Chapter 10

The Conformal Model

This is Chapter 10 of Alan Macdonald's text *Linear and Geometric Algebra*

June 20, 2016

10.1 Indefinite Inner Products

Geometric algebra is not limited to modeling n-dimensional Euclidean space with $\mathbb{G}^n$. For example, the *spacetime geometric algebra* $\mathbb{G}^{1,3}$ models the spacetime of Einstein's special relativity. It is an extension of the vector space $\mathbb{R}^{1,3}$ in much the same way as $\mathbb{G}^n$ is an extension of $\mathbb{R}^n$.

The vector space $\mathbb{R}^{1,3}$ has an *indefinite* inner product. Axioms I1-I3 in Definition 4.9 of an inner product are retained, but I4, "If $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{v} \cdot \mathbf{v} > 0$", is dropped (Exercise 4.3.12). Some nonzero vectors $\mathbf{v}$ in $\mathbb{R}^{1,3}$ have $\mathbf{v} \cdot \mathbf{v} > 0$, others $\mathbf{v} \cdot \mathbf{v} = 0$, and others $\mathbf{v} \cdot \mathbf{v} < 0$.

The vector space $\mathbb{R}^{r,s}$ with $s > 0$ has an indefinite inner product: an orthonormal basis has r $\mathbf{e}_i$'s with $\mathbf{e}_i \cdot \mathbf{e}_i = 1$ and s $\mathbf{e}_i$'s with $\mathbf{e}_i \cdot \mathbf{e}_i = -1$. According to *Sylvester's law of inertia*, r and s are independent of the orthonormal basis.

GA readily extends $\mathbb{R}^{r,s}$ to $\mathbb{G}^{r,s}$, just as it extends $\mathbb{R}^n$ to $\mathbb{G}^n$. Many properties of $\mathbb{G}^n$ remain valid in $\mathbb{G}^{r,s}$, notably, $\mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u}$, $\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$, $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu})$, $\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu})$, and $\mathbf{vu} = -\mathbf{uv}$ for orthogonal vectors $\mathbf{u}$, $\mathbf{v}$.

Exercise 10.1. Let $\mathbf{e}_+$ and $\mathbf{e}_-$ be orthogonal members of $\mathbb{R}^{r,s}$ with $\mathbf{e}_\pm^2 = \pm 1$. Show that $\mathbf{e}_+ + \mathbf{e}_-$ is *null*: $(\mathbf{e}_+ + \mathbf{e}_-) \cdot (\mathbf{e}_+ + \mathbf{e}_-) = 0$.

Direct and dual representations. The blade $\mathbf{B} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k$ in $\mathbb{G}^n$ represents the subspace span$(\mathbf{b}_1, \mathbf{b}_2, \cdots \mathbf{b}_k)$ of $\mathbb{R}^n$ (Definition 6.17). And there is a test: $\mathbf{a} \in \mathbf{B} \;\Leftrightarrow\; \mathbf{a} \wedge \mathbf{B} = \mathbf{0}$ (Theorem 7.2). We say that $\mathbf{B}$ is a *direct representation* of the subspace.

Taking the dual of $\mathbf{a} \wedge \mathbf{B} = \mathbf{0}$ gives another test: $\mathbf{a} \in \mathbf{B} \;\Leftrightarrow\; \mathbf{a} \cdot \mathbf{B}^* = \mathbf{0}$, i.e., $\mathbf{a}$ is orthogonal to the orthogonal complement of $\mathbf{B}$ (also Theorem 7.2). We say that $\mathbf{B}^*$ is a *dual representation* of the subspace.

By definition, the direct and dual representations of a geometric object are orthogonal complements.

10.2 The Conformal Model

The *conformal model* of Euclidean geometry is based on the geometric algebra $\mathbb{G}^{n+1,1}$. It is an alternative to $\mathbb{G}^n$, with many attractive features.[1]

First, extend $\mathbb{R}^n$ with vectors e_+ and e_- orthogonal to $\mathbb{R}^n$, and satisfying $e_\pm^2 = \pm 1$. Then we have $\mathbb{R}^{n+1,1}$, and with it $\mathbb{G}^{n+1,1}$. More useful than $e_\pm$ are

$$o = \tfrac{1}{2}(e_- - e_+) \quad \text{and} \quad \infty = e_- + e_+.$$

Exercise 10.2. Show that $o^2 = \infty^2 = 0$ and $o \cdot \infty = -1$. Memorize these facts.

Here is the key definition.

Definition 10.1 (Conformal point)**.** The conformal model represents the *point* at the end of the vector $\mathbf{p} \in \mathbb{R}^n$ with the *vector*

$$p = o + \mathbf{p} + \tfrac{1}{2}\mathbf{p}^2 \infty \in \mathbb{G}^{n+1,1}. \tag{10.1}$$

The unbolded vector $p \in \mathbb{G}^{n+1,1}$ corresponds to the bolded vector $\mathbf{p} \in \mathbb{R}^n$. The vector p is *normalized*: the coefficient of o is 1. But the representation is *homogeneous*: nonzero scalar multiples of p represent the same point.

The next sections will show how this representation of geometric points in $\mathbb{G}^{n+1,1}$ determines representations of geometric objects and operations in $\mathbb{G}^{n+1,1}$.

In $\mathbb{R}^n$ and $\mathbb{G}^n$ vectors do double duty, representing both points and oriented line segments (Figure 1.23). In the conformal model points and oriented line segments have different representations.

Eq. (10.1) assigns $p = o$ to $\mathbf{p} = \mathbf{0}$. So o represents the point at the origin. It is called the *origin*. Also from Eq. (10.1), $\lim_{|\mathbf{p}|\to\infty} p/\frac{1}{2}\mathbf{p}^2 = \infty$ (the vector). Thus the choice of ∞ to denote this vector. It is called *infinity*.

Figure 10.1 shows a helpful but imperfect way to visualize this for $\mathbb{R}^2$. Set a sphere on $\mathbb{R}^2$ with its South Pole on the origin o. This establishes the 1-1 correspondence shown between points $\mathbf{p}$ in the plane and points p on the sphere, except the North Pole, labeled ∞. As $\mathbf{p}$ moves farther and farther from o, p gets closer and closer to ∞.

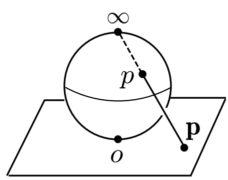

Fig. 10.1: $p \leftrightarrow \mathbf{p}$.

Exercise 10.3. a. Show that p is *null*: $p^2 = p \cdot p = 0$.

b. More generally, show that

$$p \cdot q = -\tfrac{1}{2}(\mathbf{p} - \mathbf{q})^2. \tag{10.2}$$

The *squared* Euclidean distance $(\mathbf{p} - \mathbf{q})^2$ between $\mathbf{p}$ and $\mathbf{q}$ is *linear* in p and q.

c. Show that $(\mathbf{p} - \mathbf{q})^2 = (p - q)^2$.

Exercise 10.4. Show that $p - q$ is the dual representation of the midplane between $\mathbf{p}$ and $\mathbf{q}$. This means that $\mathbf{r}$ is in the midplane if and only if $r \cdot (p - q) = 0$.

The dimension of $\mathbb{G}^{n+1,1}$ is 2^{n+2}. For $n = 3$ this is $2^{3+2} = 32$.

[1] Much of this chapter is drawn from the text *Geometric Algebra for Computer Science* by L. Dorst, D. Fontijne, and S. Mann.

10.3 Dual Representations

Notebook. A Jupyter notebook *cm3* for making calculations in the 3D conformal model is available at the book's webpage. It is also bundled with the $\mathcal{GA}$lgebra distribution. You can use it to play with the conformal model while learning and to verify calculations in this chapter. It is useful to essential when solving some of the exercises and problems to follow in this chapter.

Spheres. Equation (10.2) immediately gives an equation of the sphere with center $\mathbf{c}$ and radius ρ: $x \cdot c = -\frac{1}{2}\rho^2$. This is equivalent to $x \cdot (c - \frac{1}{2}\rho^2\infty) = 0$. Thus the *vector*

$$\sigma = c - \tfrac{1}{2}\rho^2\infty$$

is a dual representation of the *sphere*! More succinctly: σ is a `dual sphere`.

A sphere of radius zero is represented by $\sigma = c$, a point.

Exercise 10.5. Show that ρ can be extracted from σ: $\rho^2 = \sigma^2$.

Exercise 10.6. Let $p = o + \mathbf{p} + \frac{1}{2}\mathbf{p}^2\infty$ represent a point and $\sigma = c - \frac{1}{2}\rho^2\infty$ represent a sphere. Show that $2p \cdot \sigma = \rho^2 - (\mathbf{p} - \mathbf{c})^2$. In particular, $\mathbf{p}$ is inside, on, outside the circle according as $p \cdot \sigma$ is > 0, $= 0$, < 0.

We can also represent a sphere in terms of its center $\mathbf{c}$ and a point $\mathbf{p}$ on it:

$$\sigma = c - \tfrac{1}{2}\rho^2\infty = -(p \cdot \infty)\, c + (p \cdot c)\, \infty \overset{3}{=} p \cdot (c \wedge \infty). \tag{10.3}$$

Step (3) uses the identity from Problem 9.6.6a.

Planes. The vector $\mathbf{n} + \delta\infty$ is a dual representation of the plane orthogonal to $\mathbf{n}$ and at distance δ from the origin. To see this, compute

$$x \cdot (\mathbf{n} + \delta\infty) = \left(o + \mathbf{x} + \tfrac{1}{2}\mathbf{x}^2\infty\right) \cdot (\mathbf{n} + \delta\infty) = -\delta + \mathbf{x} \cdot \mathbf{n}.$$

This is zero when $\mathbf{x} \cdot \mathbf{n} = \delta$, which is a point-normal equation of a plane orthogonal to $\mathbf{n}$ at distance δ from the origin (Problem 4.1.3b).

Exercise 10.7. Extract δ from $\mathbf{n} + \delta\infty$.

A point $\mathbf{p}$ is in the plane when $p \cdot (\mathbf{n} + \delta\infty) = 0$, i.e., when $\delta = -(p \cdot \mathbf{n})/(p \cdot \infty)$. Substitute this into $\mathbf{n} + \delta\infty$ and multiply by the scalar[2] $-p \cdot \infty$ to obtain a dual representation of the plane through $\mathbf{p}$ and orthogonal to $\mathbf{n}$:

$$\pi = -(p \cdot \infty)\, \mathbf{n} + (p \cdot \mathbf{n})\, \infty \overset{3}{=} p \cdot (\mathbf{n} \wedge \infty). \tag{10.4}$$

Step (3) used the identity from Problem 9.6.6a.

Circles. We seek a dual representation of the circle with center $\mathbf{c}$, radius ρ, and normal $\mathbf{n}$. It is the intersection of the dual sphere $\sigma = c - \frac{1}{2}\rho^2\infty$ (Eq. (10.3)) and the dual plane $\pi = c \cdot (\mathbf{n} \wedge \infty)$ (Eq. (10.4)). Section 10.7 shows that a dual representation of the circle is thus $\sigma \wedge \pi = (c - \frac{1}{2}\rho^2\infty) \wedge (c \cdot (\mathbf{n} \wedge \infty))$.

Squaring the circle gives $(\sigma \wedge \pi)^2 = -\rho^2$.

Lines. A dual representation of the line containing the point $\mathbf{p}$ and orthogonal to the 3D bivector $\mathbf{B}$ is $\lambda = p \cdot (\mathbf{B}\infty)$, a 2-blade. See Section 10.6.

[2] As with points, all representations in the conformal model are homogeneous: multiplying a representation by a nonzero scalar does not change the object represented.

10.4 Direct Representations

This section describes direct representations of lines, planes, circles, and spheres. Exercise 10.16 asks you to verify them.

Lines. A direct representation of the line determined by points $\mathbf{p}$ and $\mathbf{q}$ is $L = p \wedge q \wedge \infty$. Thus a point $\mathbf{x}$ is on the line if and only if $x \wedge (p \wedge q \wedge \infty) = 0$.

Neither $\mathbf{p}$ nor $\mathbf{q}$ can be extracted from L. But the distance between them can: $(\mathbf{p} - \mathbf{q})^2 = L^2$.

If $\mathbf{x}, \mathbf{p}, \mathbf{q}$ are not collinear, then $x \wedge p \wedge q \wedge \infty \neq 0$. This leads to a measure of noncollinearity for numerical work.

Circles. A direct representation of the circle determined by noncollinear points $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ is $C = p \wedge q \wedge r$. Thus a point $\mathbf{x}$ is on the circle if and only if $x \wedge (p \wedge q \wedge r) = 0$. Its radius ρ is given by $\rho^2 = -C^2/(C \wedge \infty)^2$. Its center is at $C \infty C$. See Problem 10.6.1.

The lines $p \wedge q \wedge \infty$ are "circles through infinity".

Planes. A direct representation of the plane determined by noncollinear points $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ is $P = p \wedge q \wedge r \wedge \infty$. The area A of the triangle with vertices $\mathbf{p}, \mathbf{q}, \mathbf{r}$ is given by $A^2 = -(P/2)^2$.

The direct representation of the plane containing $\mathbf{p}$ and parallel to $\mathbf{a} \wedge \mathbf{b}$ is $p \wedge \mathbf{a} \wedge \mathbf{b} \wedge \infty$. Problem 10.6.2 asks you to show this.

Spheres. A direct representation of the sphere determined by noncoplanar points $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ is $S = p \wedge q \wedge r \wedge s$. Its radius ρ is given by $\rho^2 = S^2/(S \wedge \infty)^2$. Its center is at $S \infty S$.

The planes $p \wedge q \wedge r \wedge \infty$ are "spheres through infinity".

Note the consistency of grades between the direct representations of this section and the dual representations of the last section: Direct representations of spheres and planes are 4-vectors; their dual representations are 1-vectors, i.e., vectors. Similarly, direct representations of circles and lines are 3-vectors, trivectors; their dual representations are 2-vectors, bivectors.

Problems 10.4

10.4.1. a. Set $E = o \wedge \infty$. Show that $pE = p \cdot E + p \wedge E = (o - \frac{1}{2}|\mathbf{p}|^2 \infty) + \mathbf{p}E$. This is called a *conformal split* of p.

b. Expand $\mathbf{p}E = \sum_i p_i \mathbf{e}_i E$. Show that the trivectors $\boldsymbol{\sigma}_i = \mathbf{e}_i E$ have the properties of an orthonormal vector basis for $\mathbb{R}^n$: $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_i = 1$, and for $i \neq j$, $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = 0$.

10.5 Operations

The vectors p representing points are null (Exercise 10.3a), as are the vectors $\lambda\infty$ (Exercise 10.2). But that's it: there are no other null vectors in $\mathbb{G}^{n+1,1}$. Thus null vectors in $\mathbb{G}^{n+1,1}$ provide an *algebraic* representation of $\bar{\mathbb{R}}^n \equiv \mathbb{R}^n \cup \infty$. We will see that ∞ gives sensible results in calculations.

Orthogonal linear transformations o on $\mathbb{G}^{n+1,1}$ map $\bar{\mathbb{R}}^n$ to itself. For they preserve inner products: $p \cdot p = 0 \Rightarrow \mathsf{o}(p) \cdot \mathsf{o}(p) = 0$ (Exercise 9.20a).

Transformations on $\mathbb{R}^n$ that preserve angles are called *conformal*. Conformal transformations include rotations, translations, reflections, dilations, and inversions. Every conformal transformation is a (not unique) composition of these five. We will discuss them individually in a moment.

Here is the key to the power of the conformal model:

Conformal transformations on $\mathbb{R}^n$
are represented by
orthogonal transformations on $\mathbb{G}^{n+1,1}$.

We now represent the five aforementioned conformal transformations on $\mathbb{R}^n$ with orthogonal transformations on $\mathbb{G}^{n+1,1}$. Theorem 10.2 will show that the representations preserve inner products and so are indeed orthogonal.

Rotations. A rotation of $\mathbf{p}$ around the origin by angle $\mathbf{i}\theta$ is represented in the conformal model just as in $\mathbb{G}^n$ (Section 7.2):

$$\mathsf{R}_{\mathbf{i}\theta}(p) = \mathrm{e}^{-\mathbf{i}\theta/2}\, p\, \mathrm{e}^{\mathbf{i}\theta/2}\,.$$

Reason:

$$\begin{aligned}\mathrm{e}^{-\mathbf{i}\theta/2}\, p\, \mathrm{e}^{\mathbf{i}\theta/2} &= \mathrm{e}^{-\mathbf{i}\theta/2}\left(o + \mathbf{p} + \tfrac{1}{2}|\mathbf{p}|^2\infty\right)\mathrm{e}^{\mathbf{i}\theta/2}\\ &= o + \mathrm{e}^{-\mathbf{i}\theta/2}\,\mathbf{p}\,\mathrm{e}^{\mathbf{i}\theta/2} + \tfrac{1}{2}|\,\mathrm{e}^{-\mathbf{i}\theta/2}\,\mathbf{p}\,\mathrm{e}^{\mathbf{i}\theta/2}\,|^2\infty.\end{aligned}$$

Exercise 10.8. Verify the just used $\mathsf{R}_{\mathbf{i}\theta}(o) = o$ and $\mathsf{R}_{\mathbf{i}\theta}(\infty) = \infty$.

Translations. The translation $\mathbf{p} \mapsto \mathbf{p} + \mathbf{a}$ in $\mathbb{R}^n$ is represented in the conformal model by

$$\mathsf{T}_{\mathbf{a}}(p) = \mathrm{e}^{-\mathbf{a}\infty/2}\, p\, \mathrm{e}^{\mathbf{a}\infty/2}\,.$$

This is a "rotation around infinity". I give no proof.

Exercise 10.9. Show that $\mathrm{e}^{\pm\mathbf{a}\infty/2} = 1 \pm \mathbf{a}\,\infty/2$ (exactly).

Exercise 10.10. a. Show that $\mathsf{T}_{\mathbf{a}}(o) = a$. b. Show that $\mathsf{T}_{\mathbf{a}}(\infty) = \infty$.

Exercise 10.11. One expects that $\mathsf{T}_{\mathbf{a}}\mathsf{T}_{\mathbf{b}} = \mathsf{T}_{\mathbf{a}+\mathbf{b}}$. Prove this.

Translations in $\mathbb{R}^n$ are not linear: $(\mathbf{p}+\mathbf{q})+\mathbf{a} \neq (\mathbf{p}+\mathbf{a})+(\mathbf{q}+\mathbf{a})$. They are *linearized* in the conformal model by the linear transformations $\mathsf{T}_{\mathbf{a}}$. Linearization is a boon. To paraphrase David Hestenes: In $\mathbb{G}^n$ compositions of rotations are multiplicative while compositions of translations are additive,

$$\begin{aligned}(\mathsf{R}_{\mathbf{i}_2\theta_2} \circ \mathsf{R}_{\mathbf{i}_1\theta_1})(\mathbf{v}) &= \mathrm{e}^{-\mathbf{i}_2\theta_2/2}(\mathrm{e}^{-\mathbf{i}_1\theta_1/2}\,\mathbf{v}\,\mathrm{e}^{\mathbf{i}_1\theta_1/2})\,\mathrm{e}^{\mathbf{i}_2\theta_2/2}\\ (\mathsf{T}_{\mathbf{a}_2} \circ \mathsf{T}_{\mathbf{a}_1})(\mathbf{v}) &= (\mathbf{v}+\mathbf{a}_1)+\mathbf{a}_2,\end{aligned}$$

so combining the two destroys the simplicity of both.

Reflections. Let the normal vector $\mathbf{n}$ represent a dual plane through the origin. The reflection of the point $\mathbf{p}$ through the plane is represented in the conformal model just as in $\mathbb{G}^n$ (Theorem 7.9):

$$\mathsf{M}_{\mathbf{n}}(p) = -\mathbf{n}\, p\, \mathbf{n}^{-1}. \tag{10.5}$$

Exercise 10.12 asks you to prove this.

The "$-$" tells us that reflections reverse orientations. It also normalizes the reflection (so that the coefficient of o is one).

Exercise 10.12. Prove that $p \mapsto -\mathbf{n}\, p\, \mathbf{n}^{-1}$ represents a reflection.

Exercise 10.13. Show that the composition of two reflections is a rotation.

Dilations. The map $\mathbf{p} \mapsto \alpha\mathbf{p}$, α a scalar, is called *dilation* (by α). It is represented in the conformal model by

$$\mathsf{D}_\alpha(p) = \mathrm{e}^{-E\ln(\alpha)/2}\, p\, \mathrm{e}^{E\ln(\alpha)/2}.$$

where $E = o \wedge \infty$. Since $E^2 = +1$, a power series expansion reveals that $\mathrm{e}^{E\ln(\alpha)/2} = \cosh(\ln(\alpha)/2) + E\sinh(\ln(\alpha)/2)$. (Compare this to $\mathbf{i}^2 = -1$ and $\mathrm{e}^{\mathbf{i}\theta/2} = \cos(\theta/2) + \mathbf{i}\sin(\theta/2)$ for rotations.) To normalize the result, divide by the coefficient of o (which is $1/\alpha$).

Inversions. The map $\mathbf{p} \mapsto \mathbf{p}^{-1}$ is called an *inversion* (in the unit sphere). Points inside the sphere are mapped to the outside, and vice-versa. Inversion is not linear in $\mathbb{G}^n$ ($(\mathbf{p}+\mathbf{q})^{-1} \neq \mathbf{p}^{-1} + \mathbf{q}^{-1}$), but is linearized in the conformal model:

$$\mathsf{I}(p) = -(o - \tfrac{1}{2}\infty)p(o - \tfrac{1}{2}\infty)^{-1}. \tag{10.6}$$

Note: $(o - \frac{1}{2}\infty)^{-1} = o - \frac{1}{2}\infty$. To normalize the result, divide by the coefficient of o (which is $\mathbf{p}^2$). Inversions reverse orientations.

Problem 10.5.1 asks you to verify Eq. (10.6).

Exercise 10.14. Show that $\mathsf{I}(o) = \frac{1}{2}\infty$ and $\mathsf{I}(\infty) = 2o$.

Lines, planes, circles, and spheres map to the same under rotations, translations, reflections, and dilations. Inversions are the exception.

Under inversion lines not through the origin map to circles through the origin and vice versa (Exercise 10.17). Planes not through the origin and spheres through origin are similarly related.

Inversive geometry studies geometric objects under inversion. It is a well developed branch of geometry.

Problems 10.5

10.5.1. Show that inversion $\mathbf{p} \mapsto \mathbf{p}^{-1}$ is represented in the conformal model by $\mathsf{I}(p) = -(o - \frac{1}{2}\infty)\, p\, (o - \frac{1}{2}\infty)$. Exercise 10.14 will help.

10.5.2. Show that inversion is reflection through the hyperplane normal to e_+.

10.6 Covariance

Theorem 10.2 (Covariance)**.** Conformal operations preserve the entire algebraic structure of the conformal model: scalar multiplication and multivector addition; the geometric, inner, and outer products; and grades.[a]

[a]This is only almost true of reflections and inversions. See the footnote to the proof.

This preservation of structure by conformal operations is called *covariance*. It is very powerful in conformal geometric algebra, as you are about to see.

Proof. The conformal operations of Section 10.5 are of the form $p \mapsto \tau p \tau^{-1}$.[3] Thus the proof of Theorem 7.6 applies here. □

Translate a dual sphere. Construct a dual sphere $\sigma = p \cdot (c \wedge \infty)$ (Eq. (10.3)). We want to translate it by $\mathbf{a}$. One way is to translate p and c separately: $\mathsf{T}_{\mathbf{a}}(p) \cdot (\mathsf{T}_{\mathbf{a}}(c) \wedge \infty)$ (Eq. (10.3) again). That will get the job done. But by covariance, $\mathsf{T}_{\mathbf{a}}(\sigma) = \mathsf{T}_{\mathbf{a}}(p) \cdot (\mathsf{T}_{\mathbf{a}}(c) \wedge \infty)$ (using $\mathsf{T}_{\mathbf{a}}(\infty) = \infty$). Thus we can simply translate the sphere itself: $\mathsf{T}_{\mathbf{a}}(\sigma) = \mathrm{e}^{-\mathbf{a}\infty/2}\, \sigma\, \mathrm{e}^{\mathbf{a}\infty/2}$.

Exercise 10.15. Translate noncollinear points p, q, r with $\mathsf{T}_{\mathbf{a}}$. Show that $\mathsf{T}_{\mathbf{a}}$ also translates the circle $p \wedge q \wedge r$ through the points.

Dual line. We prove the statement from Section 10.3: $p \cdot (\mathbf{B}\infty)$ represents the dual line through the point $\mathbf{p}$ and orthogonal to the 3D bivector $\mathbf{B}$.

By covariance it is sufficient to prove this for $p = o$ and $\mathbf{B} = \mathbf{e}_1\mathbf{e}_2$:

$$\begin{aligned} x \cdot (p \cdot (\mathbf{B}\infty)) &= x \cdot (o \cdot (\mathbf{e}_1\mathbf{e}_2\infty)) = x \cdot \langle\, o\mathbf{e}_1\mathbf{e}_2\infty \,\rangle_2 = \langle\, x\mathbf{e}_1\mathbf{e}_2 o\infty \,\rangle_1 \\ &= \langle\, x\mathbf{e}_1\mathbf{e}_2(-1 + o \wedge \infty) \,\rangle_1 = -\langle\, x\mathbf{e}_1\mathbf{e}_2 \,\rangle_1 = x_2\mathbf{e}_1 - x_1\mathbf{e}_2. \end{aligned}$$

This is zero if and only if $x_1 = x_2 = 0$, i.e., $\mathbf{x}$ is on the $\mathbf{e}_3$ axis.

Exercise 10.16. Justify the four direct representations from Section 10.4. You'll need help from the cm3 notebook for some of these.

Angle between lines. We compute the angle θ between intersecting lines. Consider first lines intersecting at the origin: $\ell_1 = o \wedge p_1 \wedge \infty$ and $\ell_2 = o \wedge p_2 \wedge \infty$. Because of the simple form of ℓ_1 and ℓ_2, it is easy to compute

$$\frac{\ell_1 \cdot \ell_2}{|\ell_1|\,|\ell_2|} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{p}_1|\,|\mathbf{p}_2|} = \cos\theta. \tag{10.7}$$

Now translate the lines to an arbitrary position. By covariance, the left side of Eq. (10.7) with the new lines still gives $\cos\theta$.

Exercise 10.17. Show that under inversion lines not through the origin map to circles through the origin and vice versa. *Hint*: Use direct representations.

[3]Well, not quite: reflections and inversions are of the form $p \mapsto -\tau p \tau^{-1}$. Keep track of $-$'s separately when reflecting or inverting. "Reflect a line in a plane" below is an example.

Angle between circles. Let σ_1 and σ_2 be coplanar circles intersecting at two points. Translate the circles so that an intersection point is at the origin. Now invert. This angle preserving conformal transformation maps the circles to straight lines (Exercise 10.17). Thus the angle between the circles is given by the same formula as for the lines: $\cos\theta = (\sigma_1 \cdot \sigma_2)/(|\sigma_1|\,|\sigma_2|)$.

Translate rotations. The rotations above are around the origin. To rotate a point around $\mathbf{a}$, translate it by $-\mathbf{a}$, rotate it around the origin, and translate it back by $\mathbf{a}$. Thus a rotation around $\mathbf{a}$ is represented by $\mathsf{T}_\mathbf{a}\, \mathrm{e}^{-\mathbf{i}\theta/2}\, \mathsf{T}_\mathbf{a}^{-1}$; representations of rotations translate just as representations of geometric objects.

There is more. From Exercise 10.18, $\mathsf{T}_\mathbf{a}\, \mathrm{e}^{-\mathbf{i}\theta/2}\, \mathsf{T}_\mathbf{a}^{-1} = \mathrm{e}^{-(\mathsf{T}_\mathbf{a}\mathbf{i}\theta\mathsf{T}_\mathbf{a}^{-1})/2}$, i.e., $-\mathsf{T}_\mathbf{a}\mathbf{i}\theta\mathsf{T}_\mathbf{a}^{-1}$ specifies a rotation around $\mathbf{a}$, i.e., the bivector angle specifying the translated rotation is the translation of the angle specifying the original rotation.

Similar statements are true for reflections, dilations, and inversions.

Exercise 10.18. Show that $\mathsf{T}_\mathbf{a}\, \mathrm{e}^{-\mathbf{i}\theta/2}\, \mathsf{T}_\mathbf{a}^{-1} = \mathrm{e}^{-(\mathsf{T}_\mathbf{a}\mathbf{i}\theta\mathsf{T}_\mathbf{a}^{-1})/2}$.

Line through p parallel to a. The direct representation of the line through $\mathbf{p}$ and parallel to $\mathbf{a}$ is $p \wedge \mathbf{a} \wedge \infty$. To see this, first take $\mathbf{p} = \mathbf{0}$. Compute:

$$x \wedge (o \wedge \mathbf{a} \wedge \infty) \overset{1}{=} \mathbf{x} \wedge (o \wedge \mathbf{a} \wedge \infty) \overset{2}{=} -(\mathbf{x} \wedge \mathbf{a})(o \wedge \infty).$$

Step (1): the o factor in parentheses kills the o term in x and similarly for ∞, leaving $\mathbf{x}$. Step (2): the factors are orthogonal. Since $o \wedge \infty$ is invertible, the right side is zero if and only if $\mathbf{x}$ is a scalar multiple of $\mathbf{a}$. This establishes the result for $\mathbf{p} = \mathbf{0}$. By covariance under translation the result is true when $\mathbf{p} \neq \mathbf{0}$.

There is a similar representation in Problem 10.6.2.

Exercise 10.19. Why is $\mathbf{a}$ not translated when applying covariance above?

Reflect a line in a plane. The reflection of the point $\mathbf{p}$ in the plane through the origin with normal $\mathbf{n}_\circ$ is $-\mathbf{n}_\circ\, p\, \mathbf{n}_\circ^{-1}$ (Eq. (10.5)). This extends to the reflection of the line $L_\circ = o \wedge \mathbf{a} \wedge \infty$ in $\mathbf{n}_\circ$:

$$-\mathbf{n}_\circ L_\circ \mathbf{n}_\circ^{-1} \overset{1}{=} -\big((\mathbf{n}_\circ\, o\, \mathbf{n}_\circ^{-1}) \wedge (\mathbf{n}_\circ\, \mathbf{a}\, \mathbf{n}_\circ^{-1}) \wedge (\mathbf{n}_\circ\, \infty\, \mathbf{n}_\circ^{-1})\big) = o \wedge (-\mathbf{n}_\circ\, \mathbf{a}\, \mathbf{n}_\circ^{-1}) \wedge \infty,$$

the reflection of $L_\circ$. Step (1) applies covariance under the map $p \mapsto +\mathbf{n}_\circ\, p\, \mathbf{n}_\circ^{-1}$.

Now translate. By covariance again the reflection of the translated line L in the translated plane π is $-\mathbf{n}L\mathbf{n}^{-1}$.

Note that the reflection was obtained without determining the point where the line intersects the plane.

Problems 10.6

10.6.1. Let $C = p \wedge q \wedge r$ be the direct representation of a circle.

a. Show that its center is at $C\infty C$.

b. Show that its radius ρ is given by $\rho^2 = -C^2/(C \wedge \infty)^2$.

Hint: Start with a circle of radius ρ centered at the origin.

10.6.2. Show that $p \wedge \mathbf{a} \wedge \mathbf{b} \wedge \infty$ is a direct representation of the plane through $\mathbf{p}$ and parallel to $\mathbf{a} \wedge \mathbf{b}$. *Hint*: This is similar to the representation $p \wedge \mathbf{a} \wedge \infty$ above of the line through $\mathbf{p}$ parallel to $\mathbf{a}$.

10.7 Join and Meet

Geometric objects *join* to form higher dimensional objects. Examples: the join of two points is the line through them, the join of intersecting lines is the plane containing them, and the join of a line and a point not on it is the plane containing them.

Geometric objects *meet* in lower dimensional objects, their intersection. Examples: the meet of an intersecting line and plane is their point or line of intersection, and the meet of two intersecting planes is their line of intersection.

Geometric algebra defines the join and meet of blades (only) to represent the join and meet of the geometric objects represented by the blades.

The *join* of two blades is the span of their subspaces. If $\alpha \wedge \beta \neq 0$, then it is the join (Theorem 6.30a). There is no general formula for the join in terms of the geometric product, as there are for the inner and outer products (Eqs. (6.8) and (6.9)). However, there are efficient algorithms for computing it.

The *meet* of two blades is the intersection of their subspaces.

Theorem 10.3 (Dual representation of a meet)**.** Suppose that the join of two geometric objects is $\mathbb{R}^n$. Let α and β be their direct representations in $\mathbb{G}^n$.

Then $\alpha^* \wedge \beta^*$ is the dual representation of their meet. (Duals taken in $\mathbb{G}^n$.)

Proof. Let $x = a + b$ with $a \in \alpha$, $b \in \beta$. If $x \in \alpha^*$, then $a = 0$. If also $x \in \beta^*$, then $b = 0$, so $x = 0$. Thus $\alpha^* \wedge \beta^* = \operatorname{span}(\alpha^*, \beta^*)$ (Theorem 6.30a). Now

$$x \perp \alpha^* \wedge \beta^* \;\Leftrightarrow\; x \perp \operatorname{span}(\alpha^*, \beta^*) \;\Leftrightarrow\; x \perp \alpha^* \;\&\; x \perp \beta^* \;\Leftrightarrow\; x \in \alpha \;\&\; x \in \beta. \quad \square$$

Example. Let σ be the dual representation of a sphere and π be the dual representation of a plane through its center. Their join is $\mathbb{R}^3$, so they intersect in a circle whose dual representation in $\mathbb{G}^3$ is $\sigma \wedge \pi$.

Example. Let π_1 and π_2 be dual representations of intersecting planes $\mathbb{R}^3$. The planes span $\mathbb{R}^3$, so $\pi_1 \wedge \pi_2$ is the dual representation of their line of intersection.

Exercise 10.20. Show that parallel dual planes $\mathbf{n}$ and $\mathbf{n} + \delta\infty$ in $\mathbb{R}^3$ intersect at ∞. That is, the only $x \in \bar{\mathbb{R}}^n$ in the intersection is ∞.

Exercise 10.21. Use Theorem 10.3 to compute the intersection of the line containing the points $-\mathbf{e}_1$ and $\mathbf{e}_1$ with the line containing the points $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1 - \mathbf{e}_2$.

Problems 10.7

10.7.1. Let σ_1 and σ_2 represent dual spheres of radius ρ in $\mathbb{R}^3$ centered at $\pm\mathbf{e}_1$.

a. Compute $C = \sigma_1^* \wedge \sigma_2^*$ from Theorem 10.3. If $\rho > 1$, then C is the dual representation of the intersection of the spheres, a circle.

b. Show that $C^2 = 4(1 - \rho^2)$. Thus the spheres intersect in a circle, a point, or not at all, according as $C^2 < 0, = 0, > 0$.

This test works for all sphere pairs, not just the one of this exercise.

Conformal Model: Final Words

The representations of geometric objects and geometric operations in this chapter are only a fraction of those available. Pablo Colapinto has compiled two remarkable lists. One lists 25 different kinds of geometric objects. The other lists 200 constructions in total for them! [4]

Figure 10.2 shows a small sample. For example, the item second from last says that a direct sphere can be constructed as an outer product of a direct circle and a dual plane.

Fig. 10.2: Operations that Construct a Sphere

We now have two geometric algebras for Euclidean n-space: the vector space model $\mathbb{G}^n$ and the conformal model $\mathbb{G}^{n+1,1}$. In both, blades represent geometric objects. In the vector space model a vector represents an oriented line segment. In the conformal model it represents a point. In the vector space model an outer product of three vectors represents an oriented volume. In the conformal model it represents a circle. The semantics are different: oriented volumes are not circles. But the syntax is is the same: the same rules of geometric algebra apply. Here is the point: We need only learn the rules once to work with both.

Surveying applications of geometric algebra over the past several years shows that

The conformal model is where it is at!

[4]P. Colapinto, *VERSOR: Spatial Computing with Conformal Geometric Algebra*. See pp. 25 and 67 for the lists. Used with permission of Colapinto.