Solar System Modelling using Ordinary Differential Equations

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In this project we develop and test object oriented code to solve the n-body problem in two or three dimensions. We implement two ordinary differential equation solvers: the Forward Euler method and the velocity Verlet method. The systems studied are the *Sun-Earth*, *Sun-Earth-Jupiter*, *Solar System* and the relativistic *Sun-Mercury* system. The velocity Verlet method was found to be superior when comparing its accuracy per time steps N, and its energy conservation. Modelling the gravitational force as $1/r^{\beta}$, $\beta \in [2, 3]$ using known values for position and velocity of Earth at aphelion, we found that nature does not deviate significantly from a perfect inverse-square law and that $\beta \approx 3$ leads to especially unstable behaviour. In addition, the modified gravitational force behaves especially unphysically for. The orbital trajectories of the Solar System are presented are presented in two and three dimensions, and the perihelion precession of Mercury was found to be 0.4309.

I. INTRODUCTION

Looking up at the sky on a clear night, one can see spots of light spread across the heavens. Some of these shine brighter than the rest, and their places change faster. These are the ones we have come to know as planets. Perhaps the sky will also be lit up by the Moon, whose movements determine the tides of our oceans. As the night comes to an end, one will slowly see the light from the Sun appear at the horizon, and as the day passes the Sun will travel across the sky, spreading the light and heat that makes life on Earth possible.

The celestial bodies and their movements have always been of great interest to us humans. Not only are some of them vital for our existence, but the study of celestial bodies have also shaped how we look at ourselves and our place in the universe. Over the last centuries a great amount of scientific effort has been put into understanding the solar system our Earth is a part of, and predicting the movements of the celestial bodies around us. With the shift from a geocentric to a heliocentric world view, first put forward by Nicolas Copernicus in 1543, the Earth went from being considered the center of the universe to being one of many planets orbiting the Sun [2]. When Johannes Kepler published his laws of planetary motion, he replaced the perfectly circular planetary orbits of Copernican heliocentrism with ellipses [2]. When Isaac Newton published his law of universal gravitation in 1687, Kepler's laws could be seen as results of more general principles [2]. With his expression for the gravitational force, Newton was able to describe both falling objects on Earth and planetary motion almost perfectly with a single mathematically formulated principle. However, there were some observations that deviated slightly from the Newtonian solar system model. The most significant one was the perihelion precession of Mercury, in other words how much the angle where Mercury is closest to the sun changes over time. When Albert Einstein put forward his general theory of relativity in 1915, which today is considered the current best theory of gravitation and spacetime, he was able to explain the observed value of Mercury's perihelion precession [3]. This was an important test for the theory.

In this project, our goal is to build an object oriented numerical model that simulates the n-body problem. To do so we implement two methods for solving ordinary differential equations: the Forward Euler method and the velocity Verlet method. The aim is then to test and compare these two models for a 2-body system, and then increase to a 3-body problem while improving upon the functionality of our code. In the following we first look at the simple Earth-Sun system with Earth having a circular orbit, we use this system to do our initial tests and then we expand our numerical model to the Earth-Sun-Jupiter system. Finally we will model the whole Solar System, look closer at the gravitational inverse square law and model the perihelion precession of Mercury using relativistic corrections to the gravitational law.

The content that follows is a presentation of the methods and theory we have used in the section "Method", followed by a presentation of our results in "Results". After this we discuss our findings in "Discussion", and finally conclude with a consideration of further prospects.

II. METHOD

In this section we first present how we have set up our system of coupled differential equations. We then present the two methods we have used to solve differential equations, and the algorithms we have implemented. Then, we present the first system we consider: the Earth-Sun system. This subsection also includes some general explanations of conventions we use in project. Thereafter, we present how we implement the three-body problem consisting of Earth, Sun and Jupiter and after that how we extend our framework to the full solar system. Finally we present the algorithm we use for finding the perihelion precession of Mercury.

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For all program code, tests runs, output files and plots obtained see our GitHub-repository.

A. The coupled differential equations system

In our simulation of the solar system we have approximated the Sun and the planets as point particles with masses corresponding to the respective solar and planetary masses. The accelerations of the celestial bodies are governed only by the gravitational force $\mathbf{F}_G$ acting on each body from the other bodies in the system. This results in the following coupled differential equation for he k-th celestial body

$$\frac{\mathrm{d}^2\mathbf{r}_k}{\mathrm{d}t^2} = \frac{1}{m_k}\sum_{j\neq k}\mathbf{F}_G(\mathbf{r}_j(t), \mathbf{r}_k(t)) \tag{1}$$

where $\mathbf{r}_k$ is the position of the k-th celestial body, $\mathbf{r}_j$ is the position of celestial body $j \neq k$, t is time and m_k is the mass of the k-th body. We rewrite this second-order differential equation for each body k as two first-order equations

$$\frac{\mathrm{d}\mathbf{v}_k}{\mathrm{d}t} = \mathbf{a}_k(t) = \frac{1}{m_k}\sum_{j\neq k}\mathbf{F}_G(\mathbf{r}_j(t), \mathbf{r}_k(t)) \tag{2}$$

$$\frac{\mathrm{d}\mathbf{r}_k}{\mathrm{d}t} = \mathbf{v}_k(t) \tag{3}$$

where $\mathbf{a}_k(t)$ denotes the acceleration of the k-th body and $\mathbf{v}_k(t)$ denotes the velocity. Equations (2) and (3) are the ones we solve using numerical methods. To do this, we discretize the equations. We choose a final time T and a number of grid points N, and let the time t be given as $t_i = ih$ for $i = 0, \ldots, N$. The time step h is given by $h = T/N$. We can then consider the discretized approxmiations for $\mathbf{a}_k$, $\mathbf{v}_k$ and $\mathbf{r}_k$ for the k-th body as $\mathbf{a}_{k,i} = \mathbf{a}_k(t_i)$, $\mathbf{v}_{k,i} = \mathbf{v}_k(t_i)$ and $\mathbf{r}_{k,i} = \mathbf{r}_k(t_i)$.

The fact that the form of the equations can be written generally for all celestial bodies has allowed us to object orient our code. We construct a celestial body by creating an instance of a general celestial body class, and giving it a mass, an initial position and an initial velocity. We put these bodies in the same solar system by creating an instance of a general solar system class and adding our celestial bodies instances to it. In our general solar system class we have implemented functions that solve the coupled differential equations system for all the celestial bodies contained in the solar system. The solver functions we have implemented are Euler's forward method and the velocity Verlet method.

B. Euler's forward method

Euler's forward method, as so many of our differential equation solvers, is an algorithm derived from a simple Taylor expansion. If we expand around $\mathbf{r}(t_i)$ and $\mathbf{v}(t_i)$, moving one time step h forward, we can express $\mathbf{r}(t_i+h)$ and $\mathbf{v}(t_i+h)$ as

$$\mathbf{r}(t_i+h) = \sum_{n=0}^{\infty}\frac{\mathbf{r}^{(n)}(t_i)}{n!}h^n = \mathbf{r}(t_i) + \mathbf{r}^{(1)}(t_i)h + O(h^2) \tag{4}$$

$$\mathbf{v}(t_i+h) = \sum_{n=0}^{\infty}\frac{\mathbf{v}^{(n)}(t_i)}{n!}h^n = \mathbf{v}(t_i) + \mathbf{v}^{(1)}(t_i)h + O(h^2). \tag{5}$$

where the superscript ${}^{(n)}$ denotes the n-th derivative. Euler's forward method uses the Taylor expansion up to the second term. Using that $\mathbf{r}^{(1)}(t_i) = \mathbf{v}(t_i)$ and $\mathbf{v}^{(1)}(t_i) = \mathbf{a}(t_i)$, gives the following algorithm:

```
for i = 0,...,N:
```

$$\texttt{a} = \texttt{F(r(i))/m} \tag{6}$$

$$\texttt{r(i+1)} = \texttt{r(i) + v(i)*h} \tag{7}$$

$$\texttt{v(i+1)} = \texttt{v(i) + a*h} \tag{8}$$

As we can see from equation (4) and (5), this method has an error of $O(h^2)$ for each time step. For $N \sim 1/h$ steps, this results in a total error of $O(h)$.

We see that this algorithm requires 4 floating point operations (FLOPs) each time step, in addition to the n FLOPs required to calculate `a`, which means that it reqiures $\sim (4+n)N$ FLOPs in total.

C. The velocity Verlet method

The velocity Verlet method is based on Taylor expansion as well. In this case however, we use the Taylor expansion up to the *third* term. This gives us

$$\mathbf{r}(t_i+h) = \mathbf{r}(t_i) + \mathbf{r}^{(1)}(t_i)h + \mathbf{r}^{(2)}(t_i)\frac{h^2}{2} + O(h^3) \tag{9}$$

$$\mathbf{v}(t_i+h) = \mathbf{v}(t_i) + \mathbf{v}^{(1)}(t_i)h + \mathbf{v}^{(2)}(t_i)\frac{h^2}{2} + O(h^3). \tag{10}$$

However, we do not have an expression for $\mathbf{v}^{(2)}(t_i)$. We can approximate it as

$$\mathbf{v}^{(2)}(t_i) = \frac{\mathbf{v}^{(1)}(t_i+h) - \mathbf{v}^{(1)}(t_i)}{h} \tag{11}$$

Putting this into equation (10), we get

$$\mathbf{v}(t_i) + \mathbf{v}^{(1)}(t_i)h + \left[\frac{\mathbf{v}^{(1)}(t_i+h) - \mathbf{v}^{(1)}(t_i)}{h}\right]\frac{h^2}{2} + O(h^3) \tag{12}$$

$$= \mathbf{v}(t_i) + \left[\mathbf{v}^{(1)}(t_i+h) + \mathbf{v}^{(1)}(t_i)\right]\frac{h}{2} + O(h^3) = \mathbf{v}(t_i+h) \tag{13}$$

This gives us an algorithm where we need both the current and the new acceleration to calculate the velocity in each step. Before we calculate the new velocity, we need to use the new position to get the new acceleration. As we calculate the new acceleration in each time step, we can simply store this acceleration for the next step. This requires that we calculate the first acceleration outside the loop. We use the following algorithm:

```
a = F(r(0))/m
for i = 0,...,N:
```

$$\texttt{r(i+1)} = \texttt{r(i) + v(i)*h + a*h*h/2} \tag{14}$$

$$\texttt{a\_new} = \texttt{F(r(i+1))/m} \tag{15}$$

$$\texttt{v(i+1)} = \texttt{v(i) + (a\_new + a)*h/2} \tag{16}$$

$$\texttt{a} = \texttt{a\_new} \tag{17}$$

As we can see from equation (9) and (10), this method has an error of $O(h^3)$ for each time step. This results in a total error of $O(h^2)$, on order of h smaller than Euler's forward method.

As `h*h/2` and `h/2` are constants we can calculate outside the loop, we see that this algorithm requires 7 FLOPs each time step, excluding the FLOPs required to calculate the accelerations. As we only calculate the next acceleration in each step, we require the same number n FLOPs each time step as Euler's forward method to get the accelerations. The algorithm requires $\sim (7+n)N$ FLOPs in total.

D. The Earth-Sun system

In the first part of this project, we consider a solar system consisting only of the Earth and the Sun. We let the movements be governed by Newton's law of universal gravitation. The law describes the gravitational force $\mathbf{F}_G$ between two massive objects as

$$\mathbf{F}_G = -\frac{GMm}{|\mathbf{r}|^2}\hat{\mathbf{r}} \tag{18}$$

where M is the mass of the first object, m is the mass of the second object, G is the gravitational constant and $\mathbf{r}$ is the distance between the objects. The force is directed the opposite way of the distance vector since it is an attractive force.

The magnitude of the force between the Earth and the Sun will therefore be given by

$$F_G = \frac{GM_\odot M_{Earth}}{r^2}, \tag{19}$$

where $M_\odot$ is the solar mass, M_{Earth} is the mass of the Earth and r is the distance between the bodies.

The units we use for length, time and mass in this project are consequently astronomical units AU, years and solar mass $M_\odot$, respectively.

1. *Circular orbit and method stability*

An important first step in any computational project is to test the implemented algorithms for simple cases with known results. For this project we need to check that our velocity Verlet method gives stable and accurate results, and also the expected CPU time usage. We also test it against the forward Euler's method to see if we get the expected improvements. The simplified case we will look at is that of a simple circular orbit. We use a system consisting of the Sun and Earth, where Earth has a circular orbit with radius 1 AU. Using this we get an orbital circumference of 2π, hence, a velocity of 2π/year. Using this simplification and equation (19) we also find a neat scaling of our problem:

$$v^2 r = GM_\odot = 4\pi^2 \left[\frac{\text{AU}^3}{\text{year}^2}\right]$$

Where $v^2 r$ is the centripetal acceleration for circular motion. Since we scale all problems with the mass of the Sun this value of G will be used throughout the project. To actually implement this test in our code we use the initial condition for this simplified Earth orbit, and do several runs of both methods for an increasing number of timesteps N, hence, decreasing the timestep $\mathrm{d}t$. We then want to look at the behaviour of the orbital trajectories, the variation in distance to the Sun and the CPU time usage for each method.

2. *Conservation of energy*

The gravitational force is a conservative force. This means that the total mechanical energy of the objects should be conserved under the influence of the force. A conservative force is defined as something we can write as the negative gradient of a potential Φ

$$\mathbf{F} = -\nabla\Phi \tag{20}$$

In our case, we can find the gravitational potential U_g of our force by integration.

$$U_g = -\int_r^\infty \frac{GMm}{r^2}dr = -GMm\left[-\frac{1}{r}\right]_r^\infty \tag{21}$$

$$= -GMm\left[0 - (-\frac{1}{r})\right] = -\frac{GMm}{r}. \tag{22}$$

U_g depends on the inverse of r. In the case of the circular orbit, both the Earth-Sun distance r and the velocity of the Earth should be constant. Therefore, the potential energy $U_g = -GM_\odot M_{Earth}/r$ and the kinetic energy $K = \frac{1}{2}M_{Earth}v^2$ should be constant as well. We use this as a test to see if both our differential equation solvers manage to simulate the system in a physical way, by calculating how the kinetic and potential energy in the circular orbit evolves over time.

In the rest of the project we consequently use the velocity Verlet solver.

3. Conservation of angular momentum

We use Kepler's 2nd law to show that angular momentum should be conversed even if the orbit is an ellipse. This law states that the a line joining a planet and the Sun sweeps out equal areas during equal intervals of time. During a small amount of time $\mathrm{d}t$, a planet sweeps out an area $\mathrm{d}A = \frac{1}{2}(r)(r\mathrm{d}\theta)$. Kepler's law states that this area should be conserved, yielding a constant areal velocity

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{r^2}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}rv_\theta \tag{23}$$

The angular momentum L of an object is defined as $\vec{L} = m(\vec{r} \times \vec{v})$, where r is distance of the object from the rotational axis, and v is the velocity and m is the mass. For a point on the orbit, we will have

$$|\vec{L}| = mrv_\theta = m \cdot 2\frac{\mathrm{d}A}{\mathrm{d}t}, \tag{24}$$

putting in the expression from equation (23). Since $\frac{\mathrm{d}A}{\mathrm{d}t}$ is a constant, the angular momentum is be constant as well.

We investigate whether this holds in our solar system model by calculating angular momentum for both circular and elliptical orbits. As the mass of the Earth is constant, we only consider the specific angular momentum $|\vec{L}|/m$.

4. Escape velocity

We consider the escape velocity v_e of a planet in the gravitational field of the Sun. This is the minimum speed required for the planet to move so far away from the Sun that it is no longer affected by the Sun's gravitational field. In other words, the potential energy of the planet becomes zero. To find the escape velocity, we use the fact that the mechanical energy of the planet is conserved. We have

$$(K + U_g)_{initial} = (K + U_g)_{final}, \tag{25}$$

where K is the kinetic energy of the planet and U_g is the gravitational potential. The potential energy in the final state should be zero since the planet is no longer in the gravitational field of the Sun. The final kinetic energy should be zero as well, since the escape velocity is the *minimum* required speed to escape the gravitational field. This means that we have

$$(K + U_g)_{initial} = 0 \tag{26}$$

$$\frac{1}{2}mv_e^2 - \frac{GM_\odot m}{r_i} = 0, \tag{27}$$

where m is the mass of the planet, $M_\odot$ is the sun mass, G is the gravitational constant and r_i is the initial distance between the Sun and the planet. The analytical expression for the escape velocity is thereby

$$v_e = \sqrt{\frac{2GM_\odot}{r_i}}. \tag{28}$$

Using $GM_\odot = 4\pi^2\mathrm{AU}^3/\mathrm{yr}^2$ and $r_i = 1$ AU, we get

$$v_e = 2\pi\sqrt{2}\ \mathrm{AU/yr}. \tag{29}$$

We investigate in our model whether velocities under and above this value give allows Earth to escape from the Sun's gravitational field.

5. Variations on the gravitational force

In our Earth-Sun system we also test the implications of replacing the inverse-square gravitational force in equation (18) with a force on the form

$$\mathbf{F}_G = -\frac{GMm}{|\mathbf{r}|^\beta}\hat{\mathbf{r}} \tag{30}$$

with $\beta \in [2, 3]$. Increasing the exponent β can be understood as increasing the sensitivity of the force to the distance between the Earth and the Sun. If we consider a circular orbit where the Earth-Sun distance is $r = 1$ AU (in other words one where $v_{Earth} = 2\pi$ AU/year), the modified force should not differ from the inverse-square force. However, if we consider orbits where we have $r < 1$ AU, the modified force with $\beta > 2$ will increase faster than the inverse-square one as r becomes smaller. Equivalently, it will decrease faster than the inverse-square force for $r > 1$ AU.

We expect angular momentum to be conserved under the modified gravitational force, as this law is independent of Newton's gravitational law. We also expect the modified force to conserve the total mechanical energy, as we can find a gravitational potential for the force by integration:

$$U_{g,\beta} = -\int_r^\infty \frac{GMm}{r^\beta}dr = -GMm\left[-\frac{1}{\beta-1}\frac{1}{r^{\beta-1}}\right]_r^\infty \tag{31}$$

$$= -GMm\left[0 - \left(-\frac{1}{\beta-1}\frac{1}{r^{\beta-1}}\right)\right] = -\frac{GMm}{(\beta-1)r^{\beta-1}}. \tag{32}$$

We can see that as the gravitational potential is dependent on β, the scape velocity of the Earth will depend on β as

$$v_{e,\beta} = \sqrt{-\frac{2U_{g,\beta}}{M_{Earth}}} = \sqrt{\frac{2GM_\odot}{(\beta-1)r_i^{\beta-1}}} = \frac{v_e}{\sqrt{(\beta-1)r_i^{\beta-2}}} \tag{33}$$

where v_e is the escape velocity under the inverse-square force. For $r \geq 1$ AU and $\beta > 2$ this gives $v_{e,\beta} < v_e$.

First, we test the amount by which nature deviates from a perfect inverse-square law. We know that the Earth's orbit is not completely circular. We use values provided by NASA official Dr. David R. Williams [4] for the Earth-Sun distance at perihelion, where Earth is closest to the Sun, and aphelion, where Earth is furthest away from the Sun, and the respective velocities at these points. We use these as initial conditions and model how the Earth-Sun system evolves in time with different values of β.

Next, we check whether mechanical energy and angular momentum is conserved under the modified gravitational force. For this case we set the initial position of the Earth to be 1 AU away fom the Sun and the initial velocity to be 5 AU/yr, which results in an elliptical orbit with a considerably smaller Earth-Sun distance at perihelion. We compare the mechanical energy and specific angular momentum of the Earth, using the inverse-square force and the modified gravitational force with $\beta = 3$.

Finally, we also illustrate how the escape velocities vary under different β by using an initial velocity of 7 AU/yr.

E. The Earth-Sun-Jupiter system

One of the goals of this project is to extend our object oriented code to work for n-body problems, that is, modelling n-bodies all affecting each other with a gravitational pull. As outlined earlier, we structure our code in two class objects. The first represents a celestial body which holds all the information about "itself", such as position and velocity, mass, distance to other planets, and the forces acting on it from other bodies. The second then orchestrate the full evolution of all the bodies in the system as time passes. Hence, this class object contains all the celestial bodies, numerical integration methods, the total mass of the system, total energy and so on. This is the Solar System class. When these classes are setup we simply have to add all relevant information to the bodies, put these bodies into the general Solar system class, determine the time step and the time we wish to simulate over.

A good first test of our general Solar System class is to add three bodies, thus extending our previous 2-body problem. We do this for the Sun, Earth and Jupiter. This also serves as a good example of how much Jupiter affects Earths orbit. To really see Jupiter's effect, and also what a more general 3-body problem could look like we simulate for M_j, $M_j \cdot 10$ and $M_j \cdot 1000$, where M_j is the masss of Jupiter. We use the velocity Verlet method to solve this.

F. The full solar system

Before we move on to simulate the entire Solar System we improve the functionality of our Solar System class by implementing a function which fixes the whole system to the center-of-mass frame. We achieve this by using the expression for center of mass,

$$\mathbf{R} = \frac{1}{M_{\text{tot}}} \sum_{i=1}^{n} m_i \mathbf{r}_i$$

and then shifting all the bodies in our system by this amount to fix everything around the center of mass origin. Analogous to this we also fix the momenta of our bodies using:

$$\mathbf{V} = \frac{1}{M_{\text{tot}}} \sum_{i=1}^{n} m_i \mathbf{v}_i$$

This way the Solar System does not drift in any direction on our figures but should remain stable around the fixed origin. We use this method of fixing the origin together with our velocity Verlet solver to then calculate the orbital trajectories of the Sun and all planets, including Pluto. We also make another attempt at plotting the Sun-Earth-Jupiter system in the center-of-mass frame. We obtain all the data used for these simulations from NASA[1].

G. The perihelion precession of Mercury

Finally, we consider the perihelion precession of Mercury using both a pure Newtonian gravitational force and a gravitational force with a relativistic correction. The magnitude of this force is given as

$$F_G = \frac{GM_\odot M_{Mercury}}{r^2}\left[1 + \frac{3l^2}{r^2c^2}\right] \tag{34}$$

where r is the Mercury-Sun distance, $l = |\vec{r} \times \vec{v}|$ is the specific angular momentum of Mercury and c is the speed of light in vacuum.

The observed value of the perihelion precession of Mercury is 43 arcseconds per century when the perturbation of the orbit from other planets are subtracted. We model a solar system containing only Mercury and the Sun to see if we will get a value similar to this when using a gravitational force with a relativistic correction. We expect the pure Newtonian force to give no perihelion precession, as the inverse-square force should give closed elliptical orbits.

We find the value for the perihelion angle θ_p of Mercury in radians using

$$\tan\theta_p = \frac{y_p}{x_p} \tag{35}$$

where x_p (y_p) is the x (y) position of Mercury at perihelion. Using that 1 radian $= \frac{\pi}{648000}$ arcseconds, we find how many arcseconds the perihelion angle changes each year.

In order to make sure that the value we get for the precession is representative for Mercury on a long time scale, we simulate the movement of the Sun and Mercury over one century. Storing all the positions of the Sun and Mercury would generate very large data files, so during the simulation we only want to store the positions of Mercury at perihelion. In order to get these positions, we include a short code in our velocity Verlet solver that evaluates whether the distance between a planet and the Sun is the minimum distance in the orbit, and stores the value if this is the case. To be able to do this we need to store the Mercury-Sun distances from the two previous time steps, including the current one, as well as the planet position from the previous timestep. This is to compare the previous distance to the current one and the one before. If it is smaller than both of these, the previous planet position is at perihelion, and we can write this value to file. The algorithm can be seen below.

```
p(dim); //store previous planet position
d(3); //store distances for three steps

//fill distance vector with initial
//planet-Sun distance
for i = 0,...,2:
  d(i) = norm(Sun.position - planet.position);

while time < final_time:
  time += h;
  ...
  *calculate new planet position*
  ...
  //update the distances
  d(0) = d(1);
  d(1) = d(2);
  d(2) = norm(Sun.position - planet.position);

  if d(1) < d(0) and d(1) < d(2):
    write out time;
    write out p;
  p = planet.position
```

Here `p` and `d` are vectors of length `dim` (the spatial dimension) and 3, while `h` is the timestep. We present these perihelion angle changes for both forces together with an observed value which simply goes as $\theta_p(t) = 0.43t$.

III. RESULTS

In this section we present our results. We first present our finding from the Earth-Sun system, with the evaluations of the differential equation solver methods and the investigations into variations of the gravitational force form. We also consider escape velocity. We then present the orbits of the Earth-Sun-Jupiter system and the full solar system. Finally, we present our findings for the modelling of the perihelion precession of Mercury. Some plots are placed in the appendix to allow for larger size and thereby better readability.

A. The Earth-Sun system

1. Method stability and CPU time

We simulated the simplified case of Earth going through a circular orbit around the sun for $T = 5$ years, and $N = 10^2$ to $N = 10^5$. From this simulation we calculated the distance between the Sun and Earth, the orbital trajectories and the changes to the average distance between the Sun and Earth per choice of time step. The resulting plots are presented in figure 11. Additionally we compare the time usage of each method, which we achieved by running simulations for logarithmically increasing values of N. For each choice of N we ran the simulation 10 times and averaged over the result. In figure 12 the results from running simulations with no speedup flags is shown, while in figure 13 the results from running simulations with the "-O3" speedup flag is shown.

2. Conservation of energy

To further determine the stability of the Forward Euler method and the velocity Verlet method we calculated the kinetic, potential and total energy of Earth in the simplified circular-orbit-system. The resulting energies are presented in figure 14.

3. Variations on the gravitational force

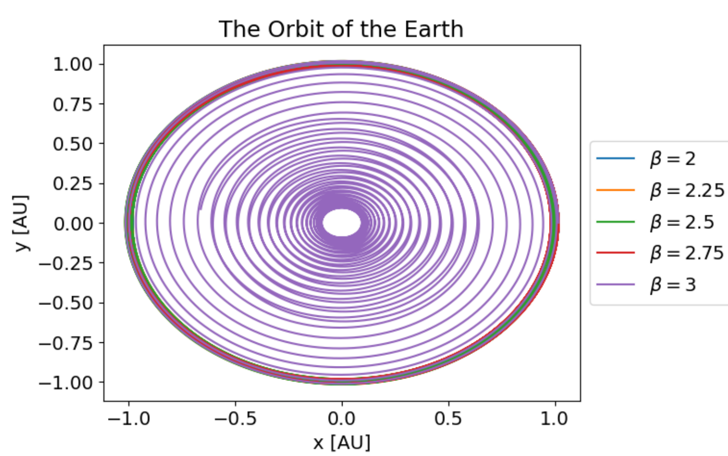

Figure 1. The orbit of the Earth beginning at aphelion with initial position $r = 1.016711$ AU and initial velocity $v = 6.78624$ AU/yr with varying β.

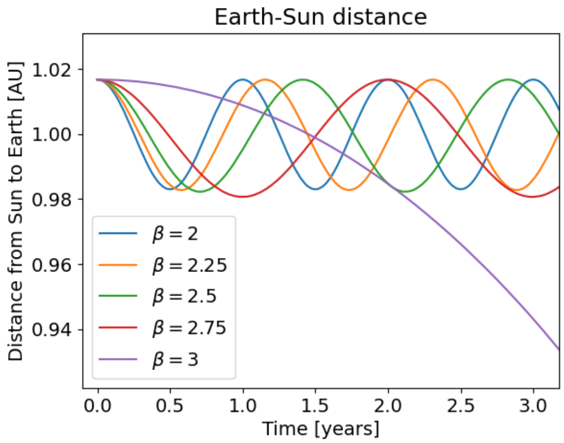

Figure 2. The Earth-Sun distance with Earth beginning at aphelion with varying β.

In figure (1) we see the orbit of the Earth over ten years for varying forms of the gravitational force. We see that most of the cases have a relatively stable orbit, while $\beta = 3$ spins into the center, very close to the Sun, and then spins out again. In figure (2) we see a zoom-in of how the distances between the Earth and the Sun varies for different β. The orbits for $2 < \beta < 3$ stay approximately within the aphelion and perihelion we see in nature, but we see that they return to these values with a slower period than the one we experience in nature, where Earth return to perihelion almost exactly once each year.

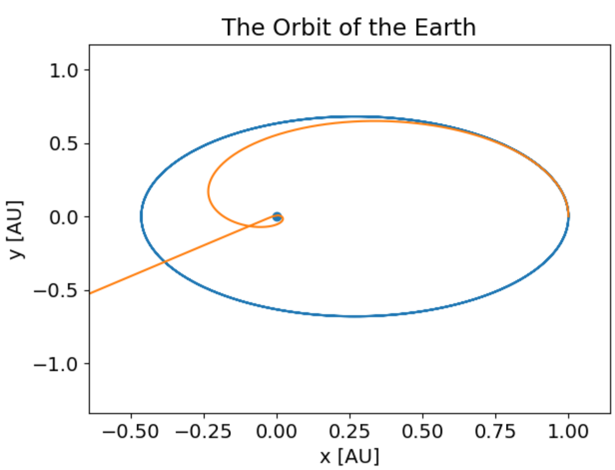

Figure 3. The orbit of the Earth with initial position $r = 1$ AU and initial velocity $v = 5$ AU/yr. One can see the orbit with $\beta = 2$ (blue line), with $\beta = 3$ (orange line) and the position of the Sun (blue dot).

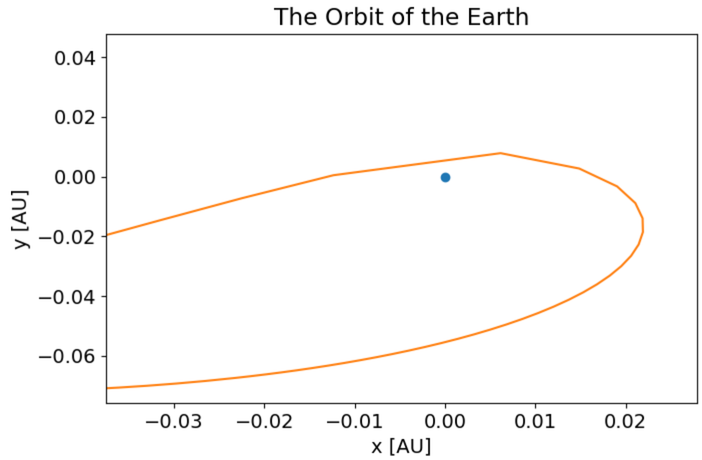

Figure 4. Close-up of the orbit of the Earth with initial position $r = 1$ AU and initial velocity $v = 5$ AU/yr. One can see the orbit with $\beta = 3$ (orange line) and the position of the Sun (blue dot).

In figure (3) we see for the low initial velocity 5 AU/yr that while the orbit for $\beta = 2$ is a closed ellipse, the Earth for $\beta = 3$ falls into the Sun, and is thrown out again. A close-up of this can be found in figure (4). Here we can see that the trajectory of the Earth appears jagged close to the Sun for $\beta = 3$.

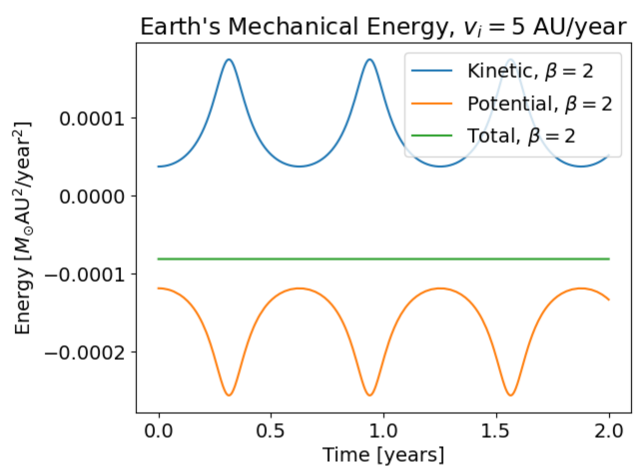

Figure 5. The mechanical energy of the Earth with initial position $r = 1$ AU and initial velocity $v = 5$ AU/yr, using the inverse-square force ($\beta = 2$).

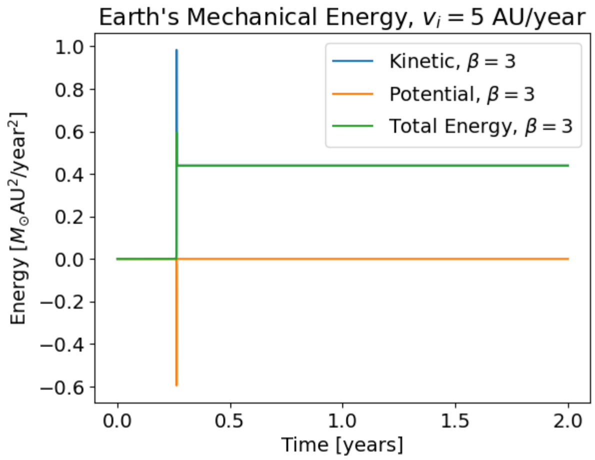

Figure 6. The mechanical energy of the Earth with initial position $r = 1$ AU and initial velocity $v = 5$ AU/yr, using the inverse-cube force ($\beta = 3$).

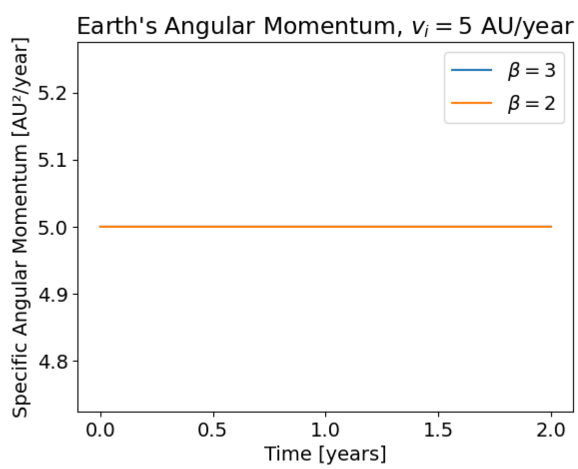

Figure 7. The angular momentum of the Earth with initial position $r = 1$ AU and initial velocity $v = 5$ AU/yr, using both the the inverse-square force ($\beta = 2$) and the inverse-cube force ($\beta = 3$).

In figure (6) we see that the mechanical energy is not conserved for this case. There is a big jump in total energy that corresponds to the same time as the when the Earth falls into the Sun. In figure (5) however, we see that the total energy of the Earth is conserved, even though the kinetic and potential energies vary. In figure (7) we see that the specific angular momentum of the Earth is conserved in both cases.

4. Escape velocity

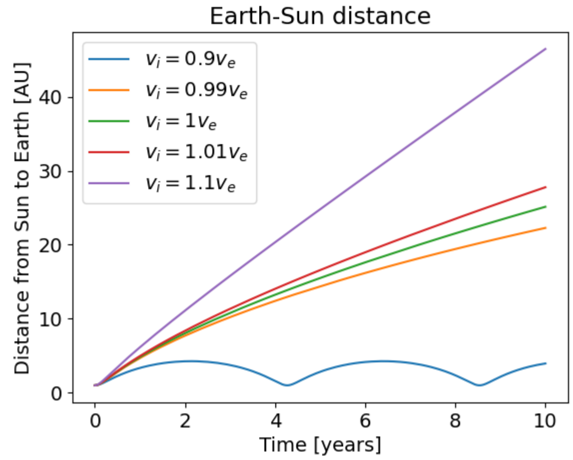

Figure 8. The Earth-Sun distance with initial position for the Earth $r = 1$ AU and different initial positions that make up different fractions of the analytical escape velocity $v_e = 2\pi\sqrt{2}$ AU/yr.

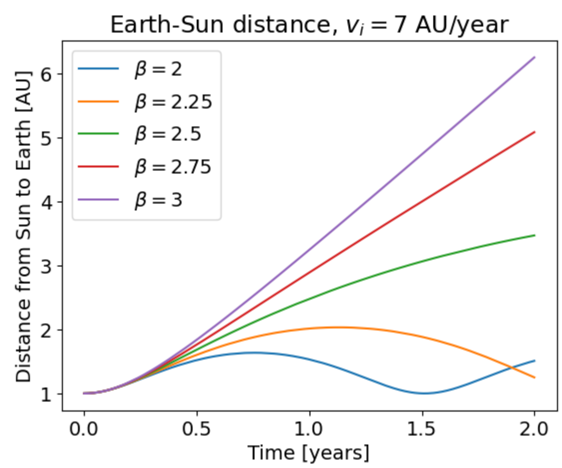

Figure 9. The Earth-Sun distance with initial position for the Earth $r = 1$ AU and initial velocity $v = 7$ AU/yr, calculated using different forms of the gravitational force $\sim 1/r^{\beta}$.

In figure (8) we see the Earth-Sun distance over ten years, using initial velocities that are under and above the analytical escape velocity $v_e = 2\pi\sqrt{2}$ AU/yr. We see that all the initial velocities except for $v_i = 0.9v_e$ seem to give an unbound Earth. In figure (9) we see the Earth-Sun distance over 2 years for simulations with the same initial velocity $v_i = 7$ AU/yr ¡ v_e, but with varying form of gravitational force $\sim 1/r^{\beta}$. We see that high β also gives an unbound Earth.

B. The Earth-Sun-Jupiter system

1. *Variations on the mass of Jupiter*

The orbital trajectories of the Sun, Earth and Jupiter were simulated for $T = 3$ years using $N = 10^5$ time steps. The mass of Jupiter was increased both by a factor of 10 and 1000 for two of the simulation. This yielded interesting results, but not really physically significant. The resulting plots are presented in figure 15.

C. The full solar system

We simulated first the orbits of the Sun, Earth and Jupiter for $T = 3$ years and $N = 10^5$ time steps. The resulting plots are presented in figure 16. Then the entire Solar System was simulated with $T = 250$ years and $N = 10^6$ time steps. The resulting plots are presented in figure 17, 18, 19 and 20.

D. The perihelion precession of Mercury

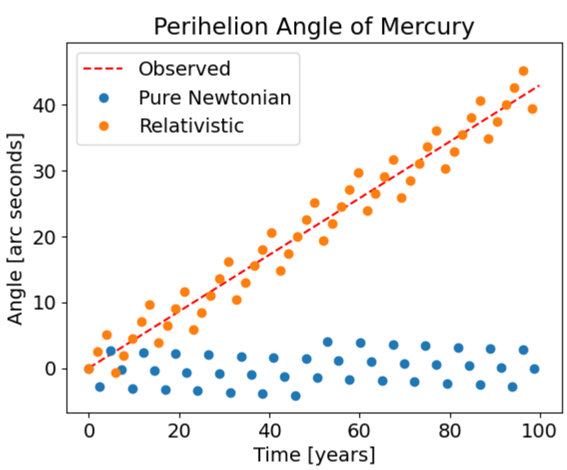

Figure 10. The perihelion angle of Mercurcy over one century, using $N = 10^8$ integration points. In the figure one sees the observed value of Mercury's perihelion (red dashed line) the perihelion angle simulated using a gravitational force with relativistic correction (orange dots) and the perihelion angle with a pure Newtonian force (blue dots).

In figure (10) we can see the perihelion angle of Mercury as it evolves over a century, both for the pure Newtonian gravitational force and for the force with a relativistic correction. We see that there is variation in which way the perihelion angle changes, giving a spread around the average change in the perihelion angle per year for both forces. Still, we see the perihelion angle change with a Newtonian force oscillates around zero, while the one with the relativisitic force seems to oscillate around the observed value. If we perform a linear regression for both datasets, we get that the pure Newtonian force angle goes as $\sim -0.0010t$ with an $R^2 = 0.001$, while the relativistic correction goes as $\sim 0.4309t$ with an $R^2 = 0.991$.

IV. DISCUSSION

In this section we discuss some of our most interesting findings. We especially focus on the stability of the differential equation solving methods, the variation on the gravitational force and the perihelion precession of Mercury.

A. Comparison of method stability and CPU time usage

As we have seen from our results presented in figure 11 the velocity Verlet method gives closed orbits and proves to be very stable in comparison with Forward Euler's method which does not behave as nicely. In the orbital trajectory plots we see that for low N Earth's orbit spirals out of control rapidly with Forward Euler, while velocity Verlet only slightly deviates. The conservation of energy was also illustrated in figure 14, which further confirms that the velocity Verlet method conserves the energy, while Forward Euler does not. Another important take away from our results was that the velocity Verlet method requires less time steps to be valid, thus, it speeds up our simulations greatly to use this method. In figure 12 and 13 we compared the CPU times for the two methods, as expected from theory both methods increase linearly with the total time steps N. We see this from the logarithmic plots since the time step dt is decreased logarithmically. We also notice that the speedup flag "-O3" does not in fact increase the performance of the ode solvers specifically.

B. The Earth-Sun system

For our Earth-Sun system we see from figure (7) that angular momentum is conserved independently of the form of Newton's gravitational force, and that it is conserved for elliptical orbits. From varying the form of gravitational force, initializing Earth at aphelion, we find that the $F_G \sim 1/r^\beta$ with $\beta > 2$ gives a delayed period for returning to aphelion, see figure (2). This is compared to how we expect Earth to behave from observations, where Earth returns to aphelion in a quite stable manner once each year. The period is as we expect for $\beta = 2$, and the delay increases with β. In other words, these findings indicate that nature does not deviate significantly from a perfect inverse-square law. In addition, the modified gravitational force behaves especially unphysically for $\beta = 3$, which spins towards the Sun in ten year hour period in a way which would end all life on Earth, see figure (1).

When we consider an elliptical orbit for the Earth that approaches the Sun more significantly, the Earth with $\beta = 3$ actually falls into the Sun. This happens around the same time as Earth with $\beta = 3$ gets a large increase in total energy, see figure (6). When the Earth comes so close to Sun, the forces become extremely large. In figure (4) we see that the Earth starts moving large distance each time step, resulting in an jagged trajectory. As the force from the Sun therefore influences the Earth over a larger distance, without the direction changing, one could imagine that the Earth would gain energy. In other words, the energy change in figure (6) is likely to be explained by the discretized nature of our model. An interesting way to develop the model further could be to implement elimination of planets that fall into the Sun. It is worth that Kepler's 2nd law is upheld even under these rather unphysical scenarios, see figure (7).

C. The perihelion precession of Mercury

The results we get for the perihelion precession of Mercury implies that implementing a relativistic correction to the gravitational force concurs well with the observed value for the perihelion precession. We suggest that the perihelion precession of Mercury can be explained by the general theory of relativity. As we expect, the perihelion angle change under the pure Newtonian force is centered around zero. The spread of the data points around the average change in perihelion angle, both for the Newtonian force and the relativistic force, are likely to be explained by unsufficient time resolution. The time resolution required to get an exact simulation is considerable, and the spread is of around the same order for both forces, which makes this a likely explanation. An point to investigate further could be find the time resolution that is required to resolve the perihelion angle practically exactly, and see if this affects the spread of the values.

V. CONCLUSION

The final simulation of our Solar System produced accurate results with stable orbits, even when simulated for $T = 250$ years. Hence, we may conclude that the velocity Verlet method is a suitable choice when modelling the n-body problem. This conclusion is further reinforced by our results showing the conservation of energy for the method. Another central result of this project is the unstable orbits resulting from changes to the exponent of the inverse-square law. These results very clearly demonstrates that gravitation does not significantly deviate from the inverse square law as discovered by Isaac Newton. Furthermore, the perihelion precession of Mercury was modelled using relativistic corrections, and our result of 0.4309 is in good agreement with observation. Future improvements could be extending the framework to function for other types of systems, such as statistical n-body systems, with added functionality for other forces than the gravitational force.

[1] NASA. Horizons web-interface, 2020. https://ssd.jpl.nasa.gov/horizons.cgi#top, accessed 26.10.2020.

[2] Prof. H. E. (Gene) Smith. A brief history of astronomy, 1999. https://casswww.ucsd.edu/archive/public/tutorial/History.html, accessed 26.10.2020.

[3] Anatoli Vankov. Einstein's paper: "explanation of the perihelion motion of mercury from general relativity theory". 20, 01 2011.

[4] Dr. David R. Williams. Earth fact sheet, 2020. https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html, accessed 26.10.2020.

Appendix: Figures

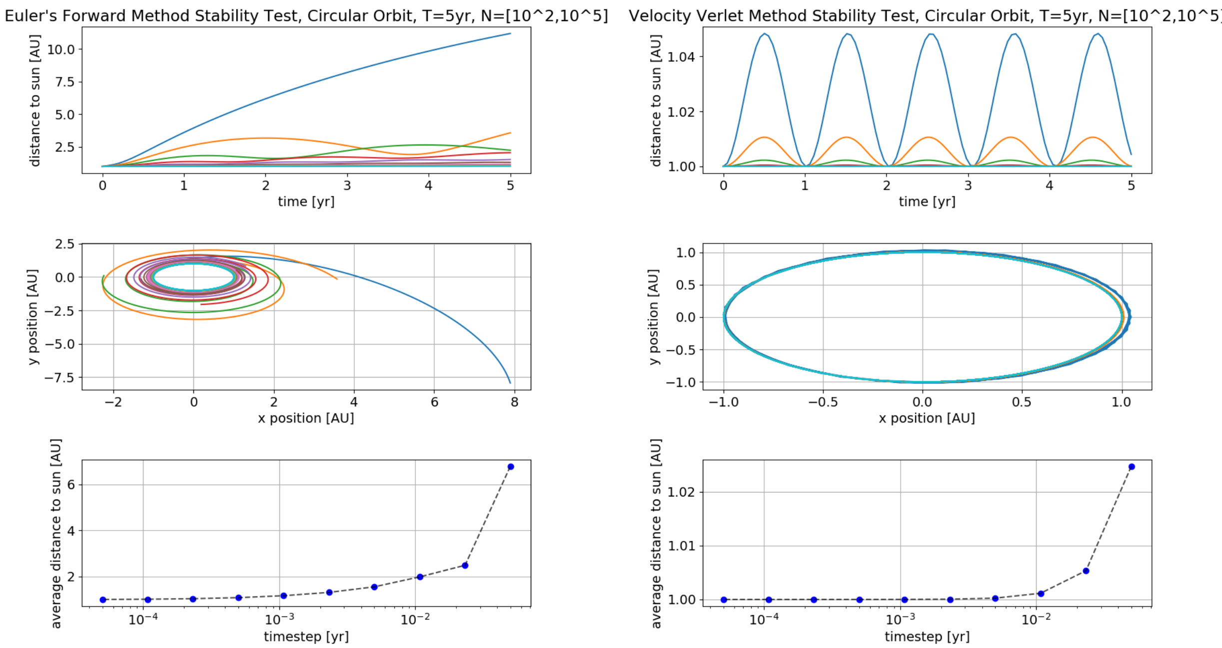

Figure 11. Comparison of Forward Euler's method and velocity Verlet method. The number of time steps was varied logarithmic from 10^2 to 10^5. The upper two plots show the Earth's distance from the Sun. The middle plots show the actual orbital trajectories. The bottom plots show Earth's average distance from the sun for each time step.

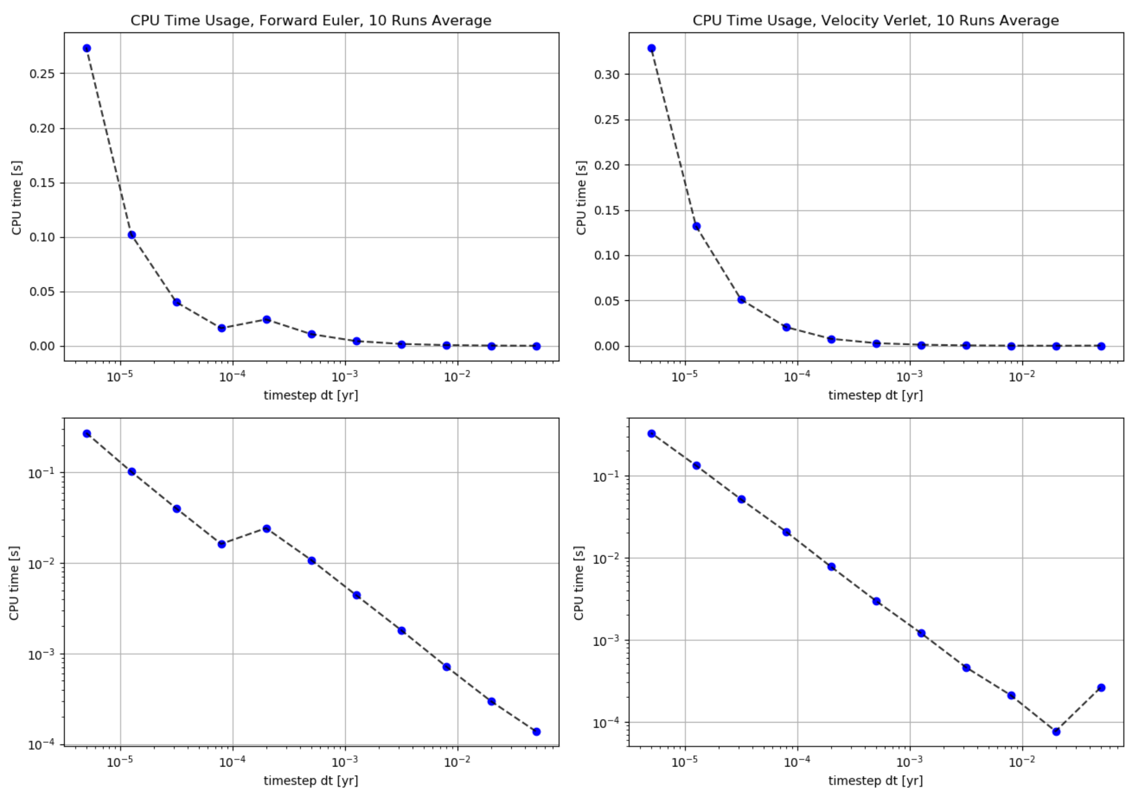

Figure 12. This figure showcases the CPU time usage of each of our methods. The bottom plots have a logarithmic y-axis. The timings were done by varying the time step and then for each step run the simulation ten times and taking the average value of this. These plots were found without using any speedup flags in the code.

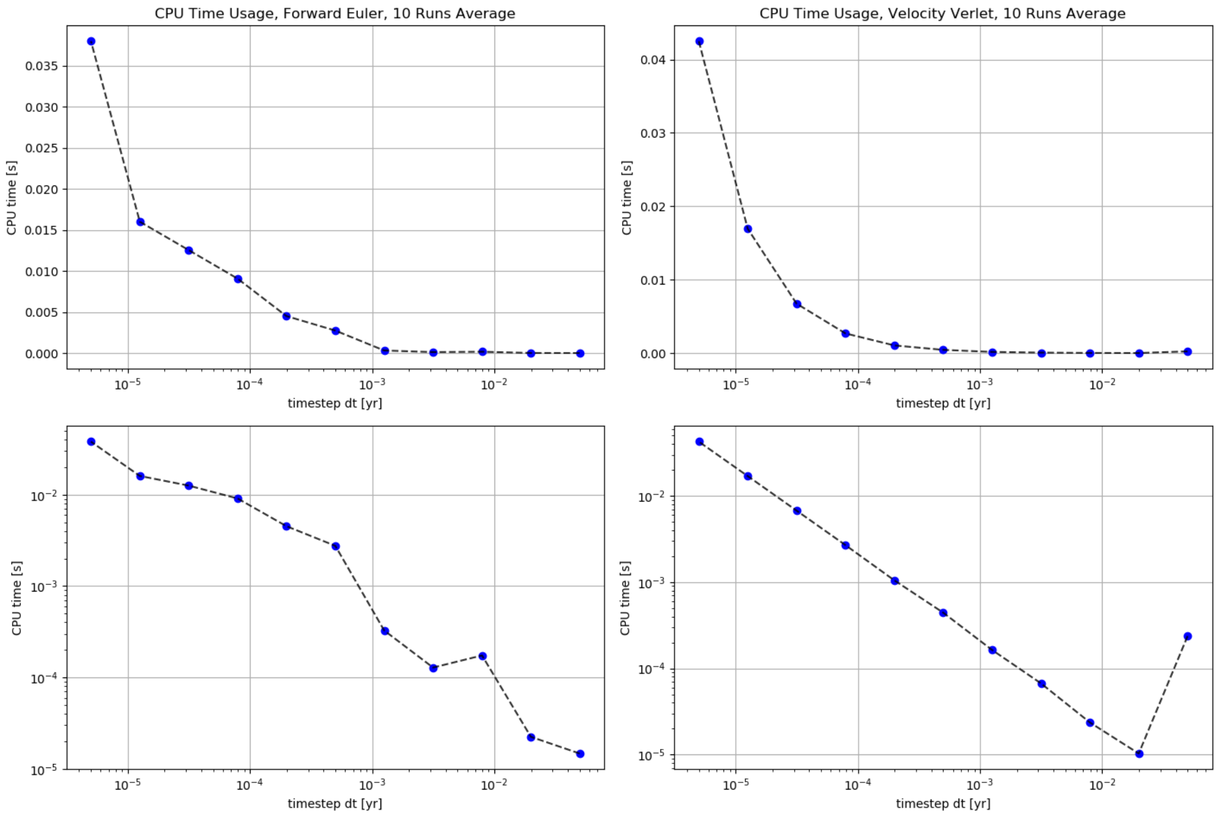

Figure 13. Same procedure as in figure 12, but now with the "-O3" speedup flag.

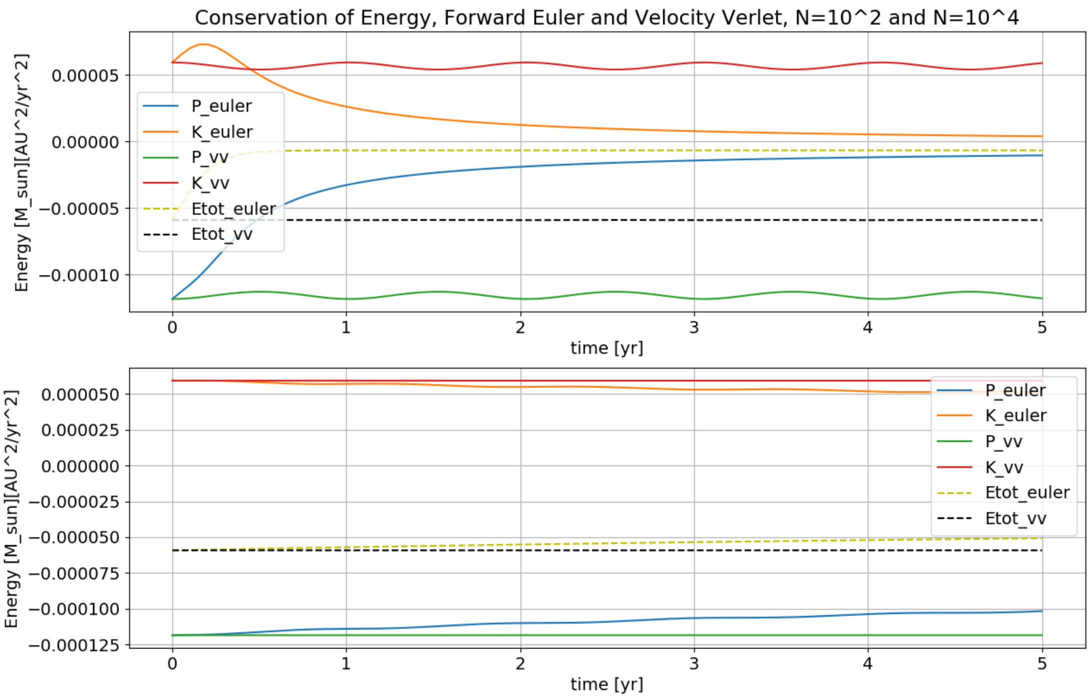

Figure 14. Plots comparing the conservation of energy for the Forward Euler method and velocity Verlet method. Plottet for $N = 10^2$ and $N = 10^4$ time steps, over $T = 5$ years. The dashed lines represents the total energy.

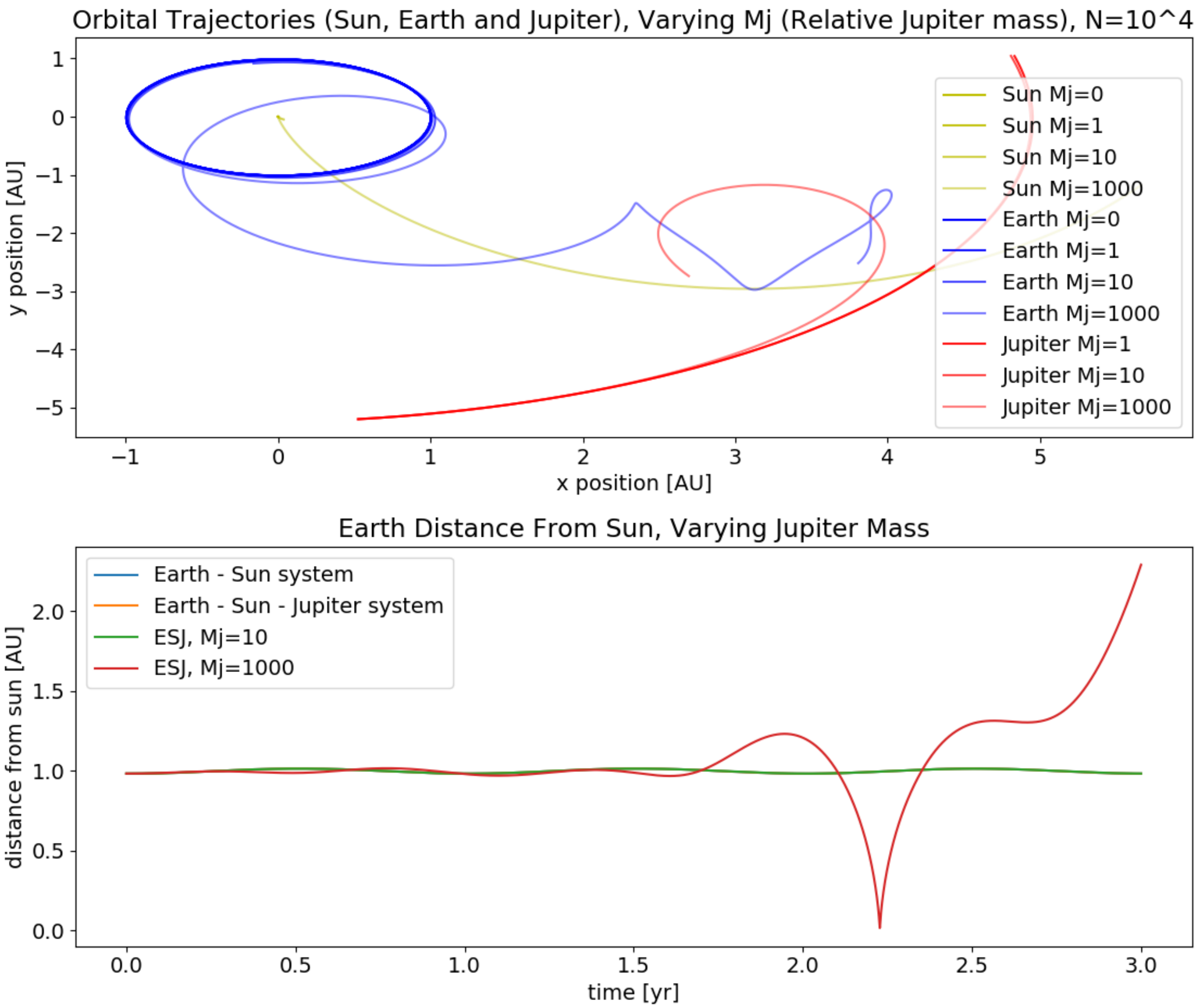

Figure 15. The upper plot shows the orbital trajectories of the Sun, Earth and Jupiter over three years. In the same plot the trajectories gotten by varying the mass of Jupiter are also shown. The lower plot shows the Earth's distance from the sun when the mass of Jupiter is varied.

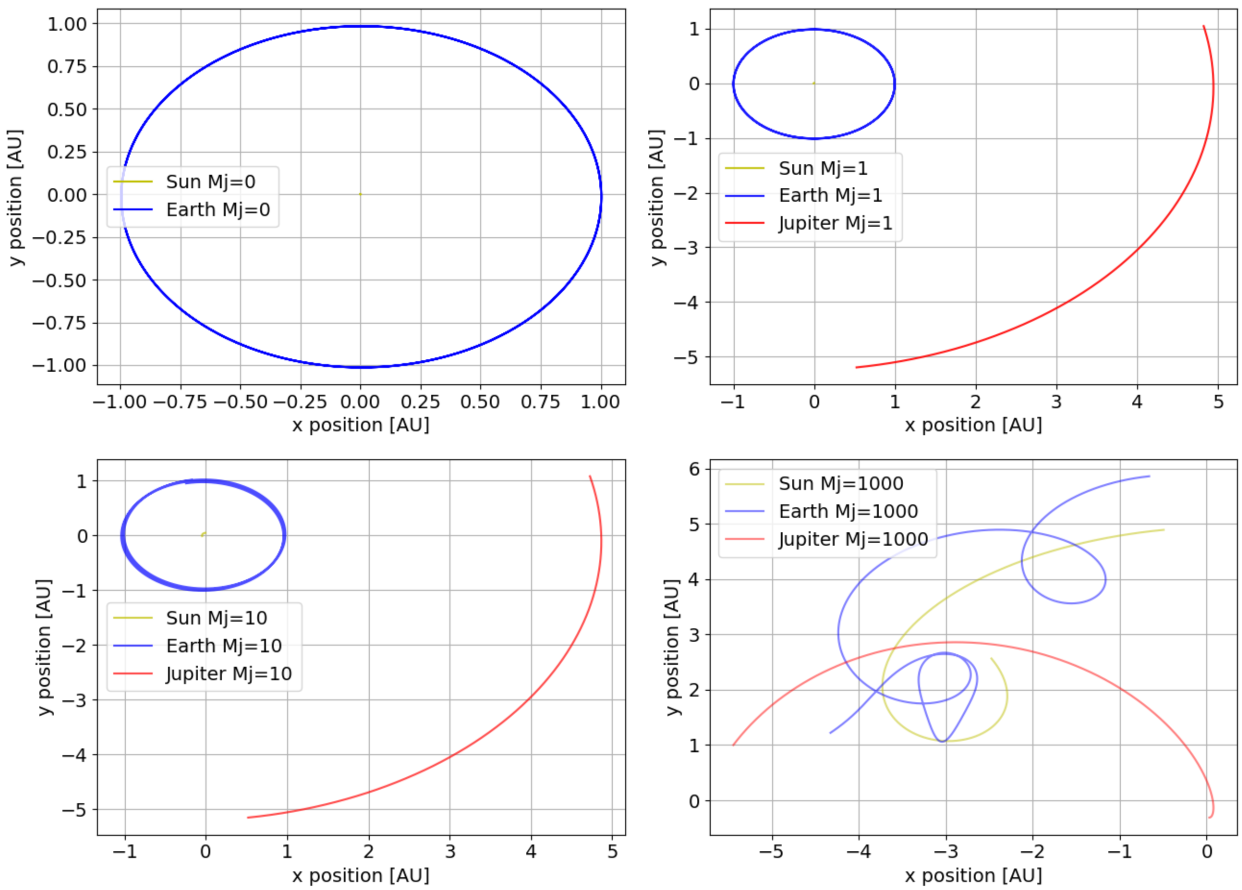

Figure 16. These plots show the orbital trajectories of the Sun, Earth and Jupiter. In these plots the system is fixed to the center of mass.

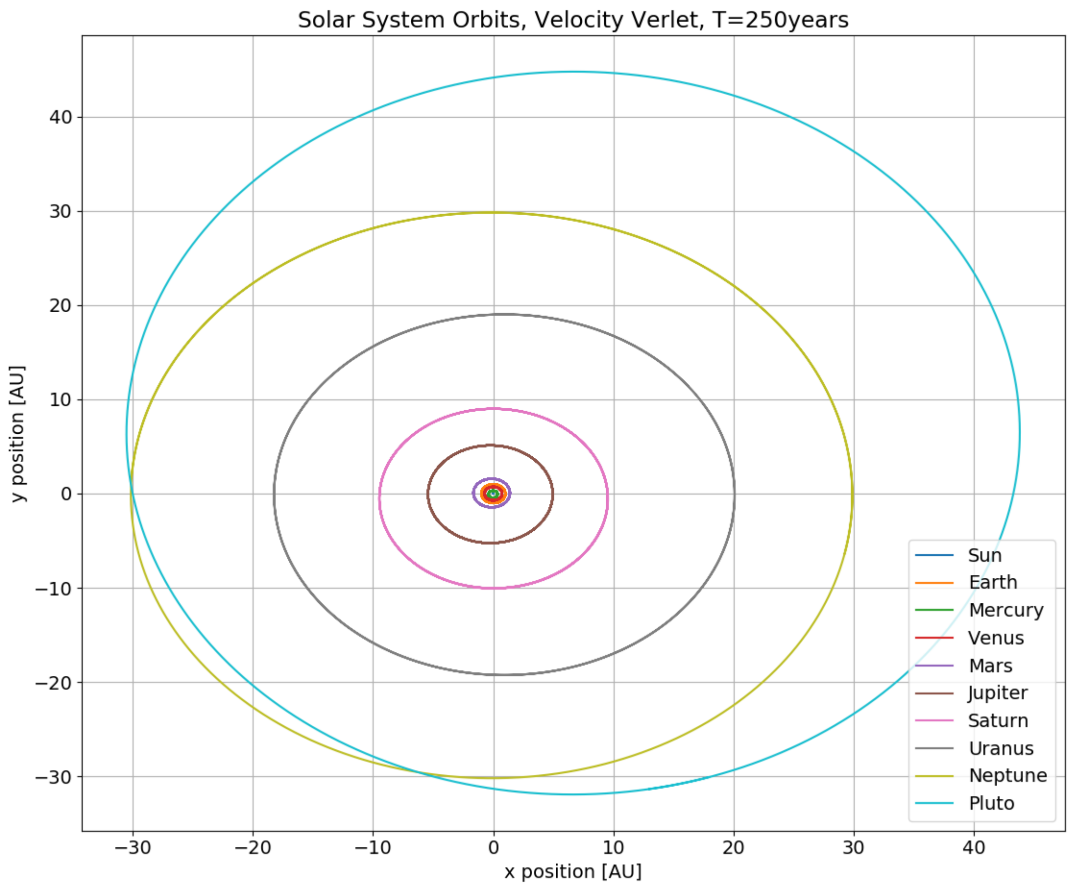

Figure 17. The orbital trajectory of the whole Solar System for $T = 250$ years. Plotted in two dimensions. In the center of mass frame. $N = 10^5$ time steps were used.

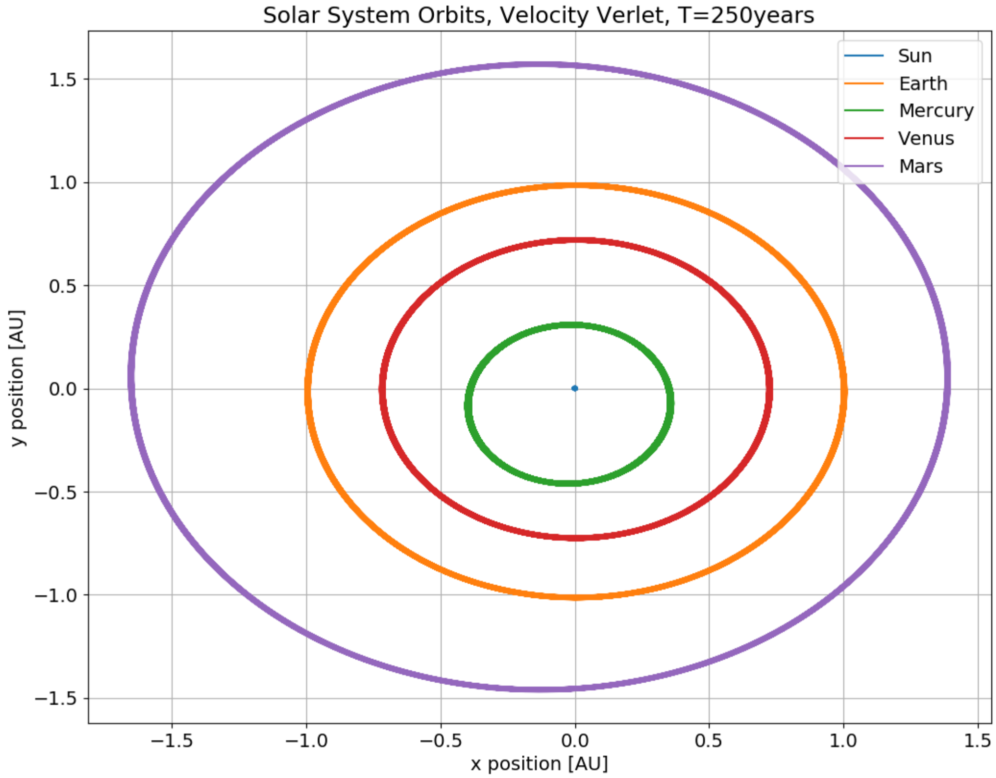

Figure 18. The orbital trajectory of the inner Solar System for $T = 250$ years. In the center of mass frame. $N = 10^5$ time steps were used.

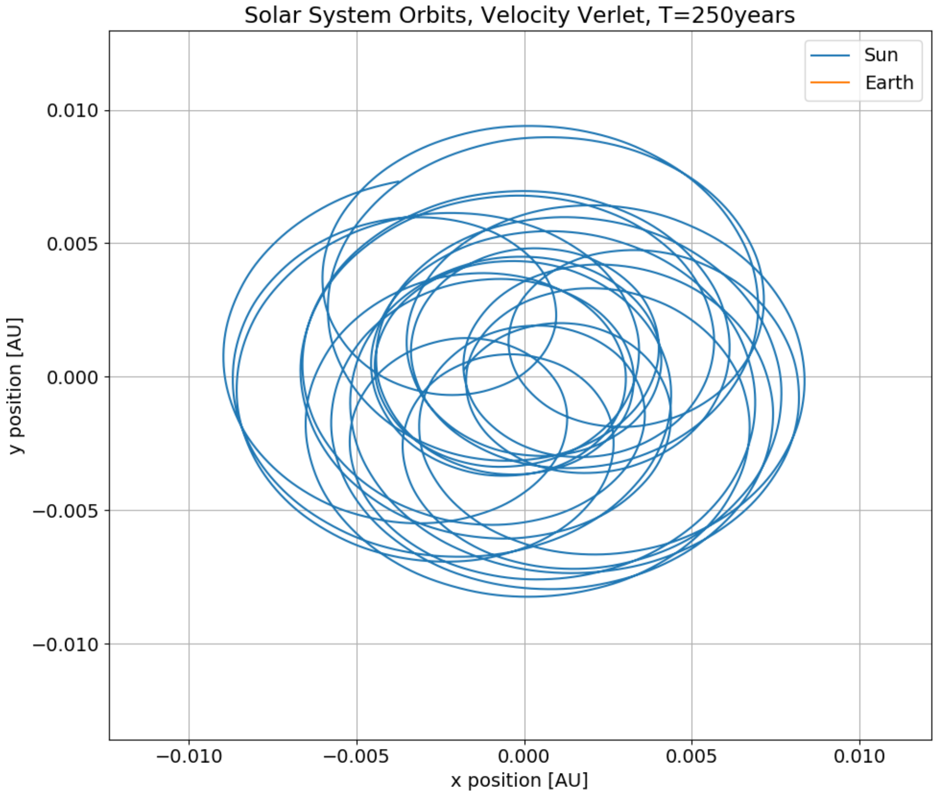

Figure 19. The orbital trajectory of the Sun for $T = 250$ years. In the center of mass frame. $N = 10^5$ time steps were used.

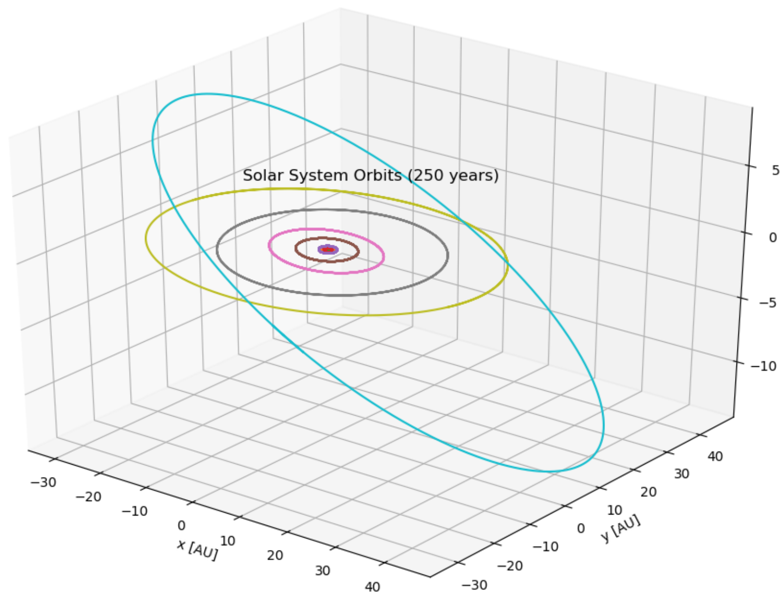

Figure 20. A three dimensional plot of the Solar System over 250 years. $N = 10^5$ time steps were used.