HW7

Solve the problem

$$4x_1 - x_2 - x_4 = 0$$
$$-x_1 + 4x_2 - x_3 - x_5 = -1$$
$$-x_2 + 4x_3 + x_5 - x_6 = 9$$
$$-x_1 + 4x_4 - x_5 - x_6 = 4$$
$$-x_2 - x_4 + 4x_5 - x_6 = 8$$
$$-x_3 - x_5 + 4x_6 = 6$$

by (a) Jacobi method, (b) Gauss-Seidel method, (c) SOR method, and (d) the conjugate gradient method.

$A\vec{x} = \vec{b}$

$$A = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 1 & -1 \\ -1 & 0 & 0 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ -1 \\ 9 \\ 4 \\ 8 \\ 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Jacobi method.

$A = D + L + U$

$(D+L+U)\vec{x} = \vec{b} \rightarrow D\vec{x} = -(L+U)\vec{x} + \vec{b}$

$\rightarrow \vec{x} = -D^{-1}(L+U)\vec{x} + D^{-1}\vec{b}$

$\equiv T\vec{x} + \vec{c}$, where $\begin{cases} T = -D^{-1}(L+U) \\ \vec{c} = D^{-1}\vec{b} \end{cases}$

$$x_i^{(k+1)} = -\sum_{j \neq i}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k+1)} + \frac{b_i}{a_{ii}}$$ ⇒用迭代

(b) Gauss-Seidel method:

$(D+L)\vec{x} = -U\vec{x} + \vec{b} \rightarrow \vec{x} = -(D+L)^{-1}U\vec{x} + (D+L)^{-1}\vec{b} \equiv T_g\vec{x} + \vec{c}$

(c) SOR method:

$D\vec{x} + \omega(D+L+U)\vec{x} = D\vec{x} + \omega\vec{b}$

$\rightarrow D\vec{x} + \omega(D+L)\vec{x} = D\vec{x} - \omega U\vec{x} + \omega\vec{b}$

$\rightarrow (D+\omega L)\vec{x} = [(1-\omega)D - \omega U]\vec{x} + \omega\vec{b}$

$\rightarrow \vec{x} = (D+\omega L)^{-1}[(1-\omega)D - \omega U]\vec{x} + \omega(D+\omega L)^{-1}\vec{b} \equiv T_\omega\vec{x} + \vec{c}$

(d) $\min_{\vec{x}} g(\vec{x}) = \min_{\vec{x}} \left(\frac{1}{2}\vec{x}^T A\vec{x} - \vec{x}^T\vec{b}\right), A=A^T$

Define $h(t) = g(\vec{x}+t\vec{v})$

Find the t and $\vec{v} \Rightarrow g(\vec{x}+t\vec{v}) \le g(\vec{x})$

Answer:

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Jacobi method:
[1.17478856 1.64317358 2.44824809 3.05598067 3.94965767 3.09947644]
Gauss-Seidel method:
 [1.17478856 1.64317358 2.44824809 3.05598067 3.94965767 3.09947644]
SOR method:
[1.17478856 1.64317358 2.44824809 3.05598067 3.94965767 3.09947644]
Conjugate Gradient method:
[1.17462188 1.64240917 2.4472713  3.0562416  3.94977191 3.09912292]
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① Jacobi method

⇒ $x_1 = 1.17478856$
$x_2 = 1.64317358$
$x_3 = 2.44824809$
$x_4 = 3.05598067$
$x_5 = 3.94965767$
$x_6 = 3.09947644$

② Gauss-Seidel method

⇒ $x_1 = 1.17478856$
$x_2 = 1.64317358$
$x_3 = 2.44824809$
$x_4 = 3.05598067$
$x_5 = 3.94965767$
$x_6 = 3.09947644$

③ SOR method

⇒ $x_1 = 1.17478856$
$x_2 = 1.64317358$
$x_3 = 2.44824809$
$x_4 = 3.05598067$
$x_5 = 3.94965767$
$x_6 = 3.09947644$

④ Conjugate gradient method

⇒ $x_1 = 1.17462188$
$x_2 = 1.64240917$
$x_3 = 2.4472713$
$x_4 = 3.0562416$
$x_5 = 3.94977191$
$x_6 = 3.09912292$