HW7

Solve the problem

$4x_1 - x_2 - x_4 = 0$

$-x_1 + 4x_2 - x_3 - x_5 = -1$

$-x_2 + 4x_3 + x_5 - x_6 = 9$

$-x_1 + 4x_4 - x_5 - x_6 = 4$

$-x_2 - x_4 + 4x_5 - x_6 = 8$

$-x_3 - x_5 + 4x_6 = 6$

by (a) Jacobi method, (b) Gauss-Seidel method, (c) SOR method, and (d) the conjugate gradient method.

$A\vec{x} = \vec{b}$

$$A = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 1 & -1 \\ -1 & 0 & 0 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ -1 \\ 9 \\ 4 \\ 8 \\ 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Jacobi method.

$A = D + L + U$

$(D+L+U)\vec{x} = \vec{b} \rightarrow D\vec{x} = -(L+U)\vec{x} + \vec{b}$

$\rightarrow \vec{x} = -D^{-1}(L+U)\vec{x} + D^{-1}\vec{b}$

$\equiv T\vec{x} + \vec{c}$, where $\begin{cases} T = -D^{-1}(L+U) \\ \vec{c} = D^{-1}\vec{b} \end{cases}$

$x_i^{(k+1)} = -\sum_{j\neq i}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k+1)} + \frac{b_i}{a_{ii}}$ ⇒用迭代，設初始值 $x_i^{(0)} = 0$

$x_1^{(k+1)} = \frac{1}{4}\left(x_2^{(k)} + x_4^{(k)}\right)$

$x_2^{(k+1)} = \frac{1}{4}\left(x_1^{(k)} + x_3^{(k)} + x_5^{(k)} - 1\right)$

$x_3^{(k+1)} = \frac{1}{4}\left(x_2^{(k)} - x_5^{(k)} + x_6^{(k)} + 9\right)$

$x_4^{(k+1)} = \frac{1}{4}\left(x_1^{(k)} + x_5^{(k)} + x_6^{(k)} + 4\right)$

$x_5^{(k+1)} = \frac{1}{4}\left(x_2^{(k)} + x_4^{(k)} + x_6^{(k)} + 8\right)$

$x_6^{(k+1)} = \frac{1}{4}\left(x_3^{(k)} + x_5^{(k)} + 6\right)$

(b) Gauss-Seidel method: → 與Jacobi類似，但每項用最新數據

$$(D+L)\vec{x} = -U\vec{x}+\vec{b} \rightarrow \vec{x} = -(D+L)^{-1}U\vec{x}+(D+L)^{-1}\vec{b} \equiv T_g\vec{x}+\vec{c}$$

$$x_1^{(k+1)} = \frac{1}{4}\left(x_2^{(k)}+x_4^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{4}\left(x_1^{(k+1)}+x_3^{(k)}+x_5^{(k)}-1\right)$$

$$x_3^{(k+1)} = \frac{1}{4}\left(x_2^{(k+1)}-x_5^{(k)}+x_6^{(k)}+9\right)$$

$$x_4^{(k+1)} = \frac{1}{4}\left(x_1^{(k+1)}+x_5^{(k)}+x_6^{(k)}+4\right)$$

$$x_5^{(k+1)} = \frac{1}{4}\left(x_2^{(k+1)}+x_4^{(k)}+x_6^{(k)}+8\right)$$

$$x_6^{(k+1)} = \frac{1}{4}\left(x_3^{(k+1)}+x_5^{(k)}+6\right)$$

(c) SOR method:

$$D\vec{x}+\omega(D+L+U)\vec{x} = D\vec{x}+\omega\vec{b}$$

$$\rightarrow D\vec{x}+\omega(D+L)\vec{x} = D\vec{x}-\omega U\vec{x}+\omega\vec{b}$$

$$\rightarrow (D+\omega L)\vec{x} = [(1-\omega)D-\omega U]\vec{x}+\omega\vec{b}$$

$$\rightarrow \vec{x} = (D+\omega L)^{-1}[(1-\omega)D-\omega U]\vec{x}+\omega(D+\omega L)^{-1}\vec{b} \equiv T_\omega\vec{x}+\vec{c}$$

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)}+\frac{\omega}{a_{ii}}\left(b_i-\sum_{j<i}a_{ij}x_j^{(k+1)}-\sum_{j>i}a_{ij}x_j^{(k)}\right)$$ → 與Gauss-Seidel類似，乘上 ω 並加上 $(1-\omega)x_i^{(k)}$

(d) $\min\limits_{\vec{x}} g(\vec{x}) = \min\limits_{\vec{x}}\left(\frac{1}{2}\vec{x}^T A\vec{x}-\vec{x}^T\vec{b}\right), A=A^T$

Define $h(t) = g(\vec{x}+t\vec{v})$

Find the t and $\vec{v}$ $\Rightarrow$ $g(\vec{x}+t\vec{v}) \le g(\vec{x})$

$$r^{(0)} = b-Ax^{(0)}$$

$$p^{(0)} = r^{(0)}$$

$$a_k = \frac{r^{(k)T}r^{(k)}}{p^{(k)T}Ap^{(k)}}$$

$$x^{(k+1)} = x^{(k)}+a_k p^{(k)}$$

$$r^{(k+1)} = r^{(k)}-a_k A p^{(k)}$$

$$\beta_k = \frac{r^{(k+1)T}r^{(k+1)}}{r^{(k)T}r^{(k)}}$$

$$p^{(k+1)} = r^{(k+1)}+\beta_k p^{(k)}$$

Answer:

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Jacobi method:
[1.17478856 1.64317358 2.44824809 3.05598067 3.94965767 3.09947644]
Gauss-Seidel method:
 [1.17478856 1.64317358 2.44824809 3.05598067 3.94965767 3.09947644]
SOR method:
[1.17478856 1.64317358 2.44824809 3.05598067 3.94965767 3.09947644]
Conjugate Gradient method:
[1.17462188 1.64240917 2.4472713  3.0562416  3.94977191 3.09912292]
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① Jacobi method

$\Rightarrow$ $x_1 = 1.17478856$

$x_2 = 1.64317358$

$x_3 = 2.44824809$

$x_4 = 3.05598067$

$x_5 = 3.94965767$

$x_6 = 3.09947644$

② Gauss-Seidel method

$\Rightarrow$ $x_1 = 1.17478856$

$x_2 = 1.64317358$

$x_3 = 2.44824809$

$x_4 = 3.05598067$

$x_5 = 3.94965767$

$x_6 = 3.09947644$

③ SOR method

$\Rightarrow$ $x_1 = 1.17478856$

$x_2 = 1.64317358$

$x_3 = 2.44824809$

$x_4 = 3.05598067$

$x_5 = 3.94965767$

$x_6 = 3.09947644$

④ Conjugate gradient method

$\Rightarrow$ $x_1 = 1.17462188$

$x_2 = 1.64240917$

$x_3 = 2.4472713$

$x_4 = 3.0562416$

$x_5 = 3.94977191$

$x_6 = 3.09912292$