Week 1

01046725 DIGITAL SIGNAL PROCESSING

Lecture #1: Introduction

Semester 2/2566

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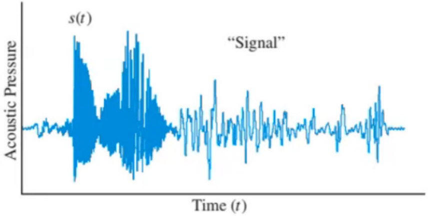
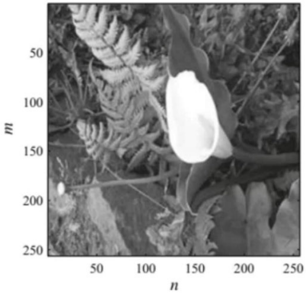
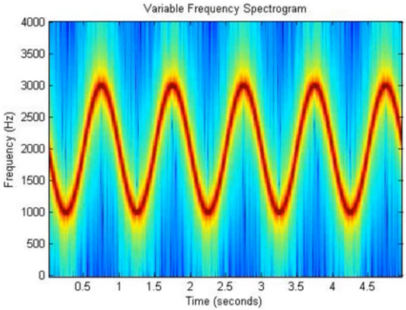
What are Signal and Signal Processing (SP)?

- **Signal** is the physical quantity that varies as a function of time (speech, biosignals), space (image) or any other variables (spectrogram). Its patterns provide information of the source.
- **Signal Processing (SP)** is the study of acquisition, representation, manipulation and transformation of the signal for needed applications.







Why DSP is so popular? (1)

When the digital computer is more powerful and widely used, the Digital Signal Processing (DSP) gains more popular than its counterpart, Analog Signal Processing (ASP).

Here are the reasons why?

1. Programmability and Reprogrammable

   ASP is implemented on Hardware (electronic circuits)

   DSP is implemented on Software (computer program)

2. Reliability

   ASP relies on the quality of electronic components and the environment while DSP does NOT.

Why DSP is so popular? (2)

When the digital computer is more powerful and widely used, the Digital Signal Processing (DSP) gains more popular than its counterpart, Analog Signal Processing (ASP).

Here are the reasons why?

3. Advancement in ADC and DAC devices

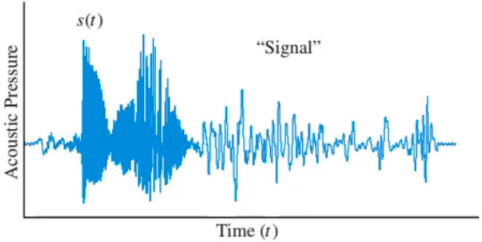
   Today ADC and DAC is faster (higher sampling rate), smoother (more bit resolution) and cheaper.

4. Can implement sophisticated signal processing function

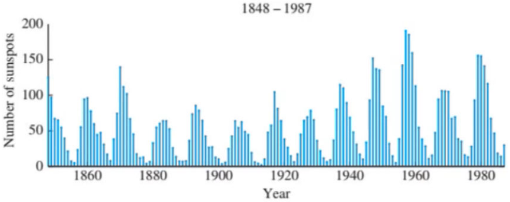
   DSP can perform the complex procedures that are difficult or unable to be implemented in the analog system, e.g. the adaptive algorithm, machine learning.

Continuous-time and Discrete-time Signals

The continuous-time (or the analog signal) signal is the function of time where its value is defined at all time. Most of signals picked up from the nature are continuous-time, e.g audio sound, ECG signal.



The discrete-time (NOT the digital signal) signal is the time series data where its value is defined only at discrete time. Some signal is collected as the discrete-time, e.g the annual mean sunspot number, the daily stock market index.

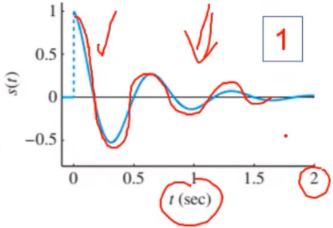


How the analog signal is changed to the digital one

$$s(t) = \begin{cases} e^{-2t}\cos(3\pi t), & t \geq 0 \\ 0, & t < 0. \end{cases}$$

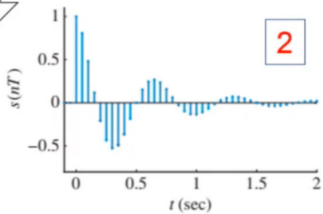
$$s[n] = s(nT) = \begin{cases} e^{-0.2n}\cos(0.3\pi n), & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Analog Signal

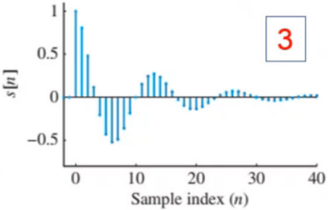


Sampling

Sampled Signal at $T = 0.1s$

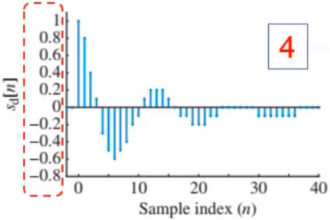


Discrete-Time Signal

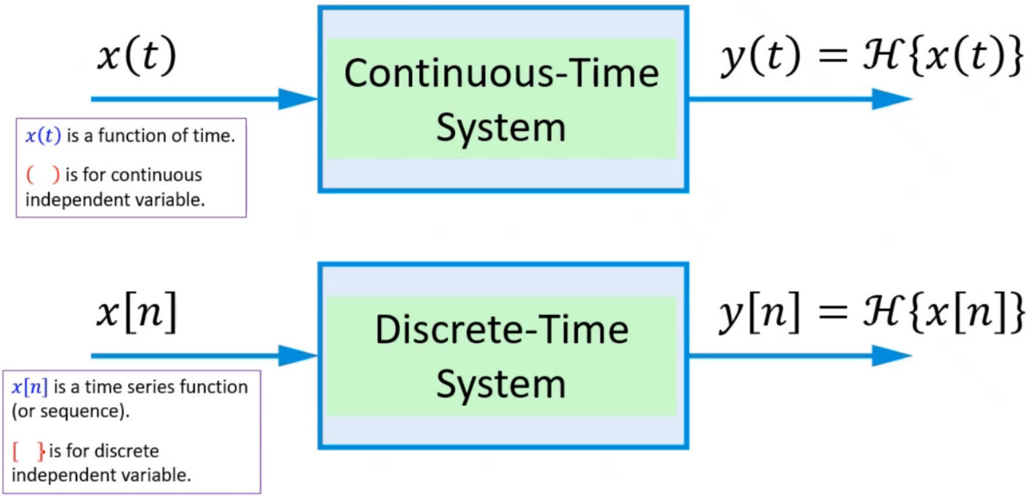


Quantizing

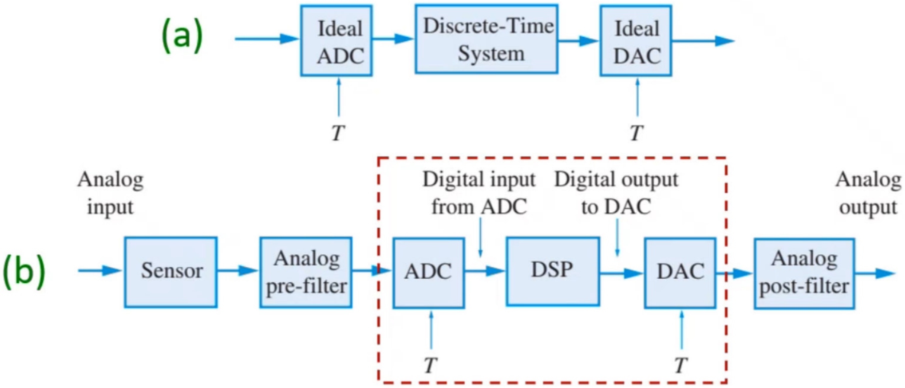
Digital Signal



Continuous-time (CT) and Discrete-time (DT) Systems

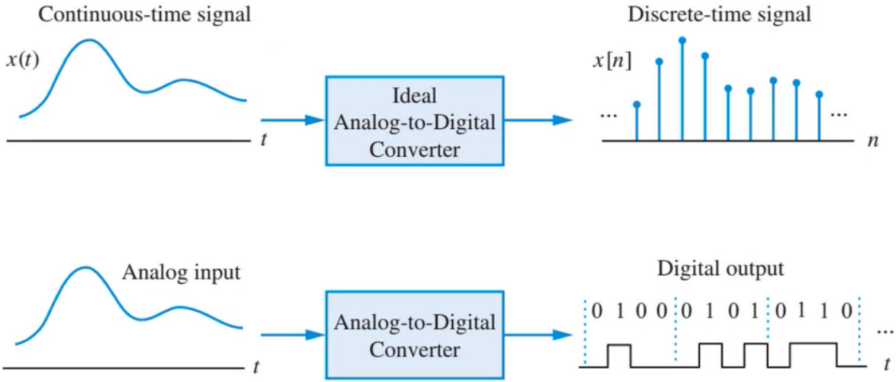


Interface Systems

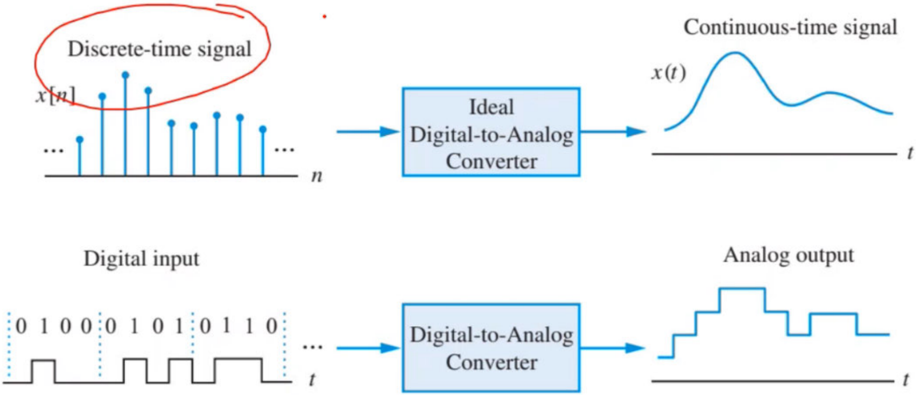


The digital processing of analog signal must have two interface systems, Analog-to-Digital Signal Converter (ADC) and Digital-to-Analog Signal Converter (DAC). (a) ideal condition and (b) practical condition.

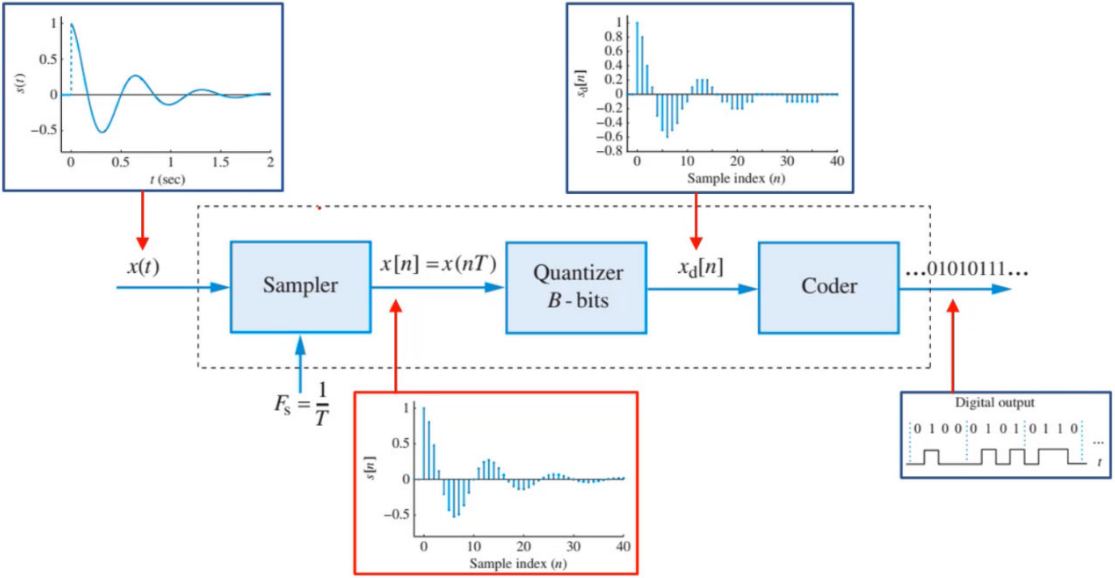
Interface System: Analog to Digital Conversion (ADC)



Interface System: Digital to Analog Conversion (DAC)



ADC Block Diagram



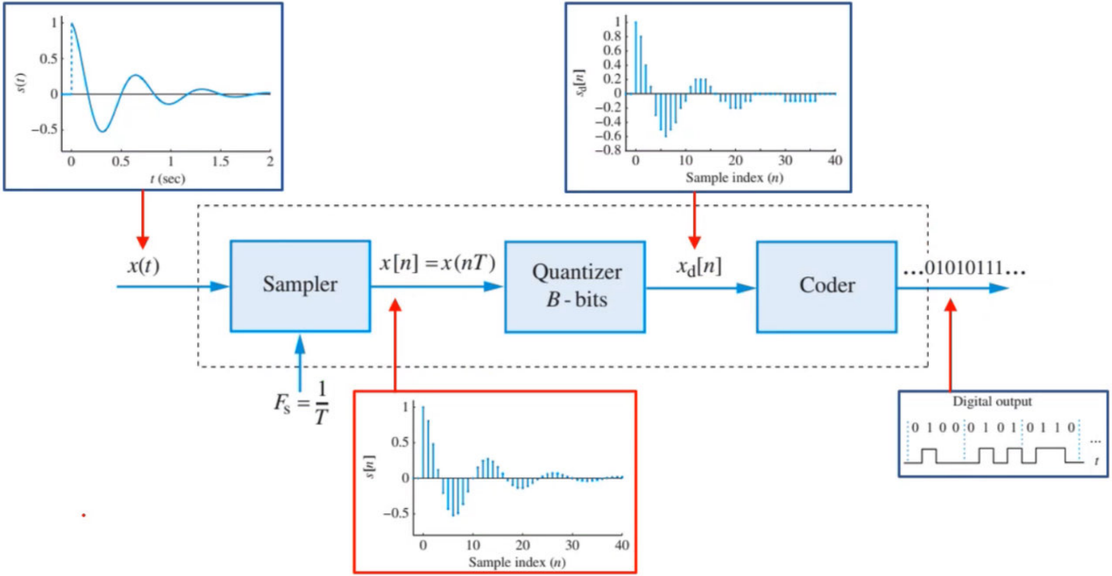
3 steps of Analog to Digital Conversion

Sampler, works as a switch, converts a continuous-time, $x(t)$ into discrete-time, $x[n]$ signal. The number of sample depends on the sampling frequency, $F_s(samples/sec)$. Theoretically this process is fully reversible.

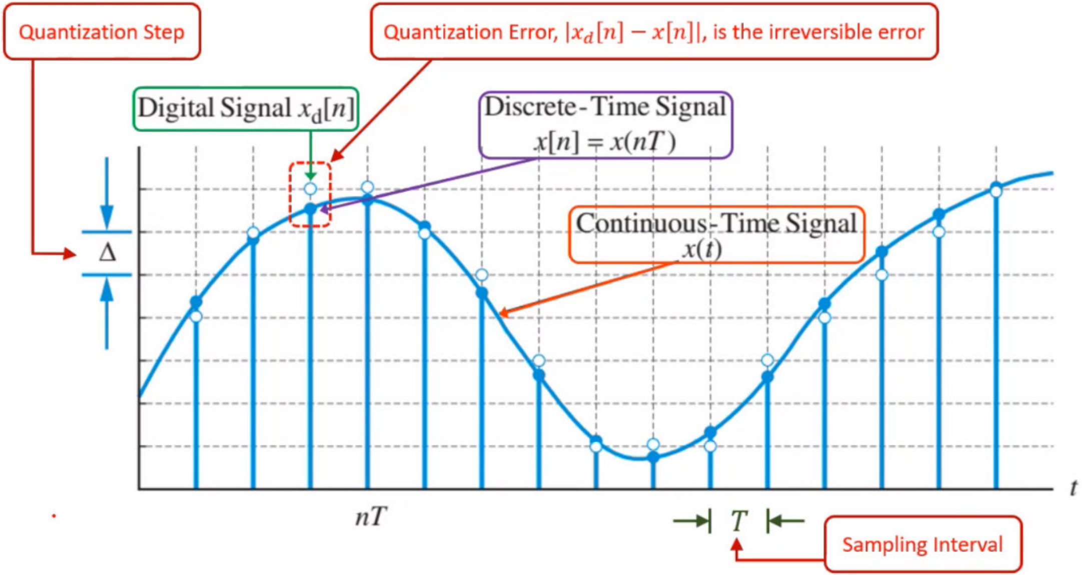
Quantizer converts a continuous-amplitude, $x[n]$ into the discrete-amplitude, $x_d[n]$ signal. The number of discrete-amplitude (quantization) level depends on the number of ADC bit resolution (in power of two). The 16-bit ADC for example has $2^{16} = 65536$ quantization levels $(0 - 65535)$. If the continuous-amplitude input voltage is $0 - 5V$, its minimum value (0V) is converted to "0" and its maximum value (5V) is changed to 65535. The quantization introduces quantization error and it is irreversible process.

Coder, the last process, converts the discrete-amplitude (quantization) level into the output binary number. The 16-bit ADC for instance generates 16 bits (2 bytes) for each sample.

ADC Block Diagram

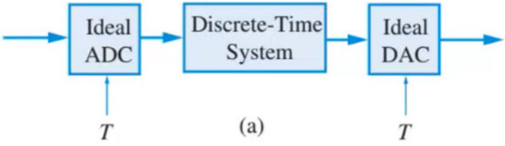


Irreversible Information Loss in Quantization Process



Discrete Time Signal Analysis

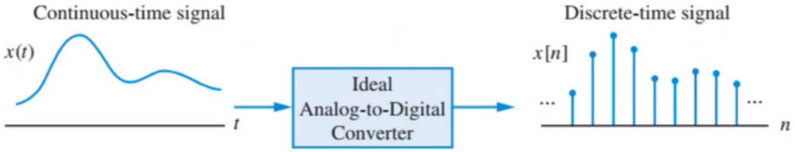
Practically, the quantization process is the nonlinear operation and introduces irreversible information loss. To avoid the analysis complexity, the Discrete-Time signal is studied instead of the Digital signal as such we are working with ideal ADC and DAC.



(a)

- To learn more about the quantization effect, students are suggested to read "Finite wordlength effects" topic in Applied Digital Signal Processing by Dimitris G. Manolakis and Vinay K. Ingle

Continuous-time to Discrete-time Signal Conversion

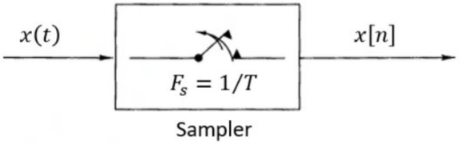


$$x(t) = x(nT) = x\left(\frac{n}{F_s}\right) \Rightarrow x[n]$$

T is Sampling Period $(sec/sample)$

F_s is Sampling Frequency $(sample/sec)$

$$T(sec/sample) = \frac{1}{F_s(sample/sec)}$$



Sampling Theorem

If the highest frequency contained in an analog signal, $x(t)$ is $F_{max} = B$ and the signal is sampled at a rate $F_s > 2B$, then $x(t)$ can be exactly recovered from the sample value, $x[n]$ using this interpolation function

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\left(2\pi B\left(t - \frac{n}{F_s}\right)\right)}{2\pi B\left(t - \frac{n}{F_s}\right)}$$

The sampling rate at the borderline, $F_s = 2B$ is called *Nyquist Frequency*.

Note that this recovery process is complex and needs infinite number of summation. This is only for theoretical interest!

Continuous-time to Discrete-time Signal Conversion

Given the fundamental signal, the sinusoidal function,

$$x(t) = A\cos(2\pi F t + \theta) = A\cos(\Omega t + \theta)$$

where F is the continuous-time (CT) sinusoidal linear frequency $(Hz\ or\ sec^{-1})$
Ω is the continuous-time (CT) sinusoidal angular frequency (rad/sec)

If the continuous-time (CT) signal is sampled at every sampling period, T

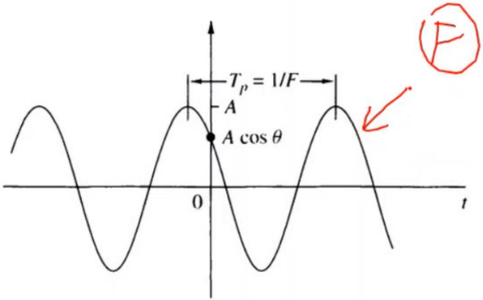
$$T(sec/sample) = \frac{1}{F_s(sample/sec)}$$

where F_s is Sampling Frequency $(sample/sec)$

Concept of Frequency of Continuous-Time (CT) Signal

$$x(t) = A\cos(\Omega t + \theta) = A\cos(2\pi F t + \theta)$$

$$T_p = \frac{1}{F} \qquad x(t + T_p) = x(t)$$



1. The CT sinusoidal signal, $x(t)$ is periodic for every value of $F(Hz)$ with the fundamental period, $T_p(sec) = \frac{1}{F}$.
2. Two CT sinusoidal signals with different frequency are different.
3. It has no limit for the highest frequency of CT sinusoidal signal. If the frequency can be increased, the rate of oscillation is higher (the period is shorter).

Continuous-time to Discrete-time Signal Conversion

Then, the discrete-time (DT) sinusoidal function is,

$$x(t) = A\cos(2\pi F t + \theta) = A\cos(\Omega t + \theta)$$

$$x(nT) = A\cos(2\pi F nT + \theta) = A\cos(\Omega nT + \theta)$$

$$x(nT) = A\cos\left(2\pi \frac{F}{F_s} n + \theta\right) = A\cos\left(\frac{\Omega}{F_s} n + \theta\right)$$

$$x[n] = A\cos(2\pi f n + \theta) = A\cos(\omega n + \theta)$$

where f is the discrete-time (DT) sinusoidal linear frequency ($sample^{-1}$)

ω is the discrete-time (DT) sinusoidal angular frequency ($radian/sample$)

Relation of Continuous-time and Discrete-time frequency

Discrete-Time Linear Frequency

Continuous-Time Linear Frequency

$$f(sample^{-1}) = \frac{F(sec^{-1})}{F_s(sample/sec)}$$

Continuous-Time Angular Frequency

$$\omega(radian/sample) = \frac{\Omega(radian/sec)}{F_s(sample/sec)}$$

Discrete-Time Angular Frequency

Sampling Frequency

f and ω are sometimes called relative or normalized frequencies

Concept of Frequency of Continuous-Time (CT) Signal

$$x(t) = A\cos(\Omega t + \theta) = A\cos(2\pi F t + \theta)$$

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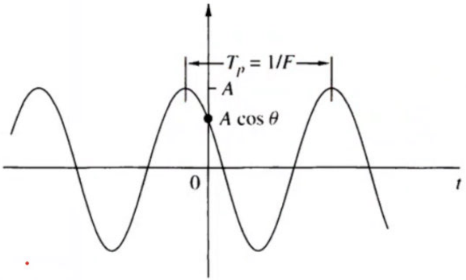

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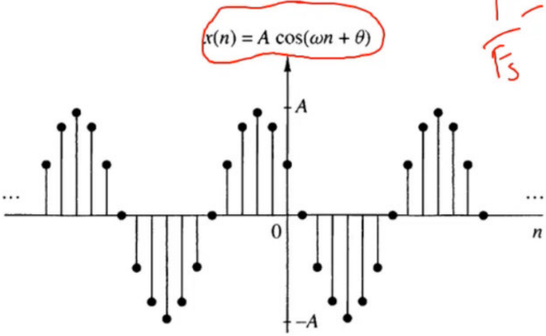
Concept of Frequency of Discrete-Time (DT) Signal

$$x[n] = A\cos(\omega n + \theta) = A\cos(2\pi f n + \theta)$$

$$f = \frac{k}{N} \qquad x[n+N] = x[n]$$

$\frac{F}{F_s} = f$



1. The DT sinusoidal signal, $x[n]$ is periodic only if f is the rational number.

   Because

   $$\cos\left(2\pi f(N+n)\right) = \cos(2\pi f n)$$

   is periodic when

   $$2\pi f N = k 2\pi$$

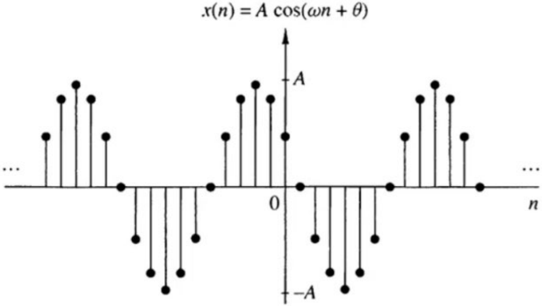
   $$\therefore f = \frac{k}{N}$$

   And the fundamental period of this signal is N $(sample)$.

Concept of Frequency of Discrete-Time (DT) Signal

$$x[n] = A\cos(\omega n + \theta) = A\cos(2\pi f n + \theta)$$

$$f = \frac{k}{N} \qquad x[n+N] = x[n]$$



2. The DT sinusoidal signals whose frequencies are over or under the original frequency by number of 2π are identical.
   Because

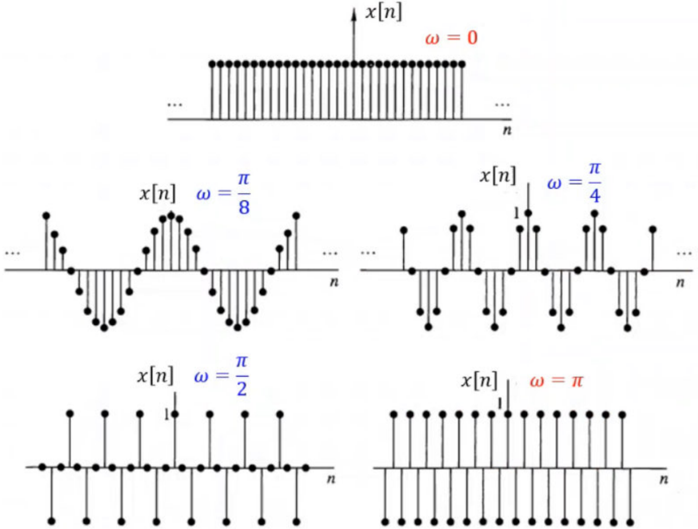
$$\cos\big((\omega \pm k2\pi)n + \theta\big) = \cos(\omega n + \theta)$$

3. That means the limit of DT sinusoidal frequency is

$$\boxed{-\pi \le \omega \le \pi} \quad \text{or} \quad \boxed{-\frac{1}{2} \le f \le \frac{1}{2}}$$

Frequency Limit of Discrete-Time (DT) Signal

$$x[n] = A\cos(\omega n + \theta) = A\cos(2\pi f n + \theta)$$



BREAK

How does the signal sampling work?

When the continuous signal, $x(t)$, is uniformly (periodically) sampled at $F_s = \frac{1}{T}$ Hz, the discrete-time signal ,$x[n]$, is generated by translating the argument from second to sample.

$$x(t) = A\cos(2\pi F t + \theta) = A\cos(\Omega t + \theta)$$

then $\quad x(nT) = A\cos(2\pi F n T + \theta) = A\cos\left(2\pi n \frac{F}{F_s} + \theta\right)$

finally $\quad x[n] = A\cos(2\pi f n + \theta) = A\cos(\omega n + \theta)$

The relation between continuous-time and discrete-time frequencies is.

$$\boxed{f(sample^{-1}) = \frac{F(sec^{-1})}{F_s(sample/sec)}} \quad \Leftrightarrow \quad \boxed{\omega(rad/sample) = \frac{\Omega(rad/sec)}{F_s(sample/sec)}}$$

How does the signal sampling work?

$$f(sample^{-1}) = \frac{F(sec^{-1})}{F_s(sample/sec)} \quad \Leftrightarrow \quad \omega(rad/sample) = \frac{\Omega(rad/sec)}{F_s(sample/sec)}$$

From this relation, $f(sample^{-1})$ and $\omega(rad/sample)$ are called relative frequencies or normalized frequencies.

The sampling process converts unlimited range of analog frequency, $F(sec^{-1})$ or $\Omega(rad/sec)$ to limited range of discrete frequency, $f(sample^{-1})$ or $\omega(rad/sample)$.

$$\Omega = 2\pi F$$
$$-\infty < \Omega(rad/sec) < \infty$$
$$-\infty < F(sec^{-1}) < \infty$$

$$\xrightarrow{f = F/F_s}$$
$$\xleftarrow[F = f.F_s]{}$$

$$\omega = 2\pi f$$
$$-\pi \leq \omega(rad/sample) \leq \pi$$
$$-\frac{1}{2} \leq f(sample^{-1}) \leq \frac{1}{2}$$

How does the signal sampling work?

The sampling process converts
unlimited range of analog frequency, $F(sec^{-1})$ or $\Omega(rad/sec)$ to
limited range of discrete frequency, $f(sample^{-1})$ or $\omega(rad/sample)$.

$$\Omega = 2\pi F$$
$$-\infty < \Omega(rad/sec) < \infty$$
$$-\infty < F(sec^{-1}) < \infty$$

$$f = F/F_s \longrightarrow$$
$$\longleftarrow F = f.F_s$$

$$\omega = 2\pi f$$
$$-\pi \leq \omega(rad/sample) \leq \infty$$
$$-\frac{1}{2} \leq f(sample^{-1}) \leq \frac{1}{2}$$

From the sampling theorem,
if $F_s > 2F_{max}$, the conversion between continuous and discrete frequency will be reversible, otherwise the aliasing effect will be present!

Aliasing Effect

If $x(t) = A\cos(2\pi F_0 t + \theta)$ is sampled to $x[n] = A\cos(2\pi f_0 n + \theta)$ at $F_s = 1/T$,

$$x(t) = A\cos(2\pi F_0 t + \theta)$$

$$x(nT) = A\cos(2\pi F_0 nT + \theta)$$

$$x(nT) = A\cos\left(2\pi \frac{F_0}{F_s} n + \theta\right)$$

$$x[n] = A\cos(2\pi f_0 n + \theta); \qquad f_0 = F_0/F_s$$

If there is another continuous signal with different frequency,

$x_k(t) = A\cos(2\pi F_k t + \theta)$ where $F_k = F_0 + kF_s; k \in I$ and sampled at $F_s = 1/T$,

$$x_k(t) = A\cos(2\pi F_k t + \theta) = A\cos[2\pi(F_0 + kF_s)t + \theta]$$

$$x_k(nT) = A\cos[2\pi(F_0 + kF_s)nT + \theta] = A\cos[2\pi F_0 nT + 2\pi k F_s nT + \theta]$$

$$x_k(nT) = A\cos\left[2\pi \frac{F_0}{F_s} n + 2\pi k \frac{F_s}{F_s} n + \theta\right]$$

$$x_k[n] = A\cos[2\pi f_0 n + 2\pi k n + \theta] = A\cos[2\pi f_0 n + \theta] = x[n]$$

Examples of Aliasing Effect

If $x(t) = \sin\left(\frac{\pi}{4}t\right) = \sin\left(2\pi.\frac{1}{8}t\right) \Rightarrow F_0 = \frac{1}{8}$ Hz

is sampled at $F_s = 1$ sample/sec $\Rightarrow$ $\mathrm{T} = 1/F_s = 1$ sec/sample,

$$x(nT) = \sin\left(2\pi.\frac{1}{8}n.T\right) = \sin\left(2\pi.\frac{1}{8}n.1\right)$$

$$x(nT) = \sin\left(2\pi.\frac{1}{8}n.\frac{1}{F_s}\right) = \sin\left(2\pi.\frac{1}{8}n.\frac{1}{1}\right)$$

$$x[n] = \sin\left(2\pi.\frac{1}{8}n\right) = \sin\left(\frac{\pi}{4}n\right) \Rightarrow f_0 = \frac{F_0}{F_s} = \frac{1/8}{1} = \frac{1}{8} \text{ sample}^{-1}$$

Examples of Aliasing Effect

If $x_k(t) = \sin\left(-\frac{7\pi}{4}t\right) = \sin\left(2\pi.\left(-\frac{7}{8}\right)t\right) \Rightarrow F_k = -\frac{7}{8}$ Hz

is sampled at $F_s = 1$ sample/sec $\Rightarrow$ T $= 1/F_s = 1$ sec/sample,

$$x_k(nT) = \sin\left(2\pi.\left(-\frac{7}{8}\right)n.T\right) = \sin\left(2\pi.\left(-\frac{7}{8}\right)n.1\right)$$

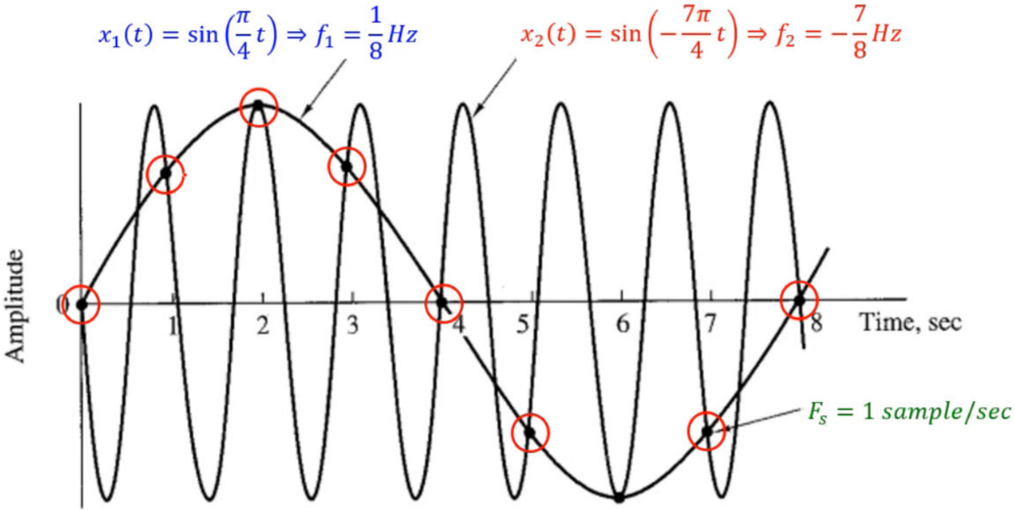
$$x_k(nT) = \sin\left(2\pi.\left(-\frac{7}{8}\right)n.\frac{1}{F_s}\right) = \sin\left(2\pi.\left(-\frac{7}{8}\right)n.\frac{1}{1}\right)$$

$$x_k[n] = \sin\left(2\pi.\left(-\frac{7}{8}\right)n\right) = \sin\left(-\frac{7\pi}{4}n\right) = \sin\left[\left(2\pi - \frac{7\pi}{4}\right)n\right] = \sin\left(\frac{\pi}{4}n\right) = x[n]$$

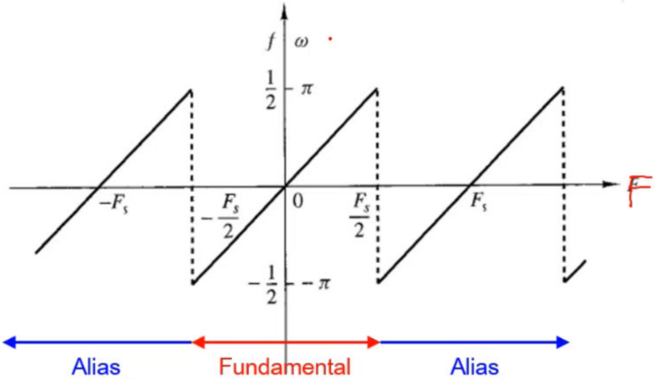
$$f_k = \frac{F_k}{F_s} = \frac{-7/8}{1} = -\frac{7}{8} \text{ sample}^{-1}$$

$$F_k = -\frac{7}{8} = F_0 + kF_s = \frac{1}{8} + 1.(-1) = -\frac{7}{8}$$

Examples of Aliasing Effect

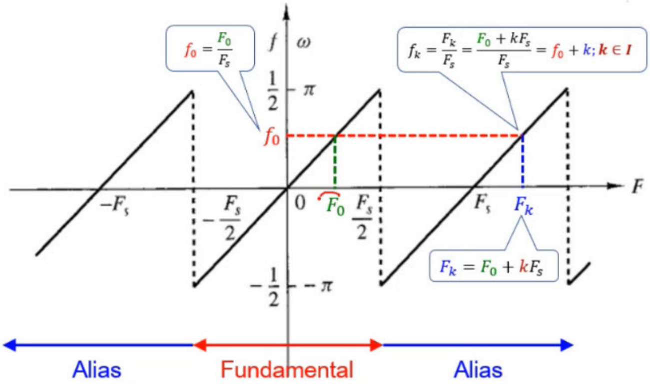


Continuous-Time Frequency ⇔ Discrete-Time Frequency



Relationship between Continuous-Time Frequency, F or Ω
and Discrete-Time Frequency, f or ω
through the sampling frequency, F_s

Continuous-Time Frequency ⇔ Discrete-Time Frequency



Relationship between Continuous-Time Frequency, F or Ω
and Discrete-Time Frequency, f or ω
through the sampling frequency, F_s

Sampling Frequency and Signal Conversion

The sampling theorem says that:

"If the highest frequency contained in an analog signal, $x_a(t)$ is $F_{max} = B$ and the signal is sampled at a rate $F_s > 2B$ then $x_a(t)$ can be exactly recovered from its sampled values, $x[n]$ using the interpolation function, $g(t) = \mathrm{sinc}(2\pi B t)$"

$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n].\, g\left(t - \frac{n}{F_s}\right)$$

if $F_s > 2F_{max}$, the conversion between continuous and discrete frequency will be reversible, otherwise the aliasing effect will be present!

Example of Signal Conversion at $F_s > 2F_{max}$

$$x(t) = 5\sin(300\pi t) = 5\sin(2\pi.150t) \Rightarrow F = 150\text{Hz}$$

If $F_s = 600\text{Hz}$ (sample/sec) $\Rightarrow T = \frac{1}{600}$ (sec/sample)

$$F_s = 600 > 2F$$

$$x(nT) = 5\sin(2\pi.150nT) = 5\sin\left(2\pi.\frac{150}{600}n\right) \Rightarrow x[n] = 5\sin\left(\frac{\pi}{2}n\right)$$

Restore $x(t)$ from $x[n]$ at $F_s = 600\text{Hz}$ (sample/sec)

$$x[n] = 5\sin\left(\frac{\pi}{2}n\right) = 5\sin\left(2\pi.\frac{1}{4}n\right) \Rightarrow f = \frac{1}{4} \text{ (sample}^{-1}\text{)}$$

$$F = f.F_s = \frac{1}{4}.600 = 150\text{Hz}$$

$$x_r(t) = 5\sin\left[2\pi\left(\frac{1}{4}\right)600t\right] = 5\sin(300\pi t)$$

Example of Signal Conversion at $F_s < 2F_{max}$

$$x(t) = 5\sin(300\pi t) = 5\sin(2\pi.150t) \Rightarrow F = 150\text{Hz}$$

If $F_s = 200\text{Hz}$ (sample/sec) $\Rightarrow T = \frac{1}{200}$ (sec/sample)

$$F_s = 200 < 2F \quad .$$

$$x(nT) = 5\sin(2\pi.150nT) = 5\sin\left(2\pi.\frac{150}{200}n\right) \Rightarrow x[n] = 5\sin\left(\frac{3\pi}{2}\,n\right)$$

Restore $x(t)$ from $x[n]$ at $F_s = 200\text{Hz}$ (sample/sec)

$$x[n] = 5\sin\left(\frac{3\pi}{2}n\right) = 5\sin\left(2\pi.\frac{3}{4}n\right) \Rightarrow f = \frac{3}{4} \text{ (sample}^{-1}\text{)}$$

$$x[n] = 5\sin\left(\frac{3\pi}{2}n\right) = 5\sin\left(-\frac{\pi}{2}n\right) = 5\sin\left[2\pi\left(-\frac{1}{4}\right)n\right] \Rightarrow f = -\frac{1}{4} \text{ (sample}^{-1}\text{)}$$

$$F = f.F_s = \left(-\frac{1}{4}\right).200 = -50\text{Hz}\ ; \quad \boxed{|f| \leq \frac{1}{2}}$$

$$x_r(t) = 5\sin\left[2\pi\left(-\frac{1}{4}\right)200t\right] = 5\sin(-100\pi t) = \boxed{-5\sin(100\pi t) \neq 5\sin(300\pi t)}$$

Example of Signal Conversion at $F_s = 2F_{max}$

$$x(t) = 5\cos(300\pi t) = 5\cos(2\pi.150t) \Rightarrow F = 150\text{Hz}$$

If $F_s = 300\text{Hz}$ (sample/sec) $\Rightarrow T = \frac{1}{300}$ (sec/sample)

$$F_s = 300 = 2F$$

$$x(nT) = 5\cos(2\pi.150nT) = 5\cos\left(2\pi.\frac{150}{300}n\right) \Rightarrow x[n] = 5\cos(\pi n)$$

Restore $x(t)$ from $x[n]$ at $F_s = 300\text{Hz}$ (sample/sec)

$$x[n] = 5\cos(\pi n) = 5\cos\left(2\pi.\frac{1}{2}n\right) \Rightarrow f = \frac{1}{2} \text{ (sample}^{-1}\text{)}$$

$$x[n] = 5\cos(\pi n) = \{5, -5, 5, -5, \dots\} \neq 0$$

$$F = f.F_s = \frac{1}{2}.300 = 150\text{Hz}$$

$$x_r(t) = 5\cos\left[2\pi\left(\frac{1}{2}\right)300t\right] = 5\cos(300\pi t)$$

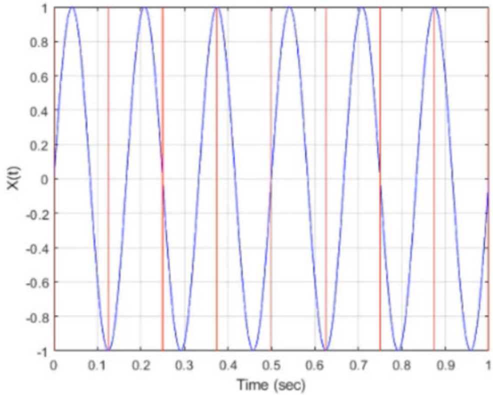
MATLAB Code Example of Signal Conversion

```
t=linspace(0,1,1001);   % Time Scale in second (Continuous Time)
F=6;                    % Continuous Frequency (1/sec)
xt=sin(2*pi*F*t);       % Continuous Time Sinusoidal signal

disp(['The continuous signal frequency, F is ',num2str(F),'Hz']);
figure(1);clf;          % Display Original Continuous Time Signal
A=axes;
plot(t,xt,'b');
grid on
xlabel("Time (sec)");
ylabel("X(t)");

FS=8;                   % Sampling Frequency (Samples/Sec)
T=1/FS;                 % Sampling Period (Sec/Sample)
N=[0:T:1];
YL=get(A,'YLim');
for a=1:length(N)
    line(N(a)*[1 1],YL,'Color','r');  % Sampling Points
end
disp(['The Sampling frequency, FS is ',num2str(FS),'samples/sec']);
```

```
The continuous signal frequency, F is 6Hz
```




```
The Sampling frequency, FS is 8samples/sec
```

```
t=linspace(0,2,1001);
F=4;
xt=2*cos(2*pi*F*t);

disp(['The continuous signal frequency is ',num2str(F),'Hz']);
figure(1);
clf;
A=axes;
plot(t,xt,'b');
grid off
xlabel("Time (sec)");
ylabel("X(t)");
```

The continuous signal frequency is 4Hz