

Exercises 3.15

Exercise 3.15.1

Let p , p_i , q , q_i be density functions on \mathbb{R} and $\alpha \in \mathbb{R}$. Show that the cross-entropy satisfies the following properties:

- a. $S(p_1 + p_2, q) = S(p_1, q) + S(p_2, q)$;
- b. $S(\alpha p, q) = \alpha S(p, q) = S(p, q^\alpha)$;
- c. $S(p, q_1 q_2) = S(p, q_1) + S(p, q_2)$.

Exercise 3.15.2

Show that the cross entropy satisfies the following inequality

$$S(p, q) \geq 1 - \int p(x)q(x)dx$$

Exercise 3.15.3

Let p a fixed density. Show that the symmetric relative entropy

$$D_{KL}(p||q) + D_{KL}(q||p)$$

reaches its minimum for $p = q$, and the minimum is equal to zero.

Exercise 3.15.4

Consider two exponential densities, $p_1 = \xi^1 e^{\xi^1 x}$ and $p_2 = \xi^2 e^{\xi^2 x}$, $x \geq 0$.

- a. Show that $D_{KL}(p_1||p_2) = \frac{\xi^2}{\xi^1} - \ln \frac{\xi^2}{\xi^1} - 1$.
- b. Verify $D_{KL}(p_1||p_2) \neq D_{KL}(p_2||p_1)$.
- c. Show that the triangle inequality doesn't hold for three arbitrary densities.

Exercise 3.15.5

Let X be a discrete random variable. Show the inequality

$$H(X) \geq 0.$$

Exercise 3.15.6

Prove that if p and q are the densities of two discrete random variables, then $D_{KL}(p||q) \leq S(p, q)$

Exercise 3.15.7

We assume the target variable Z is \mathcal{E} -measurable. What is mean squared error function in this case?

Exercise 3.15.8

Assume that a neural network has an input-output function $f_{w,b}$ linear in w and b . Show that the cost function (3.3.1) reaches its minimum for a unique pair (w^*, b^*) , which can be computed explicitly.

Exercise 3.15.9

Show that the Shannon entropy can be retrieved from the Reyni entropy as

$$H(p) = \lim_{\alpha \rightarrow 1} H_\alpha(x).$$

Exercise 3.15.10

Let $\phi_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$. Consider the convolution operation $(f * g)(x) := \int f(t)g(x-t)dt$.

- Show that $\phi_\sigma * \phi_\sigma = \phi_{\sigma\sqrt{2}}$;
- Find $\phi_\sigma * \phi_{\sigma'}$ in the case $\sigma \neq \sigma'$.

Exercise 3.15.11

Consider two probability densities, $p(x)$ and $q(x)$. The Cauchy-Schwartz divergence is defined by

$$D_{CS}(p, q) := -\ln\left(\frac{\int p(x)q(x)dx}{\sqrt{\int p(x)^2 dx} \sqrt{\int q(x)^2 dx}}\right)$$

Show the following:

- $D_{CS}(p, q) = 0$ if and only if $p = q$;
- $D_{CS}(p, q) \geq 0$;
- $D_{CS}(p, q) = D_{CS}(q, p)$;
- $D_{CS}(p, q) = -\ln \int pq dx - \frac{1}{2}H_2(p) - \frac{1}{2}H_2(q)$, where $H_2(\cdot)$ denotes the quadratic Reyni entropy.

Exercise 3.15.12

- Show that for any function $f \in L^1[0, 1]$ we have the inequality $\|\tanh(f)\|_1 \leq \|f\|_1$.
- Show that for any function $f \in L^2[0, 1]$ we have the inequality $\|\tanh(f)\|_2 \leq \|f\|_2$.

Exercise 3.15.13

Consider two distributions on the sample space $\mathcal{X} = \{x_1, x_2\}$ given by

$$p = \begin{pmatrix} x_1 & x_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad q = \begin{pmatrix} x_1 & x_2 \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix}$$

Consider the function $\phi : \mathcal{X} \rightarrow \mathbb{R}^2$ defined by $\phi(x_1) = (0, 1)$ $\phi(x_2) = (1, 0)$. Find the maximum mean discrepancy between p and q .

SOLUTIONS

3.15.1 (a)

The claim follows from the linearity of the integral operator. In symbols we have:

$$\begin{aligned} S(p_1 + p_2, q) &= - \int_{\mathbb{R}} (p_1(x) + p_2(x)) \ln q(x) dx = - \int_{\mathbb{R}} p_1(x) \ln q(x) dx - \int_{\mathbb{R}} p_2(x) \ln q(x) dx \\ &= S(p_1, q) + S(p_2, q). \end{aligned}$$

□

3.15.1 (b)

From the linearity of the integral operator, and the property $c \ln(x) = \ln(x^c)$ we have:

$$\begin{aligned} S(\alpha p, q) &= - \int_{\mathbb{R}} \alpha p(x) \ln q(x) dx = -\alpha \int_{\mathbb{R}} p(x) \ln q(x) dx = \alpha S(p, q) \\ &= - \int_{\mathbb{R}} \alpha p(x) \ln q(x) dx = - \int_{\mathbb{R}} p(x) \ln q(x)^\alpha dx = S(p, q^\alpha). \end{aligned}$$

□

3.15.1 (c)

Using the addition identity for the logarithm, we get:

$$\begin{aligned} S(p, q_1 q_2) &= - \int_{\mathbb{R}} p(x) \ln q_1(x) q_2(x) dx = - \int_{\mathbb{R}} p(x) \ln q_1(x) dx - \int_{\mathbb{R}} p(x) \ln q_2(x) dx \\ &= S(p, q_1) + S(p, q_2). \end{aligned}$$

□

3.15.2

By the inequality $\ln(x) \leq x - 1$, $\forall x \in \mathbb{R}^+$, and the definition of cross-entropy follows:

$$\begin{aligned} S(p, q) &= - \int_{\mathbb{R}} p(x) \ln q(x) dx \geq - \int_{\mathbb{R}} p(x) (q(x) - 1) dx \\ &\geq - \int_{\mathbb{R}} -p(x) dx - \int_{\mathbb{R}} p(x) q(x) dx = 1 - \int_{\mathbb{R}} p(x) q(x) dx. \end{aligned}$$

□

3.15.3

From proposition 3.5.1 follows that $D_{KL}(p||q) \geq 0$, $D_{KL}(q||p) \geq 0$, then $D_{KL}(p||q) + D_{KL}(q||p) \geq 0$. Clearly the value 0 is a minimum. Let's now prove that this minimum is attained when $p = q$. It is well known from the cross-entropy definition $S(p, p) = H(p)$ and $S(q, q) = H(q)$ then:

$D_{KL}(p||q) = D_{KL}(p||p) = S(p, p) - H(p) = 0$ and $D_{KL}(q||p) = D_{KL}(q||q) = S(q, q) - H(q) = 0$, which in turn imply $D_{KL}(p||q) + D_{KL}(q||p) = 0$. □

3.15.4 (a)

By direct calculation we find:

$$\begin{aligned}
D_{KL}(p_1 \| p_2) &= S(p_1, p_2) - H(p_1) = - \int_{\mathbb{R}} \xi^1 e^{-\xi^1 x} \ln(\xi^2 e^{-\xi^2 x}) dx - \int_{\mathbb{R}} \xi^1 e^{-\xi^1 x} \ln(\xi^1 e^{-\xi^1 x}) \\
&= - \int_{\mathbb{R}} \xi^1 e^{-\xi^1 x} \ln(\xi^2) dx + \int_{\mathbb{R}} \xi^1 e^{-\xi^1 x} \xi^2 x dx + \int_{\mathbb{R}} \xi^1 e^{-\xi^1 x} \ln(\xi^1) dx - \int_{\mathbb{R}} \xi^1 e^{-\xi^1 x} \xi^1 x dx \\
&= -(\ln(\xi^2) - \ln(\xi^1)) \int_{\mathbb{R}} \xi^1 e^{-\xi^1 x} dx + (\xi^2 - \xi^1) \int_{\mathbb{R}} \xi^1 x e^{-\xi^1 x} dx \\
&= -(\ln(\xi^2) - \ln(\xi^1)) \mathbb{E}_{X \sim \exp(\xi^1)} [1] + (\xi^2 - \xi^1) \mathbb{E}_{X \sim \exp(\xi^1)} [X] = -\ln \frac{\xi^2}{\xi^1} + (\xi^2 - \xi^1) \frac{1}{\xi^1} \\
&= -\ln \frac{\xi^2}{\xi^1} + \frac{\xi^2}{\xi^1} - 1
\end{aligned}$$

□

3.15.4 (b)

Suppose the equality $D_{KL}(p \| p) = D_{KL}(q \| p)$ holds and $\xi^1 \neq \xi^2$, then from exercise 3.14.4.a it follows:
 $-\ln \frac{\xi^2}{\xi^1} + \frac{\xi^2}{\xi^1} - 1 = -\ln \frac{\xi^1}{\xi^2} + \frac{\xi^1}{\xi^2} - 1 \implies \frac{\xi^2}{\xi^1} = \frac{\xi^1}{\xi^2}$. The later implies $\frac{\xi^1}{\xi^2} = 1$ or equivalently $\xi^1 = \xi^2$, which is a contradiction.

3.15.4 (c)

Let $p_1 = \exp(2)$, $p_2 = \exp(3)$, $p_3 = \exp(4)$. Suppose the triangle inequality holds for these three arbitrary exponential distributions. This is:

$D_{KL}(p_1 \| p_3) \leq D_{KL}(p_1 \| p_2) + D_{KL}(p_2 \| p_3)$. By exercise 3.15.4.b we would have:

$$\begin{aligned}
D_{KL}(p_1 \| p_3) &= \frac{4}{2} - \ln \frac{4}{2} - 1 \leq D_{KL}(p_1 \| p_2) + D_{KL}(p_2 \| p_3) = \frac{3}{2} - \ln \frac{3}{2} - 1 + \frac{4}{3} - \ln \frac{4}{3} - 1 \\
2 &\leq \frac{3}{2} + \frac{4}{3} - 1 = \frac{17}{6} - 1 = \frac{11}{6} = \frac{12}{6} - \frac{1}{6} = 2 - \frac{1}{6} \text{ (contradiction!)}
\end{aligned}$$

□

3.15.5 (a)

Given that $p(x)$ is a distribution, it follows that $p(X)$ as a r.v satisfies the inequality $0 \leq p(X) \leq 1$. This means $p(x) \leq 1, \forall x \in \text{sup}(X)$. Taking natural logs on both sides of the inequality $p(x) \leq 1$ and multiplying by -1 , we obtain: $\ln p(x) \geq 0$; Multiplying by $p(x)$ and summing over the support of X , we get:

$$\mathbb{E}[-\ln p(X)] = H(X) = \sum_{x \in \text{sup}(X)} -p(x) \ln(p(x)) \geq 0.$$

□

3.15.6

This is an immediate consequence of exercise 3.15.5. Indeed, we have:

$$D_{KL}(p \| q) = S(p, q) - H(p) \leq S(p, q) - 0 \leq S(p, q).$$

□

3.15.7

If the target variable Z happens to be \mathcal{E} -measurable, then Y is independent of the sigma algebra \mathcal{E} . From this follows that $C(\omega, b) = d(Z, Y)^2 = \mathbb{E}[(Z - \mathbb{E}[Z|\mathcal{E}])^2] = \mathbb{E}[(Z - Z)^2] = 0$.

3.15.8

In this case $f_{\omega, b}(\mathbf{x}) = \omega \cdot \mathbf{x} + b$, defined on a compact subset of \mathbb{R}^n . Therefore, the cost function is given by:

$C(\omega, b) := \sum_{0 \leq i \leq n} (\omega \cdot \mathbf{x}^i + b - \phi(\mathbf{x}^i))^2$. Obviously we have $0 \leq C(\omega, b)$. This means the function attains such

minimum inside the compact set; Let \mathbf{x}^i the n -dimensional observations, i.e $\mathbf{x}^i = (x_1^i, \dots, x_n^i)$. Then, the normal equations for the ω_k (the components of the vector ω), $\forall k \in [n]$ and the bias parameter b are:

$$\begin{cases} \sum_{0 \leq j \leq n} \omega_j \sum_{0 \leq i \leq n} x_j^i x_k^i + b \sum_{0 \leq i \leq n} x_k^i = \sum_{0 \leq i \leq n} \phi(\mathbf{x}^i) x_k^i, \forall k \in [n] \\ \sum_{0 \leq j \leq n} \omega_j \sum_{0 \leq i \leq n} x_j^i + nb = \sum_{0 \leq i \leq n} \phi(\mathbf{x}^i) \end{cases} \quad (1)$$

This system of equations has the following matricial expression:

$$\begin{bmatrix} \sum_{0 \leq i \leq n} x_1^i x_1^i & \sum_{0 \leq i \leq n} x_2^i x_1^i & \cdots & \sum_{0 \leq i \leq n} x_m^i x_1^i & \cdots & \sum_{0 \leq i \leq n} x_1^i \\ \sum_{0 \leq i \leq n} x_1^i x_2^i & \sum_{0 \leq i \leq n} x_2^i x_2^i & \cdots & \sum_{0 \leq i \leq n} x_m^i x_2^i & \cdots & \sum_{0 \leq i \leq n} x_2^i \\ \vdots & \vdots & & \vdots & & \vdots \\ \sum_{0 \leq i \leq n} x_1^i x_k^i & \sum_{0 \leq i \leq n} x_2^i x_k^i & \cdots & \sum_{0 \leq i \leq n} x_m^i x_k^i & \cdots & \sum_{0 \leq i \leq n} x_k^i \\ \vdots & \vdots & & \vdots & & \vdots \\ \sum_{0 \leq i \leq n} x_1^i & \sum_{0 \leq i \leq n} x_2^i & \cdots & \sum_{0 \leq i \leq n} x_m^i & \cdots & n \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \\ \vdots \\ b \end{bmatrix} = \begin{bmatrix} \sum_{0 \leq i \leq n} \phi(\mathbf{x}^i) x_1^i \\ \sum_{0 \leq i \leq n} \phi(\mathbf{x}^i) x_2^i \\ \vdots \\ \sum_{0 \leq i \leq n} \phi(\mathbf{x}^i) x_m^i \\ \vdots \\ \sum_{0 \leq i \leq n} \phi(\mathbf{x}^i) \end{bmatrix} \quad (2)$$

Note that the entries of the matrix in equation 2, are exactly the partial derivatives $\partial_{\omega_k \omega_k}^2 C(\omega, b)$, $\partial_{\omega_k b}^2 C(\omega, b)$, $\partial_{\omega_k \omega_j}^2 C(\omega, b)$, $\partial_{\omega_k b}^2 C(\omega, b)$ i.e such matrix is the hessian-matrix $\mathcal{H}_{C(\omega, b)}$; Let $\{v_m\}_{m \in [n+1]}$ be a collection of vectors in \mathbb{R}^n defined as follows: $\forall m, 0 \leq m \leq n, v_m := (x_m^1, \dots, x_m^n)$. For $m = n+1$ we define: $v_{n+1} := \mathbf{1} = (1, \dots, 1)$, and a vector $\varphi := (\phi(\mathbf{x}^1), \dots, \phi(\mathbf{x}^n))$. The system of equations can be written as:

$$\begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_m & \cdots & v_1 \cdot v_{n+1} \\ v_1 \cdot v_2 & v_2 \cdot v_2 & \cdots & v_2 \cdot v_m & \cdots & v_2 \cdot v_{n+1} \\ \vdots & \vdots & & \vdots & & \vdots \\ v_m \cdot v_1 & v_m \cdot v_2 & \cdots & v_m \cdot v_m & \cdots & v_m \cdot v_{n+1} \\ \vdots & \vdots & & \vdots & & \vdots \\ v_{n+1} \cdot v_1 & v_{n+1} \cdot v_2 & \cdots & v_{n+1} \cdot v_m & \cdots & v_{n+1} \cdot v_{n+1} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \\ \vdots \\ b \end{bmatrix} = \begin{bmatrix} \varphi \cdot v_1 \\ \varphi \cdot v_2 \\ \vdots \\ \varphi \cdot v_m \\ \vdots \\ \varphi \cdot v_{n+1} \end{bmatrix} \quad (3)$$

This is $\mathcal{H}_{C(\omega, b)} = G(v_1, \dots, v_{n+1})$ i.e the matrix in system 3 is a Gramm matrix, which by the postively semidefiniteness property of the Gramm matrices implies this matrix is postively defined. Then, the solution (ω^*, b^*) to such system is unique and this solution gives indeed a minimum for $C(\omega, b)$. Furthermore, the values of the pair (ω^*, b^*) are explicitly computable because the system is linear. □

3.15.9

Note that the Reyni entropy can be expressed as follows: $H_\alpha(p(X)) = \frac{\ln \mathbb{E}[(p(X))^{\alpha-1}]}{1-\alpha} = \frac{\ln \int_{\sup(X)} (p(x))^{\alpha-1} dP}{\alpha-1}$. In the last expression $\frac{dP}{dx} = p(x)$; This representation is a consequence of the Radon-Nikodym's Theorem. Now, let's analyse the following function defined as a parametric integral. Let $I(\alpha)$ defined as:

$$I(\alpha) = \int_{\sup(X)} (p(x))^{\alpha-1} dP$$

CASE $0 < p(x) < 1$:

Let $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ a monotonically decreasing sequence of reals tending to 1. With this we can construct an increasing sequence of P-integrable functions of the form $\{(p(x))^{\alpha_n-1}\}_{n \in \mathbb{N}}$. Then, by construction the point limit will be the function $f(x) = 1$ which is P-integrable. Applying the Lebeque's monotone convergence theorem:

$$\lim_{k \rightarrow \infty} I(\alpha_k) = \lim_{k \rightarrow \infty} \int_{\sup(X)} (p(x))^{\alpha_k-1} dP = \int_{\sup(X)} \lim_{k \rightarrow \infty} (p(x))^{\alpha_k-1} dP = I(\lim_{k \rightarrow \infty} \alpha_k) = \lim_{\alpha \rightarrow 1^+} I(\alpha) = 1.$$

This proves $I(\alpha)$ is right-continuous

Taking $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ a monotonically increasing sequence of reals tending to 1, let's construct the sequence of decreasing P-integrable functions of the form $\{(p(x))^{\alpha_n-1}\}_{n \in \mathbb{N}}$. Once again the point limit function is $f(x) = 1$. Applying the monotone convergence

$$\lim_{k \rightarrow \infty} I(\alpha_k) = \lim_{k \rightarrow \infty} \int_{\sup(X)} (p(x))^{\alpha_k-1} dP = \int_{\sup(X)} \lim_{k \rightarrow \infty} (p(x))^{\alpha_k-1} dP = I(\lim_{k \rightarrow \infty} \alpha_k) = \lim_{\alpha \rightarrow 1^-} I(\alpha) = 1.$$

This proves $I(\alpha)$ is left-continuous.

CASE $p(x) > 1$:

The same is true in this case. It is worth to note that if the sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ is chosen to be decreasing then the sequence of functions is increasing, the contrary situation still holds.

Because the function $f(\alpha, x) = (p(x))^{\alpha-1}$ is derivable, and its derivative is continuous (this can be proved by the monotone convergence theorem), the above can be then used to compute the limit $\lim_{\alpha \rightarrow 1} H_\alpha(p(X))$.

By a simple application $\alpha \rightarrow 1$ we find an indetermination of the type 0/0; Using L'Hôpital's rule we get:

$$\lim_{\alpha \rightarrow 1} H_\alpha(p(X)) = \lim_{\alpha \rightarrow 1} \frac{1}{\alpha-1} \frac{1}{\int_{\sup(X)} (p(x))^{\alpha-1} dP} \int_{\sup(X)} (p(x))^{\alpha-1} \ln(p(x)) dP = \int_{\sup(X)} -\ln(p(x)) dP = H(p).$$

□