Exercises 4.17

Exercise 4.17.1

Let $f(x_1, x_2) = e^{x_1} \sin(x_2)$, with $(x_1, x_2) \in (0, 1) \times (0, \frac{\pi}{2})$.

a. Show that f is a harmonic function;

b. Find $\|\nabla f\|$;

c. Show that the equation $\nabla f = 0$ does not have any solutions;

d. Find the maxima and minima for the function f.

Exercise 4.17.2

Consider the quadratic function $Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - b\mathbf{x}$, with A nonsingular square matrix of order n.

a. Find the gradient $\|\nabla Q\|$;

b. Write the gradient descent iteration;

c. Find the Hessian H_Q;

d. Write the iteration by Newton's formula and compute its limit.

Exercise 4.17.3

Let A be a nonsingular square matrix of order n and $b \in \mathbb{R}^n$ a given vector. Consider the linear system $A\mathbf{x} = b$. The solution can be approximated using the following steps:

a. Associate the cost function $C(\mathbf{x}) = \frac{1}{2}\|A\mathbf{x} - b\|^2$. Find its gradient, $\nabla C(\mathbf{x})$, and Hessian $H_C(\mathbf{x})$;

b. Write the gradient descent algorithm iteration which converges to the system solution $\mathbf{x}$ with the inital value $\mathbf{x}^0 = 0$;

c. Write Newton's iteration which converges to the system solution $\mathbf{x}$ with the initial value $\mathbf{x}^0 = 0$.

Exercise 4.17.4

a. Let $(a_n)_n$ be a sequence with $a_0 > 0$ satisfying the inequality $a_{n+1} \leq \mu a_n + K,\ \forall n \geq 1$, with $\mu \in (0, 1)$ and $K > 0$. Show that the sequence $(a_n)_n$ is bounded from above.

b. Consider the momentum method equations (4.4.16) − (4.4.17), and asume that the function f has a bounded gradient $\|\nabla f\| \leq M$. Show that the sequence of velocities, $(v^n)_n$ is bounded.

Exercise 4.17.5

a. Let f and g two integrable functions. Verify that

$$\int (f \star g)(x)dx = \int f(x)dx \int g(x)dx$$

b. Show that $\|f \star g\| \leq \|f\|_1 \|g\|_1$

c. Let $f_\sigma := g \star G_\sigma$ where $G_\sigma = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$. Prove that $\|f_\sigma\|_1 \leq \|f\|_1$ for $\sigma > 0$

Exercise 4.17.6

Show that the convolution of two Gaussians is also a Gaussian:

$$G_{\sigma_1} \star G_{\sigma_2} = G_\sigma, \text{ where } \sigma = \sqrt{\sigma_1^2 + \sigma_2^2}.$$

Exercise 4.17.7

Show that the if n have the sum equal to s,

$$\sigma_1 + \ldots + \sigma_n = s,$$

then the numbers for which the sum of their squares, $\sum_{j=1}^n \sigma_j^2$, its minimum occurs for the case when all the numbers are equal to $\frac{s}{n}$.

SOLUTIONS

Exercise 4.17.1 (a)

By definition a function is harmonic when satisffies the condition $\nabla^2 f = 0$. Let's corroborate this is indeed fullfilied by the function $f(x_1, x_2) = e^{x_1}\sin(x_2)$. For $\frac{\partial^2}{\partial x_1^2} f$ and $\frac{\partial^2}{\partial x_2^2} f$ we have:

$$\frac{\partial^2}{\partial x_1^2} e^{x_1}\sin(x_2) = \sin(x_2)\frac{\partial^2}{\partial x_1^2} e^{x_1} = e^{x_1}\sin(x_2)$$
$$\frac{\partial^2}{\partial x_2^2} e^{x_1}\sin(x_2) = e^{x_1}\frac{\partial^2}{\partial x_2^2}\sin(x_2) = -e^{x_1}\sin(x_2)$$

From the latter follows $\nabla^2 f = \dfrac{\partial^2}{\partial x_1^2} f + \dfrac{\partial^2}{\partial x_2^2} f = 0$ i.e the function f is harmonic. $\square$

Exercise 4.17.1 (b)

By the pythagorean identity between the trigonometric functions sin and cos follows:

$$\|\nabla f\| = \sqrt{\nabla f \cdot \nabla f} = \sqrt{(e^{x_1}\cos(x_2))^2 + (e^{x_1}\sin x_2)^2} = e^{x_1}$$

Exercise 4.17.1 (c)

$$\nabla f = (e^{x_1}\cos(x_2), e^{x_1}\sin(x_2)) = 0 \iff e^{x_1} = 0$$

Such equation ($e^x = 0$) is known to not have a solution. Therefore, $\nabla f = 0$ is not solvable. $\square$

Exercise 4.17.1 (d)

Let's define the extension of f over the compact $K := [0,1] \times [0, \frac{\pi}{2}]$ with the same association rule as above. On this set f is also harmonic. Then, f reaches its minimun and maximum on the boundaries of the set K; note that both functions e^{x_1} and $\sin(x_2)$ are increasing, then the maxima is met at the point when both functions reach their maximum.

This means the maximum of the function f is met on the point $(1, \frac{\pi}{2})$. Likewise, the minimum is reached at the point $(0, 0)$. $\square$