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1. Use Gaussian elimination and pivoting technique to solve

$$1.19x_1 + 2.11x_2 - 100x_3 + x_4 = 1.12$$

$$14.2x_1 - 0.112x_2 + 12.2x_3 - x_4 = 3.44$$

$$100x_2 - 99.9x_3 + x_4 = 2.15$$

$$15.3x_1 + 0.110x_2 - 13.1x_3 - x_4 = 4.16$$

$$\begin{bmatrix} 1.19 & 2.11 & -100 & 1 \\ 14.2 & -0.112 & 12.2 & -1 \\ 0 & 100 & -99.9 & 1 \\ 15.3 & 0.110 & -13.1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.12 \\ 3.44 \\ 2.15 \\ 4.16 \end{bmatrix}$$

$$\xrightarrow[\text{with } 1^{st} \text{ row}]{\text{Substitute } 4^{th} \text{ row}} \left[\begin{array}{cccc|c} 15.3 & 0.110 & -13.1 & -1 & 4.16 \\ 14.2 & -0.112 & 12.2 & -1 & 3.44 \\ 0 & 100 & -99.9 & 1 & 2.15 \\ 1.19 & 2.11 & -100 & 1 & 1.12 \end{array}\right]$$

$$\xrightarrow[\text{for } i=2\sim4]{\text{Minus by } \frac{a_{i1}}{a_{11}} \times 1^{st} \text{ row}} \left[\begin{array}{cccc|c} 15.3 & 0.110 & -13.1 & -1 & 4.16 \\ 0 & -0.2141 & 24.3582 & -0.0719 & -0.4209 \\ 0 & 100 & -99.9 & 1 & 2.15 \\ 0 & 2.10144 & -98.9811 & 1.0778 & 0.79644 \end{array}\right]$$

$$\xrightarrow[\text{with } 2^{nd} \text{ row}]{\text{Substitute } 3^{rd} \text{ row}} \left[\begin{array}{cccc|c} 15.3 & 0.110 & -13.1 & -1 & 4.16 \\ 0 & 100 & -99.9 & 1 & 2.15 \\ 0 & -0.2141 & 24.3582 & -0.0719 & -0.4209 \\ 0 & 2.10144 & -98.9811 & 1.0778 & 0.79644 \end{array}\right]$$

$$\xrightarrow[\text{for } i=3\sim4]{\text{Minus by } \frac{a_{i2}}{a_{22}} \times 2^{st} \text{ row}} \left[\begin{array}{cccc|c} 15.3 & 0.110 & -13.1 & -1 & 4.16 \\ 0 & 100 & -99.9 & 1 & 2.15 \\ 0 & 0 & 24.1443 & -0.06976 & -0.4163 \\ 0 & 0 & -96.8818 & 1.0568 & 0.75126 \end{array}\right]$$

$$\xrightarrow{\text{No need to substitute } 4^{th} \text{ row}}$$

$$\xrightarrow[\text{for } i=4]{\text{Minus by } \frac{a_{i3}}{a_{33}} \times 3^{st} \text{ row}} \left[\begin{array}{cccc|c} 15.3 & 0.110 & -13.1 & -1 & 4.16 \\ 0 & 100 & -99.9 & 1 & 2.15 \\ 0 & 0 & 24.1443 & -0.06976 & -0.4163 \\ 0 & 0 & 0 & 0.77688 & -0.9192 \end{array}\right]$$

$$x_n = \frac{C_n}{a_{nn}} ;\ x_i = \frac{1}{a_{ii}}\left[C_i - \sum_{j=i+1}^{n} a_{ij} x_j\right],\ i=n-1\sim1$$

$$x_4 = \frac{C_4}{a_{44}} = \frac{-0.9192}{0.77688} = -1.1832$$

$$x_3 = \frac{1}{a_{33}}\left[C_3 - \sum_{j=4}^{4} a_{3j} x_j\right] = \frac{1}{a_{33}}\left[C_3 - a_{34}x_4\right]$$

$$= \frac{1}{24.1443}\left[-0.4163 - (-0.06976)(-1.1832)\right] = -0.020661$$

$$x_2 = \frac{1}{a_{22}}\left[C_2 - \sum_{j=3}^{4} a_{2j} x_j\right] = \frac{1}{a_{22}}\left[C_2 - a_{23}x_3 - a_{24}x_4\right]$$

$$= \frac{1}{100}\left[2.15 - (-99.9)(-0.020661) - (1)(-1.1832)\right] = 0.012692$$

$$x_1 = \frac{1}{a_{11}}\left[C_1 - \sum_{j=2}^{4} a_{1j} x_j\right] = \frac{1}{a_{11}}\left[C_1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4\right]$$

$$= \frac{1}{15.3}\left[4.16 - (0.110)(0.012692) - (-13.1)(-0.020661) - (-1)(-1.1832)\right] = 0.17678$$

2. Find the inverse of the matrix A where $A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & -1 & 6 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$

$$AA^{-1} = A^{-1}A = I$$

$$\begin{array}{r} \\ -\frac{1}{4}\times 1^{st}\text{ row} \\ +\frac{1}{4}\times 1^{st}\text{ row} \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 4 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 6 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \begin{array}{r} \\ \\ +\frac{3}{11}\times 2^{nd} \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 4 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{11}{4} & -\frac{3}{4} & 0 & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & -\frac{3}{4} & \frac{23}{4} & 2 & \frac{1}{4} & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{r} 1^{st}\times\frac{1}{4} \\ 2^{nd}\times\frac{4}{11} \\ 3^{rd}\times\frac{11}{61} \\ 4^{st}\times\frac{61}{261} \end{array} \left[\begin{array}{cccc|cccc} 4 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{11}{4} & -\frac{3}{4} & 0 & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & 0 & \frac{61}{11} & 2 & \frac{2}{11} & \frac{3}{11} & 1 & 0 \\ 0 & 0 & 0 & \frac{261}{61} & -\frac{4}{61} & -\frac{6}{61} & -\frac{22}{61} & 1 \end{array}\right] \leftarrow \begin{array}{r} \\ \\ -\frac{22}{61}\times 3^{rd} \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 4 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{11}{4} & -\frac{3}{4} & 0 & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & 0 & \frac{61}{11} & 2 & \frac{2}{11} & \frac{3}{11} & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{r} \\ \\ -\frac{22}{61}\times 4^{st} \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{11} & 0 & -\frac{1}{11} & \frac{4}{11} & 0 & 0 \\ 0 & 0 & 1 & \frac{22}{61} & \frac{2}{61} & \frac{3}{61} & \frac{11}{61} & 0 \\ 0 & 0 & 0 & 1 & -\frac{4}{261} & -\frac{2}{87} & -\frac{22}{261} & \frac{61}{261} \end{array}\right] \rightarrow \begin{array}{r} +\frac{1}{4}\times 3^{rd} \\ +\frac{3}{11}\times 3^{rd} \\ \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{11} & 0 & -\frac{1}{11} & \frac{4}{11} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{10}{261} & \frac{5}{87} & \frac{55}{261} & -\frac{22}{261} \\ 0 & 0 & 0 & 1 & -\frac{4}{261} & -\frac{2}{87} & -\frac{22}{261} & \frac{61}{261} \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{73}{261} & -\frac{7}{87} & \frac{10}{261} & -\frac{4}{261} \\ 0 & 1 & 0 & 0 & -\frac{7}{87} & \frac{11}{29} & \frac{5}{87} & -\frac{2}{87} \\ 0 & 0 & 1 & 0 & \frac{10}{261} & \frac{5}{87} & \frac{55}{261} & -\frac{22}{261} \\ 0 & 0 & 0 & 1 & -\frac{4}{261} & -\frac{2}{87} & -\frac{22}{261} & \frac{61}{261} \end{array}\right] \leftarrow \begin{array}{r} -\frac{1}{4}\times 2^{rd} \\ \\ \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & \frac{1}{4} & 0 & 0 & \frac{271}{1044} & \frac{5}{348} & \frac{55}{1044} & -\frac{11}{522} \\ 0 & 1 & 0 & 0 & -\frac{7}{87} & \frac{11}{29} & \frac{5}{87} & -\frac{2}{87} \\ 0 & 0 & 1 & 0 & \frac{10}{261} & \frac{5}{87} & \frac{55}{261} & -\frac{22}{261} \\ 0 & 0 & 0 & 1 & -\frac{4}{261} & -\frac{2}{87} & -\frac{22}{261} & \frac{61}{261} \end{array}\right]$$

$$\therefore\ A^T = \begin{bmatrix} \frac{73}{261} & -\frac{7}{87} & \frac{10}{261} & -\frac{4}{261} \\ -\frac{7}{87} & \frac{11}{29} & \frac{5}{87} & -\frac{2}{87} \\ \frac{10}{261} & \frac{5}{87} & \frac{55}{261} & -\frac{22}{261} \\ -\frac{4}{261} & -\frac{2}{87} & -\frac{22}{261} & \frac{61}{261} \end{bmatrix} \approx \begin{bmatrix} 0.2797 & -0.08046 & 0.03831 & -0.01533 \\ -0.08046 & 0.3793 & 0.05747 & -0.02299 \\ 0.03831 & 0.05747 & 0.2107 & -0.08429 \\ -0.01533 & -0.02299 & -0.08429 & 0.2337 \end{bmatrix}$$

3. Use Crout factorization for a tri-diagonal system to solve the problem

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{Bmatrix}.$$

Crout factorization →

$n=4$

$$A = LU = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & 0 & 0 \\ l_{21} & l_{21}u_{12}+l_{22} & l_{22}u_{23} & 0 \\ 0 & l_{32} & l_{32}u_{23}+l_{33} & l_{33}u_{34} \\ 0 & 0 & l_{43} & l_{43}u_{34}+l_{44} \end{bmatrix}$$

$$\left.\begin{aligned}
& l_{11}=3,\ l_{21}=-1,\ l_{32}=-1,\ l_{43}=-1 \\
& l_{11}u_{12}=-1 \Rightarrow u_{12}=-\tfrac{1}{3} \\
& l_{21}u_{12}+l_{22}=3 \Rightarrow l_{22}=3-(-1)(-\tfrac{1}{3})=\tfrac{8}{3} \\
& l_{22}u_{23}=-1 \Rightarrow u_{23}=\frac{-1}{8/3}=-\frac{3}{8} \\
& l_{32}u_{23}+l_{33}=3 \Rightarrow l_{33}=3-(-1)(-\tfrac{3}{8})=\tfrac{21}{8} \\
& l_{33}u_{34}=-1 \Rightarrow u_{34}=\frac{-1}{21/8}=-\frac{8}{21} \\
& l_{43}u_{34}+l_{44}=3 \Rightarrow l_{44}=3-(-1)(-\tfrac{8}{21})=\tfrac{55}{21}
\end{aligned}\right\} \Rightarrow L=\begin{bmatrix} 3 & 0 & 0 & 0 \\ -1 & \frac{8}{3} & 0 & 0 \\ 0 & -1 & \frac{21}{8} & 0 \\ 0 & 0 & -1 & \frac{55}{21} \end{bmatrix}$$

$$U=\begin{bmatrix} 1 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{3}{8} & 0 \\ 0 & 0 & 1 & -\frac{8}{21} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} A\vec{x}=\vec{b} \\ U\vec{x}=\vec{y} \\ L\vec{y}=\vec{b} \end{cases} \Rightarrow$$

① $L\vec{y}=\vec{b}$:

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -1 & \frac{8}{3} & 0 & 0 \\ 0 & -1 & \frac{21}{8} & 0 \\ 0 & 0 & -1 & \frac{55}{21} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} & 3y_1=2 \to y_1=\tfrac{2}{3} \\ & -y_1+\tfrac{8}{3}y_2=3 \to y_2=(3+\tfrac{2}{3})\times\tfrac{3}{8}=\tfrac{11}{8} \\ & -y_2+\tfrac{21}{8}y_3=4 \to y_3=(4+\tfrac{11}{8})\times\tfrac{8}{21}=\tfrac{43}{21} \\ & -y_3+\tfrac{55}{21}y_4=1 \to y_4=(1+\tfrac{43}{21})\times\tfrac{21}{55}=\tfrac{64}{55} \end{aligned}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{11}{8} \\ \frac{43}{21} \\ \frac{64}{55} \end{bmatrix}$$

② $U\vec{x}=\vec{y}$:

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{3}{8} & 0 \\ 0 & 0 & 1 & -\frac{8}{21} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{11}{8} \\ \frac{43}{21} \\ \frac{64}{55} \end{bmatrix} \Rightarrow \begin{aligned} & x_4=\tfrac{64}{55} \\ & x_3-\tfrac{8}{21}x_4=\tfrac{43}{21} \to x_3=\tfrac{43}{21}+\tfrac{8}{21}\times\tfrac{64}{55}=\tfrac{137}{55} \\ & x_2-\tfrac{3}{8}x_3=\tfrac{11}{8} \to x_2=\tfrac{11}{8}+\tfrac{3}{8}\times\tfrac{137}{55}=\tfrac{127}{55} \\ & x_1-\tfrac{1}{3}x_2=\tfrac{2}{3} \to x_1=\tfrac{2}{3}+\tfrac{1}{3}\times\tfrac{127}{55}=\tfrac{79}{55} \end{aligned}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{79}{55} \\ \frac{127}{55} \\ \frac{137}{55} \\ \frac{64}{55} \end{bmatrix} = \begin{bmatrix} 1.4364 \\ 2.3091 \\ 2.4909 \\ 1.1636 \end{bmatrix}$$