

Exercises 2.5

Exercise 2.5.1

- Show that the logistic function σ satisfies the inequality $0 < \sigma'(x) \leq \frac{1}{4}$, for all $x \in \mathbb{R}$.
- How does the inequality change in the case of the functions σ_c ?

Exercise 2.5.2

Let $S(x)$ and $H(x)$ denote the bipolar step function and the Heaviside function, respectively. Show that:

- $S(x) = 2H(x) - 1$
- $\text{ReLU}(x) = \frac{1}{2}x(S(x) + 1)$

Exercise 2.5.3

Show that the softplus function, $sp(x)$, satisfies the following properties:

- $sp'(x) = \sigma(x)$, where $\sigma(x) = \frac{1}{1+e^{-x}}$
- Show that $sp(x)$ is invertible with inverse $sp^{-1}(x) = \ln(e^x - 1)$
- Use the softplus function to show the formula $\sigma(x) = 1 - \sigma(-x)$

Exercise 2.5.4

Show that $\tanh(x) = 2\sigma(2x) - 1$

Exercise 2.5.5

Show that the softsign function, $so(x)$, satisfies the following properties:

- It is strictly increasing;
- It is onto $(-1, 1)$, with the inverse $so^{-1}(x) = \frac{x}{1-|x|}$, for $|x| < 1$.
- $so(|x|)$ is subadditive, i.e., $so(|x + y|) \leq so(|x|) + so(|y|)$.

Exercise 2.5.6

Show that the softmax function is invariant with respect to the addition of constant vectors $\mathbf{c} = (c_1 \dots c_n)^T$, i.e.,

$$\text{softmax}(y + \mathbf{c}) = \text{softmax}(y).$$

This property is used in practice by replacing $\mathbf{c} = -\max_i y_i$, fact that leads to a more stable numerically variant of this function.

Exercise 2.5.7

Let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $\rho(y) \in \mathbb{R}^n$, with $\rho(y)_i = \frac{y_i^2}{\|y\|}$. Show that:

- a. $0 \leq \rho(y)_i \leq 1$ and $\sum_i \rho(y)_i = 1$.
- b. The function ρ is invariant with to multiplication by nonzero constant, i.e., $\rho(\lambda y) = \rho(y)$ for any $\lambda \in \mathbb{R}/0$. Taking $\lambda = \frac{1}{\max_i y_i}$ leads in practice to a more stable version of this function.

Exercise 2.5.8 (cosine squasher)

Show that the function $\varphi(x) = \frac{1}{2}(1 + \cos(x + \frac{3\pi}{2}))1_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x) + 1_{(\frac{\pi}{2}, \infty)}(x)$ is a squashing function.

Exercise 2.5.9

- a. Show that any squashing function is a sigmoidal function.
- b. Give an example of a sigmoidal function which is not a squashing function.

SOLUTIONS

2.5.1 (a)

Computing the derivative of σ we find: $\sigma'(x) = \frac{d}{dx} \frac{1}{1+e^{-x}} = \frac{d}{dx} \frac{e^x}{1+e^x} = \frac{e^x}{(1+e^x)^2}$. From the inequality $1 \leq (1+e^x)^2$ and the non-negativeness of the exponential function follows that $0 \leq \frac{e^x}{(1+e^x)^2}$.

Now let's prove that in $x = 0$ the function has a local maximum in $[-1, 1]$, this will imply $0 \leq \frac{e^x}{(1+e^x)^2} \leq \sigma'(0)$, $\sigma'(0) = \frac{1}{4}$. By computing the first derivative of σ' we find: $\sigma''(x) = e^x \frac{1-e^x}{(1+e^x)^3}$. The critical will be found by solving the equation $\sigma''(x) = 0$.

From $\sigma''(x) = e^x \frac{1-e^x}{(1+e^x)^3} = 0$ follows that $1 - e^x = 0$, it is straightforward to check that the solution is $x = 0$. It rests to determine the nature of the extremizing point. To achieve this goal is necessary to calculate the second derivative of σ' .

$$\begin{aligned} \sigma'''(x) &= \frac{d}{dx} \frac{e^x - e^{2x}}{(1+e^x)^3} = \frac{(e^x - 2e^{2x})(1+e^x)^3 - 3(1+e^x)^2 e^x (e^x - e^{2x})}{(1+e^x)^6} \\ &= \frac{e^x \{1 - 4e^x + e^{2x}\} (1+e^x)^2}{(1+e^x)^6} = \frac{e^x \{1 - 4e^x + e^{2x}\}}{(1+e^x)^4} \end{aligned}$$

We clearly have $\sigma'''(0) < 0$, then $x = 0$ is a local maximum for σ' , i.e. $\forall x \in [-1, 1]$, $\sigma'(x) \leq \frac{1}{4}$. On the other hand, the function σ' decreases on the intervals $(-\infty, -1)$ and $(1, \infty)$ this implies that:

$$\sup_{x \in (1, \infty)} \sigma'(x) = \frac{e}{(1+e)^2} = \frac{e^{-1}}{(1+e^{-1})^2} = \sup_{x \in (-\infty, -1)} \sigma'(x). \text{ From the fact that } \frac{e}{(1+e)^2} < \frac{1}{4} \text{ follows that } 0 \leq \sigma'(x) \leq \frac{1}{4} \text{ is valid } \forall x \in \mathbb{R}. \quad \square$$

2.5.1 (b)

The inequality changes to: $0 \leq \sigma'_c(x) \leq \frac{c}{4}$, $\forall x \in \mathbb{R}$. From the expression $\sigma_c(x) = \frac{1}{1+e^{-cx}}$, $c > 0$ one finds that $\sigma'_c(x) = \frac{d}{dx} \frac{e^{cx}}{1+e^{cx}} = c \frac{e^{cx}}{(1+e^{cx})^2}$. By the chain rule it can be easily verified that all the computations made for $\sigma'(x)$ in 2.5.1.a, can be applied to $\sigma'_c(x)$, having in mind the relationship $\sigma'_c(x) = c\sigma'(cx)$.

Then, one finds: $\sigma''_c(x) = c^2 e^{cx} \frac{1-e^{cx}}{(1+e^{cx})^3}$, this implies that $x = 0$ is a critical point. Using the same relationship is clear that $\sigma'''_c(x) \Big|_{x=0} = c^3 \frac{e^{cx} \{1-4e^{cx}+e^{2cx}\}}{(1+e^{cx})^4} \Big|_{x=0} < 0$. Then, $x = 0$ is a maximum.

Arguing like in 2.5.1.a, on the interval $[-1, 1]$, $\sigma'_c(0) = \frac{c}{4}$ is a local maximum. More over, the function σ'_c decreases on the intervals $(-\infty, -1)$ and $(1, \infty)$, implying:

$$\sup_{x \in (1, \infty)} \sigma'_c(x) = \frac{ce^c}{(1+e^c)^2} = \frac{ce^{-c}}{(1+e^{-c})^2} = \sup_{x \in (-\infty, -1)} \sigma'_c(x)$$

Let's now prove the inequality $\frac{ce^c}{(1+e^c)^2} < \frac{c}{4}$. We have:

$$\begin{aligned} \frac{ce^c}{(1+e^c)^2} &= \frac{c}{\frac{(1+e^c)^2}{e^{\frac{c}{2}}}} = \frac{c}{\frac{(1+e^{\frac{c}{2}})^2}{e^{\frac{c}{2}}}} \\ &= \frac{c}{(e^{-\frac{c}{2}} + e^{\frac{c}{2}})^2} < \frac{c}{(1 - \frac{c}{2} + 1 + \frac{c}{2})^2} = \frac{c}{4} \end{aligned}$$

Where we have used the inequality $1+x \leq e^x$, $\forall x \in \mathbb{R}$. The latter shows $\sigma'_c(0)$ is a global maximum, i.e. $0 \leq \sigma'_c(x) \leq \frac{c}{4}$ is valid $\forall x \in \mathbb{R}$. \square

2.5.2 (a)

From the Heaviside function definition one has:

$$\begin{aligned}
2H(x) - 1 &= \begin{cases} 2 - 1 & \text{if } x > 0 \\ 2(0) - 1 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{otherwise} \end{cases} = S(x). \quad \square
\end{aligned}$$

2.5.2 (b)

We know $ReLU(x) := \max(0, x)$. Consider the identities $\max(0, x) = \frac{1}{2}\{x + |x|\}$,

$$|x| = \begin{cases} 1x & \text{if } x > 0 \\ -1x & \text{otherwise} \end{cases} = xS(x). \text{ Substituting the last identity into the first one yields:}$$

$$ReLU(x) = \frac{1}{2}(x + xS(x)) = \frac{1}{2}x(1 + S(x)). \quad \square$$

2.5.3 (a)

The identity immediately follows from the application of the chain rule to the function $\ln(1 + e^x)$. In fact, we have: $sp'(x) = \frac{d}{dx} \ln(1 + e^x) = \frac{e^x}{1+e^x} = \frac{1}{1+e^{-x}} = \sigma(x)$. \square

2.5.3 (b)

The function e^x is well known to never be zero, then $1 + e^x > 0, \forall x \in \mathbb{R}$. This implies $sp'(x) \neq 0, \forall x \in \mathbb{R}$. Then by the inverse function theorem the function is invertible. We can now compute its inverse, which is given by:

$$sp(x) = y = \ln(e^x + 1) \implies e^y = e^x + 1 \implies x = sp^{-1}(y) = \ln(e^y - 1). \quad \square$$

2.5.3 (c)

Let $F(x) := x + sp(-x) - sp(x)$. It happens that derivative of F is 0, then by the chain rule, the linearity of the derivative operator and the relationship proved in 2.5.3.a yield:

$\frac{d}{dx} [x + sp(-x)] = 1 - \sigma(-x) = \frac{d}{dx} sp(x) = \sigma(x)$. Lets now prove the claim aforementioned to complete the proof. We have:

$$\begin{aligned}
\frac{d}{dx} F(x) &= \frac{d}{dx} [x + sp(-x) - sp(x)] = \frac{d}{dx} \left[x + \ln\left(\frac{e^{-x} + 1}{e^x + 1}\right) \right] \\
&= 1 + \frac{e^x + 1}{e^{-x} + 1} \frac{d}{dx} \left[\frac{e^{-x} + 1}{e^x + 1} \right] = 1 + \frac{e^x + 1}{e^{-x} + 1} \frac{-e^{-x}(e^x + 1) - e^x(1 + e^{-x})}{(e^x + 1)^2} \\
&= 1 + \frac{e^x + 1}{e^{-x} + 1} \frac{-e^{-x}(e^x + 1) - (1 + e^x)}{(e^x + 1)^2} = 1 + \frac{e^x + 1}{e^{-x} + 1} \frac{-(e^x + 1)(1 + e^x)}{(e^x + 1)^2} = 1 - 1 = 0. \quad \square
\end{aligned}$$

2.5.4 (a)

From the tanh definition we have:

$$\begin{aligned}
 \tanh(x) &:= \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x}{e^x + e^{-x}} - \frac{e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}}{e^{2x} + 1} - \frac{e^{-x}}{e^x + e^{-x}} \\
 &= \sigma(2x) - \frac{e^{-x}}{e^{-x}(1 + e^{2x})} = \sigma(2x) - \frac{1}{1 + e^{2x}} \\
 &= \sigma(2x) - \frac{1 + e^{2x} - e^{2x}}{1 + e^{2x}} = \sigma(2x) - \left\{1 - \frac{e^{2x}}{1 + e^{2x}}\right\} \\
 &= \sigma(2x) - \{1 - \sigma(2x)\} = 2\sigma(2x) - 1. \quad \square
 \end{aligned}$$

2.5.5 (a)

CASE $x > 0$:

Taking the derivative of $so(x)$ we find: $\frac{d}{dx}so(x) = \frac{d}{dx}\frac{x}{1+x} = \frac{1(1+x)-x(1)}{(1+x)^2} = \frac{1}{(1+x)^2} > 0$. This implies $so(x)$ is strictly increasing on the interval $(0, \infty)$.

CASE $x < 0$:

Taking the derivative of $so(x)$: $\frac{d}{dx}so(x) = \frac{d}{dx}\frac{x}{1-x} = \frac{1(1-x)+x(1)}{(1-x)^2} = \frac{1}{(1-x)^2} > 0$. Therefore, the function $so(x)$ is strictly increasing on the interval $(-\infty, 0)$.

CASE $x < y, x < 0, y > 0$:

Let $u, u' \in (x, y)$ with $u < 0, u' > 0$. Then is clear that $u < u'$. From the condition $u < |u|$ follows that $uu' = u|u'| < |u||u'|$. Summing this inequality to the inequality $u < u'$ we get: $u + u|u'| = u(1 + |u'|) < u' + |u||u'| = u'(1 + |u|)$ which implies $\frac{u}{1+|u|} < \frac{u'}{1+|u'|}$, i.e so is strictly increasing on intervals of the type $(x, y), x < 0, y > 0$.

The aforementioned 3 cases imply $so(x), \forall x \in \mathbb{R}$ is strictly increasing. \square

2.5.5 (b)

From $x < 1 + |x|$ follows $\frac{x}{1+|x|} < 1, \forall x \in \mathbb{R}^{>0}$. To get the inequality $-1 < \frac{x}{1+|x|}$ apply the second inequality to u and then multiply by -1 . In summary, we have that the image of $so(x)$ is the interval $(-1, 1)$

On the other hand, note that $S(x) = S(so(x))$. Now, let $u \in (-1, 1), u > 0$. Suppose there is a x such that $so(x) = u$, we have:

$$u = \frac{x}{1+|x|} \implies x = u + |x|u = u(1+x) \implies x(1-u) = u \implies x = \frac{u}{1-u} = so^{-1}(u)$$

If in the contrary, $u < 0$, suppose it exists an x such that $so(x) = u$:

$$u = \frac{x}{1+|x|} \implies x = u - xu = u(1-x) \implies x(1+u) = u \implies x = \frac{u}{1+u} = so^{-1}(u)$$

Both cases can be compactly written as: $x = so^{-1}(u) = \frac{u}{1-|u|}$. \square

2.5.5 (c)

From 2.5.5.a and by the triangle inequality ($|x+y| < |x| + |y|$) follows:

$$\begin{aligned}
 so(|x+y|) &\leq so(|x| + |y|) = \frac{|x| + |y|}{1 + |x| + |y|} = \frac{|x|}{1 + |x| + |y|} + \frac{|y|}{1 + |x| + |y|} \\
 &\leq \frac{|x|}{1 + |x|} + \frac{|y|}{1 + |y|} = so(|x|) + so(|y|). \quad \square
 \end{aligned}$$

2.5.6 (a)

To be more consistent with notation lets write $\text{softmax}(\mathbf{c}; y) := \frac{(e^{c_1+y}, \dots, e^{c_j+y}, \dots, e^{c_n+y})}{\sum_{j=0}^n e^{c_j+y}}$ i.e softmax

with a scalar shift, instead of $\text{softmax}(y + \mathbf{c})$. Because otherwise, one should be precise to define objects of the type $y + \mathbf{c}$ with $y \in \mathbb{R}$, $\mathbf{c} \in \mathbb{R}^n$. That said, $\text{softmax}(\mathbf{c}; 0) = \text{softmax}(\mathbf{c})$; Continuing with the proof, from the functional form of the function $\text{softmax}(y; \mathbf{c})$ it is clear that:

$$\begin{aligned} \text{softmax}(\mathbf{c}; y) &= \frac{(e^{c_1+y}, \dots, e^{c_j+y}, \dots, e^{c_n+y})}{\sum_{j=0}^n e^{c_j+y}} \\ &= \frac{(e^{c_1} e^y, \dots, e^{c_j} e^y, \dots, e^{c_n} e^y)}{e^y \sum_{j=0}^n e^{c_j}} = \frac{e^y (e^{c_1}, \dots, e^{c_j}, \dots, e^{c_n})}{e^y \sum_{j=0}^n e^{c_j}} \\ &= \frac{(e^{c_1}, \dots, e^{c_j}, \dots, e^{c_n})}{\sum_{j=0}^n e^{c_j}} = \text{softmax}(\mathbf{c}; 0). \quad \square \end{aligned}$$