Week 5

Penguin.

01046725 DIGITAL SIGNAL PROCESSING

Design of Nonrecursive Digital Filter

Semester 2/2566

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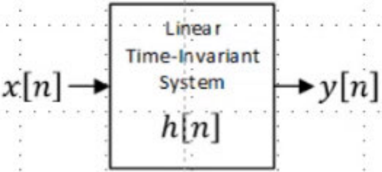
Digital Filter Design

Most of DSP system is known to be a "filter" because it transmits or suppresses some specific frequencies.

Digital Filter Design of the LTI system is the process to compute the coefficients of LCCDE in Time Domain to give its desired Frequency Response, $H(\omega)$, characteristic in Frequency Domain.

Time Domain Processing



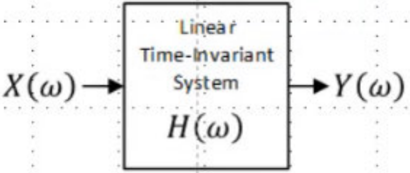
$$y[n] = x[n] * h[n] = h[n] * x[n]$$

where $h[n]$ is the unit impulse response

and $\quad y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k].h[n-k]$

If $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

Frequency Domain Processing



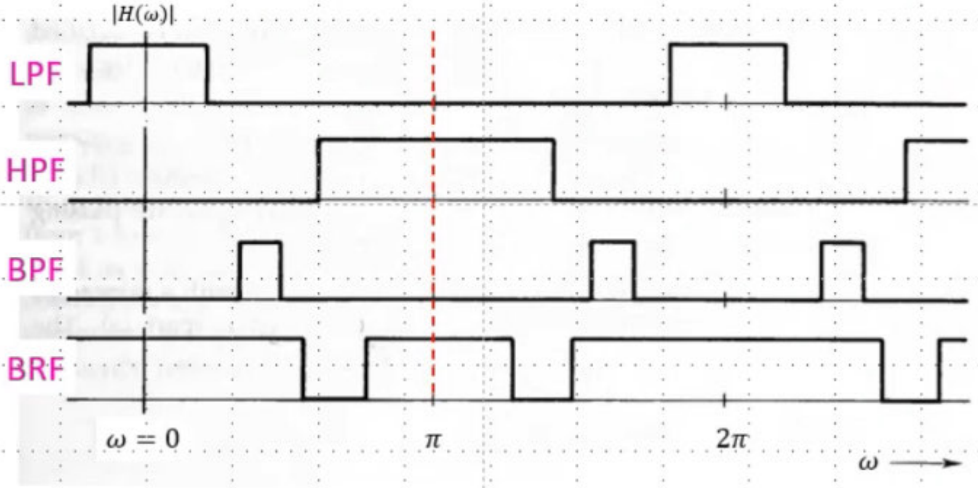
$$Y(\omega) = X(\omega).H(\omega) = H(\omega).X(\omega)$$

where $H(\omega)$ is the frequency response

and $Y(\omega) = X(\omega).H(\omega) = |X(\omega)|.|H(\omega)|\measuredangle X(\omega) + \measuredangle H(\omega)$

If $x[n] = \delta[n] \rightleftharpoons X(\omega) = \delta(\omega) = 1 \Rightarrow Y(\omega) = H(\omega)$

Four different types of (ideal) Digital Filter



LCCDE to Digital Filters

Discrete-Time Linear Time Invariant (LTI) System can be expressed as the Linear Constant Coefficient Difference Equation (LCCDE). Its transformation is the Filter.

Order of System

$$\sum_{k=0}^{N} a_k . y[n-k] = \sum_{k=0}^{M} b_k . x[n-k]$$

Recursive Part

$$y[n] = \boxed{-\sum_{k=1}^{N} a_k . y[n-k]} + \sum_{k=0}^{M} b_k . x[n-k]$$

LCCDE to Digital Filters

$$\sum_{k=0}^{N} a_k . y[n-k] = \sum_{k=0}^{M} b_k . x[n-k]$$

$$\sum_{k=0}^{N} a_k . Y(\omega) . exp(-jk\omega) = \sum_{k=0}^{M} b_k . X(\omega) . exp(-jk\omega)$$

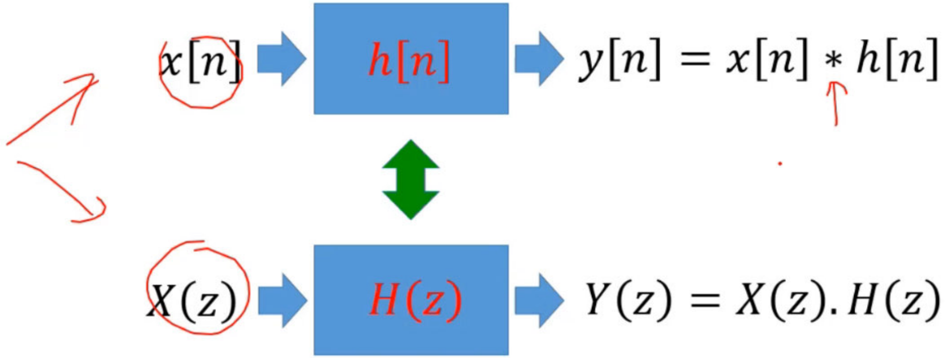
$$\sum_{k=0}^{N} a_k . Y(z) . z^{-k} = \sum_{k=0}^{M} b_k . X(\omega) . z^{-k}$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = x[n] * h[n]$$

$$\Updownarrow$$

$$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow Y(\omega) = X(\omega).H(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k.exp(-jk\omega)}{\sum_{k=0}^{N} a_k.exp(-jk\omega)}$$



$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k . z^{-k}}{\sum_{k=0}^{N} a_k . z^{-k}} = \frac{K(z - z_1)(z - z_2)(z - z_3)\ \ldots}{(z - p_1)(z - p_2)(z - p_3)\ \ldots}$$

Stability of the system

Nonrecursive Digital Filters is called FIR filter

$$y[n] = -\cancel{\sum_{k=1}^{N} a_k . y[n-k]} + \sum_{k=0}^{M} b_k . x[n-k]$$

b_k is $h[n]$

Finite Impulse Response

FIR length

This is the Convolution Sum operation.

$$\boxed{y[n] = \sum_{k=0}^{M} b_k . x[n-k]}$$

$$y[n] = \sum_{k=0}^{M} b_k . x[n-k]$$

$$\Updownarrow$$

$$Y(\omega) = \sum_{k=0}^{M} b_k . X(\omega) . exp(-jk\omega)$$

Frequency Response

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \sum_{k=0}^{M} b_k . exp(-jk\omega)}$$

$$y[n] = \sum_{k=0}^{M} b_k . x[n-k]$$

$$\Updownarrow$$

$$Y(z) = \sum_{k=0}^{M} b_k . X(z) . z^{-k}$$

Transfer Function

"Zeros" ONLY!

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k . z^{-k}}$$

Design of Nonrecursive (FIR) Digital Filters

Designing Nonrecursive filter is to find the finite number of the coefficients of Impulse Response, $h[n]$, which has the desired Frequency Response, $H(\omega)$, characteristic.

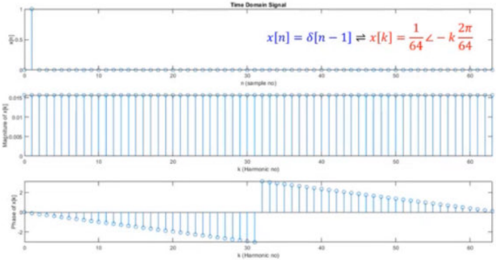
Usually, Nonrecursive filter is slower than Recursive one since it needs a large number of the coefficients of Impulse Response. However today computer is much faster, the discrepancy is now diminished.

Advantages of Nonrecursive Digital Filters

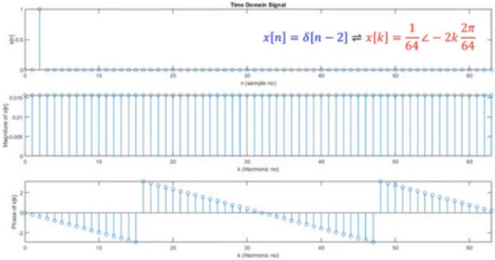
- The Nonrecursive (FIR) filter is inherently (BIBO) STABLE since it has ONLY ZEROs Transfer Function , $H(z)$.
- The Nonrecursive filter can be made symmetrical (non causal form) because it has a limited samples (FIR). This leads to the Linear-Phase characteristic where the phase response is a linear function of the frequency and results in "NO Phase Distortion"

In signal processing, phase distortion or phase-frequency distortion is distortion, that is, change in the shape of the waveform, that occurs when a filter's phase response is not linear over the frequency range of interest, that is, the phase shift introduced by a circuit or device is not directly proportional to frequency.

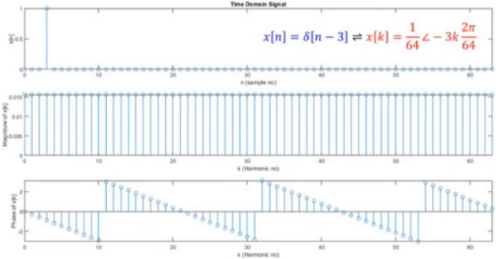
Example 4 : Linear Phase Characteristic



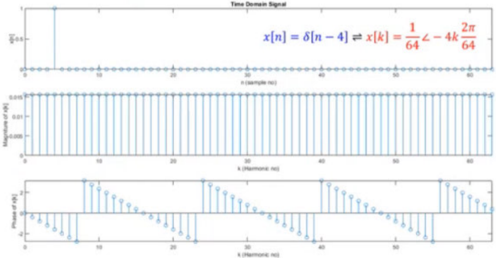
Example 4 : Linear Phase Characteristic



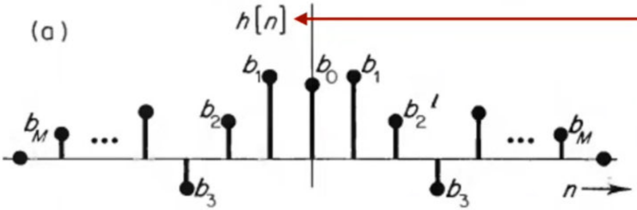
Example 4 : Linear Phase Characteristic



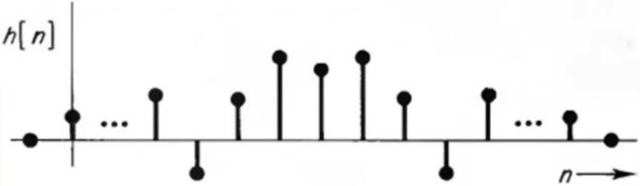
Example 4 : Linear Phase Characteristic



Linear Phase Characteristics



$h[n]$ is EVEN,
$H(\omega)$ is REAL
$(\measuredangle\phi(\omega) = 0)$



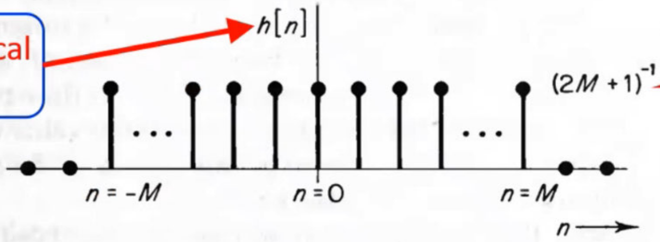
$h[n]$ is Causal,
$H(\omega)$ is Complex
$\measuredangle\phi(\omega)$ is linear function

Simple Moving Average Filters

Nonrecursive (FIR) filter can be easily found in many applications. The most popular FIR filter is the Simple Moving Average Filter as shown below.

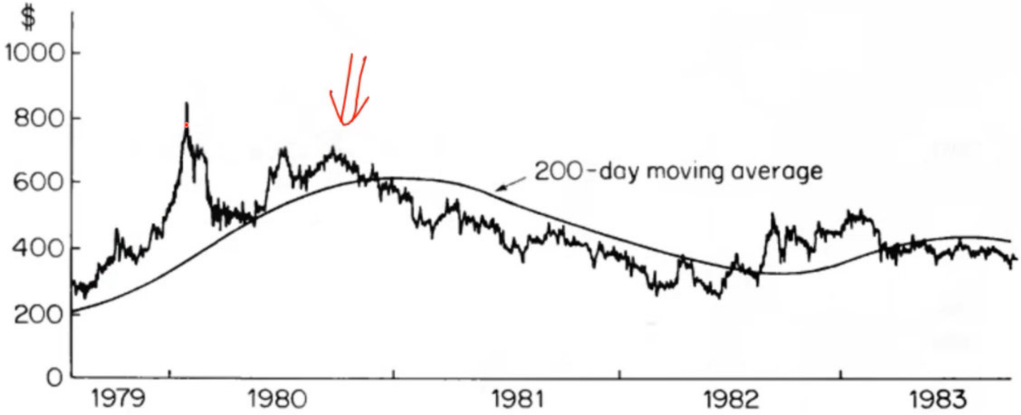
$h[n]$ is symmetrical but noncausal

Amplitude

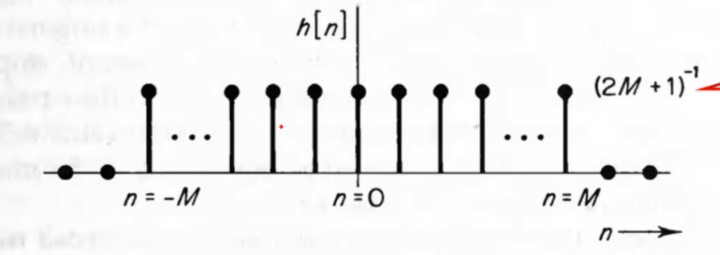


Impulse response of a noncausal, low-pass, moving-average filter.

Example of Simple Moving Average Filters



Spectrum of Simple Moving Average Filters



Impulse response of a noncausal, low-pass, moving-average filter.

Amplitude

Even Function

$$h[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} \delta[n-k]$$

Real Function

$$H(\omega) = \frac{1}{2M+1} \{1 + 2\cos\omega + 2\cos 2\omega + \cdots 2\cos M\omega\}$$

Discrete-Time (Forward) Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] exp(-j\omega n)$$

$$X(\omega) = \sum_{n=-1}^{-\infty} x[n] exp(-j\omega n) + x[0]. exp(0) + \sum_{n=1}^{\infty} x[n] exp(-j\omega n)$$

$$X(\omega) = \sum_{n=1}^{\infty} x[-n] exp(+j\omega n) + x[0] + \sum_{n=1}^{\infty} x[n] exp(-j\omega n)$$

$$X(\omega) = x[0] + \sum_{n=1}^{\infty} \{x[-n] exp(+j\omega n) + x[n] exp(-j\omega n)\}$$

When $x[n]$ is even function then $X(\omega)$ is real

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] exp(-j\omega n)$$

$$X(\omega) = x[0] + \sum_{n=1}^{\infty} \{x[-n] exp(+j\omega n) + x[n] exp(-j\omega n)\}$$

$$X(\omega) = x[0] + \sum_{n=1}^{\infty} \{x[-n]\cos(\omega n) + j.x[-n]\sin(\omega n) + x[n]\cos(\omega n) - j.x[n]\sin(\omega n)\}$$

If $x[n]$ is even function then $x[n] = x[-n]$

This is the Real Function

$$X(\omega) = x[0] + 2\sum_{n=1}^{\infty} \{x[n]\cos(\omega n)\}$$

There is NO imaginary part.

Zero Locations of Simple Moving Average Filters ($M = 2$)

$$h[n] = \frac{1}{2(2)+1}\sum_{k=-2}^{2}\delta[n-k] = 0.2\sum_{k=-2}^{2}\delta[n-k]$$

$$\boxed{h[n] = \delta[n-k] \rightleftharpoons H(z) = z^{-k}}$$

$$h[n] = 0.2\sum_{k=-2}^{2}\delta[n-k] \rightleftharpoons H(z) = 0.2[z^{2}+z^{1}+z^{0}+z^{-1}+z^{-2}]$$

$$H(z) = \frac{Y(z)}{X(z)} = 0.2\left[\frac{z^{4}+z^{3}+z^{2}+z^{1}+z^{0}}{z^{2}}\right]$$

4th zero order

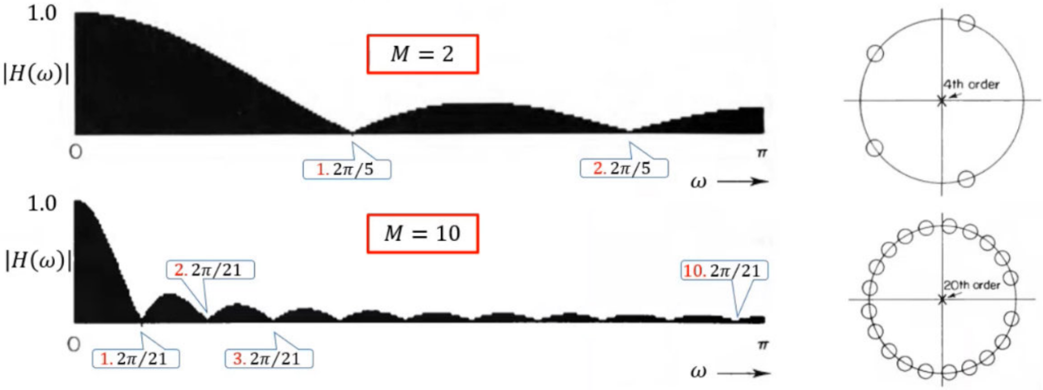
```
z=[1 1 1 1 1];
Z=roots(z)
angle(Z)*180/pi
```

```
Z = 4×1 complex
   0.3090 + 0.9511i
   0.3090 - 0.9511i
  -0.8090 + 0.5878i
  -0.8090 - 0.5878i

ans = 4×1
   72.0000
  -72.0000
  144.0000
 -144.0000
```

$$y[n+2] = 0.2\sum_{k=0}^{4}x[n+k] \Rightarrow y[n] = 0.2\sum_{k=-2}^{2}x[n+k]$$

Spectrum of Moving Average Filters ($M = 2, M = 10$)



Moving Average Filter shows the Low Pass Filter (LPF) characteristic. The magnitude of spectrum, $|H(\omega)|$ is displayed in Half period $[0, \pi]$.

Design of Nonrecursive HPF and BPF

- To build the HPF or BPF filters, the center frequency, ω_o, is shifted over the range $0 \leq \omega \leq \pi$.
- This is done by convoluting the LPF with an impulse (delta function) at the desired center frequency.
- Convolution in Frequency domain is Multiplication in Time domain (Modulation Property)

$$x[n].y[n] \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega).Y(\omega-\lambda)d\lambda$$

Modulation Property with Cosine function

$$FT\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega.n} = X(\omega)$$

$$FT\{x[n].e^{j\omega_0 n}\} = \sum_{n=-\infty}^{\infty} x[n].e^{j\omega_0 n}e^{-j\omega.n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0).n} = X(\omega - \omega_0)$$

$$\cos(\omega_o.n) = \frac{1}{2}\left(e^{j\omega_o.n} + e^{-j\omega_o.n}\right)$$

$$FT\{x[n].\cos(\omega_o.n)\} = \frac{1}{2}\{X(\omega - \omega_o) + X(\omega + \omega_o)\}$$

Design of Nonrecursive HPF and BPF

If $h[n] \rightleftharpoons H(\omega)$ is the Low Pass Filter (LPF),

then $h[n].\cos(\omega_0 n) \rightleftharpoons H_{BPF}(\omega) = \frac{1}{2}[H(\omega - \omega_o) + H(\omega + \omega_o)]$

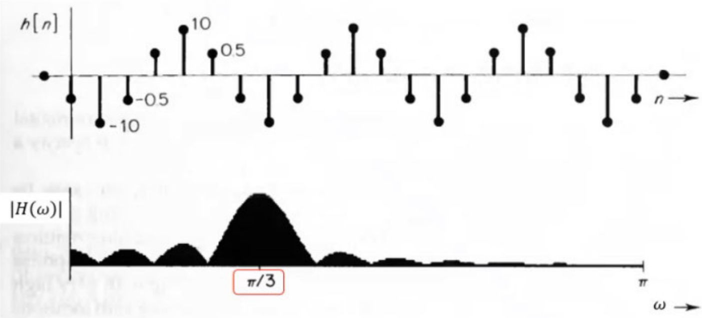
The Band Pass Filter (BPF) is simply designed by multiplying the impulse response of Low Pass Filter, $h[n]$ to $\cos(\omega_0 n)$.

This shifts the peak (center) frequency from $0\ rad/sample$ of LPF to the new center frequency, $\omega_0\ rad/sample$ of BPF or HPF.

It is LPF when $\omega_0 = 0$, BPF when $0 < \omega_0 < \pi$ and HPF when $\omega_0 = \pi$.

Example of Nonrecursive BPF Digital Filters

When the impulse response of Low Pass Filter $(\omega_o = 0\ rad/sample)$ is designed as $h[n]_{LPF}$,
the impulse response of Band Pass Filter at $\omega_o = \pi/3\ rad/sample$ is
, $h[n]_{BPF} = h[n]_{LPF} . \cos\left(n.\frac{\pi}{3}\right)$.

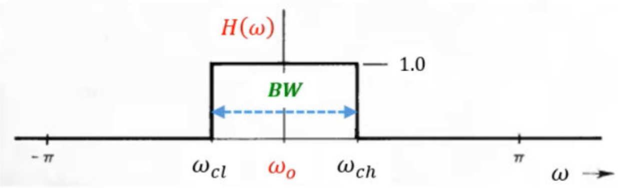


BREAK

Nonrecursive Filter Design by Fourier Transform Method

- To systematically design the digital filter, the Cutoff Frequency, ω_c, must be specified in addition to the Center Frequency, ω_o.
- The filter Bandwidth (BW) is the pass band of frequency between Low and High Cutoff Frequencies, $[\omega_{cl}, \omega_{ch}]$.

$$BW = |\omega_{ch} - \omega_{cl}| \text{ or } \frac{BW}{2} = |\omega_{ch} - \omega_o| = |\omega_o - \omega_{cl}|$$



Nonrecursive Filter Design by Fourier Transform Method

From the ideal Frequency Response, $H(\omega)$, of LPF $(\omega_o = 0\, rad/sample)$ with the Cutoff Frequency $\omega_c\ rad/sample$, its impulse response, $h[n]$,is calculated from the Inverse Fourier Transform.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] exp(-j\omega n)$$

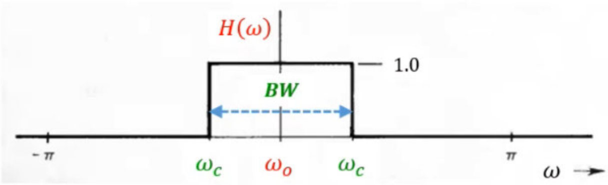
$$\Updownarrow$$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega) exp(j\omega n) d\omega$$

Synthesis Equation (Inverse Transform)

$$\boxed{x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(\omega) exp(j\omega n) d\omega}$$

Nonrecursive Filter Design by Fourier Transform Method



$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega) exp(j\omega n) d\omega$$

$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} 1.\, exp(j\omega n) d\omega$$

Nonrecursive Filter Design by Fourier Transform Method

$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} 1.\, exp(j\omega n)d\omega$$

$$h[n] = \frac{1}{2\pi}\left[\frac{exp(j\omega n)}{jn}\right]\Bigg|_{-\omega_c}^{+\omega_c}$$

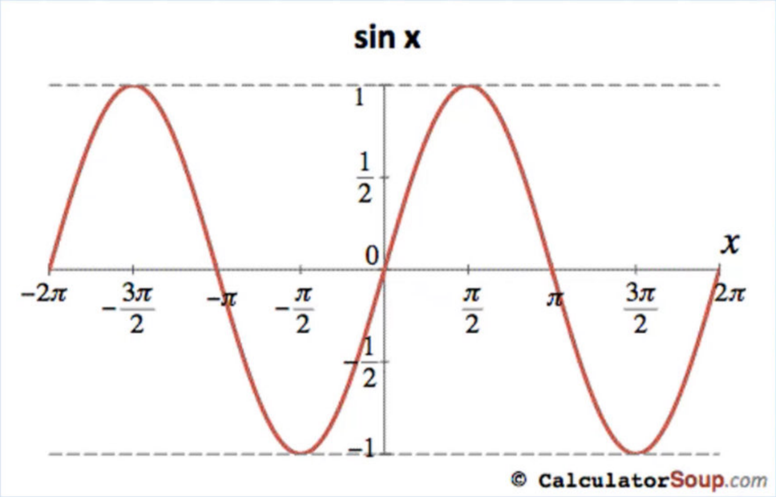
$$h[n] = \frac{exp(j\omega_c n) - exp(-j\omega_c n)}{2\pi.jn}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$h[n] = \frac{\sin(\omega_c n)}{n\pi} = \frac{\omega_c}{\pi}.\frac{\sin(\omega_c n)}{\omega_c n} = \frac{\omega_c}{\pi}.\operatorname{sinc}(\omega_c n)$$

$$\boxed{\operatorname{sinc}\theta = \frac{\sin\theta}{\theta}}$$

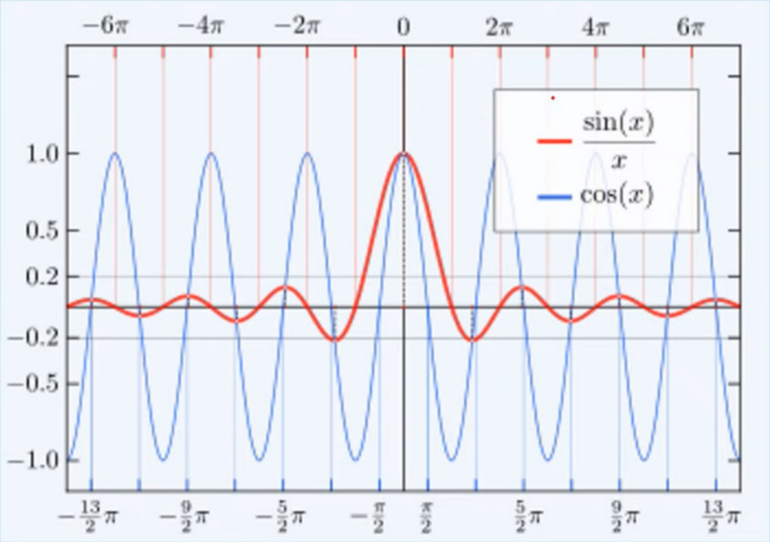
Sine Function



Sinc Function

$$f(x) = \frac{\sin x}{x}$$

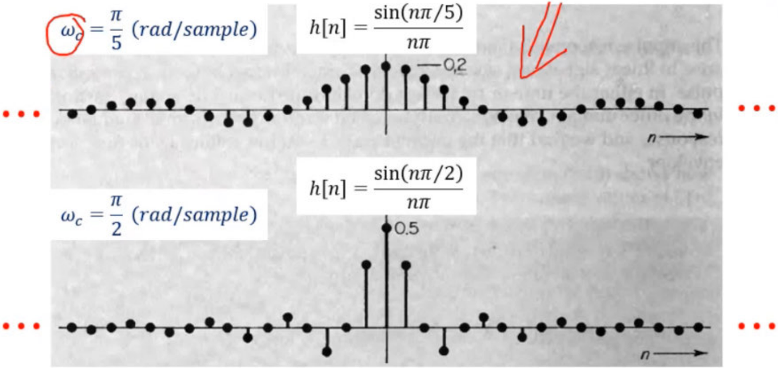
Sinc Function vs Cosine Function



Nonrecursive Filter Design by Fourier Transform Method

The impulse response of ideal nonrecursive LPF is the infinite "sinc" (even) function, $h[n] = \frac{\omega_c}{\pi}.\,\mathrm{sinc}(\omega_c n)$ as shown in 2 examples below.

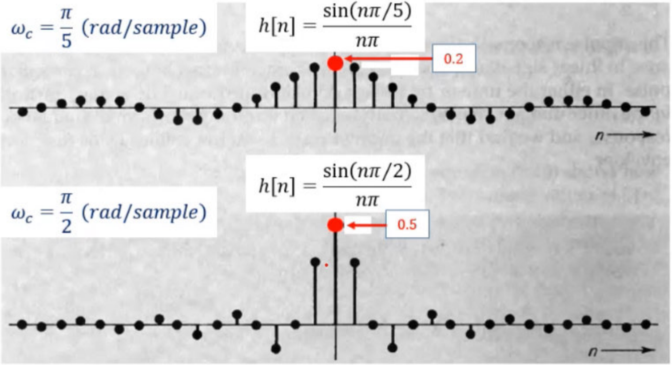
ω



l'Hospital's rule for Sinc Function

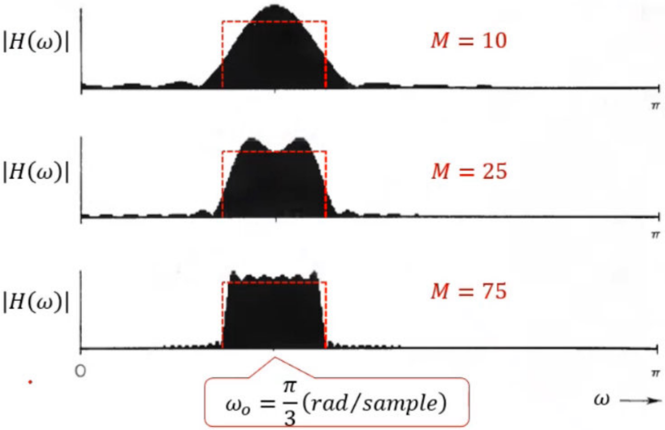
Note that $h[n=0]$ is computed from l'Hospital's rule.

$$h[0] = \frac{\frac{d}{dn}(\sin \omega_c n)}{\frac{d}{dn}(n\pi)}\Bigg|_{n=0} = \frac{\omega_c . \cos \omega_c n}{\pi}\Bigg|_{n=0} = \frac{\omega_c}{\pi}$$



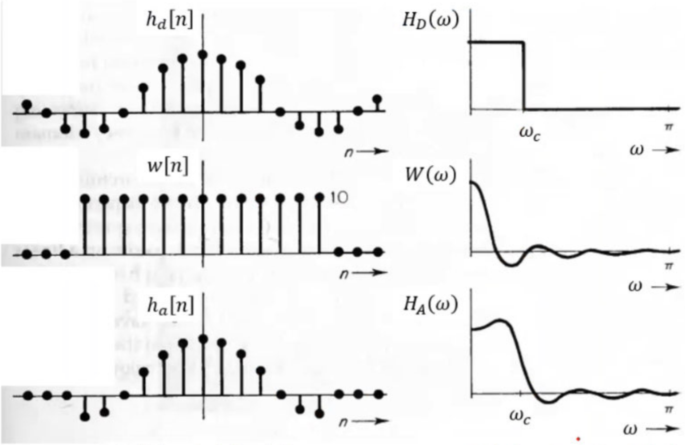
Nonrecursive Filter Design by Fourier Transform Method

The infinite Sinc function (IIR) must be truncated to become a practical Finite Impulse Response (FIR).



The longer $h[n]$ (M is larger) is, the closer $|H(\omega)|$ is to the ideal filter

Nonrecursive Filter Design by Fourier Transform Method



$h_a[n] = h_d[n].w[n]$

$H_A(\omega) = H_D(\omega) * W(\omega)$

- When the infinite sinc function is truncated, it is equivalent to multiplying with the finite-length pulse function in Time-Domain.
- From Modulation property, this is the convolution of those spectrums in Frequency Domain.
- This causes the ripple in the transition period called "Gibb's phenomena"

Summary of Filter Design by Fourier Transform Method

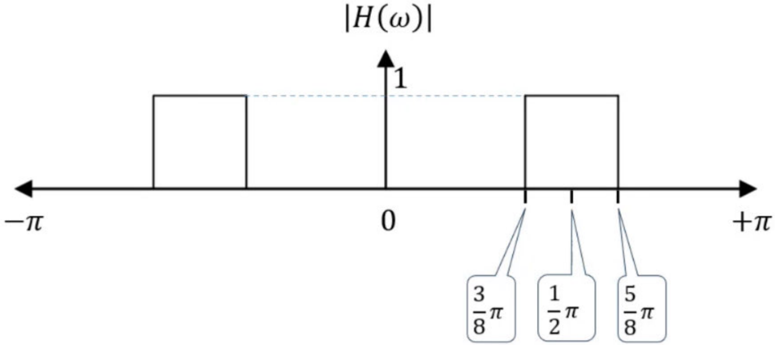
1. Define the center Frequency, $\omega_o\ (rad/sample)$
2. Define the Cutoff Frequency, ω_c, or Bandwidth (BW) in $rad/sample$
3. Generate (noncausal) Infinite Impulse Response, $h_d[n]$,from the equation

$$h_d[n] = \frac{2}{n\pi}\sin(n\omega_c)\,.\cos(n\omega_o)$$

4. Define the size (in samples) of $h[n]$ and truncate it symmetrically to become Finite Impulse Response (Even function), $h_a[n]$.
5. Shift $h_a[n]$ to the right until it starts at $n = 0$ to make it as a "Causal System".

Exercise for DSP course 2/2566

1. Design nonrecursive FIR Band Pass Filter, $h[n]$, with $M = 10$ ($h[n] = 21$ samples) with $\omega_o = 0.5\pi$ rad/sample and $\omega_c = \frac{\pi}{8}$ rad/sample (Bandwidth $= \frac{\pi}{4}$ rad/sample)
2. Plot $h[n]$ for $n = -M\ to + M$ samples
3. Plot $|H(\omega)|$ for $\omega = -3\pi\ to + 3\pi$
4. Redesign from 1 to 3 with $M = 50$ ($h[n] = 101$ samples) and $M = 100$ ($h[n] = 201$ samples)



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