1. The initial-value problem

$y' = 1 + (y/t) + (y/t)^2$, $1 \le t \le 2$, $y(1) = 0$ has the exact solution $y(t) = t\tan(\ln t)$.

a. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

b. Use Taylor's method of order 2 with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

1\. a. $h=0.1 = t_{i+1} - t_i$, $y(t_{i+1}) = y(t_i) + f[t_i, y(t_i)]h = y(t_i) + \frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i} h$ —(1)

$\Rightarrow t_{i+1} = t_i + 0.1$ $\Rightarrow y(t_0) = y(1) = 0$

$t_i = 1 + 0.1i$

$f(t,y) = 1 + \left(\frac{y}{t}\right) + \left(\frac{y}{t}\right)^2 \Rightarrow f(t_i, y_i) = 1 + \frac{y_i}{t_i} + \left(\frac{y_i}{t_i}\right)^2 \Rightarrow f(t_i, w_i) = 1 + \frac{w_i}{1+0.i} + \left(\frac{w_i}{1+0.1i}\right)^2$ —(2)

(1) & (2)

$w_{i+1} = w_i + (0.1)\left[1 + \frac{w_i}{1+0.1i} + \left(\frac{w_i}{1+0.1i}\right)^2\right]$, $i = 1, 2, 3, \cdots, 9, 10$

$\therefore$ By $\begin{cases} t_i = t_0 + ih = 1 + 0.2i, & i = 0, 1, 2, \cdots, 10 \\ w_{i+1} = w_i + (0.1)\left[1 + \frac{w_i}{1+0.1i} + \left(\frac{w_i}{1+0.1i}\right)^2\right] \end{cases}$

i	t_i	w_i (approximate)	y_i (exact)	Δy absolute error	ε relative error
0	1.0	0	0	0	0
1	1.1	0.1	0.1052	0.00520	0.0494
2	1.2	$\frac{301}{1440} = 0.2099$	0.2212	0.0113	0.0511
3	1.3	0.3305	0.3491	0.0186	0.0533
4	1.4	0.4624	0.4897	0.0273	0.0557
5	1.5	0.6063	0.6439	0.0376	0.0584
6	1.6	0.7630	0.8128	0.0498	0.0613
7	1.7	0.9335	0.9975	0.0640	0.0642
8	1.8	1.1185	1.1994	0.0809	0.0675
9	1.9	1.3193	1.4201	0.101	0.0710
10	2.0	1.5369	1.6613	0.124	0.0749

b. Taylor's $\rightarrow$ $y(t_{i+1}) = y(t_i) + hT^{(n)}[t_i, y(t_i)]$, $y' = 1 + \left(\frac{y}{t}\right) + \left(\frac{y}{t}\right)^2$

Order 2 $\rightarrow$ $T^{(2)} = f + \frac{h}{2!}\frac{df}{dt} = f + \frac{h}{2}\left[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right]$

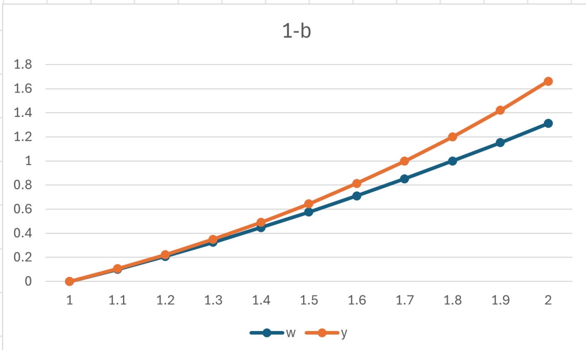
$h = 0.1$

令 $w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$

Let $f(t,y) = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2$

$\frac{df}{dt} = \frac{\partial f}{\partial t} + f\frac{df}{dy}$

$= -\frac{y}{t^2} - \frac{2y^2}{t^3} + \left[1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2\right]\left[\frac{1}{t} + \frac{2y}{t^2}\right]$



$$= -\cancel{\frac{y}{t^2}} - \cancel{\frac{2y^2}{t^3}} + \frac{1}{t} + \frac{2y}{t^2} + \cancel{\frac{y}{t^2}} + \cancel{\frac{2y^2}{t^3}} + \frac{y^2}{t^3} + \frac{2y^3}{t^4} = \frac{1}{t} + \frac{2y}{t^2} + \frac{y^2}{t^3} + \frac{2y^3}{t^4}$$

$$\rightarrow \begin{cases} w_{i+1} = w_i + h\left[\frac{1}{t_i} + \frac{2w_i}{t_i^2} + \frac{w_i^2}{t_i^3} + \frac{2w_i^3}{t_i^4}\right] = w_i + (0.1)\left[\frac{1}{1+0.1i} + \frac{2w_i}{(1+0.1i)^2} + \frac{w_i^2}{(1+0.1i)^3} + \frac{2w_i^3}{(1+0.1i)^4}\right] \\ t_i = 1+0.1i \end{cases}$$

i	t_i	w_i (approximate)	y_i (exact)	Δy absolute error	ε relative error
0	1.0	0	0	0	0
1	1.1	0.1	0.1052	0.00520	0.0494
2	1.2	0.2083	0.2212	0.0129	0.0582
3	1.3	0.3239	0.3491	0.0251	0.0720
4	1.4	0.4463	0.4897	0.0433	0.0884
5	1.5	0.5752	0.6439	0.0686	0.1066
6	1.6	0.7104	0.8128	0.1024	0.1260
7	1.7	0.8517	0.9975	0.1458	0.1462
8	1.8	0.9990	1.1994	0.2004	0.1671
9	1.9	1.1523	1.4201	0.2678	0.1886
10	2.0	1.3116	1.6613	0.3497	0.2105

2. The system of initial-value problems

$$u_1' = 9u_1 + 24u_2 + 5\cos t - \frac{1}{3}\sin t, \quad \underline{u_1(0) = \frac{4}{3}}, \qquad u_1' = f_1$$

$$u_2' = -24u_1 - 52u_2 - 9\cos t + \frac{1}{3}\sin t, \quad \underline{u_2(0) = \frac{2}{3}}, \qquad u_2' = f_2$$

has the unique solution

$$u_1 = 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t, \quad u_2 = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t.$$

Try $h = 0.05$ and $h = 0.1$ in Runge-Kutta method, and compare their results with the exact value.

Runge-Kutta method $\rightarrow$

Let $0 \le t \le 1$

$u_1'(t) = f_1(t, u_1, u_2, \cdots, u_m)$

$u_2'(t) = f_2(t, u_1, u_2, \cdots, u_m)$

$\vdots$

$u_m'(t) = f_m(t, u_1, u_2, \cdots, u_m)$

u_j: $f(t, u_1, u_2)$

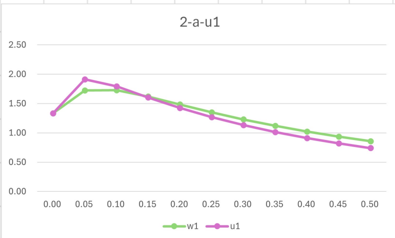
$k_{1,j} = hf_1(t_i, w_{i,1}, w_{i,2})$

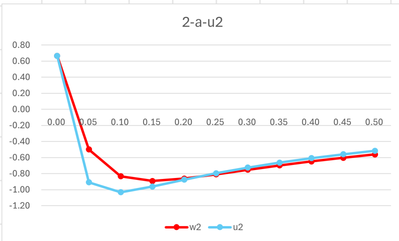
$k_{2,j} = hf_1(t_i + \frac{h}{2}, w_{i,1} + \frac{1}{2}k_{1,1}, w_{i,2} + \frac{1}{2}k_{1,2})$

$k_{3,j} = hf_1(t_i + \frac{h}{2}, w_{i,1} + \frac{1}{2}k_{2,1}, w_{i,2} + \frac{1}{2}k_{2,2})$

$k_{4,j} = hf_1(t_{i+1}, w_{i,1} + k_{3,1}, w_{i,2} + k_{3,2})$

$w_{i+1,j} = w_{i,j} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$





$h = 0.05$

u_1: $k_{1,1} = h[9w_{i,1} + 24w_{i,2} + 5\cos t_i - \frac{1}{3}\sin t_i]$

$= (0.05)[9w_{i,1} + 24w_{i,2} + 5\cos(0.05i) - \frac{1}{3}\sin(0.05i)]$

$k_{2,1} = h[9(w_{i,1} + \frac{1}{2}k_{1,1}) + 24(w_{i,2} + \frac{1}{2}k_{1,2}) + 5\cos(t_i + \frac{h}{2}) - \frac{1}{3}\sin(t_i + \frac{h}{2})]$

$= (0.05)[9(w_{i,1} + \frac{1}{2}k_{1,1}) + 24(w_{i,2} + \frac{1}{2}k_{1,2}) + 5\cos(0.05i + 0.025) - \frac{1}{3}\sin(0.05i + 0.025)]$

$k_{3,1} = h[9(w_{i,1} + \frac{1}{2}k_{2,1}) + 24(w_{i,2} + \frac{1}{2}k_{2,2}) + 5\cos(t_i + \frac{h}{2}) - \frac{1}{3}\sin(t_i + \frac{h}{2})]$

$= (0.05)[9(w_{i,1} + \frac{1}{2}k_{2,1}) + 24(w_{i,2} + \frac{1}{2}k_{2,2}) + 5\cos(0.05i + 0.025) - \frac{1}{3}\sin(0.05i + 0.025)]$

$k_{4,1} = h[9(w_{i,1} + k_{3,1}) + 24(w_{i,2} + k_{3,2}) + 5\cos(t_{i+1}) - \frac{1}{3}\sin(t_{i+1})]$

$= (0.05)[9(w_{i,1} + k_{3,1}) + 24(w_{i,2} + k_{3,2}) + 5\cos(0.05i) - \frac{1}{3}\sin(0.05i)]$

$w_{i+1,1} = w_{i,1} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$

u_2: $k_{1,2} = h[-24 w_{i,1} - 52 w_{i,2} - 9\cos t_i + \frac{1}{3}\sin t_i]$

$$= (0.05)[-24 w_{i,1} - 52 w_{i,2} - 9\cos(0.05i) + \tfrac{1}{3}\sin(0.05i)]$$

$$k_{2,2} = (0.05)[-24(w_{i,1} + \tfrac{1}{2}k_{1,1}) - 52(w_{i,2} + \tfrac{1}{2}k_{1,2}) - 9\cos(0.05i + 0.025) + \tfrac{1}{3}\sin(0.05i + 0.025)]$$

$$k_{3,2} = (0.05)[-24(w_{i,1} + \tfrac{1}{2}k_{2,1}) - 52(w_{i,2} + \tfrac{1}{2}k_{2,2}) - 9\cos(0.05i + 0.025) + \tfrac{1}{3}\sin(0.05i + 0.025)]$$

$$k_{4,2} = (0.05)[-24(w_{i,1} + k_{3,1}) - 52(w_{i,2} + k_{3,2}) - 9\cos(0.05i) + \tfrac{1}{3}\sin(0.05i)]$$

$$w_{i+1,2} = w_{i,2} + \tfrac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$$

t_i	j	$k_{1,j}$	$k_{2,j}$	$k_{3,j}$	$k_{4,j}$	$w_{i,j}$
0	1	1.65	−0.2492	1.6808	−2.1806	1.3333
	2	−3.783	0.1456	−3.8225	4.1381	0.6667
0.05	1	0.4240	−0.2107	0.4426	−0.8562	1.7221
	2	−1.2155	0.1114	−1.2328	1.4585	−0.4998
0.10	1	0.02464	−0.1826	0.03807	−0.3927	1.7273
	2	−0.3528	0.09272	−0.3621	0.5429	−0.8331
0.15	1	−0.09674	−0.1599	−0.08598	−0.22285	1.6178
	2	−0.06668	0.08028	−0.07285	0.2259	−0.8912
0.20	1	−0.1258	−0.1409	−0.1167	−0.1541	1.4826
	2	0.02490	0.07077	0.02019	0.1125	−0.8622
0.25	1	−0.1251	−0.1245	−0.1174	−0.1210	1.3501
	2	0.05137	0.06299	0.04750	0.06873	−0.8090
0.30	1	−0.1156	−0.1104	−0.1091	−0.1012	1.2284
	2	0.05645	0.0564	0.05316	0.04920	−0.7521
0.35	1	−0.1049	−0.09815	−0.09906	−0.08699	1.1191
	2	0.05487	0.0508	0.05205	0.03842	−0.6980
0.40	1	−0.09438	−0.08758	−0.08934	−0.07565	1.0214
	2	0.0515	0.04606	0.04911	0.03107	−0.6481
0.45	1	−0.08500	−0.07844	−0.08063	−0.06611	0.9341
	2	0.04808	0.04200	0.04598	0.02529	−0.6027
0.50	1	−0.07679	−0.07053	−0.07301	−0.05789	0.8559
	2	0.04494	0.03847	0.04313	0.02042	−0.5611

Compare with exact value:

t_i	$w_{i,1}$	$u_1(t_i)$	δy absolute error	ε relative error	$w_{i,2}$	$u_2(t_i)$	δy absolute error	ε relative error
0	1.3333	1.3333	0	0	0.6667	0.6667	0	0
0.05	1.7221	1.9121	0.19	0.09937	−0.4998	−0.9091	−0.4092	0.4501
0.10	1.7273	1.7931	0.0658	0.03670	−0.8331	−1.0320	−0.1989	0.1927
0.15	1.6178	1.6020	−0.0158	0.009863	−0.8912	−0.9615	−0.07025	0.07306
0.20	1.4826	1.4239	−0.0587	0.04122	−0.8622	−0.8747	−0.01248	0.01427
0.25	1.3501	1.2676	−0.0825	0.06508	−0.8090	−0.7952	0.01376	0.01730
0.30	1.2284	1.1316	−0.0968	0.08554	−0.7521	−0.7250	0.02714	0.03743
0.35	1.1191	1.0130	−0.1061	0.1047	−0.6980	−0.6631	0.03493	0.05268
0.40	1.0214	0.9094	−0.112	0.1232	−0.6481	−0.6082	0.03994	0.06567
0.45	0.9341	0.8186	−0.1155	0.1411	−0.6027	−0.5594	0.04327	0.07735
0.50	0.8559	0.7388	−0.1171	0.1585	−0.5611	−0.5157	0.04546	0.08815

$h=0.1$:

u_1: $k_{1,1}=(0.1)[9w_{i,1}+24w_{i,2}+5\cos(0.1i)-\frac{1}{3}\sin(0.1i)]$

$k_{2,1}=(0.1)[9(w_{i,1}+\frac{1}{2}k_{1,1})+24(w_{i,2}+\frac{1}{2}k_{1,2})+5\cos(0.1i+0.05)-\frac{1}{3}\sin(0.1i+0.05)]$

$k_{3,1}=(0.1)[9(w_{i,1}+\frac{1}{2}k_{2,1})+24(w_{i,2}+\frac{1}{2}k_{2,2})+5\cos(0.1i+0.05)-\frac{1}{3}\sin(0.1i+0.05)]$

$k_{4,1}=(0.1)[9(w_{i,1}+k_{3,1})+24(w_{i,2}+k_{3,2})+5\cos(0.1i)-\frac{1}{3}\sin(0.1i)]$

$w_{i+1,1}=w_{i,1}+\frac{1}{6}(k_{1,1}+2k_{2,1}+2k_{3,1}+k_{4,1})$

u_2: $k_{1,2}=(0.1)[-24w_{i,1}-52w_{i,2}-9\cos(0.1i)+\frac{1}{3}\sin(0.1i)]$

$k_{2,2}=(0.1)[-24(w_{i,1}+\frac{1}{2}k_{1,1})-52(w_{i,2}+\frac{1}{2}k_{1,2})-9\cos(0.1i+0.05)+\frac{1}{3}\sin(0.1i+0.05)]$

$k_{3,2}=(0.1)[-24(w_{i,1}+\frac{1}{2}k_{2,1})-52(w_{i,2}+\frac{1}{2}k_{2,2})-9\cos(0.1i+0.05)+\frac{1}{3}\sin(0.1i+0.05)]$

$k_{4,2}=(0.1)[-24(w_{i,1}+k_{3,1})-52(w_{i,2}+k_{3,2})-9\cos(0.1i)+\frac{1}{3}\sin(0.1i)]$

$w_{i+1,2}=w_{i,2}+\frac{1}{6}(k_{1,1}+2k_{2,1}+2k_{3,1}+k_{4,1})$

t_i	j	$k_{1,j}$	$k_{2,j}$	$k_{3,j}$	$k_{4,j}$	(divergent) $w_{i,j}$
0	1	3.3	-4.2973	11.1433	-43.3008	1.3333
	2	-7.5667	8.1495	-23.5957	88.3872	0.6667
0.1	1	19.3191	-20.3597	59.7297	-222.808	-3.0515
	2	-40.3063	41.3144	-123.285	457.424	8.988
0.2	1	101.867	-104.072	311.188	-1153.888	-23.8429
	2	-209.810	213.467	-639.926	2370.957	51.1841
0.3	1	529.474	-538.432	1614.603	-5981.630	-130.141
	2	-1088.5	1106.2	-3318.4	12292.4	269.222
0.4	1	2746.2	-2790.7	8372.3	-31012.7	-680.110
	2	-5644.0	5734.9	-17205.8	63732.9	1399.12
0.5	1	14240	-14469	43409	-160794	-3530.68
	2	-29263	29734	-89209	330442	7256.96

t_i	$w_{i,1}$	$u_1(t_i)$	δy absolute error	ε relative error	$w_{i,2}$	$u_2(t_i)$	δy absolute error	ε relative error
0	1.3333	1.3333	0	0	0.6667	0.6667	0	0
0.1	-3.0515	1.7931	4.8445	2.7018	8.9880	-1.0320	-10.02	9.7093
0.2	-23.843	1.4239	25.267	17.745	51.184	-0.8745	-52.06	59.517
0.3	-130.141	1.1316	131.27	116.01	269.222	-0.7250	-269.95	372.34
0.4	-680.110	0.9094	681.02	748.86	1399.12	-0.6082	-1399.7	2301.38
0.5	-3530.68	0.7388	3531.4	4780.0	7257.0	-0.5157	-7257.5	14074.2

$\Rightarrow$ $\because h$ too big $\therefore w_i$ is divergent.