1.1

Let V be a pre-Hilbert space. Use $0 \leq \langle \alpha x + \beta y, \alpha x + \beta y \rangle$ for $\alpha, \beta \in \mathbb{R}$ and $x, y \in V$ to prove the Cauchy–Schwarz inequality.

SOLUTION:

Let $x, y \in \mathbb{V}, \alpha, \beta \in \mathbb{R}$, $0 \leq \langle \alpha x + \beta y, \alpha x + \beta y \rangle$. Without loss of generality take $\beta = 1$. Then, by the linearity of $\langle \star, \cdot \rangle$ follows:

$$\begin{aligned} 0 &\leq \langle \alpha x + y, \alpha x + y, \rangle \\ &= \langle \alpha x, \alpha x \rangle + \langle \alpha x, y \rangle + \langle y, \alpha x \rangle + \langle y, y \rangle \\ &= \alpha^2 \langle x, x \rangle + \alpha \langle x, y \rangle + \alpha \langle y, x \rangle + \langle y, y \rangle \\ &= \alpha^2 \|x\|^2 + 2\alpha \langle x, y \rangle + \|y\|^2 \end{aligned}$$

The minimum (as a function of α) is attained when:

$$\alpha^2 \|x\|^2 + 2\alpha \langle x, y \rangle + \|y\|^2 = 0$$

Note that because this polynomial as a function of α is a quadratic polynomial, it has at most one real root. Therefore the discriminant must be non-positive, i.e. $4\langle x, y \rangle^2 - 4\|x\|^2\|y\|^2 \leq 0$, which is equivalent to the Cauchy-Schwarz-Bunyakowsky inequality.

1.2

Let V be a normed space with norm $\| \cdot \| : V \to \mathbb{R}$. Show that V is a pre-Hilbert space and that the norm comes from an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ if and only if the parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in V$$

holds, and that in this case the inner product satisfies

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 \right), \quad x, y \in V.$$

SOLUTION:
$\Leftarrow]$

1.3

Let V be a pre-Hilbert space. Show that the norm induced by the inner product satisfies $\|x + y\| < 2$ for all $x \neq y \in V$ with $\|x\| = \|y\| = 1$.

SOLUTION: We know that $\langle \star, \star \rangle^{\frac{1}{2}} = \| \star \|$. By ex. 1.2 this implies the internal product can be expressed in

1.4

Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite and let $C \in \mathbb{R}^{n \times m}$. Show:

1. the matrix $C^\top AC$ is positive semi-definite,
2. $\operatorname{rank}(C^\top AC) = \operatorname{rank}(C)$,
3. the matrix $C^\top AC$ is positive definite if and only if $\operatorname{rank}(C) = m$.

1.5

Show that a matrix $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite if and only if it is of the form $A = BB^\top$ with an invertible matrix $B \in \mathbb{R}^{n \times n}$.

1.6

Let $A, B, C \in \mathbb{R}^{2 \times 2}$ with $C = AB$. Let

$$p = (a_{11} + a_{22})(b_{11} + b_{22}), \quad q = (a_{21} + a_{22})b_{11},$$

$$r = a_{11}(b_{12} - b_{22}), \quad s = a_{22}(b_{21} - b_{11}),$$

$$t = (a_{11} + a_{12})b_{22}, \quad u = (a_{21} - a_{11})(b_{11} + b_{12}),$$

$$v = (a_{12} - a_{22})(b_{21} + b_{22}).$$

Show that the elements of C can then be computed via

$$c_{11} = p + s - t + v, \quad c_{12} = r + t,$$

$$c_{21} = q + s, \quad c_{22} = p + r - q + u.$$

Compare the number of multiplications and additions for this method with the number of multiplications and additions for the standard method of multiplying two 2×2 matrices.

Finally, show that if the above method is recursively applied to matrices $A, B \in \mathbb{R}^{n \times n}$ with $n = 2^k$, then the method requires 7^k multiplications and $6 \cdot 7^k - 6 \cdot 2^{2k}$ additions and subtractions.