

## Exercises 1.9

### Exercise 1.9.1

A factory has  $n$  suppliers that produce quantities  $x_1 \dots x_n$  per day. The factory is connected with suppliers by a system of roads, which can be at variable capacities  $c_1 \dots c_n$ , so that the factory is supplied daily the amount  $x = c_1x_1 + \dots + c_nx_n$ .

- Given that the factory production process starts when the supply reaches the critical daily level  $b$ , write a formula for the daily factory revenue.
- Formulate the problem as a learning problem.

### Exercise 1.9.2

A number of financial institutions, each having a wealth  $x_i$ , deposit amounts of money in a fund, at some adjustable rates of deposit  $w_i$ , so the money in the fund is given by  $x = x_1w_1 + \dots + x_nw_n$ . The fund is set up to function as in the following: as long as the fund has less than a certain reserve fund  $M$ , the fund manager does not invest. Only the money exceeding the reserve fund  $M$  is invested. Let  $k = e^{rt}$ , where  $r$  and  $t$  denote the investment rate of return and time of investment, respectively.

- Find the formula for the investment.
- Formulate the problem as a learning problem.

### Exercise 1.9.3

- Given a continuous function  $f : [0, 1] \rightarrow \mathbb{R}$ , find a linear function  $L(x) = ax + b$  with  $L(0) = f(0)$  and such that  $\frac{1}{2} \int_0^1 (L(x) - f(x))^2 dx$  is minimized.
- Given a continuous function  $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ , find a linear function  $L(x, y) = ax + by + c$  with  $L(0, 0) = f(0, 0)$  and such that the error  $\frac{1}{2} \int_{[0, 1]^2} (L(x, y) - f(x, y))^2 dx$  is minimized.

### Exercise 1.9.4

For any compact  $K \subset \mathbb{R}^n$  we associate the symmetric matrix  $\rho_{ij} = \int_K x_i x_j dx_1 \dots dx_n$ . The invertibility of the matrix  $(\rho_{ij})$  depends both on the shape of  $K$  and the dimension  $n$ .

- Show that if  $n = 2$  then  $\det(\rho_{ij}) \neq 0$ , for any compact  $K \subset \mathbb{R}^2$ .
- Assume  $K = [0, 1]^n$ . Show that  $\det(\rho_{ij}) \neq 0$ , for any  $n \geq 1$ .

# SOLUTIONS

## 1.9.1 (a)

Let  $\mathbf{c} := (c_1, \dots, c_n)$  and  $\mathbf{p} := (x_1, \dots, x_n)$  the roads variable capacities and the produced quantities, respectively. Then  $x = \mathbf{c} \cdot \mathbf{p}$ . Suppose the cost of product per item is  $k$ , so if the production starts after the critical daily level  $b$  is meet. This is,  $x - b > 0$ . It is clear that the revenue  $L_r$  will be given by the formula:

$$L_r(\mathbf{p}; \mathbf{c}, b) = \begin{cases} k(\mathbf{c} \cdot \mathbf{p} - b), & \text{if } \mathbf{c} \cdot \mathbf{p} - b > 0 \\ 0, & \text{otherwise.} \end{cases}$$

## 1.9.1 (b)

The learning problem can be stated like this: "Given a vector of road variable capacities  $\mathbf{c}$  and a daily critical level  $b$ , provided that the production of the  $n$  factories is expressed by the vector  $\mathbf{p}$ . The goal is find a vector  $\mathbf{p}^*$  and a scalar  $b^*$  such that the ideal revenue  $r(\mathbf{p}) = \mathbf{c} \cdot \mathbf{p}$  is close that provided by the data  $L_r(\mathbf{p}; \mathbf{c}, b)$  (obtained in 1.9.1 (a))". In other words, the pair  $(\mathbf{c}^*, b^*)$  minimizes the distance (in the  $\mathcal{L}_2$  sense) between  $r(\mathbf{p})$  and  $L_r(\mathbf{p}; \mathbf{c}, b)$ . In symbols:

$$(\mathbf{c}^*, b^*) = \arg \min_{\mathbf{c} \in \mathbb{R}^n, b \in \mathbb{R}} \int_{\mathcal{K}} (r(\mathbf{p}) - L_r(\mathbf{p}; \mathbf{c}, b))^2 d\mathbf{p}.$$

## 1.9.2 (a)

If  $\mathbf{w} := (w_1 \dots w_n)$  encodes the adjustable rates of deposit corresponding to each of the  $n$  financial institutions and  $\mathbf{x} := (x_1 \dots x_n)$  the wealth of each of the  $n$  institutions, the money in the fund is expressed by  $x = \mathbf{w} \cdot \mathbf{x}$ . It's known that the fund is set to function if the revenue exceeds a given capital  $M$  and that the investment grows proportional to  $e^{rt}$  (profit per investment), then it is clear that the investment is given by the formula:

$$L_I(\mathbf{x}; \mathbf{w}, M) = \begin{cases} e^{rt}(\mathbf{w} \cdot \mathbf{x} - M), & \text{if } \mathbf{w} \cdot \mathbf{x} > M \\ 0, & \text{otherwise.} \end{cases}$$

## 1.9.2 (b)

Let  $I(\mathbf{x}) = x = \mathbf{w} \cdot \mathbf{x}$  be the ideal investment,  $L_I(\mathbf{x}; \mathbf{w}, M)$  given in 1.9.2 (b). Then the learning problem can be stated as follows: "Given the vector of wealth of the  $n$  institutions  $\mathbf{x}$ , the vector of adjustable rates of deposit  $\mathbf{w}$  and that inversions are placed in the fund if the capital exceeds the quantity  $M$ . It is needed to find a tuple  $(\mathbf{w}^*, M)$  that minimizes the distance (in the  $\mathcal{L}_2$  sense) between the functions  $I(\mathbf{x})$  and  $L_I(\mathbf{x}; \mathbf{w}, M)$ "

In mathematical terms this means that:

$$(\mathbf{w}^*, M^*) = \arg \min_{\mathbf{w} \in \mathbb{R}^n, M \in \mathbb{R}} \int_{\mathcal{K}} (I(\mathbf{x}) - L_I(\mathbf{x}; \mathbf{w}, M))^2 d\mathbf{x}.$$

## 1.9.3 (a)

It follows from the constrain  $f(0) = L(0)$  that  $b = f(0)$ . From now on the notation  $L(x; a)$  will be used. That said, the task is now to find an  $a^*$  such that  $a^* = \arg \min_{a \in \mathbb{R}} \|f(x) - L(x; a)\|_{\mathcal{L}_2([0,1])}^2$ . Now, let  $C(a) := \|f(x) - L(x; a)\|_{\mathcal{L}_2([0,1])}^2$ , then  $C(a)$  has the most explicit form:

$$C(a) = \int_{[0,1]} (f(x) - ax - f(0))^2 dx.$$

The extremizing  $a^*$  can be found by computing critical points and applying the second derivative test to determine the nature of the critical point. Then one has:

$\frac{d}{da}C(a) = \int_{[0,1]} -2(f(x) - ax - f(0))x dx = a \int_{[0,1]} 2x^2 dx + \int_{[0,1]} 2(f(0) - f(x))x dx$ . To find the critical point  $a^*$  we solve the equation  $\frac{d}{da}C(a) = 0$ . Solving for  $a$  one finds that such extrema is meet at:

$$a^* = \frac{\int_{[0,1]} (f(x) - f(0))x dx}{\int_{[0,1]} x^2 dx}. \text{ Now, lets compute the second derivative to determine the nature}$$

of this critical point. A straight-forward calculation for finding the second derivative yields:

$\frac{d^2}{dx^2}C(a) = 2 \int_{[0,1]} x^2 dx$ . Because the even function  $2x^2$  satisfies  $2x^2 > 0, \forall x \in [0, 1]$  and the interval of integration is not symetric, is clear that  $\frac{d^2}{dx^2}C(a) = 2 \int_{[0,1]} x^2 dx > 0, \forall a \in \mathbb{R}$ . From this follows that  $a^*$  is a

minimum. Then,  $L(x) = \frac{\int_{[0,1]} (f(t) - f(0))t dt}{\int_{[0,1]} t^2 dt} x + f(0)$ .

### 1.9.3 (b)

The condition  $f(0, 0) = L(0, 0)$  implies  $b = f(0, 0)$ . Lets be more explicit by writing  $L(x, y; a, b)$ . The goal is to find  $(a^*, b^*)$  satisfying  $(a^*, b^*) = \arg \min_{a \in \mathbb{R}, b \in \mathbb{R}} \|f(x, y) - L(x, y; a, b)\|_{\mathcal{L}_2([0,1]^2)}^2$