Exercises 1.9

Exercise 1.9.1

A factory has n suppliers that produce quantities $x_1 \dots x_n$ per day. The factory is connected with suppliers by a system of roads, which can be at variable capacities $c_1 \dots c_n$, so that the factory is supplied daily the amount $x = c_1 x_1 + \dots + c_n x_n$.

a. Given that the factory production process starts when the supply reaches the critical daily level b, write a formula for the daily factory revenue.

b. Formulate the problem as a learning problem.

Exercise 1.9.2

A number of finantial institutions, each having a wealth x_i, deposit amounts of money in a fund, at some adjustable rates of deposit w_i, so the money in the fund is given by $x = x_1 w_1 + \dots + x_n w_n$. The fund is set up to function as in the following: as long a sthe fund has less than a certain reserve fund M, the fund manager does not invest. Only the money exceeding the reserve found M is invested. Let $k = e^{rt}$, where r and t denote the investment rate of return and time of investment, respectively.

a. Find the formula for the investment.

b. Formulate the problem as a learning problem.

Exercise 1.9.3

a. Given a continous function $f : [0,1] \to \mathbb{R}$, find a linear function $L(x) = ax + b$ with $L(0) = f(0)$ and such that $\frac{1}{2}\int_0^1 (L(x) - f(x))^2 dx$ is minimized.

b. Given a continous function $f : [0,1] \times [0,1] \to \mathbb{R}$, find a linear function $L(x,y) = ax + by + c$ with $L(0,0) = f(0,0)$ and such that the error $\frac{1}{2}\int_{[0,1]^2} (L(x,y) - f(x,y))^2 dx$ is minimized

Exercise 1.9.4

For any compact $K \subset \mathbb{R}^n$ we associate the symetric matrix $\rho_{ij} = \displaystyle\int_K x_i x_j dx_1 \dots dx_n$ The invertiblity of the matrix (ρ_{ij}) depends both on the shape of K and the dimension n.

a. Show that if $n = 2$ then $\det(\rho_{ij}) \neq 0$, for any compact $K \subset \mathbb{R}^2$.

b. Asume $K = [0,1]^n$. Show that $\det(\rho_{ij}) \neq 0$, for any $n \geq 1$.

SOLUTIONS

1.9.1 (a)

Let $\mathbf{c} := (c_1, \ldots, c_n)$ and $\mathbf{p} := (x_1, \ldots, x_n)$ the roads variable capacities and the produced quantities, respectively. Then $x = \mathbf{c} \cdot \mathbf{p}$. Suppose the cost of product per item is k, so if the production starts after the critical daily level b is meet. This is, $x - b > 0$. It is clear that the revenue L_r will be given by the formula:

$$L_r(\mathbf{p}; \mathbf{c}, b) = \begin{cases} k(\mathbf{c} \cdot \mathbf{p} - b), & \text{if } \mathbf{c} \cdot \mathbf{p} - b > 0 \\ 0, & \text{otherwise.} \end{cases}$$

1.9.1 (b)

The learning problem can be stated like this: "Given a vector of road variable capacities **c** and a daily critical level b, provided that the production of the n factories is expressed by the vector $\mathbf{p}$. The goal is find a vector $\mathbf{p}^\star$ and a scalar $b^\star$ such that the ideal revenue $r(\mathbf{p}) = \mathbf{c} \cdot \mathbf{p}$ is close that provided by the data $L_r(\mathbf{p}; \mathbf{c}, b)$ (obtained in 1.9.1 (a))". In other words, the pair $(\mathbf{c}^\star, b^\star)$ minimizes the distance between $r(\mathbf{p})$ and $L_r(\mathbf{p}; \mathbf{c}, b)$. In symbols:

$$(\mathbf{c}^\star, b^\star) = \underset{\mathbf{c} \in \mathbb{R}^n,\ p \in \mathbb{R}}{\arg\min} \int_{\mathcal{K}} (r(\mathbf{p}) - L_r(\mathbf{p}; \mathbf{c}, b))^2 d\mathbf{p}.$$

1.9.2 (a)

If $\mathbf{w} := (w_1 \ldots w_n)$ encodes the adjustable rates of deposit corresponding to each of the n finantial institutions and $\mathbf{x} := (x_1 \ldots x_n)$ the wealth of each of the n institutions, the money in the fund is expressed by $x = \mathbf{w} \cdot \mathbf{x}$. It's known that the fund is set to function if the revenue exceds a given capital M and that the investment grows proportional to e^{rt}(profit per investment), then it is clear that the invesment is given by the formula:

$$L_I(\mathbf{x}; \mathbf{w}, M) = \begin{cases} e^{rt}(\mathbf{w} \cdot \mathbf{x} - M), & \text{if } \mathbf{w} \cdot \mathbf{x} > M \\ 0, & \text{otherwise.} \end{cases}$$

1.9.2 (b)

Let $I(\mathbf{x}) = x = \mathbf{w} \cdot \mathbf{x}$ be the ideal investment, $L_I(\mathbf{x}; \mathbf{w}, M)$ the