

Week 5



Penguin.

01046725 DIGITAL SIGNAL PROCESSING

Design of Nonrecursive Digital Filter

Semester 2/2566

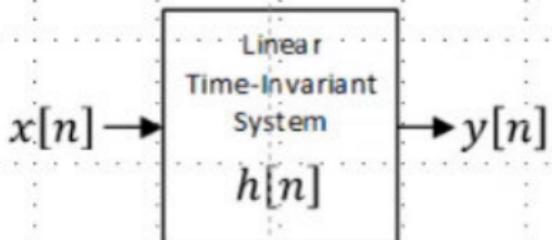
Asst. Prof. Suradej Tretriluxana, PhD.

Digital Filter Design

Most of DSP system is known to be a “**filter**” because it **transmits** or **suppresses** some specific frequencies.

Digital Filter Design of the **LTI** system is the process to compute the **coefficients** of LCCDE **in Time Domain** to give its desired Frequency Response, $H(\omega)$, characteristic **in Frequency Domain**.

Time Domain Processing



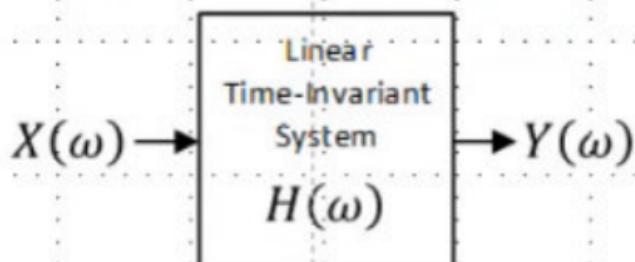
$$y[n] = x[n] * h[n] = h[n] * x[n]$$

where $h[n]$ is the unit impulse response

and $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k].h[n-k]$

If $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

Frequency Domain Processing



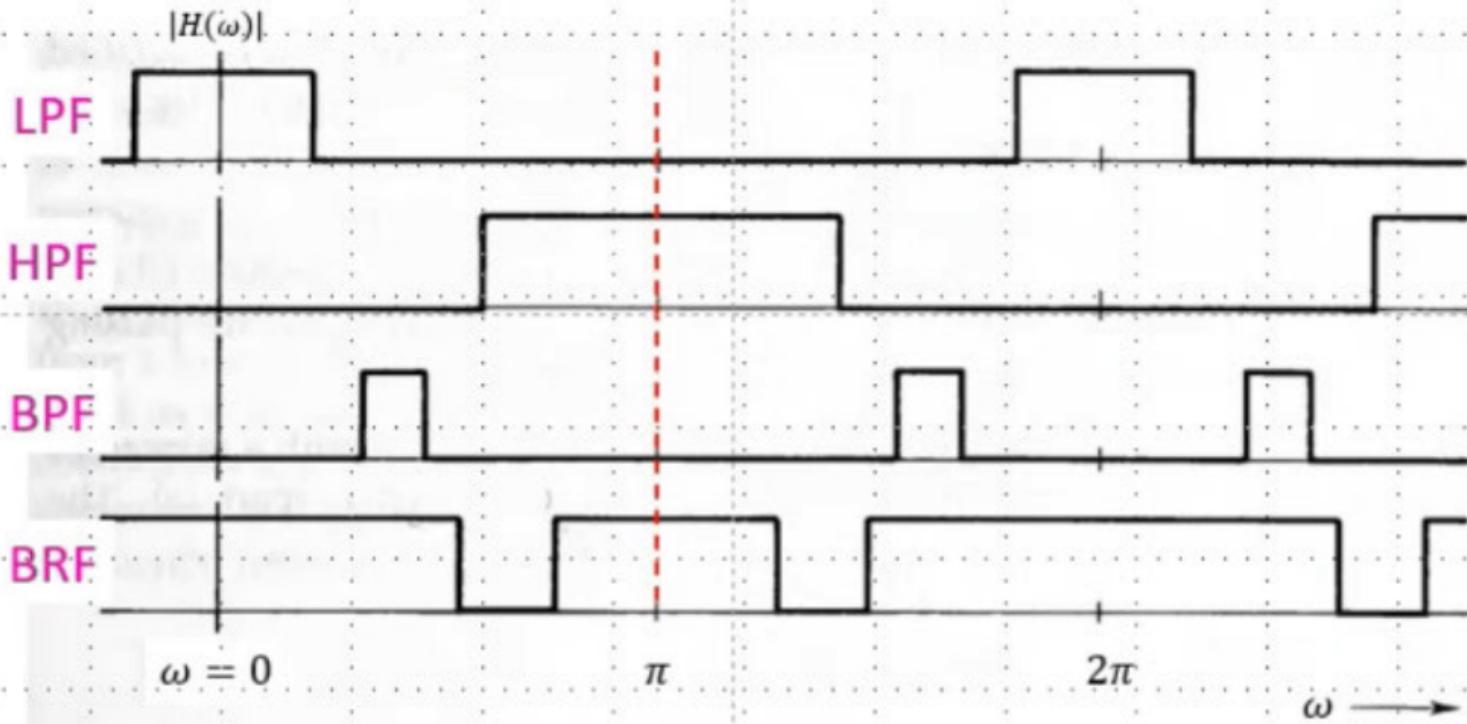
$$Y(\omega) = X(\omega) \cdot H(\omega) = H(\omega) \cdot X(\omega)$$

where $H(\omega)$ is the frequency response

and $Y(\omega) = X(\omega) \cdot H(\omega) = |X(\omega)| \cdot |H(\omega)| \angle X(\omega) + \angle H(\omega)$

If $x[n] = \delta[n] \Leftrightarrow X(\omega) = \delta(\omega) = 1 \Rightarrow Y(\omega) = H(\omega)$

Four different types of (ideal) Digital Filter



LCCDE to Digital Filters

Discrete-Time Linear Time Invariant (LTI) System can be expressed as the Linear Constant Coefficient Difference Equation (LCCDE). Its transformation is the Filter.

Order of System

$$\sum_{k=0}^N a_k \cdot y[n - k] = \sum_{k=0}^M b_k \cdot x[n - k]$$

Recursive Part

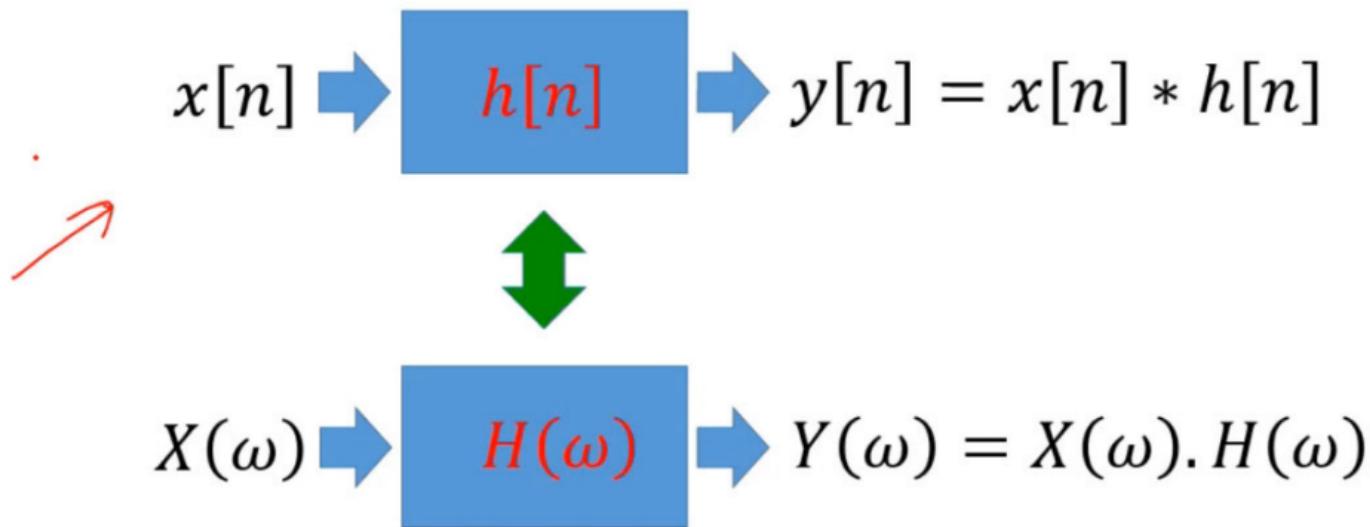
$$y[n] = - \sum_{k=1}^N a_k \cdot y[n - k] + \sum_{k=0}^M b_k \cdot x[n - k]$$

LCCDE to Digital Filters

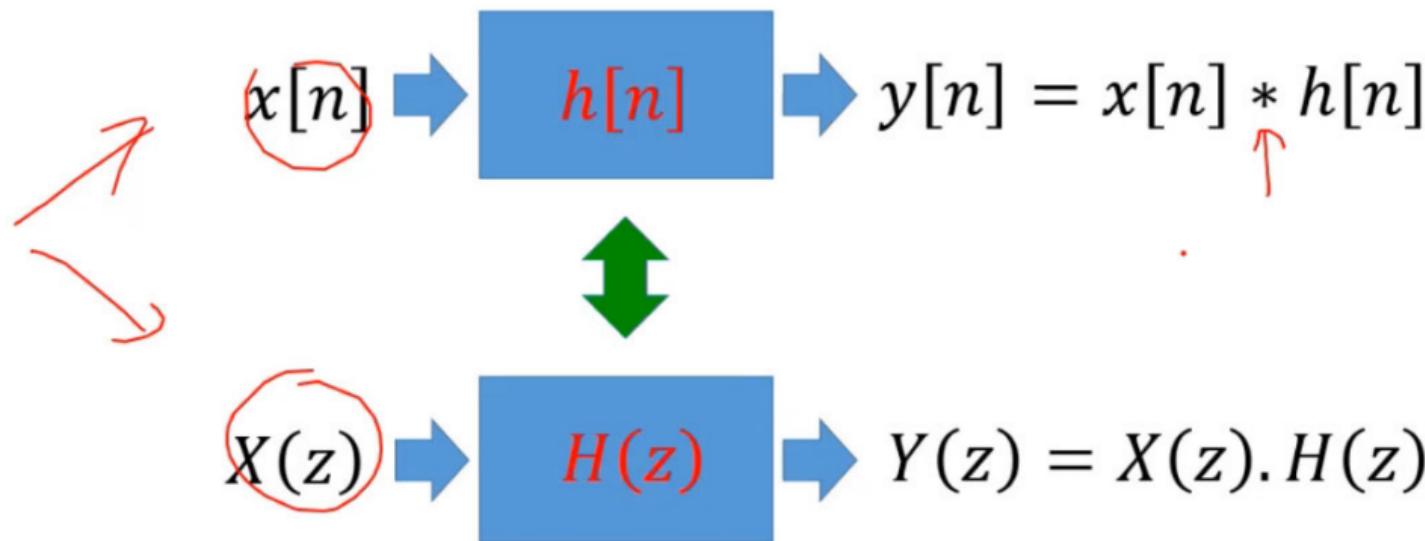
$$\sum_{k=0}^N a_k \cdot y[n - k] = \sum_{k=0}^M b_k \cdot x[n - k]$$

$$\sum_{k=0}^N a_k \cdot Y(\omega) \cdot \exp(-jk\omega) = \sum_{k=0}^M b_k \cdot X(\omega) \cdot \exp(-jk\omega)$$

$$\sum_{k=0}^N a_k \cdot Y(z) \cdot z^{-k} = \sum_{k=0}^M b_k \cdot X(\omega) \cdot z^{-k}$$



$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^M b_k \cdot \exp(-jk\omega)}{\sum_{k=0}^N a_k \cdot \exp(-jk\omega)}$$



$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k \cdot z^{-k}}{\sum_{k=0}^N a_k \cdot z^{-k}} = \frac{K(z - z_1)(z - z_2)(z - z_3) \dots}{(z - p_1)(z - p_2)(z - p_3) \dots}$$

Stability of the system

Nonrecursive Digital Filters is called **FIR filter**

$$y[n] = - \sum_{k=1}^N a_k \cdot y[n-k] + \sum_{k=0}^M b_k \cdot x[n-k]$$

FIR length

b_k is $h[n]$
Finite Impulse
Response

This is the
Convolution
Sum operation.

$$y[n] = \sum_{k=0}^M b_k \cdot x[n-k]$$

$$y[n] = \sum_{k=0}^M b_k \cdot x[n - k]$$



Frequency
Response

$$Y(\omega) = \sum_{k=0}^M b_k \cdot X(\omega) \cdot \exp(-jk\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \sum_{k=0}^M b_k \cdot \exp(-jk\omega)$$

$$y[n] = \sum_{k=0}^M b_k \cdot x[n - k]$$



$$Y(z) = \sum_{k=0}^M b_k \cdot X(z) \cdot z^{-k}$$

Transfer
Function

"Zeros"
ONLY!

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^M b_k \cdot z^{-k}$$

Design of Nonrecursive (FIR) Digital Filters

Designing Nonrecursive filter is to find the finite number of the coefficients of Impulse Response, $h[n]$, which has the desired Frequency Response, $H(\omega)$, characteristic.

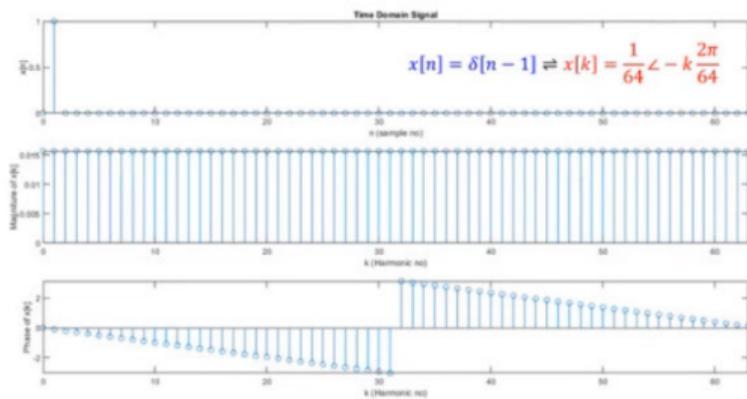
Usually, Nonrecursive filter is slower than Recursive one since it needs a large number of the coefficients of Impulse Response. However today computer is much faster, the discrepancy is now diminished.

Advantages of Nonrecursive Digital Filters

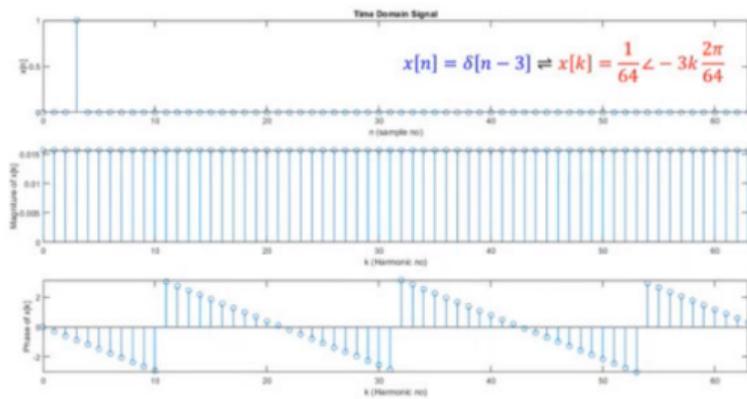
- The Nonrecursive (FIR) filter is inherently (BIBO) STABLE since it has ONLY ZEROS Transfer Function , $H(z)$.
- The Nonrecursive filter can be made symmetrical (non causal form) because it has a limited samples (FIR). This leads to the Linear-Phase characteristic where the phase response is a linear function of the frequency and results in “NO Phase Distortion”

In signal processing, phase distortion or phase-frequency distortion is distortion, that is, change in the shape of the waveform, that occurs when a filter's phase response is not linear over the frequency range of interest, that is, the phase shift introduced by a circuit or device is not directly proportional to frequency.

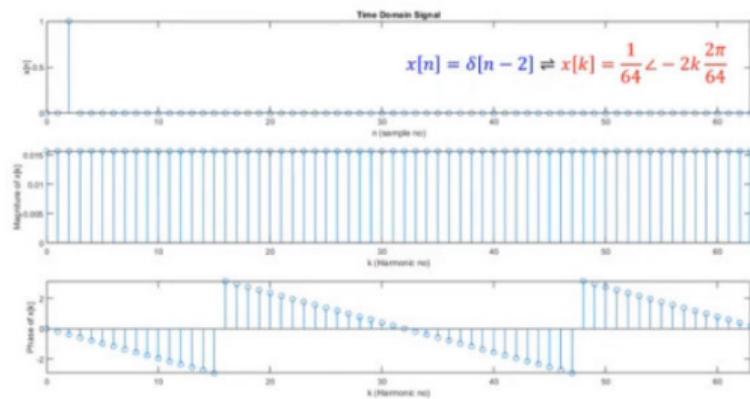
Example 4 : Linear Phase Characteristic



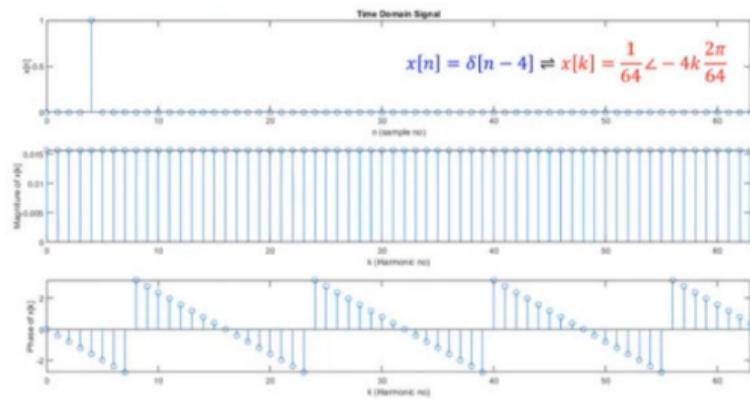
Example 4 : Linear Phase Characteristic



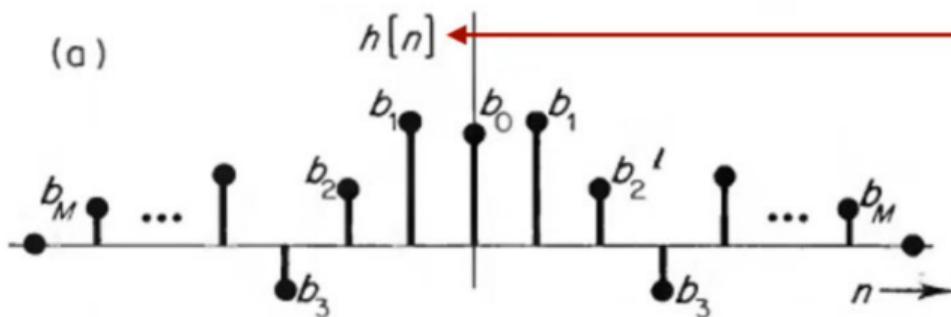
Example 4 : Linear Phase Characteristic



Example 4 : Linear Phase Characteristic

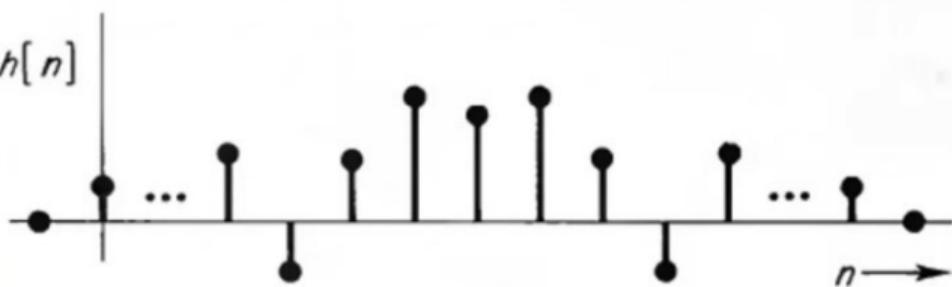


Linear Phase Characteristics



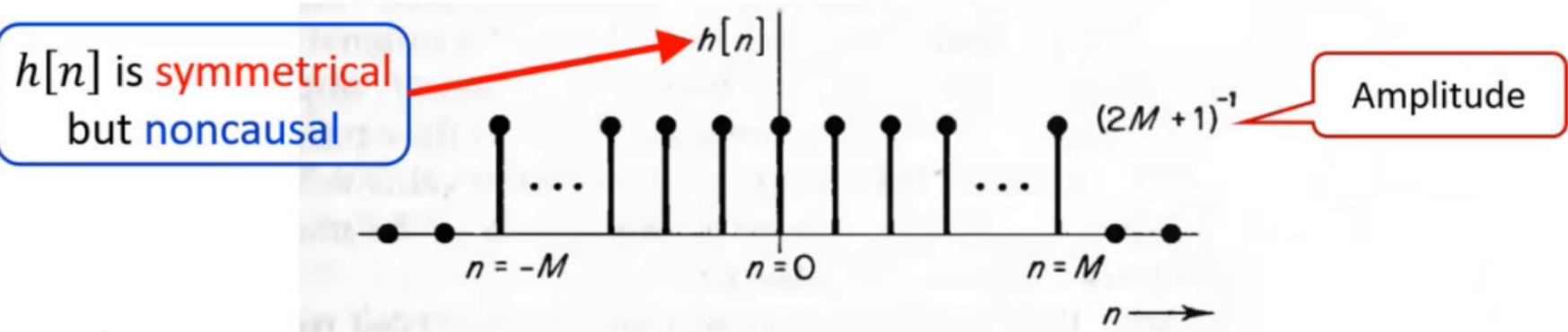
$h[n]$ is EVEN,
 $H(\omega)$ is REAL
($\phi(\omega) = 0$)

$h[n]$ is Causal,
 $H(\omega)$ is Complex
 $\phi(\omega)$ is linear function



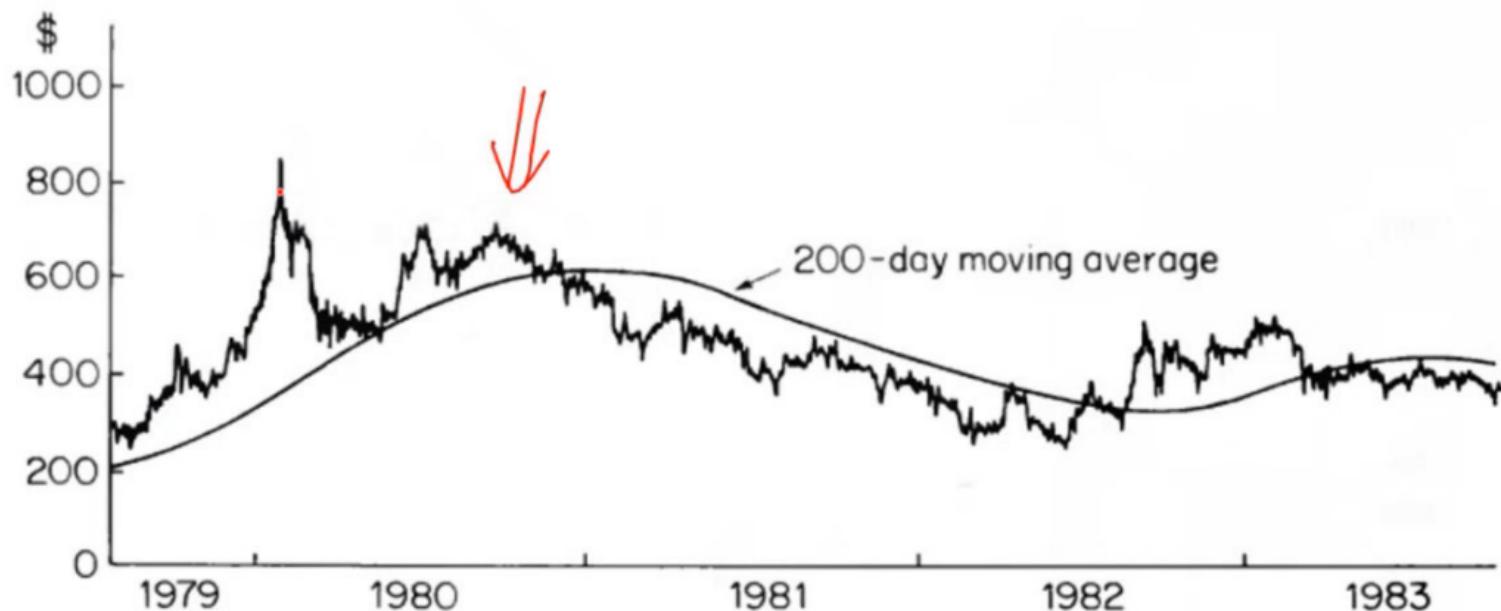
Simple Moving Average Filters

Nonrecursive (FIR) filter can be easily found in many applications. The most popular FIR filter is the **Simple Moving Average Filter** as shown below.

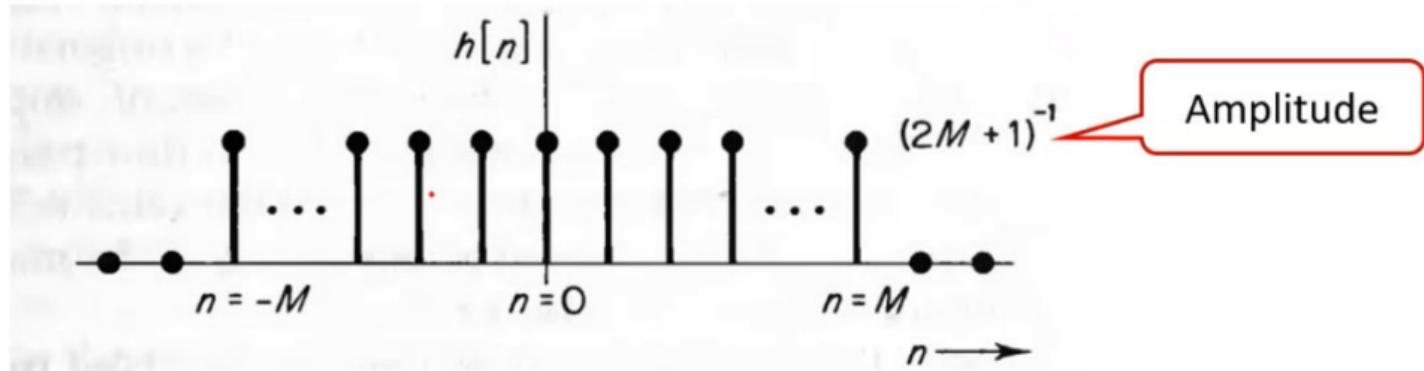


Impulse response of a noncausal, low-pass, moving-average filter.

Example of Simple Moving Average Filters



Spectrum of Simple Moving Average Filters



Impulse response of a noncausal, low-pass, moving-average filter.

Even Function

$$h[n] = \frac{1}{2M+1} \sum_{k=-M}^M \delta[n-k]$$

Real Function

$$H(\omega) = \frac{1}{2M+1} \{1 + 2 \cos \omega + 2 \cos 2\omega + \dots + 2 \cos M\omega\}$$

Discrete-Time (Forward) Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\omega n)$$

$$X(\omega) = \sum_{n=-1}^{-\infty} x[n] \exp(-j\omega n) + x[0] \cdot \exp(0) + \sum_{n=1}^{\infty} x[n] \exp(-j\omega n)$$

$$X(\omega) = \sum_{n=1}^{\infty} x[-n] \exp(+j\omega n) + x[0] + \sum_{n=1}^{\infty} x[n] \exp(-j\omega n)$$

$$\therefore X(\omega) = x[0] + \sum_{n=1}^{\infty} \{x[-n] \exp(+j\omega n) + x[n] \exp(-j\omega n)\}$$

When $x[n]$ is even function then $X(\omega)$ is real

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\omega n)$$

$$X(\omega) = x[0] + \sum_{n=1}^{\infty} \{x[-n] \exp(+j\omega n) + x[n] \exp(-j\omega n)\}$$

$$X(\omega) = x[0] + \sum_{n=1}^{\infty} \{x[-n] \cos(\omega n) + j \cdot x[-n] \sin(\omega n) + x[n] \cos(\omega n) - j \cdot x[n] \sin(\omega n)\}$$

If $x[n]$ is even function then $x[n] = x[-n]$

This is the Real Function

$$X(\omega) = x[0] + 2 \sum_{n=1}^{\infty} \{x[n] \cos(\omega n)\}$$

There is NO imaginary part.

Zero Locations of Simple Moving Average Filters ($M = 2$)

$$h[n] = \frac{1}{2(2) + 1} \sum_{k=-2}^2 \delta[n - k] = 0.2 \sum_{k=-2}^2 \delta[n - k]$$

$$h[n] = \delta[n - k] \Leftrightarrow H(z) = z^{-k}$$

$$h[n] = 0.2 \sum_{k=-2}^2 \delta[n - k] \Leftrightarrow H(z) = 0.2[z^2 + z^1 + z^0 + z^{-1} + z^{-2}]$$

$$H(z) = \frac{Y(z)}{X(z)} = 0.2 \left[\frac{z^4 + z^3 + z^2 + z^1 + z^0}{z^2} \right]$$

4th zero order

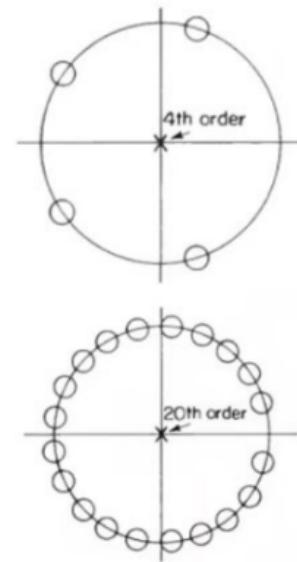
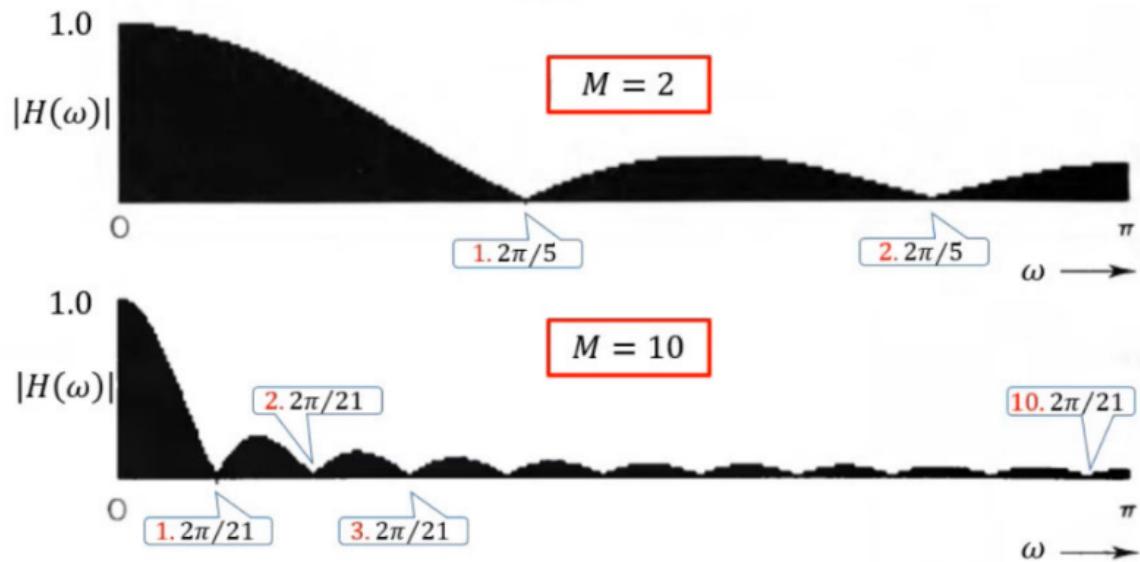
```
z=[1 1 1 1 1];
Z=roots(z)
angle(Z)*180/pi
```

```
Z = 4x1 complex
  0.3090 + 0.9511i
  0.3090 - 0.9511i
  -0.8090 + 0.5878i
  -0.8090 - 0.5878i
```

```
a = 4x1
  72.0000
 -72.0000
 144.0000
 -144.0000
```

$$y[n+2] = 0.2 \sum_{k=0}^4 x[n+k] \Rightarrow y[n] = 0.2 \sum_{k=-2}^2 x[n+k]$$

Spectrum of Moving Average Filters ($M = 2, M = 10$)



Moving Average Filter shows the **Low Pass Filter (LPF)** characteristic. The magnitude of spectrum, $|H(\omega)|$ is displayed in **Half period** $[0, \pi]$.

Design of Nonrecursive HPF and BPF

- To build the HPF or BPF filters, the center frequency, ω_o , is shifted over the range $0 \leq \omega \leq \pi$.
- This is done by convoluting the LPF with an impulse (delta function) at the desired center frequency.
- Convolution in Frequency domain is Multiplication in Time domain (Modulation Property)

$$x[n].y[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega).Y(\omega - \lambda)d\lambda$$

Modulation Property with Cosine function

$$FT\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega.n} = X(\omega)$$

$$FT\{x[n].e^{j\omega_0 n}\} = \sum_{n=-\infty}^{\infty} x[n].e^{j\omega_0 n}e^{-j\omega.n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0).n} = X(\omega - \omega_0)$$

$$\cos(\omega_o.n) = \frac{1}{2}(e^{j\omega_o.n} + e^{-j\omega_o.n})$$

$$FT\{x[n].\cos(\omega_o.n)\} = \frac{1}{2}\{X(\omega - \omega_o) + X(\omega + \omega_o)\}$$

Design of Nonrecursive HPF and BPF

If $h[n] \Leftrightarrow H(\omega)$ is the Low Pass Filter (LPF),

then $h[n].\cos(\omega_0 n) \Leftrightarrow H_{BPF}(\omega) = \frac{1}{2} [H(\omega - \omega_0) + H(\omega + \omega_0)]$

The Band Pass Filter (BPF) is simply designed by multiplying the impulse response of Low Pass Filter, $h[n]$ to $\cos(\omega_0 n)$.

This shifts the peak (center) frequency from 0 rad/sample of LPF to the new center frequency, $\omega_0 \text{ rad/sample}$ of BPF or HPF.

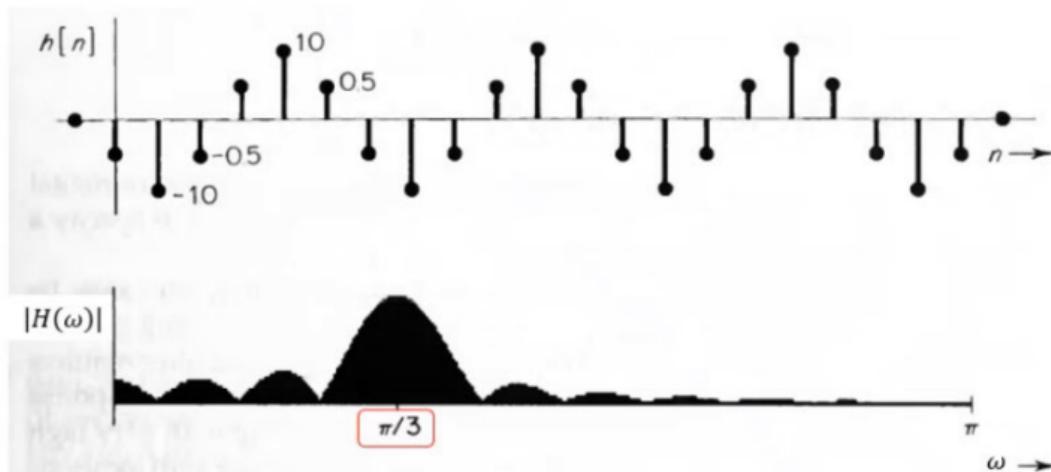
It is LPF when $\omega_0 = 0$, BPF when $0 < \omega_0 < \pi$ and HPF when $\omega_0 = \pi$.

Example of Nonrecursive BPF Digital Filters

When the impulse response of Low Pass Filter ($\omega_o = 0 \text{ rad/sample}$) is designed as $h[n]_{LPF}$,

the impulse response of Band Pass Filter at $\omega_o = \pi/3 \text{ rad/sample}$ is

$$, h[n]_{BPF} = h[n]_{LPF} \cdot \cos\left(n \cdot \frac{\pi}{3}\right).$$

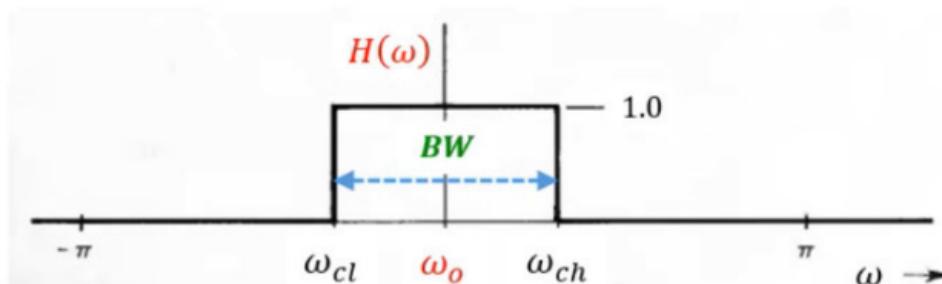


BREAK

Nonrecursive Filter Design by Fourier Transform Method

- To systematically design the digital filter, the **Cutoff Frequency**, ω_c , must be specified in addition to the **Center Frequency**, ω_o .
- The filter Bandwidth (BW) is the pass band of frequency between Low and High Cutoff Frequencies, $[\omega_{cl}, \omega_{ch}]$.

$$BW = |\omega_{ch} - \omega_{cl}| \text{ or } \frac{BW}{2} = |\omega_{ch} - \omega_o| = |\omega_o - \omega_{cl}|$$



Nonrecursive Filter Design by Fourier Transform Method

From the **ideal** Frequency Response, $H(\omega)$, of LPF ($\omega_o = 0 \text{ rad/sample}$) with the Cutoff Frequency $\omega_c \text{ rad/sample}$, its impulse response, $h[n]$, is calculated from the Inverse Fourier Transform.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] \exp(-j\omega n)$$

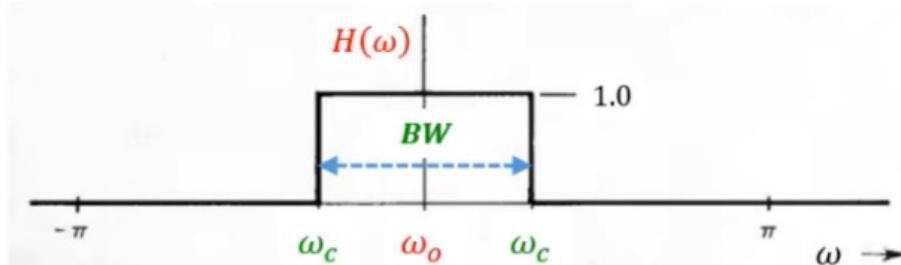
Synthesis Equation (Inverse Transform)



$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) \exp(j\omega n) d\omega$$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) \exp(j\omega n) d\omega$$

Nonrecursive Filter Design by Fourier Transform Method



$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) \exp(j\omega n) d\omega$$

$$h[n] = \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} 1. \exp(j\omega n) d\omega$$

Nonrecursive Filter Design by Fourier Transform Method

$$h[n] = \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} 1 \cdot \exp(j\omega n) d\omega$$

$$h[n] = \frac{1}{2\pi} \left[\frac{\exp(j\omega n)}{jn} \right] \Big|_{-\omega_c}^{+\omega_c}$$

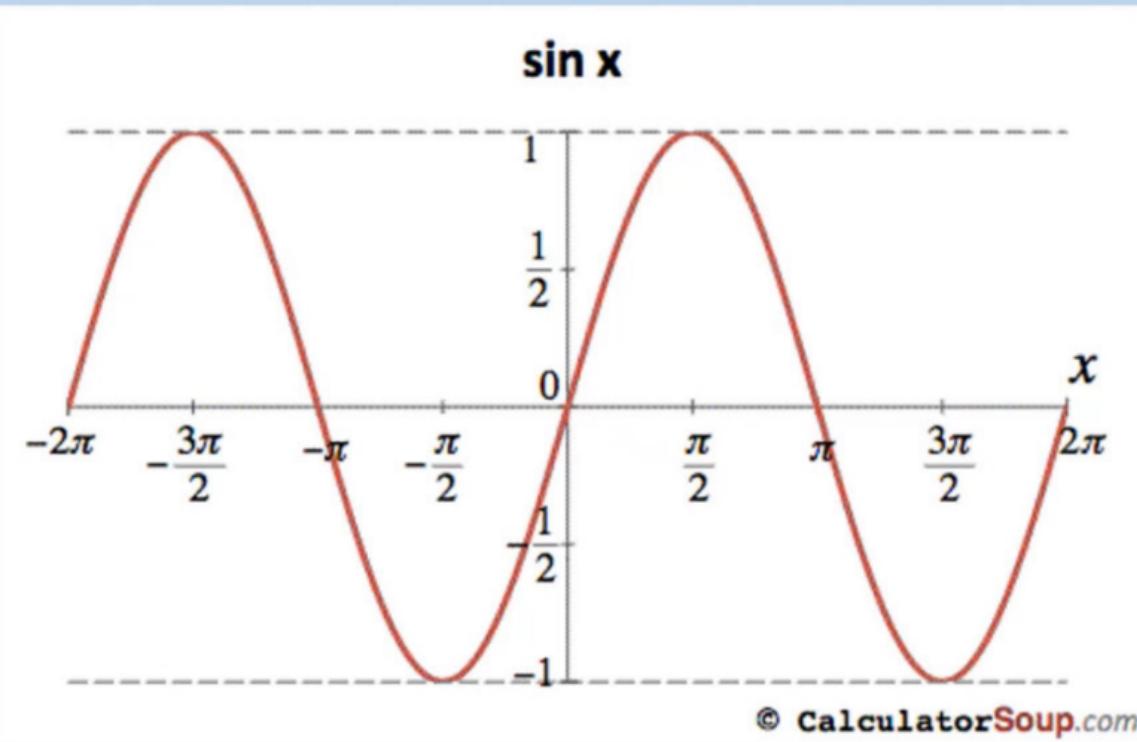
$$h[n] = \frac{\exp(j\omega_c n) - \exp(-j\omega_c n)}{2\pi \cdot jn}$$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$h[n] = \frac{\sin(\omega_c n)}{n\pi} = \frac{\omega_c}{\pi} \cdot \frac{\sin(\omega_c n)}{\omega_c n} = \frac{\omega_c}{\pi} \cdot \text{sinc}(\omega_c n)$$

$$\text{sinc } \theta = \frac{\sin \theta}{\theta}$$

Sine Function

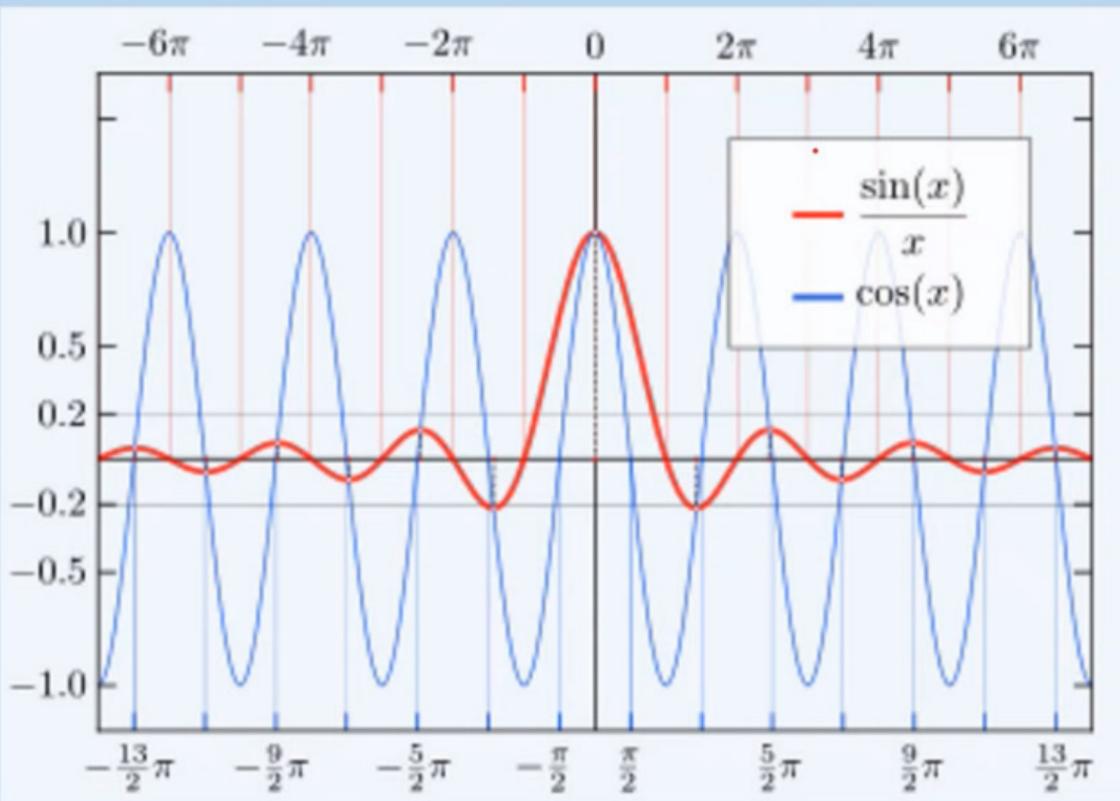


Sinc Function



$$f(x) = \frac{\sin x}{x}$$

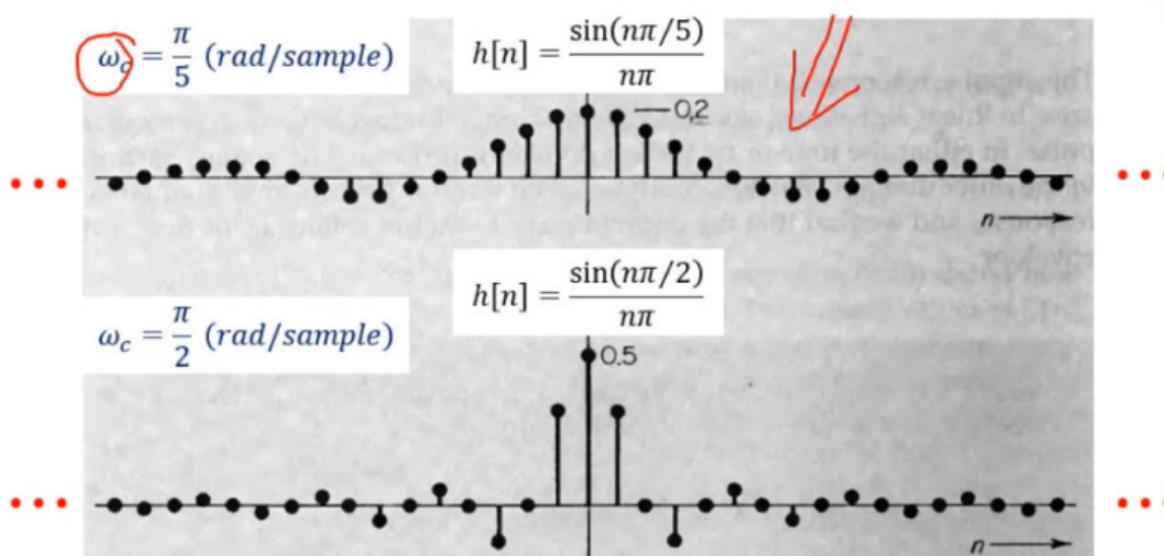
Sinc Function vs Cosine Function



Nonrecursive Filter Design by Fourier Transform Method

The impulse response of ideal nonrecursive LPF is the infinite “sinc” (even) function, $h[n] = \frac{\omega_c}{\pi} \cdot \text{sinc}(\omega_c n)$ as shown in 2 examples below.

ω

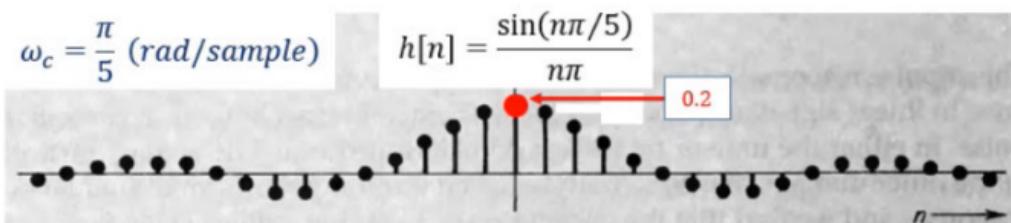


l'Hospital's rule for Sinc Function

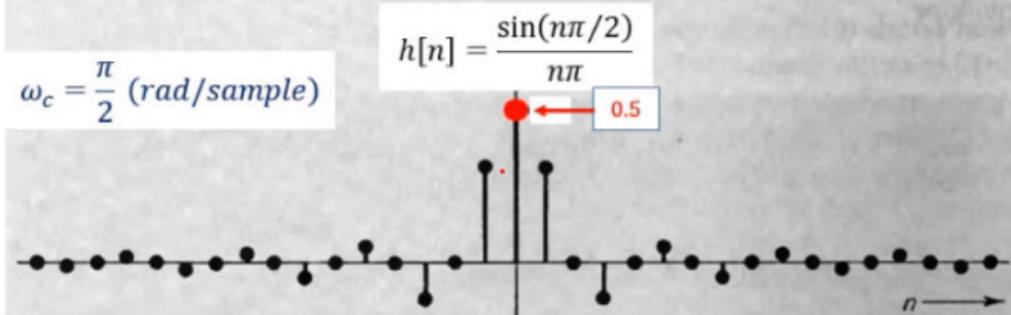
Note that $h[n = 0]$
is computed from
l'Hospital's rule.

$$h[0] = \frac{\frac{d}{dn}(\sin \omega_c n)}{\frac{d}{dn}(n\pi)} \Big|_{n=0} = \frac{\omega_c \cdot \cos \omega_c n}{\pi} \Big|_{n=0} = \frac{\omega_c}{\pi}$$

$$\omega_c = \frac{\pi}{5} \text{ (rad/sample)}$$

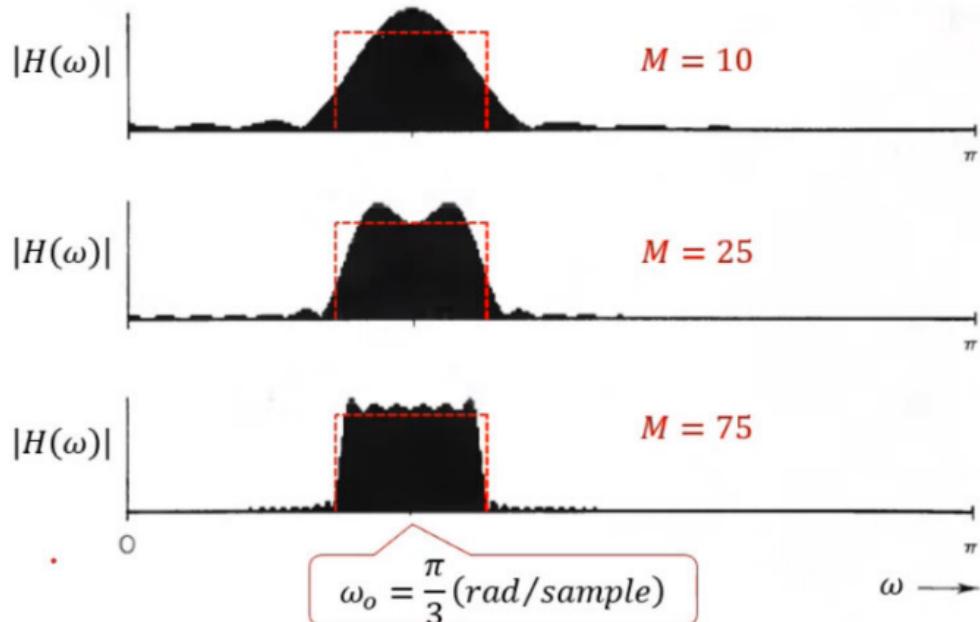


$$\omega_c = \frac{\pi}{2} \text{ (rad/sample)}$$



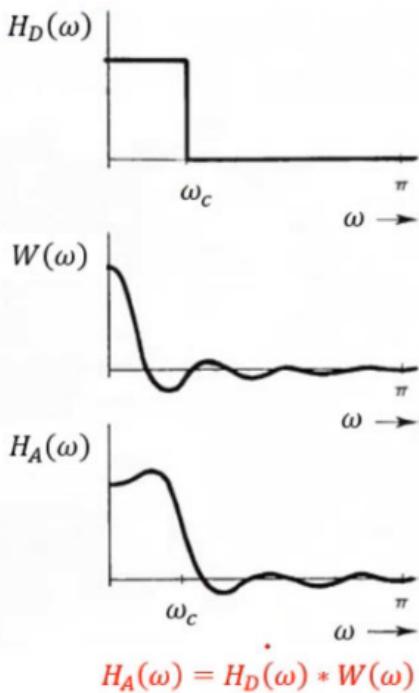
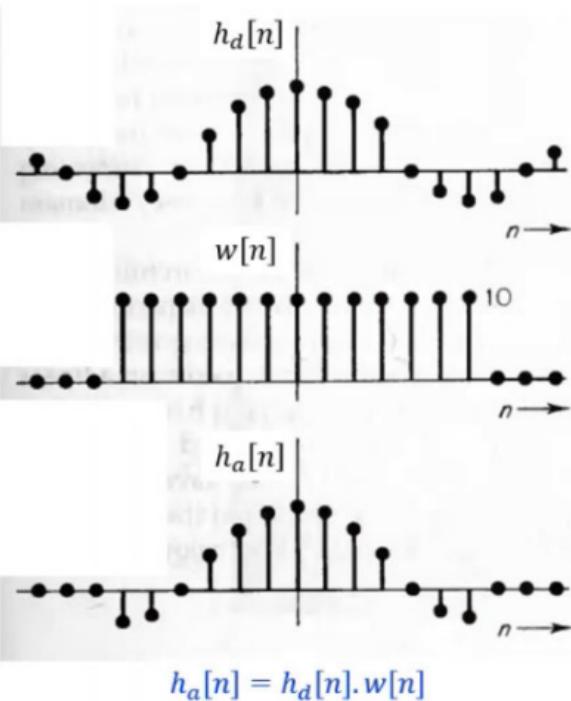
Nonrecursive Filter Design by Fourier Transform Method

The infinite Sinc function (IIR) must be truncated to become a practical Finite Impulse Response (FIR).



The longer $h[n]$ (M is larger) is, the closer $|H(\omega)|$ is to the ideal filter

Nonrecursive Filter Design by Fourier Transform Method



- When the infinite sinc function is truncated, it is equivalent to multiplying with the finite-length pulse function in Time-Domain.
- From Modulation property, this is the convolution of those spectrums in Frequency Domain.
- This causes the ripple in the transition period called "Gibb's phenomena"

Summary of Filter Design by Fourier Transform Method

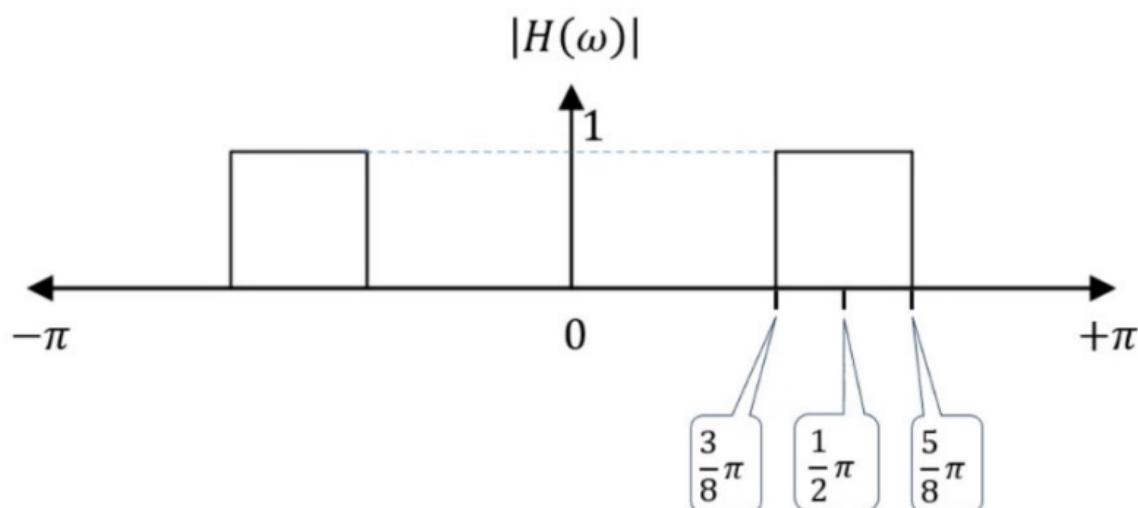
1. Define the center Frequency, ω_o (*rad/sample*)
2. Define the Cutoff Frequency, ω_c , or Bandwidth (*BW*) in *rad/sample*
3. Generate (*noncausal*) Infinite Impulse Response, $h_d[n]$, from the equation

$$h_d[n] = \frac{1}{n\pi} \sin(n\omega_c) \cdot \cos(n\omega_o)$$

4. Define the size (*in samples*) of $h[n]$ and truncate it *symmetrically* to become Finite Impulse Response (*Even function*), $h_a[n]$.
5. Shift $h_a[n]$ to the right until it starts at $n = 0$ to make it as a “Causal System”.

Exercise for DSP course 2/2566

1. Design nonrecursive FIR Band Pass Filter, $h[n]$, with $M = 10$ ($h[n] = 21$ samples) with $\omega_o = 0.5\pi$ rad/sample and $\omega_c = \frac{\pi}{8}$ rad/sample (Bandwidth = $\frac{\pi}{4}$ rad/sample)
2. Plot $h[n]$ for $n = -M$ to $+M$ samples
3. Plot $|H(\omega)|$ for $\omega = -3\pi$ to $+3\pi$
4. Redesign from 1 to 3 with $M = 50$ ($h[n] = 101$ samples) and $M = 100$ ($h[n] = 201$ samples)



END