

Exercises 4.17

Exercise 4.17.1

Let $f(x_1, x_2) = e^{x_1} \sin(x_2)$, with $(x_1, x_2) \in (0, 1) \times (0, \frac{\pi}{2})$.

- Show that f is a harmonic function;
- Find $\|\nabla f\|$;
- Show that the equation $\nabla f = 0$ does not have any solutions;
- Find the maxima and minima for the function f .

Exercise 4.17.2

Consider the quadratic function $Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A \mathbf{x} - b\mathbf{x}$, with A nonsingular square matrix of order n .

- Find the gradient $\|\nabla Q\|$;
- Write the gradient descent iteration;
- Find the Hessian H_Q ;
- Write the iteration by Newton's formula and compute its limit.

Exercise 4.17.3

Let A be a nonsingular square matrix of order n and $b \in \mathbb{R}^n$ a given vector. Consider the linear system $A\mathbf{x} = b$. The solution can be approximated using the following steps:

- Associate the cost function $C(\mathbf{x}) = \frac{1}{2}\|A\mathbf{x} - b\|^2$. Find its gradient, $\nabla C(\mathbf{x})$, and Hessian $H_C(\mathbf{x})$;
- Write the gradient descent algorithm iteration which converges to the system solution \mathbf{x} with the initial value $\mathbf{x}^0 = 0$;
- Write Newton's iteration which converges to the system solution \mathbf{x} with the initial value $\mathbf{x}^0 = 0$.

Exercise 4.17.4

- Let $(a_n)_n$ be a sequence with $a_0 > 0$ satisfying the inequality $a_{n+1} \leq \mu a_n + K$, $\forall n \geq 1$, with $\mu \in (0, 1)$ and $K > 0$. Show that the sequence $(a_n)_n$ is bounded from above.
- Consider the momentum method equations (4.4.16) – (4.4.17), and assume that the function f has a bounded gradient $\|\nabla f\| \leq M$. Show that the sequence of velocities, $(v^n)_n$ is bounded.

Exercise 4.17.5

- Let f and g two integrable functions. Verify that

$$\int (f \star g)(x) dx = \int f(x) dx \int g(x) dx$$

- Show that $\|f \star g\| \leq \|f\|_1 \|g\|_1$

- Let $f_\sigma := g \star G_\sigma$ where $G_\sigma = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$. Prove that $\|f_\sigma\|_1 \leq \|f\|_1$ for $\sigma > 0$

Exercise 4.17.6

Show that the convolution of two Gaussians is also a Gaussian:

$$G_{\sigma_1} \star G_{\sigma_2} = G_{\sigma}, \text{ where } \sigma = \sqrt{\sigma_1^2 + \sigma_2^2}.$$

Exercise 4.17.7

Show that the if n have the sum equal to s ,

$$\sigma_1 + \dots + \sigma_n = s,$$

then the numbers for which the sum of their squares, $\sum_{j=1}^n \sigma_j^2$, its minimum occurs for the case when all the numbers are equal to $\frac{s}{n}$.

SOLUTIONS

Exercise 4.17.1 (a)

By definition a function is harmonic when satisfies the condition $\nabla^2 f = 0$. Let's corroborate this is indeed fulfilled by the function $f(x_1, x_2) = e^{x_1} \sin(x_2)$. For $\frac{\partial^2}{\partial x_1^2} f$ and $\frac{\partial^2}{\partial x_2^2} f$ we have:

$$\begin{aligned}\frac{\partial^2}{\partial x_1^2} e^{x_1} \sin(x_2) &= \sin(x_2) \frac{\partial^2}{\partial x_1^2} e^{x_1} = e^{x_1} \sin(x_2) \\ \frac{\partial^2}{\partial x_2^2} e^{x_1} \sin(x_2) &= e^{x_1} \frac{\partial^2}{\partial x_2^2} \sin(x_2) = -e^{x_1} \sin(x_2)\end{aligned}$$

From the latter follows $\nabla^2 f = \frac{\partial^2}{\partial x_1^2} f + \frac{\partial^2}{\partial x_2^2} f = 0$ i.e the function f is harmonic.

□