Week 3

01046725 DIGITAL SIGNAL PROCESSING

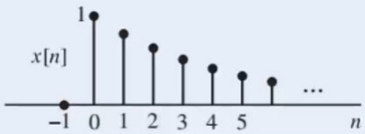
Discrete-Time Signals and Systems,

Time Domain Analysis

Semester 2/2566

Asst. Prof. Suradej Tretriluxana, PhD.

Discrete-time Signal Representation

Representation	Example
Functional	$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$
Tabular	$\begin{array}{c\|ccccccccc} n & \ldots & -2 & -1 & 0 & 1 & 2 & 3 & \ldots \\ \hline x[n] & \ldots & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \ldots \end{array}$
Sequence	$x[n] = \{ \ldots \; 0 \; \underset{\uparrow}{1} \; \frac{1}{2} \; \frac{1}{4} \; \frac{1}{8} \; \ldots \}$
Pictorial	

[1] The symbol ↑ denotes the index $n = 0$; it is omitted when the table starts at $n = 0$.

Fundamental Discrete-time Signals

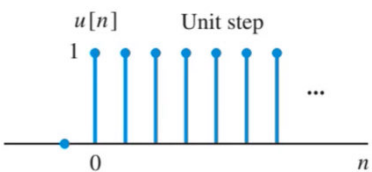
Unit Impulse Signal

$$\delta[n] \triangleq \begin{cases} 1, & n = 0 \\ 0. & n \neq 0 \end{cases}$$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

Unit Step Signal

$$u[n] \triangleq \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Fundamental Discrete-time Signal: Sinusoidal Sequence

Amplitude

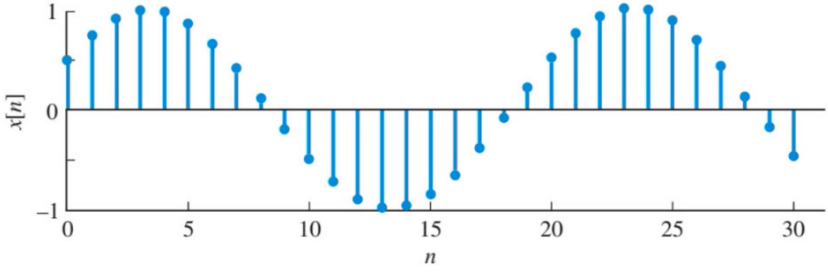
Function {sine, cosine}

Phase (*radian*)

$$x[n] = A\cos(\omega n + \phi) = A\cos(2\pi f n + \phi)$$

Angular Frequency (*radian/sample*)

Linear Frequency ($sample^{-1}$)



Fundamental Discrete-time Signal: Exponential Sequence

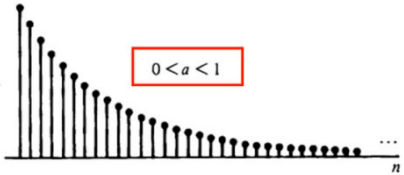
Base Power

$$x[n] = a^n$$

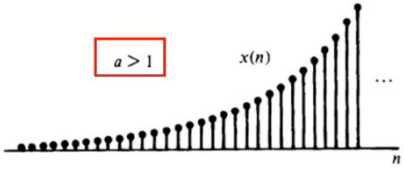
If the base, a is real number, the function can be displayed in 4 conditions according to the range of base.

Convergent

$0 < a < 1$



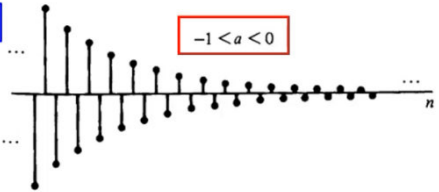
$a > 1$



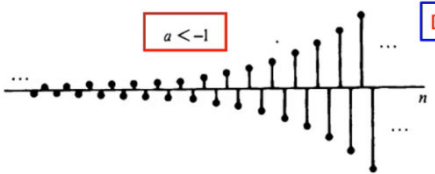
Divergent

Convergent

$-1 < a < 0$



$a < -1$



Divergent

Fundamental Discrete-time Signal: Exponential Sequence

Base Power

$$x[n] = a^n$$

If the base is complex number, $a = re^{j\theta}$,

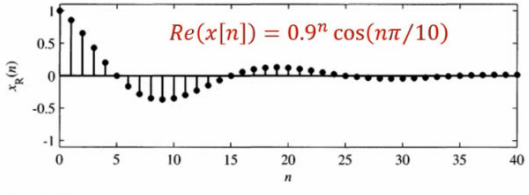
$x[n]$ will also be the complex number

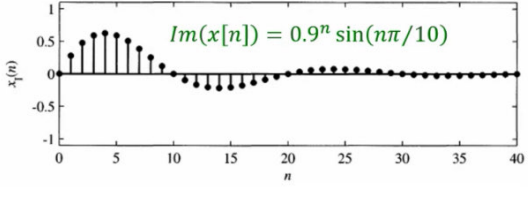
$$x[n] = \left(re^{j\theta}\right)^n = r^n e^{j\theta n} = r^n[\cos(\theta n) + j\sin(\theta n)]$$

$$x[n] = r^n \cos(\theta n) + jr^n \sin(\theta n)$$

Both real and imaginary parts are the product of sinusoidal and the real exponential functions.

Example: if $a = 0.9e^{j\pi/10}$





Fundamental Discrete-time Signal: Exponential Sequence

Base Power

$$x[n] = a^n$$

If the base is complex number, $a = re^{j\theta}$,

$x[n]$ will also be the complex number

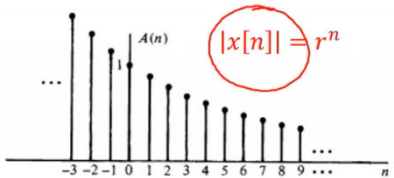
$$x[n] = a^n = r^n e^{j\theta n}$$

$$|x[n]| = r^n$$

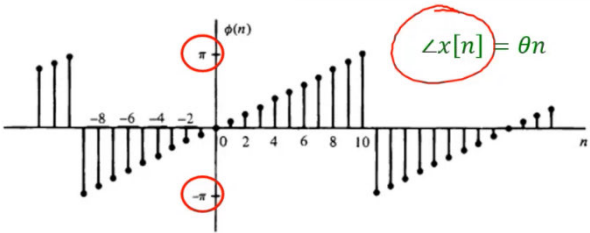
$$\angle x[n] = \theta n$$

Its magnitude, $|x[n]|$ and phase, $\angle$ are usually plotted separately as shown here.

Example: if $a = 0.9e^{j\pi/10}$



(a) Graph of $A(n) = r^n$, $r = 0.9$



Energy and Power Signals

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

E_x is the energy of $x[n]$. $x[n]$ can be either real or complex number.
If $0 < E_x < \infty$ then $x[n]$ is called Energy signal.

$$P_x = \lim_{N\to\infty} \left[\frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \right]$$

P_x is the power of $x[n]$. $x[n]$ can be either real or complex number.
If $E_x = 0$ then $P_x = 0$. If $E_x = \infty$ then P_x is either finite or infinite.
If $0 < P_x < \infty$ then $x[n]$ is called Power signal.

Periodic Signal

If $x[n+N] = x[n]$ for all n , then $x[n]$ is called periodic signal.

The smallest $N; N \in I^+$ is called the fundamental period.

If $x[n]$ is a sinusoidal function with the discrete-time frequency, f $(sample^{-1})$

Then N is the period of $x[n]$,

when f is written in the simplest form of the ratio number, $f = \frac{k}{N}$

Even and Odd Signals

If $x[n] = x[-n]$, then $x[n]$ is called even signal, $x_e[n]$.

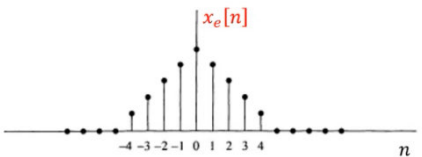
If $x[n] = -x[-n]$, then $x[n]$ is called odd signal, $x_o[n]$.

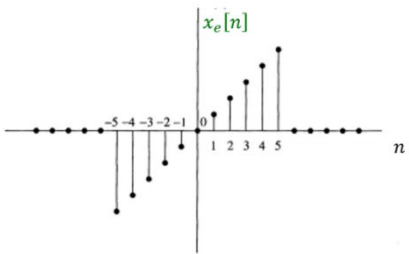
Note that $x_o[0]$ must be zero.

A signal, $x[n]$ can be decomposed into the summation of an odd and even signals

$$x[n] = x_e[n] + x_o[n]$$

where $x_e[n] = \frac{x[n]+x[-n]}{2}$ and $x_o[n] = \frac{x[n]-x[-n]}{2}$





Discrete-Time Signal Operations

```
x1=[1 4 16 32]

x1 = 1×4
     1     4    16    32

x2=[2 4 6 8]

x2 = 1×4
     2     4     6     8
```

$$y[n] = x_1[n] + x_2[n],$$
$$y[n] = x_1[n] - x_2[n],$$
$$y[n] = x_1[n] \cdot x_2[n],$$
$$y[n] = x_1[n]/x_2[n],$$

```
yn=x1+x2

yn = 1×4
     3     8    22    40

yn=x1-x2

yn = 1×4
    -1     0    10    24

yn=x1.*x2

yn = 1×4
     2    16    96   256

yn=x1./x2

yn = 1×4
    0.5000    1.0000    2.6667    4.0000
```

Array Operations

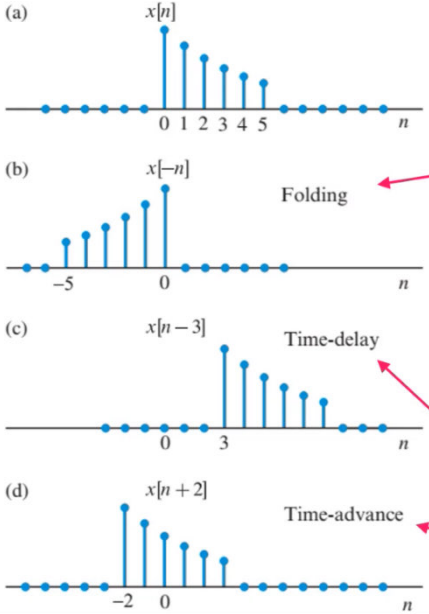
$$y[n] = a \cdot x_2[n].$$

```
yn=2*x2

yn = 1×4
     4     8    12    16
```

Scalar Operation

Discrete-Time Signal Operations



Folding Operation, FD

$$y[n] = FD(x[n]) = x[-n]$$

It changes only "n" to "$-n$".

Time Shift Operation, TD_k where $k \in I$

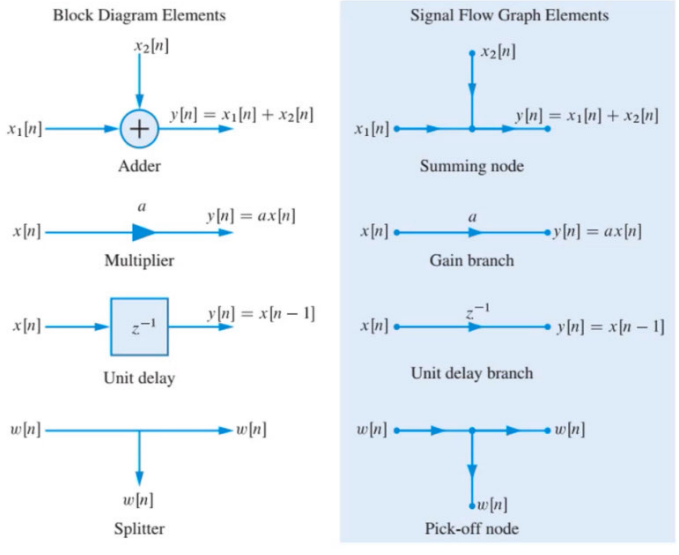
$$y[n] = TD_k(x[n]) = x[n-k]$$

It changes "n" to "$n-k$".

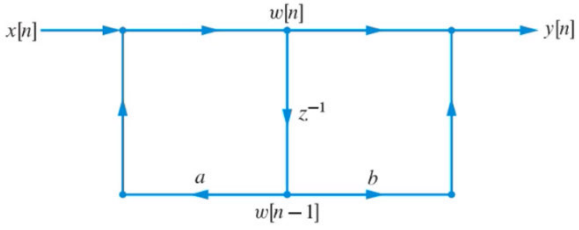
If $k > 0$ e.g. $TD_3(x[n]) = x[n-3]$

If $k < 0$ e.g. $TD_{-2}(x[n]) = x[n+2]$

Block Diagrams and Signal Flow Graphs



Example of Signal Flow Graphs



$$w[n] = x[n] + aw[n-1] \quad (1)$$

$$y[n] = w[n] + bw[n-1] \quad (2)$$

$$\therefore y[n] = x[n] + (a+b)w[n-1] \quad (3)$$

$$\therefore y[n] = x[n] + bx[n-1] + ay[n-1]$$

From (1); $x[n] = w[n] - aw[n-1]$

then $x[n-1] = w[n-1] - aw[n-2]$

and $bx[n-1] = bw[n-1] - abw[n-2]$ (4)

From (2); $y[n-1] = w[n-1] + bw[n-2]$

And $ay[n-1] = aw[n-1] + abw[n-2]$ (5)

(4) + (5); $bx[n-1] + ay[n-1] = (a+b)w[n-1]$ (6)

Discrete-Time System

Discrete-Time (DT) system is a computational process or algorithm that transforms the DT input signal, $x[n]$ into the DT output sequence, $y[n]$. This input-output relation can be denoted as $y[n] = \mathcal{H}\{x[n]\}$ and depicted below.

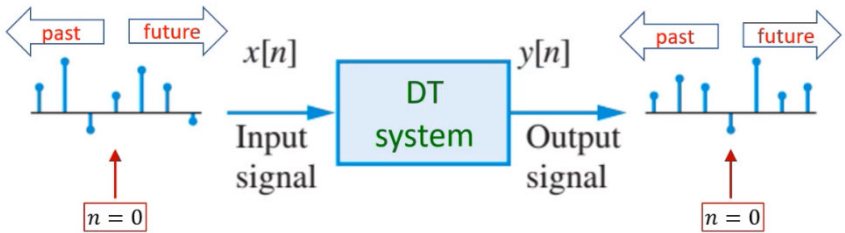
The DT system is usually written as a mathematic equation e.g. $y[n] = 2x[n] + x[n-1]$. It can be a simple expression or very complex algorithm. The DT system in this study is confined to the following properties (causality, stability, linearity and time invariant).

Causal and Noncausal System

The system is causal when its output is NOT computed from the future value of its input, $y[n_0] = \mathcal{H}\{x[n]; n \le n_0\}$.

With this property, the system can be implemented real-time.

$$y[n_0] = \mathcal{H}\{x[n]; n \le n_0\}$$



$y[n] = 2x[n] + x[n-1]$ This is the causal system.

$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$ This is NOT the causal system.

Stable and Unstable System

The system is "Bounded-Input Bounded-Output (BIBO) stable", if every bounded input signal results in a bounded output signal.

$$|x[n]| \leq M_x < \infty \Rightarrow |y[n]| \leq M_y < \infty$$

$$y[n] = \sum_{k=0}^{\infty} x[n-k]$$

This system is NOT (BIBO) stable.

If $\lim_{n\to\infty} x[n] = u[n]$ then $x[\infty] = u[\infty] = 1$

but $y[n] = y[\infty] = \infty$

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$

This system is (BIBO) stable.

If $|x[n]| \leq M_x < \infty$ for all n

then $|y[n]| \leq M_y < \infty$ for all n

Time Invariant and Time-Varying System

The system is time invariant when its characteristic does NOT change with time, otherwise it is time-varying system.

$$y[n] = \mathcal{H}\{x[n]\} \Rightarrow y[n-k] = \mathcal{H}\{x[n-k]\}.$$

If the system, $y[n] = x[n]\sin(\omega n)$ is time invariant system.

If input is $x[n] \Rightarrow y_1[n] = x[n]\sin(\omega n)$

If input is $x[n-k] \Rightarrow y_2[n] = x[n-k]\sin(\omega n)$

Compute $y[n-k] = x[n-k]\sin\big(\omega(n-k)\big)$

When $y[n-k] \neq y_2[n]$, the system is time-varying.

If the system, $y[n] = 2x[n] + 1$ is time invariant system.

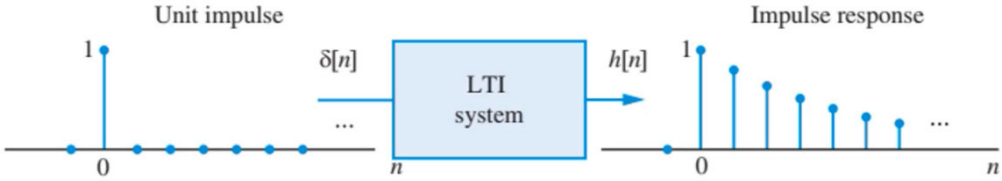
If input is $x[n] \Rightarrow y_1[n] = 2x[n] + 1$

If input is $x[n-k] \Rightarrow y_2[n] = 2x[n-k] + 1$

Compute $y[n-k] = 2x[n-k] + 1$

When $y[n-k] = y_2[n]$, the system is time invariant.

Discrete-Time System (Processor)

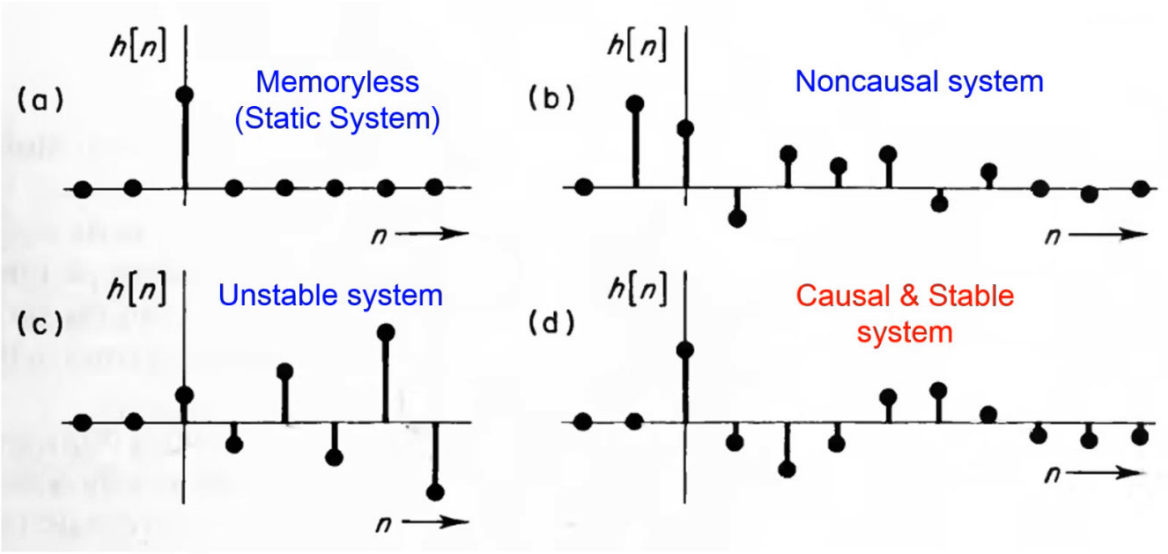


The System's characteristic is defined by the unit impulse response, $h[n]$. It is the system response when the input is impulse signal, $\delta[n]$.

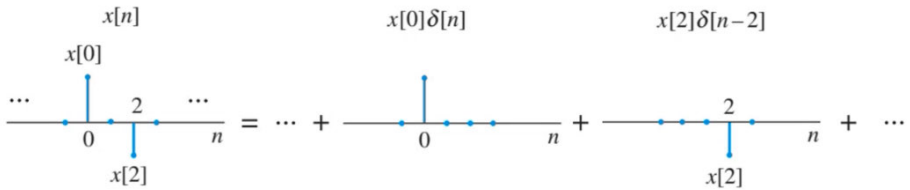
If $h[n]$ is finite length, it is called Finite Impulse Response (FIR)

If $h[n]$ is infinite length, it is called Infinite Impulse Response (IIR)

Examples of Unit Impulse Response, $h[n]$



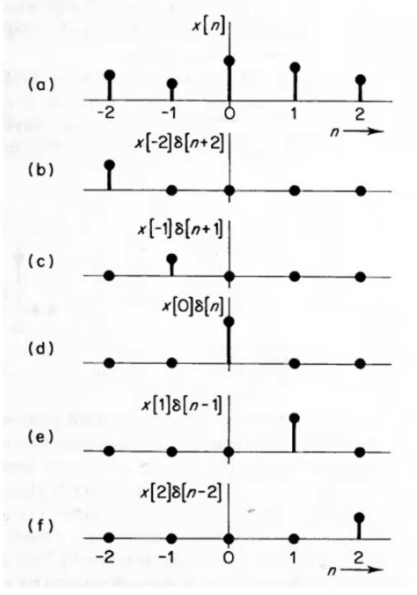
Sifting Operation



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \quad -\infty < n < \infty$$

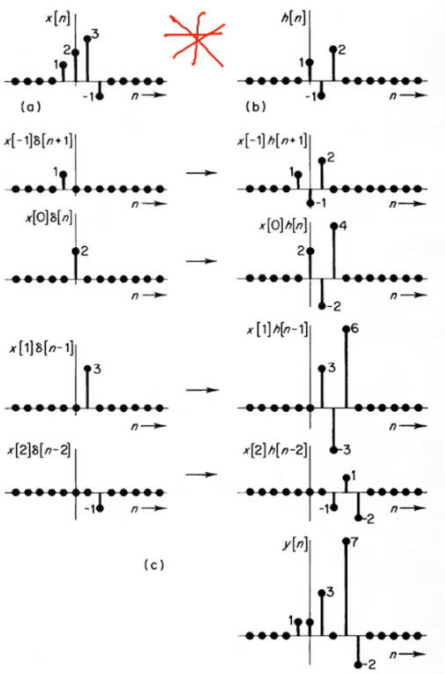
Any discrete-time domain signal can be decomposed into the summation of weighted, shifted unit impulse signals.

Unit Impulse Signal and Sifting Property



$$x[n] = \cdots + x[-2]\,\delta[n+2] + x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2] + \cdots$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$



Digital Convolution

$$x[n] = \{1,2,3,-1\}$$

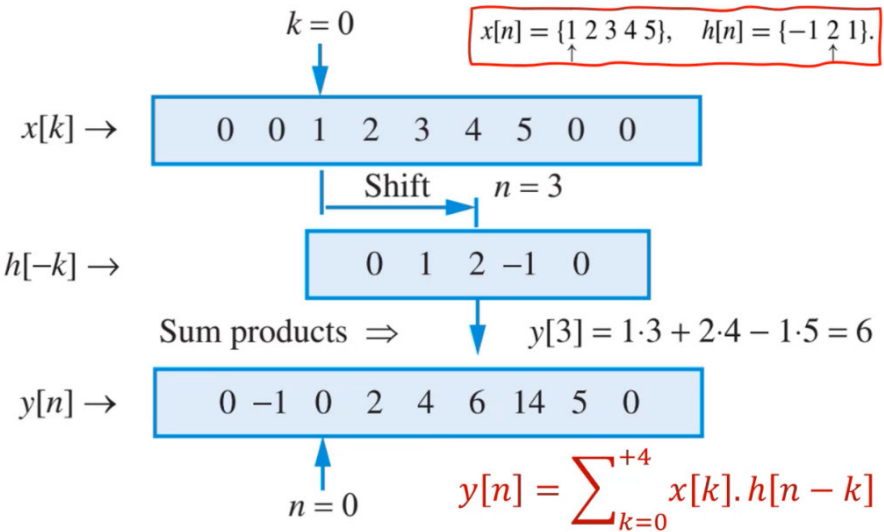
$$h[n] = \{1,-1,2\}$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k].h[n-k] = \sum_{k=-1}^{+2} x[k].h[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k].x[n-k] = \sum_{k=0}^{+2} h[k].x[n-k]$$

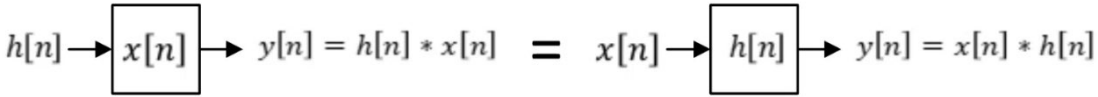
$$y[n] = \{1,1,3,0,7,-2\} \; ; -1 \le n \le 4$$

Digital Convolution (another example)

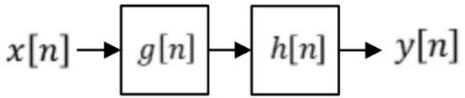


Convolution Properties

Commutative Property

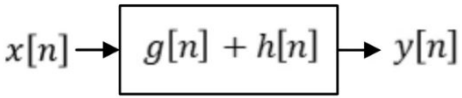


Associative Property



$$y[n] = \{x[n] * g[n]\} * h[n]$$
$$y[n] = x[n] * \{g[n] * h[n]\}$$

Distributive Property



$$y[n] = x[n] * \{g[n] + h[n]\}$$
$$y[n] = \{x[n] * g[n]\} + \{x[n] * h[n]\}$$

Convolution and Difference Equation (DE)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k].h[n-k] = \sum_{k=-\infty}^{+\infty} h[k].x[n-k]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} h[k].x[n-k] = \sum_{k=-\infty}^{+\infty} a_k.x[n-k]$$

Convolution is a subset of Difference Equation (DE) when the output $y[n]$ is the function of only input signal $x[n]$ (Nonrecursive Function).

LTI system described by Difference Equation (DE)

The general form of the Linear Time-Invariant (LTI) system is described by Linear Constant Coefficient Difference Equation (LCCDE) in this form.

$$\sum_{k=0}^{N} a_k \, . \, y[n-k] = \sum_{k=0}^{M} b_k \, . \, x[n-k]$$

Where N is the order of the system,

a_k and b_k are the constant coefficients of the output, $y[n]$ and input, $x[n]$ terms respectively.

If $N = 0$, the Difference Equation (DE) is reduced to ,

$$y[n] = \frac{1}{a_0} \sum_{k=0}^{M} b_k \, . \, x[n-k]$$

It is a convolution sum of input, $x[n]$,
with the Finite Impulse Response (FIR), $M + 1$ terms.

Impulse Response $h[n]$, $x[n] = \delta[n]$ and Initially Relaxed

Impulse Response $h[n]$ is calculated recursively from the difference equation by giving system input signal as the unit impulse $\delta[n]$ and zero initial condition(s).

$$x[n] = \delta[n] = \{1,0,0,0, \dots\} \text{ and } y[-1] = 0 \text{ (initially relaxed)}$$

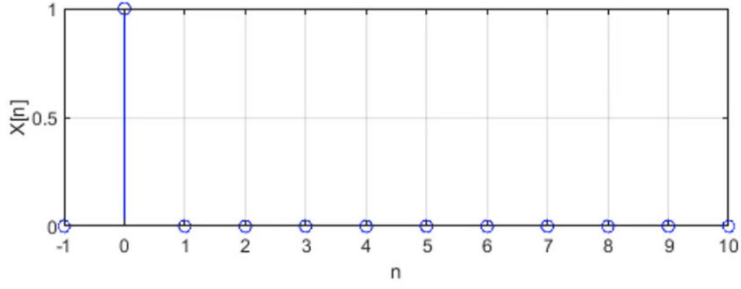
$$y[n] = 0.5y[n-1] + 0.5x[n]$$

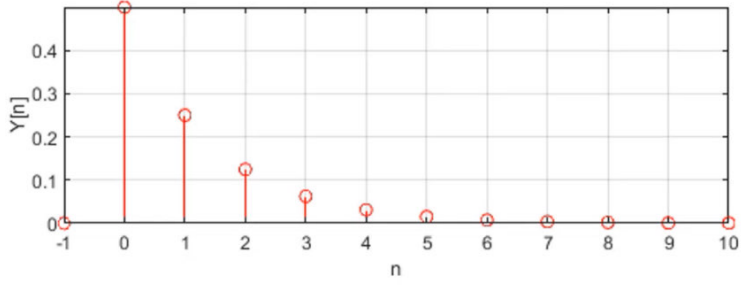
$$h[n] = 0.5h[n-1] + 0.5\delta[n]$$

$$h[0] = 0.5h[-1] + 0.5\delta[0] = 0.5(0) + 0.5(1) = 0.5$$
$$h[1] = 0.5h[0] + 0.5\delta[1] = 0.5(0.5) + 0.5(0) = 0.25$$
$$h[2] = 0.5h[1] + 0.5\delta[2] = 0.5(0.25) + 0.5(0) = 0.125$$
$$h[3] = 0.5h[2] + 0.5\delta[3] = 0.5(0.125) + 0.5(0) = 0.0625$$
$$\vdots$$

$$h[n] = \{0.5,0.25,0.125,0.0625, \dots\}$$ ⟵ Infinite Impulse Response

Impulse Response $h[n]$, $x[n] = \delta[n]$ and Initially Relaxed





n	X[n]	Y[n]
-1	0	0
0	1	0.5
1	0	0.25
2	0	0.125
3	0	0.0625
⋮	⋮	⋮

Zero State Response $y_{zs}[n]$ (Initially relaxed)

The Response $y[n]$ is calculated recursively from the difference equation by giving system input signal $x[n]$ and the initial condition(s).

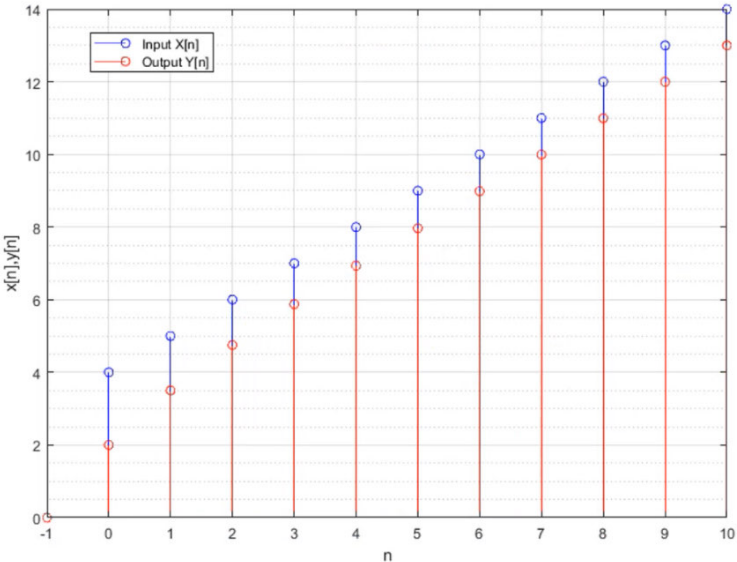
$x[n] = \{4,5,6, \dots\}$ and $y[-1] = 0$ (initially relaxed)

$$y[n] = 0.5y[n-1] + 0.5x[n]$$

$$y[0] = 0.5y[-1] + 0.5x[0] = 0.5(0) + 0.5(4) = 2$$
$$y[1] = 0.5y[0] + 0.5x[1] = 0.5(2) + 0.5(5) = 3.5$$
$$y[2] = 0.5y[1] + 0.5x[2] = 0.5(3.5) + 0.5(6) = 4.75$$
$$y[3] = 0.5y[2] + 0.5x[3] = 0.5(4.75) + 0.5(7) = 5.875$$
$$y[4] = 0.5y[3] + 0.5x[4] = 0.5(5.875) + 0.5(8) = 6.9375$$
$$\vdots$$

$y[n] = \{2,3.5,4.75,5.875,6.9375, \dots\}$ ⟵ The output is similar to input

Zero State Response $y_{zs}[n]$ (Initially relaxed)



n	X[n]	Y[n]
-1	0	0
0	4	2
1	5	3.5
2	6	4.75
3	7	5.875
4	8	6.9375
⋮	⋮	⋮

Zero Input Response $y_{zi}[n]$ $(x[n] = \{0,0,0, \dots\})$

The Response $y[n]$ is calculated recursively from the difference equation by giving system input signal $x[n]$ and the initial condition(s).

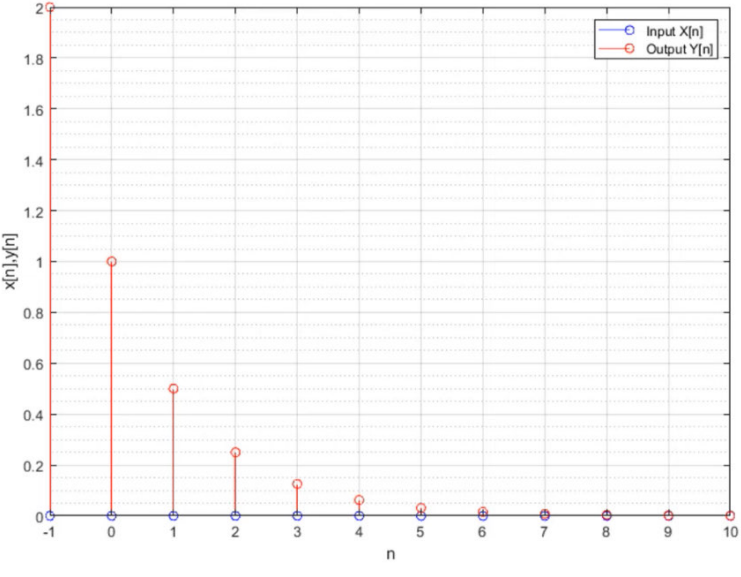
$x[n] = \{0,0,0, \dots\}$ (zero input) and $y[-1] = 2$ (nonzero IC)

$$y[n] = 0.5y[n-1] + 0.5x[n]$$

$$y[0] = 0.5y[-1] + 0.5x[0] = 0.5(2) + 0.5(0) = 1$$
$$y[1] = 0.5y[0] + 0.5x[1] = 0.5(1) + 0.5(0) = 0.5$$
$$y[2] = 0.5y[1] + 0.5x[2] = 0.5(0.5) + 0.5(0) = 0.25$$
$$y[3] = 0.5y[2] + 0.5x[3] = 0.5(0.25) + 0.5(0) = 0.125$$
$$y[4] = 0.5y[3] + 0.5x[4] = 0.5(0.125) + 0.5(0) = 0.0625$$
$$\vdots$$

$y[n] = \{1, 0.5, 0.25, 0.125, 0.0625, \dots\}$ ← The output is converse!

Zero Input Response $y_{zi}[n]$ $(x[n] = \{0,0,0, \dots\})$



n	X[n]	Y[n]
-1	0	2
0	0	1
1	0	0.5
2	0	0.25
3	0	0.125
4	0	0.0625
⋮	⋮	⋮

Complete Response $y[n] = y_{zs}[n] + y_{zi}[n]$

The Response $y[n]$ is calculated recursively from the difference equation by giving system input signal $x[n]$ and the initial condition(s).

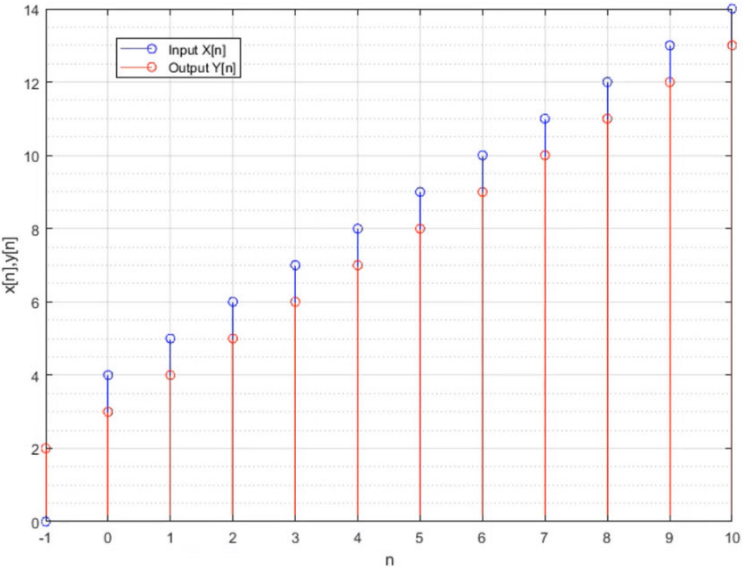
$$x[n] = \{4,5,6, \dots\} \text{ and } y[-1] = 2 \text{ (nonzero IC)}$$

$$y[n] = 0.5y[n-1] + 0.5x[n]$$

$$\begin{aligned}
y[0] &= 0.5y[-1] + 0.5x[0] = 0.5(2) + 0.5(4) = 3 \\
y[1] &= 0.5y[0] + 0.5x[1] = 0.5(3) + 0.5(5) = 4 \\
y[2] &= 0.5y[1] + 0.5x[2] = 0.5(4) + 0.5(6) = 5 \\
y[3] &= 0.5y[2] + 0.5x[3] = 0.5(5) + 0.5(7) = 6 \\
y[4] &= 0.5y[3] + 0.5x[4] = 0.5(6) + 0.5(8) = 7 \\
&\vdots
\end{aligned}$$

$y[n] = \{3,4,5,6,7, \dots\}$ ← The output is similar to input

Complete Response $y[n] = y_{zs}[n] + y_{zi}[n]$



n	X[n]	Y[n]
-1	0	2
0	4	3
1	5	4
2	6	5
3	7	6
4	8	7
⋮	⋮	⋮

Response $y[n]$ from Input $x[n]$ is Constant

The Response $y[n]$ is calculated recursively from the difference equation by giving system input signal $x[n]$ and the initial condition(s).

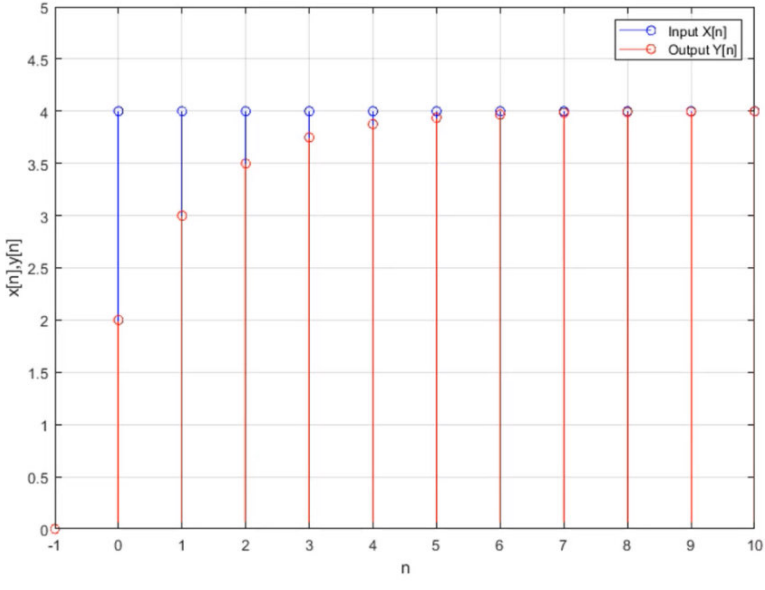
$x[n] = \{4,4,4, \dots\}$ (Constant) and $y[-1] = 0$ (initially relaxed)

$$y[n] = 0.5y[n-1] + 0.5x[n]$$

$$y[0] = 0.5y[-1] + 0.5x[0] = 0.5(0) + 0.5(4) = 2$$
$$y[1] = 0.5y[0] + 0.5x[1] = 0.5(2) + 0.5(4) = 3$$
$$y[2] = 0.5y[1] + 0.5x[2] = 0.5(3) + 0.5(4) = 3.5$$
$$y[3] = 0.5y[2] + 0.5x[3] = 0.5(3.5) + 0.5(4) = 3.75$$
$$y[4] = 0.5y[3] + 0.5x[4] = 0.5(3.75) + 0.5(4) = 3.875$$
$$\vdots$$

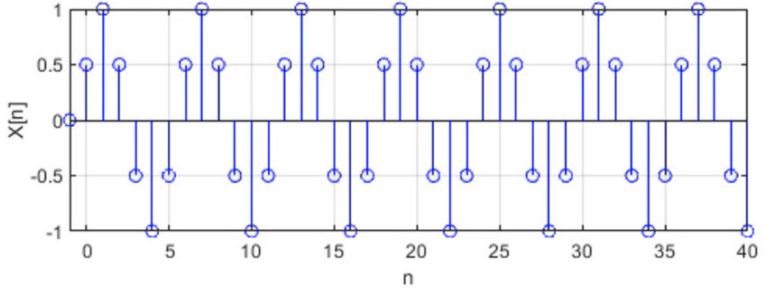
$y[n] = \{2,3, 3.5,3.75,3.875, \dots\}$ ← The output is moving to constant!

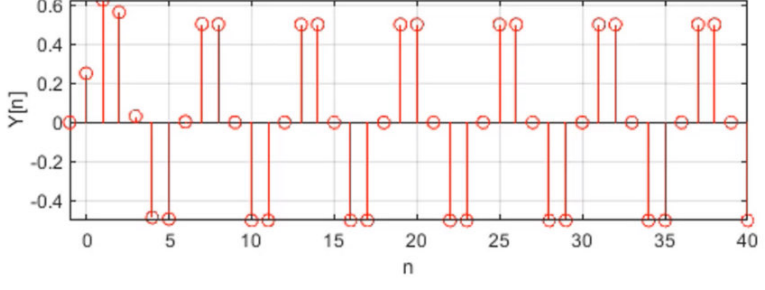
Response $y[n]$ from Input $x[n]$ is Constant



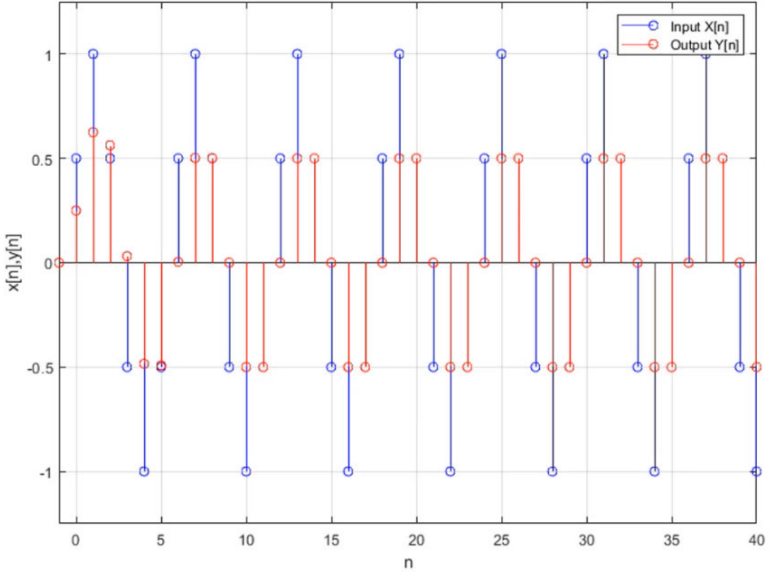
n	X[n]	Y[n]
-1	0	0
0	4	2
1	4	3
2	4	3.5
3	4	3.75
4	4	3.875
⋮	⋮	⋮

$x[n] = \sin\left(\frac{2\pi n}{6}\right)$ (Sinusoidal) and $y[-1] = 0$ (initially relaxed)

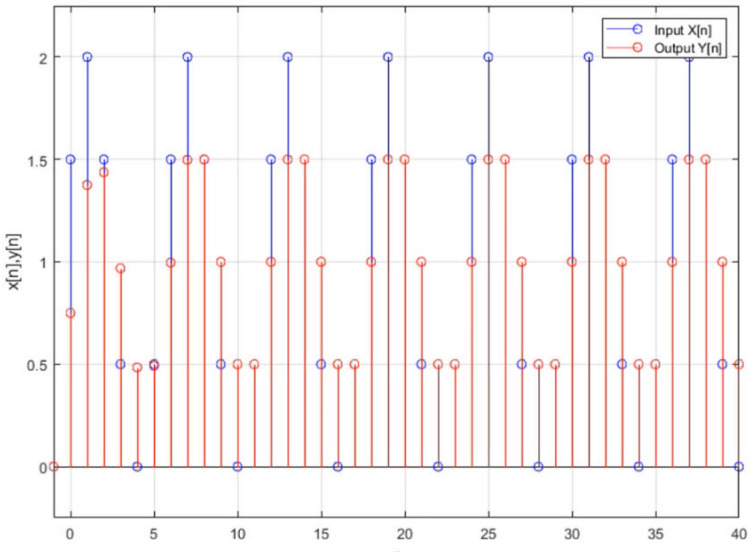




$x[n] = \sin\left(\frac{2\pi n}{6}\right)$ (Sinusoidal) and $y[-1] = 0$ (initially relaxed)



$x[n] = \sin\left(\frac{2\pi n}{6}\right) + 1$ (AC+DC) and $y[-1] = 0$ (initially relaxed)



END