Exercises 1.9

Exercise 1.9.1

A factory has n suppliers that produce quantities $x_1 \dots x_n$ per day. The factory is connected with suppliers by a system of roads, which can be at variable capacities $c_1 \dots c_n$, so that the factory is supplied daily the amount $x = c_1 x_1 + \cdots + c_n x_n$.

a. Given that the factory production process starts when the supply reaches the critical daily level b, write a formula for the daily factory revenue.

b. Formulate the problem as a learning problem.

Exercise 1.9.2

A number of finantial institutions, each having a wealth x_i, deposit amounts of money in a fund, at some adjustable rates of deposit w_i, so the money in the fund is given by $x = x_1 w_1 + \cdots + x_n w_n$. The fund is set up to function as in the following: as long as the fund has less than a certain reserve fund M, the fund manager does not invest. Only the money exceeding the reserve found M is invested. Let $k = e^{rt}$, where r and t denote the investment rate of return and time of investment, respectively.

a. Find the formula for the investment.

b. Formulate the problem as a learning problem.

Exercise 1.9.3

a. Given a continous function $f : [0,1] \to \mathbb{R}$, find a linear function $L(x) = ax + b$ with $L(0) = f(0)$ and such that $\frac{1}{2}\int_0^1 (L(x) - f(x))^2 dx$ is minimized.

b. Given a continous function $f : [0,1] \times [0,1] \to \mathbb{R}$, find a linear function $L(x,y) = ax + by + c$ with $L(0,0) = f(0,0)$ and such that the error $\frac{1}{2}\int_{[0,1]^2} (L(x,y) - f(x,y))^2 dx$ is minimized

Exercise 1.9.4

For any compact $K \subset \mathbb{R}^n$ we associate the symetric matrix $\rho_{ij} = \displaystyle\int_K x_i x_j dx_1 \dots dx_n$ The invertiblity of the matrix (ρ_{ij}) depends both on the shape of K and the dimension n.

a. Show that if $n = 2$ then $\det(\rho_{ij}) \neq 0$, for any compact $K \subset \mathbb{R}^2$.

b. Asume $K = [0,1]^n$. Show that $\det(\rho_{ij}) \neq 0$, for any $n \geq 1$.

SOLUTIONS

1.9.1 (a)

Let $\mathbf{c} := (c_1, \ldots, c_n)$ and $\mathbf{p} := (x_1, \ldots, x_n)$ the roads variable capacities and the produced quantities, respectively. Then $x = \mathbf{c} \cdot \mathbf{p}$. Suppose the cost of product per item is k, so if the production starts after the critical daily level b is meet. This is, $x - b > 0$. It is clear that the revenue L_r will be given by the formula:

$$L_r(\mathbf{p}; \mathbf{c}, b) = \begin{cases} k(\mathbf{c} \cdot \mathbf{p} - b), & \text{if } \mathbf{c} \cdot \mathbf{p} - b > 0 \\ 0, \text{ otherwise.} \end{cases}$$

1.9.1 (b)

The learning problem can be stated like this: "Given a vector of road variable capacities $\mathbf{c}$ and a daily critical level b, provided that the production of the n factories is expressed by the vector $\mathbf{p}$. The goal is find a vector $\mathbf{p}^\star$ and a scalar $b^\star$ such that the ideal revenue $r(\mathbf{p}) = \mathbf{c} \cdot \mathbf{p}$ is close that provided by the data $L_r(\mathbf{p}; \mathbf{c}, b)$ (obtained in 1.9.1 (a))". In other words, the pair $(\mathbf{c}^\star, b^\star)$ minimizes the distance (in the $\mathcal{L}_2$ sense) between $r(\mathbf{p})$ and $L_r(\mathbf{p}; \mathbf{c}, b)$. In symbols:

$$(\mathbf{c}^\star, b^\star) = \underset{\mathbf{c} \in \mathbb{R}^n,\ p \in \mathbb{R}}{\arg\min} \int_{\mathcal{K}} (r(\mathbf{p}) - L_r(\mathbf{p}; \mathbf{c}, b))^2 d\mathbf{p}.$$

1.9.2 (a)

If $\mathbf{w} := (w_1 \ldots w_n)$ encodes the adjustable rates of deposit corresponding to each of the n finantial institutions and $\mathbf{x} := (x_1 \ldots x_n)$ the wealth of each of the n institutions, the money in the fund is expressed by $x = \mathbf{w} \cdot \mathbf{x}$. It's known that the fund is set to function if the revenue exceds a given capital M and that the investment grows proportional to e^{rt}(profit per investment), then it is clear that the invesment is given by the formula:

$$L_I(\mathbf{x}; \mathbf{w}, M) = \begin{cases} e^{rt}(\mathbf{w} \cdot \mathbf{x} - M), \text{ if } \mathbf{w} \cdot \mathbf{x} > M \\ 0, \text{ otherwise.} \end{cases}$$

1.9.2 (b)

Let $I(\mathbf{x}) = x = \mathbf{w} \cdot \mathbf{x}$ be the ideal investment, $L_I(\mathbf{x}; \mathbf{w}, M)$ given in 1.9.2 (b). Then the learning problem can be stated as follows: "Given the vector of wealth of the n institutions $\mathbf{x}$, the vector of adjustable rates of deposit $\mathbf{w}$ and that inversions are placed in the fund if the capital exceeds the quantity M. It is needed to find a tuple $(\mathbf{w}^\star, M)$ that minimizes the distance (in the $\mathcal{L}_2$ sense) between the functions $I(\mathbf{x})$ and $L_I(\mathbf{x}; \mathbf{w}, M)$"

In mathematical terms this means that:

$$(\mathbf{w}^\star, M^\star) = \underset{\mathbf{w} \in \mathbb{R}^n,\ M \in \mathbb{R}}{\arg\min} \int_{\mathcal{K}} (I(\mathbf{x}) - L_I(\mathbf{x}; \mathbf{c}, M))^2 d\mathbf{x}.$$

1.9.3 (a)

It follows from the constrain $f(0) = L(0)$ that $b = f(0)$. From now on the notation $L(x; a)$ will be used. That said, the task is now to find an $a^\star$ such that
$a^\star = \underset{a \in \mathbb{R}}{\arg\min} \|f(x) - L(x; a)\|^2_{\mathcal{L}_2([0,1])}$. Now, let $C(a) := \|f(x) - L(x; a)\|^2_{\mathcal{L}_2([0,1])}$, then $C(a)$ has the most explicit form:

$$C(a) = \int_{[0,1]} (f(x) - ax - f(0))^2 dx.$$

The extremizing $a^\star$ can be found by computing critical points and applying the second derivative test to determine the nature of the critical point. Then one has:

$\frac{d}{da}C(a) = \int_{[0,1]} -2(f(x) - ax - f(0))xdx = a\int_{[0,1]} 2x^2dx + \int_{[0,1]} 2(f(0) - f(x))xdx$. To find the critical point $a^\star$ we solve the equation $\frac{d}{da}C(a) = 0$. Solving for a one finds that such extrema is meet at:

$a^\star = \dfrac{\int_{[0,1]} (f(x) - f(0))xdx}{\int_{[0,1]} x^2dx}$. Now, lets compute the second derivative to determine the nature of this critical point. A straight-forward calculation for finding the second derivative yields:

$\frac{d^2}{dx^2}C(a) = 2\int_{[0,1]} x^2dx$. Because the even function $2x^2$ satisfies $2x^2 > 0$, $\forall x \in [0,1]$ and the interval of integration is not symetric, is clear that $\frac{d^2}{dx^2}C(a) = 2\int_{[0,1]} x^2dx > 0$, $\forall a \in \mathbb{R}$. From this follows that $a^\star$ is a minimum. Then, $L(x) = \dfrac{\int_{[0,1]} (f(t) - f(0))tdt}{\int_{[0,1]} t^2dt}x + f(0)$.

1.9.3 (b)

The condition $f(0,0) = L(0,0)$ implies $b = f(0,0)$.Lets be more explicit by writing $L(x,y;a,b)$. The goal is to find $(a^\star, b^\star)$ satisfying $(a^\star, b^\star) = \underset{a\in\mathbb{R}, b\in\mathbb{R}}{\arg\min} \|f(x,y) - L(x,y;a,b)\|^2_{\mathcal{L}_2([0,1]^2)}$. If $C(a,b) = \|f(x,y) - L(x,y;a,b)\|^2_{\mathcal{L}_2([0,1]^2)}$, it's necessary to solve the linear system:

$$\begin{cases} \dfrac{\partial}{\partial a}C(a,b) = 0 \\ \dfrac{\partial}{\partial b}C(a,b) = 0. \end{cases} \tag{1}$$

Lets now compute both partial derivatives. For a we have:

$$\begin{aligned} \frac{\partial}{\partial a}C(a,b) &= \int_{[0,1]^2} -2(f(x,y) - ax - by - f(0,0))xdxdy \\ &= a\int_{[0,1]^2} 2x^2dxdy + b\int_{[0,1]^2} 2xydxdy + \int_{[0,1]^2} 2(f(0,0) - f(x,y))xdxdy \end{aligned}$$

Likewise, for b we find:

$$\begin{aligned} \frac{\partial}{\partial b}C(a,b) &= \int_{[0,1]^2} -2(f(x,y) - ax - by - f(0,0))ydxdy \\ &= a\int_{[0,1]^2} 2xydxdy + b\int_{[0,1]^2} 2y^2dxdy + \int_{[0,1]^2} 2(f(0,0) - f(x,y))ydxdy. \end{aligned}$$

System (1) is equivalent to:

$$\begin{cases} a\int_{[0,1]^2} 2x^2dxdy + b\int_{[0,1]^2} 2xydxdy = \int_{[0,1]^2} 2(f(x,y) - f(0,0))xdxdy \\ a\int_{[0,1]^2} 2xydxdy + b\int_{[0,1]^2} 2y^2dxdy = \int_{[0,1]^2} 2(f(x,y) - f(0,0))ydxdy \end{cases} \tag{2}$$

Because the function $L(x, y; a, b)$ is continous, then the mixed partial derivatives coincide. It's easy to verify $\frac{\partial^2}{\partial a \partial b} C(a, b) = \int_{[0,1]^2} 2xy dx dy$. Similarly, is straight-forward to see that $\frac{\partial^2}{\partial a^2} C(a, b) = \int_{[0,1]^2} 2x^2 dx dy$ and $\frac{\partial^2}{\partial b^2} C(a, b) = \int_{[0,1]^2} 2y^2 dx dy$. Then, system (2) can be represented in the following matricial form:

$$\begin{bmatrix} \int_{[0,1]^2} 2x^2 dx dy & \int_{[0,1]^2} 2xy dx dy \\ \int_{[0,1]^2} 2xy dx dy & \int_{[0,1]^2} 2y^2 dx dy \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \int_{[0,1]^2} 2(f(x,y) - f(0,0)) x dx dy \\ \int_{[0,1]^2} 2(f(x,y) - f(0,0)) y dx dy \end{bmatrix} \tag{3}$$

Lets denote H_C the matrix on the l.h.s of the equation (3). By the Cauchy–Bunyakovsky–Schwarz inequality follows that:

$\det(H_C) = 4 \int_{[0,1]^2} x^2 dx dy \int_{[0,1]^2} y^2 dx dy - 4 (\int_{[0,1]^2} xy dx dy)^2 > 0$. Since $\det(H_C) \neq 0$ there's a unique solution to the system (3). Such solutions are given explicitly by:

$$\begin{aligned} a^\star &= \frac{\det \begin{bmatrix} \int_{[0,1]^2} x^2 dx dy & \int_{[0,1]^2} (f(x,y) - f(0,0)) x dx dy \\ \int_{[0,1]^2} xy dx dy & \int_{[0,1]^2} (f(x,y) - f(0,0)) y dx dy \end{bmatrix}}{\int_{[0,1]^2} x^2 dx dy \int_{[0,1]^2} y^2 dx dy - (\int_{[0,1]^2} xy dx dy)^2} \\ b^\star &= \frac{\det \begin{bmatrix} \int_{[0,1]^2} (f(x,y) - f(0,0)) x dx dy & \int_{[0,1]^2} xy dx dy \\ \int_{[0,1]^2} (f(x,y) - f(0,0)) y dx dy & \int_{[0,1]^2} y^2 dx dy \end{bmatrix}}{\int_{[0,1]^2} x^2 dx dy \int_{[0,1]^2} y^2 dx dy - (\int_{[0,1]^2} xy dx dy)^2} \end{aligned} \tag{4}$$

11.7 Note that $(1, 1)$ minor of $\det(H_C)$ also has positive determinant, then by Sylvester's criterion H_C is positively defined, and the critical point $(a^\star, b^\star)$ is a minimum.

1.9.4 (a)

For the $n = 2$ case we have:

$\mathbf{P} = \begin{bmatrix} \int_K x_1^2 dx_1 dx_2 & \int_K x_1 x_2 dx_1 dx_2 \\ \int_K x_1 x_2 dx_1 dx_2 & \int_K x_2^2 dx_1 dx_2 \end{bmatrix}$. Once again, by the Cauchy–Bunyakovsky–Schwarz inequality follows that $\det(\mathbf{P}) = \int_K x_1^2 dx_1 dx_2 \int_K x_2^2 dx_1 dx_2 - (\int_K x_1 x_2 dx_1 dx_2)^2 > 0$. The equality can not occur because that would imply K is a section of line which contradicts the arbitrairiness of K. $\square$

1.9.4 (b)

Note that for $\forall i, j \in \{1 \dots n\}$ if $i = j$, $\mathbf{P}_{ii} = \int_0^1 \cdots \int_0^1 x_i^2 dx_1 \dots dx_n$. By Fubini's theorem the integral reduces to:
$\mathbf{P}_{ii} = \int_0^1 x_i^2 dx_i \int_0^1 \prod_{k \neq i}^{n} dx_k = \frac{1}{3}$. On the other hand, for $i \neq j$ one has:

$$\mathbf{P}_{ij} = \int_0^1 x_i x_j dx_i \int_0^1 dx_1 \dots dx_n = \int_0^1 x_i dx_i \int_0^1 x_j dx_j \int_0^1 \cdots \int_0^1 \prod_{k \neq i,\ k \neq j}^{n} dx_k = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

This implies the matrix $\mathbf{P}$ is represented by:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \cdots & \frac{1}{4} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{3} \end{bmatrix}$$

It's easy due to the simple expression of $\mathbf{P}$. The quantity $\det(\mathbf{P})$ will be found using elementary transformations on the row, to which the determinant is invariant. Having said that, if all the other $n-1$ rows are added to the first one gets:

$$\det(\mathbf{P}) = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \cdots & \frac{1}{4} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{3} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{3} + (n-1)\frac{1}{4} & \frac{1}{3} + (n-1)\frac{1}{4} & \cdots & \frac{1}{3} + (n-1)\frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \cdots & \frac{1}{4} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{3} \end{bmatrix}$$

$$= (\frac{1}{3} + (n-1)\frac{1}{4}) \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{4} & \frac{1}{3} & \cdots & \frac{1}{4} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{3} \end{bmatrix}$$

Substracting 1/4 times the first row to every other column the later transform into:

$$\det(\mathbf{P}) = (\frac{1}{3} + (n-1)\frac{1}{4}) \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{4} & \frac{1}{3} & \cdots & \frac{1}{4} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{3} \end{bmatrix} = (\frac{1}{3} + (n-1)\frac{1}{4}) \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & \frac{1}{3} - \frac{1}{4} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{3} - \frac{1}{4} \end{bmatrix}$$

$$= (\frac{1}{3} + (n-1)\frac{1}{4})(\frac{1}{3} - \frac{1}{4})^{n-1} \neq 0.$$

$\square$