439	0.0376	0.0584
6	1.6	0.7630	0.8128	0.0498	0.0613
7	1.7	0.9335	0.9975	0.0640	0.0642
8	1.8	1.1185	1.1994	0.0809	0.0675
9	1.9	1.3193	1.4201	0.101	0.0710
10	2.0	1.5369	1.6613	0.124	0.0749

b. Taylor's $\rightarrow$ $y(t_{i+1}) = y(t_i) + hT^{(n)}[t_i, y(t_i)]$, $y' = 1 + \left(\dfrac{y}{t}\right) + \left(\dfrac{y}{t}\right)^2$

Order 2 $\rightarrow$ $T^{(2)} = f + \dfrac{h}{2!}\dfrac{df}{dt} = f + \dfrac{h}{2}\left[\dfrac{\partial f}{\partial t} + f\dfrac{\partial f}{\partial y}\right]$
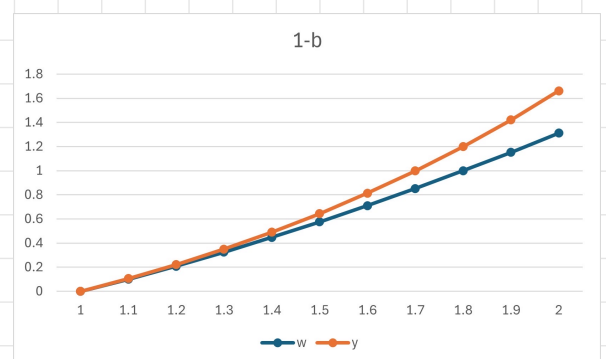
$h = 0.1$

$\therefore w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$

Let $f(t, y) = 1 + \dfrac{y}{t} + \left(\dfrac{y}{t}\right)^2$

$\dfrac{df}{dt} = \dfrac{\partial f}{\partial t} + f\dfrac{df}{dy}$

$= -\dfrac{y}{t^2} - \dfrac{2y^2}{t^3} + \left[1 + \dfrac{y}{t} + \left(\dfrac{y}{t}\right)^2\right]\left[\dfrac{1}{t} + \dfrac{2y}{t^2}\right]$

1-b

$$= -\frac{y}{t^2} - \frac{2y^2}{t^3} + \frac{1}{t} + \frac{2y}{t^2} + \frac{y}{t^2} + \frac{2y^2}{t^3} + \frac{y^2}{t^3} + \frac{2y^3}{t^4} = \frac{1}{t} + \frac{2y}{t^2} + \frac{y^2}{t^3} + \frac{2y^3}{t^4}$$

$$\rightarrow \begin{cases} w_{i+1} = w_i + h\left[\frac{1}{t_i} + \frac{2w_i}{t_i^2} + \frac{w_i^2}{t_i^3} + \frac{2w_i^3}{t_i^4}\right] = w_i + (0.1)\left[\frac{1}{1+0.1i} + \frac{2w_i}{(1+0.1i)^2} + \frac{w_i^2}{(1+0.1i)^3} + \frac{2w_i^3}{(1+0.1i)^4}\right] \\ t_i = 1 + 0.1i \end{cases}$$

i	t_i	w_i (approximate)	y_i (exact)	Δy absolute error	ε relative error
0	1.0	0	0	0	0
1	1.1	0.1	0.1052	0.00520	0.0494
2	1.2	0.2083	0.2212	0.0129	0.0582
3	1.3	0.3239	0.3491	0.0251	0.0720
4	1.4	0.4463	0.4897	0.0433	0.0884
5	1.5	0.5752	0.6439	0.0686	0.1066
6	1.6	0.7104	0.8128	0.1024	0.1260
7	1.7	0.8517	0.9975	0.1458	0.1462
8	1.8	0.9990	1.1994	0.2004	0.1671
9	1.9	1.1523	1.4201	0.2678	0.1886
10	2.0	1.3116	1.6613	0.3497	0.2105

2. The system of initial-value problems

$$u_1' = 9u_1 + 24u_2 + 5\cos t - \frac{1}{3}\sin t, \quad u_1(0) = \frac{4}{3}, \quad u_1' = f_1$$

$$u_2' = -24u_1 - 52u_2 - 9\cos t + \frac{1}{3}\sin t, \quad u_2(0) = \frac{2}{3}, \quad u_2' = f_2$$

has the unique solution

$$u_1 = 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t, \quad u_2 = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t.$$

Try $h = 0.05$ and $h = 0.1$ in Runge-Kutta method, and compare their results with the exact value.
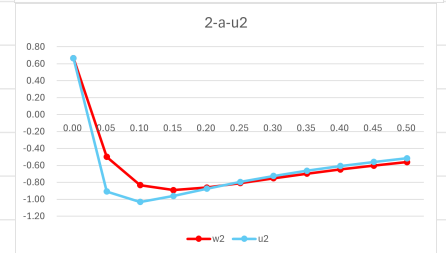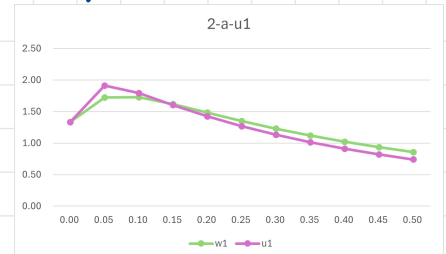

2-a-u1

2-a-u2

$$u_j: \quad f(t, u_1, u_2)$$

Runge – Kutta method $\rightarrow$
$$k_{1,j} = h f_1(t_i, w_{i,1}, w_{i,2})$$
$$k_{2,j} = h f_1\left(t_i + \frac{h}{2}, w_{i,1} + \frac{1}{2}k_{1,1}, w_{i,2} + \frac{1}{2}k_{1,2}\right)$$

Let, $0 \le t \le 1$
$$u_1(t) = f_1(t, u_1, u_2, \cdots, u_m)$$
$$k_{3,j} = h f_1\left(t_i + \frac{h}{2}, w_{i,1} + \frac{1}{2}k_{2,1}, w_{i,2} + \frac{1}{2}k_{2,2}\right)$$
$$u_2'(t) = f_2(t, u_1, u_2, \cdots, u_m)$$
$$k_{4,j} = h f_1\left(t_{i+1}^{t_i+h}, w_{i,1} + k_{3,1}, w_{i,2} + k_{3,2}\right)$$
$$w_{i+1,j} = w_{i,j} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$$

$$\vdots$$

$$u_m'(t) = f_m(t, u_1, u_2, \cdots, u_m)$$

$h = 0.05$

$u_1:$
$$k_{1,1} = h\left[9w_{i,1} + 24w_{i,2} + 5\cos t_i - \frac{1}{3}\sin t_i\right]$$
$$= (0.05)\left[9w_{i,1} + 24w_{i,2} + 5\cos(0.05i) - \frac{1}{3}\sin(0.05i)\right]$$

$$k_{2,1} = h\left[9\left(w_{i,1} + \frac{1}{2}k_{1,1}\right) + 24\left(w_{i,2} + \frac{1}{2}k_{1,2}\right) + 5\cos\left(t_i + \frac{h}{2}\right) - \frac{1}{3}\sin\left(t_i + \frac{h}{2}\right)\right]$$
$$= (0.05)\left[9\left(w_{i,1} + \frac{1}{2}k_{1,1}\right) + 24\left(w_{i,2} + \frac{1}{2}k_{1,2}\right) + 5\cos(0.05i + 0.025) - \frac{1}{3}\sin(0.05i + 0.025)\right]$$

$$k_{3,1} = h\left[9\left(w_{i,1} + \frac{1}{2}k_{2,1}\right) + 24\left(w_{i,2} + \frac{1}{2}k_{2,2}\right) + 5\cos\left(t_i + \frac{h}{2}\right) - \frac{1}{3}\sin\left(t_i + \frac{h}{2}\right)\right]$$
$$= (0.05)\left[9\left(w_{i,1} + \frac{1}{2}k_{2,1}\right) + 24\left(w_{i,2} + \frac{1}{2}k_{2,2}\right) + 5\cos(0.05i + 0.025) - \frac{1}{3}\sin(0.05i + 0.025)\right]$$

$$k_{4,1} = h\left[9\left(w_{i,1} + k_{3,1}\right) + 24\left(w_{i,2} + k_{3,2}\right) + 5\cos(t_{i+1}) - \frac{1}{3}\sin(t_{i+1})\right]$$
$$= (0.05)\left[9\left(w_{i,1} + k_{3,1}\right) + 24\left(w_{i,2} + k_{3,2}\right) + 5\cos(0.05i + 0.05) - \frac{1}{3}\sin(0.05i + 0.05)\right]$$

$$w_{i+1,1} = w_{i,1} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$$

u_2: $k_{1,2} = h[-24 W_{i,1} - 52 W_{i,2} - 9\cos t_i + \frac{1}{3}\sin t_i]$

$= (0.05)[-24 W_{i,1} - 52 W_{i,2} - 9\cos(0.05i) + \frac{1}{3}\sin(0.05i)]$

$k_{2,2} = (0.05)[-24(W_{i,1} + \frac{1}{2}k_{1,1}) - 52(W_{i,2} + \frac{1}{2}k_{1,2}) - 9\cos(0.05i + 0.025) + \frac{1}{3}\sin(0.05i + 0.025)]$

$k_{3,2} = (0.05)[-24(W_{i,1} + \frac{1}{2}k_{2,1}) - 52(W_{i,2} + \frac{1}{2}k_{2,2}) - 9\cos(0.05i + 0.025) + \frac{1}{3}\sin(0.05i + 0.025)]$

$k_{4,2} = (0.05)[-24(W_{i,1} + k_{3,1}) - 52(W_{i,2} + k_{3,2}) - 9\cos(0.05i + 0.05) + \frac{1}{3}\sin(0.05i + 0.05)]$

$W_{i+1,2} = W_{i,2} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$

t_i	j	$k_{1,j}$	$k_{2,j}$	$k_{3,j}$	$k_{4,j}$	$W_{i,j}$
0	1	1.65	−0.2492	1.6808	−2.1817	1.3333
	2	−3.7833	0.1456	−3.8225	4.1395	0.6667
0.05	1	0.4242	−0.2107	0.4428	−0.8581	1.7219
	2	−1.2159	0.1114	−1.2331	1.4614	−0.4996
0.10	1	0.0251	−0.1826	0.0385	−0.3955	1.7269
	2	−0.3536	0.0927	−0.3629	0.5474	−0.8326
0.15	1	−0.0960	−0.1599	−0.0853	−0.2265	1.6172
	2	−0.0681	0.0803	−0.0743	0.2320	−0.8904
0.20	1	−0.1248	−0.1408	−0.1157	−0.1586	1.4817
	2	0.0230	0.0708	0.0182	0.1203	−0.8610
0.25	1	−0.1239	−0.1244	−0.1161	−0.1263	1.3489
	2	0.0489	0.0630	0.0450	0.0781	−0.8075
0.30	1	−0.1143	−0.1103	−0.1096	−0.1074	1.2271
	2	0.0534	0.0564	0.0501	0.0602	−0.7503
0.35	1	−0.1031	−0.0981	−0.0972	−0.0940	1.1175
	2	0.0513	0.0508	0.0485	0.0510	−0.6959
0.40	1	−0.0923	−0.0875	−0.0873	−0.0835	1.0195
	2	0.0475	0.0460	0.0450	0.0452	−0.6457
0.45	1	−0.0827	−0.0783	−0.0783	−0.0747	0.9320
	2	0.0435	0.0419	0.0414	0.0409	−0.5999
0.50	1	−0.0742	−0.0704	−0.0704	−0.0217	0.8535
	2	0.0399	0.0384	0.0380	−0.0376	−0.5581

Compare with exact value:

t_i	$W_{i,1}$	$u_1(t_i)$	absolute error δy	relative error ε	$W_{i,2}$	$u_2(t_i)$	absolute error δy	relative error ε
0	1.3333	1.3333	0	0	0.6667	0.6667	0	0
0.05	1.7219	1.9121	0.1902	0.0995	−0.4996	−0.9091	−0.4095	0.4504
0.10	1.7269	1.7931	0.0665	0.0369	−0.8326	−1.0320	−0.1994	0.1932
0.15	1.6172	1.6020	−0.0152	−0.00948	−0.8904	−0.9615	−0.0711	0.0739
0.20	1.4817	1.4239	−0.0578	−0.04058	−0.8610	−0.8747	−0.01364	0.0156
0.25	1.3489	1.2676	−0.0813	−0.06413	−0.8075	−0.7952	0.01228	−0.0155
0.30	1.2271	1.1316	−0.0955	−0.08438	−0.7503	−0.7250	0.0253	−0.0350
0.35	1.1175	1.0130	−0.1045	−0.1031	−0.6959	−0.6631	0.0328	−0.0495
0.40	1.0195	0.9094	−0.1101	−0.1211	−0.6457	−0.6082	0.0375	−0.0617
0.45	0.9320	0.8186	−0.1134	−0.1385	−0.5999	−0.5594	0.0405	−0.0725
0.50	0.8535	0.7388	−0.1148	−0.1553	−0.5581	−0.5157	0.0424	−0.0823

$h = 0.1$:

u_1: $k_{1,1} = (0.1)[9w_{i,1} + 24w_{i,2} + 5\cos(0.1i) - \frac{1}{3}\sin(0.1i)]$

$\quad k_{2,1} = (0.1)[9(w_{i,1} + \frac{1}{2}k_{1,1}) + 24(w_{i,2} + \frac{1}{2}k_{1,2}) + 5\cos(0.1i + 0.05) - \frac{1}{3}\sin(0.1i + 0.05)]$

$\quad k_{3,1} = (0.1)[9(w_{i,1} + \frac{1}{2}k_{2,1}) + 24(w_{i,2} + \frac{1}{2}k_{2,2}) + 5\cos(0.1i + 0.05) - \frac{1}{3}\sin(0.1i + 0.05)]$

$\quad k_{4,1} = (0.1)[9(w_{i,1} + k_{3,1}) + 24(w_{i,2} + k_{3,2}) + 5\cos(0.1i + 0.1) - \frac{1}{3}\sin(0.1i + 0.1)]$

$\quad w_{i+1,1} = w_{i,1} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$

u_2: $k_{1,2} = (0.1)[-24w_{i,1} - 52w_{i,2} - 9\cos(0.1i) + \frac{1}{3}\sin(0.1i)]$

$\quad k_{2,2} = (0.1)[-24(w_{i,1} + \frac{1}{2}k_{1,1}) - 52(w_{i,2} + \frac{1}{2}k_{1,2}) - 9\cos(0.1i + 0.05) + \frac{1}{3}\sin(0.1i + 0.05)]$

$\quad k_{3,2} = (0.1)[-24(w_{i,1} + \frac{1}{2}k_{2,1}) - 52(w_{i,2} + \frac{1}{2}k_{2,2}) - 9\cos(0.1i + 0.05) + \frac{1}{3}\sin(0.1i + 0.05)]$

$\quad k_{4,2} = (0.1)[-24(w_{i,1} + k_{3,1}) - 52(w_{i,2} + k_{3,2}) - 9\cos(0.1i + 0.1) + \frac{1}{3}\sin(0.1i + 0.1)]$

$\quad w_{i+1,2} = w_{i,2} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$ (divergent)

t_i	j	$k_{1,j}$	$k_{2,j}$	$k_{3,j}$	$k_{4,j}$	$w_{i,j}$
0	1	3.3	-4.2973	11.1433	-43.307	1.3333
	2	-7.5667	8.1495	-23.5957	88.395	0.6667
0.1	1	19.321	-20.362	59.736	-222.84	-3.0524
	2	-40.311	41.319	-123.30	457.49	8.989
0.2	1	101.88	-104.09	311.24	-1154.1	-23.848
	2	-209.84	213.5	-640.03	2371.4	51.193
0.3	1	529.57	-538.52	1614.9	-5982.7	-130.17
	2	-1088.7	1106.4	-3319.0	12294	269.27
0.4	1	2746.7	-2791.2	8373.8	-31018	-680.23
	2	-5645.0	5735.9	-17209	63744	1399.4
0.5	1	14242	-14471	43417	-16022	-3531.3
	2	-29269	29739	-89225	330499.97	7258.2

t_i	$w_{i,1}$	$u_1(t_i)$	δy absolute error	ε relative error	$w_{i,2}$	$u_2(t_i)$	δy absolute error	ε relative error
0	1.3333	1.3333	0	0	0.6667	0.6667	0	0
0.1	-3.0524	1.7931	4.8455	2.7024	8.9893	-1.0320	-10.021	9.7105
0.2	-23.848	1.4239	25.272	17.748	51.193	-0.8745	-52.067	59.527
0.3	-130.165	1.1316	131.30	116.03	269.27	-0.7250	-269.99	372.41
0.4	-680.231	0.9094	681.14	748.99	1399.4	-0.6082	-1399.98	2301.78
0.5	-3531.30	0.7388	3532.0	4780.9	7258.2	-0.5157	-7258.8	14076.7

$\Rightarrow$ ∴ h too big ∴ w_i is divergent.