Continuous-Galerkin Finite Element Method (CGFEM) and its implementation in `Firedrake`

A short introduction

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CGFEM procedure

1. Derive the weak formulation
   - Multiply a PDE by an (arbitrary) test function
   - Integrate over the entire domain or
     integrate over each element and then take the sum
   - Integration by parts and apply the boundary conditions

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   - Assemble the matrix (globally)
4. Solve the algebraic system

Poisson's equation in a unit square

Example

For a given function f, seek u such that in $\Omega = [0,1] \times [0,1]$

$$-\nabla^2 u = f, \tag{1a}$$
$$u(0,y) = u(1,y) = 0, \tag{1b}$$
$$\partial_y u(x,0) = \partial_y u(x,1) = 0. \tag{1c}$$

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Firedrake implementation

"Firedrake is an automated system for the solution of partial differential equations using the finite element method (FEM). Firedrake uses *sophisticated code generation* to provide mathematicians, scientists, and engineers with a very high productivity way to create sophisticated high performance simulations."

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Example

Choose f to be

$$f = 2\pi^2 \sin(\pi x)\cos(\pi y), \tag{2}$$

which yields the exact solution:

$$u_{\mathrm{ex}} = \sin(\pi x)\cos(\pi y). \tag{3}$$

Demonstration

- Mathematical and numerical modelling: **CGFEM**
- Computational modelling: **Firedrake**
  `https://www.firedrakeproject.org/`
- Post-Processing
  - Visualising the results: **ParaView**
    `https://www.paraview.org/`
  - Verification and validation
    e.g. convergence analysis based on the L^2 error:

$$L^2(\Delta x) = \sqrt{\sum_i \left(u_{h,i} - u_{\mathrm{ex},i}\right)^2}$$

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Differential equations and minimisation problems

Poisson's equation in a unit square

For a given function f, consider the PDE with homogeneous boundary conditions in $\Omega = [0,1] \times [0,1]$:

$$-\nabla^2 u = f, \tag{4a}$$

$$u|_{\Gamma_1} = 0, \tag{4b}$$

$$\nabla u \cdot \mathbf{n}|_{\Gamma_2} = 0, \tag{4c}$$

where the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$.
Then the solution u minimises the functional

$$I[u] = \iint_\Omega \frac{1}{2}|\nabla u|^2 - uf \,\mathrm{d}\Omega \tag{5}$$

over the space $\Sigma = \{u \text{ smooth} \mid u|_{\Gamma_1 = 0}\}$.
On the other hand, if u minimises (5) then u satisfies (4).

From the minimisation problem to the PDE[1]

Let $\hat{u}$ minimise (5), consider a set of functions $u(x, y) = \hat{u}(x, y) + \epsilon\eta(x, y)$, where ϵ is a parameter and η is an arbitrary function satisfying $\eta|_{\Gamma_1} = 0$. A necessary condition for the existence of a minimum of (5) at $\epsilon = 0$ is

$$\delta I \equiv \left.\frac{\mathrm{d}I}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \lim_{\epsilon\to 0} \frac{I[\hat{u}(x, y) + \epsilon\,\eta(x, y)] - I[\hat{u}(x, y)]}{\epsilon} = 0. \tag{6}$$

$$\Rightarrow \qquad \iint_\Omega \nabla\hat{u}\cdot\nabla\eta - \eta f \,\mathrm{d}\Omega = 0 \tag{7}$$

Using integration by parts and Gauss' theorem:

$$\Rightarrow \quad -\iint_\Omega \eta\left(\nabla^2\hat{u} + f\right)\,\mathrm{d}\Omega + \cancelto{0}{\int_{\Gamma_1} \eta\nabla\hat{u}\cdot\mathbf{n}\,\mathrm{d}\Gamma} + \cancelto{0}{\int_{\Gamma_2} \eta\nabla\hat{u}\cdot\mathbf{n}\,\mathrm{d}\Gamma} = 0 \tag{8}$$

Instead of solving (4), alternatively, we can solve the weak formulation (7).

[1] J. van Kan (2014) *Numerical methods in Scientific Computing*. Delft Academic Press, The Netherlands. Chapter 5.

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Mathematical model

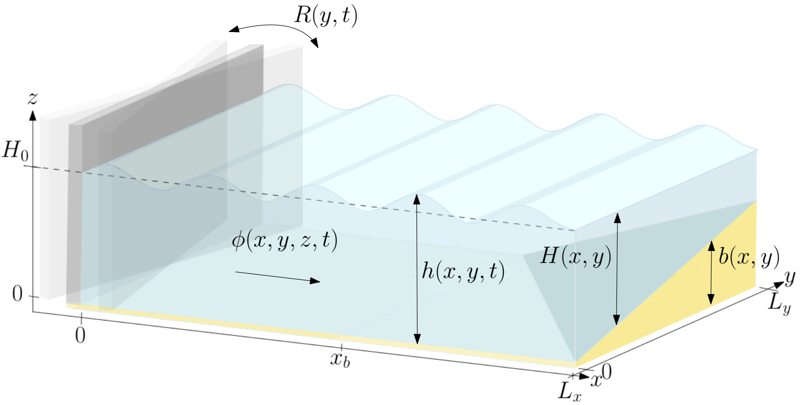


Figure: Schematic of the numerical wave tank. Waves are generated by a vertical piston wavemaker oscillating horizontally at $x = R(y,t)$ around $x = 0$. The depth at rest $H(x,y)$ varies in space due to the nonuniform seabed topography $b(x,y)$.

Mathematical model

In this study, the nonlinear potential-flow equations (PFE)

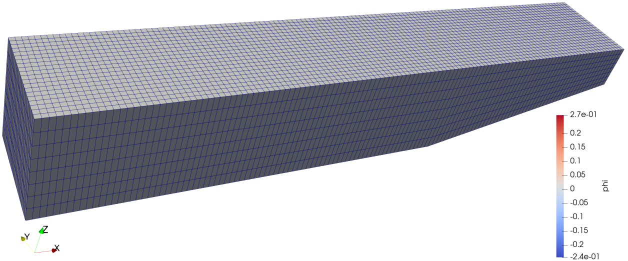
$$
\begin{aligned}
\delta\phi:&\ \nabla^2\phi = 0, \quad \text{in } \Omega, && \text{(9a)}\\
(\delta\phi)|_{z=b+h}:&\ \partial_t h + \nabla(h+b)\cdot\nabla\phi - \partial_z\phi = 0, \quad \text{at } z = b+h, && \text{(9b)}\\
\delta h:&\ \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(b+h-H_0) = 0, \quad \text{at } z = b+h, && \text{(9c)}\\
(\delta\phi)|_{x=R}:&\ \partial_x\phi - \partial_y\phi\,\partial_y R = \partial_t R, \quad \text{at } x = R, && \text{(9d)}
\end{aligned}
$$

are obtained from Luke's variational principle [2]:

$$
0 = \delta\int_0^T\iint_{\Omega_h}\int_{b(x,y)}^{b(x,y)+h(x,y,t)} \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(z-H_0)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \quad (10)
$$

[2] Luke, J. (1967). A variational principle for a fluid with a free surface. Journal of Fluid Mechanics, 27(2), 395-397.

Numerical results



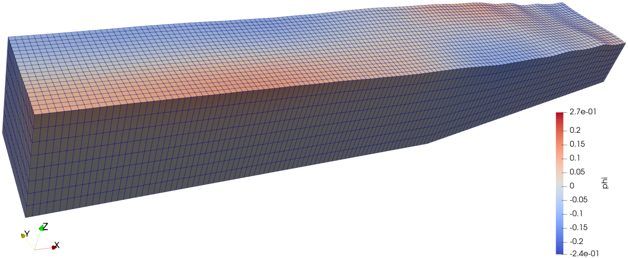


Figure: Velocity potential fields at $t = t_0$ (top) and $t = t_{\text{end}}$ (bottom).

Thank you!
Questions?