Exercises 2.5

Exercise 2.5.1

a. Show that the logistic function σ satisfies the inequality $0 < \sigma'(x) \leq \frac{1}{4}$, for all $x \in \mathbb{R}$.

b. How does the inequality changes in the case of the functions σ_c?

Exercise 2.5.2

Let $S(x)$ and $H(x)$ denote the bipolar step fucntion and the Heaviside function,respectively. Show that:

a. $S(x) = 2H(x) - 1$

b. $ReLU(x) = \frac{1}{2}x(S(x) + 1)$

Exercise 2.5.3

Show that the softplus function,$sp(x)$, satisfies the following properties:

a. $sp'(x) = \sigma(x)$, where $\frac{1}{1+e^-x}$

b. Show that $sp(x)$ is invertible with inverse $sp^-1(x) = \ln(e^x - 1)$

c. Use the softplus function to show teh formula $\sigma(x) = 1 - \sigma(-x)$

Exercise 2.5.4

Show that $\tanh(x) = 2\sigma(2x) - 1$

Exercise 2.5.5

Show that the softsign function,$so(x)$, satisfies the following properties:

a. Its sctrictly increasing;

b. Its is onto $(-1, 1)$, with the inverse $so^-1(x) = \frac{1}{1-|x|}$, for $|x| < 1$.

c. $so(|x|)$ is subadditive, i.e., $so(|x + y|) \leq so(|x|) + so(|y|)$.

Exercise 2.5.6

Show that the softmax function is invariant with respect to the addition of constant vectors $\boldsymbol{c} = (c_1 \dots c_n)^T$, i.e.,

$$softmax(y + \boldsymbol{c}) = softmax(y).$$

This property is used in practice by replacing $\boldsymbol{c} = -\max_i y_i$, fact that leads to a more stable numerically variant of this function.

Exercise 2.5.7

Let $\rho : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\rho(y) \in \mathbb{R}^n$, with $\rho(y)_i = \frac{y_i^2}{\|y\|}$. Show that:

a. $0 \leq \rho(y)_i \leq 1$ and $\sum_i \rho(y)_i = 1$.

b. The function ρ is invariant with to multiplication by nonzero constant, i.e., $\rho(\lambda y) = \rho(y)$ for any $\lambda \in \mathbb{R}/0$. Taking $\lambda = \frac{1}{\max_i y_i}$ leads in practice to a more stable version of this function.

Exercise 2.5.8 (cosine squasher)

Show that the function $\varphi(x) = \frac{1}{2}(1 + \cos(x + \frac{3\pi}{2}))1_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x) + 1_{(\frac{\pi}{2}, \infty)}(x)$ is a squashing function.

Exercise 2.5.9

a. Show that any squashinf function is a sigmoidal function.

b. Give an example of a sigmoidal function which is not a squashing function.