EC 603
ADVANCED MACROECONOMICS I
Fall 2016
Problem Set-4 Solutions

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Incomplete Market - Huggett(1993)

This problem intends to improve the command over the solution methods to the heterogeneous-agent incomplete market models.

Consider the following incomplete-market environment by Huggett (1993). There is a unit measure of agents, each of whom maximizes her expected discounted life-time utility. Specifically, she maximizes:

$$max_{C_t,n_t,k_{t+1}} \sum \beta^t u(c_t, n_t)$$

where β refers to the discount rate and $u(.)$ is a utility function that is strictly concave, continuous and twice differentiable.

Agents are subject to uncertain idiosyncratic employment shocks. In each period agents can either be employed, (denoted as e), or become unemployed (denoted as u). In other words, agents' state $s_t \in S = \{e, u\}$. This is the main source of heterogeneity in this model. Employed agents earn a higher income than the unemployed agents, i.e. $y(e) > y(u)$.

The employment shocks are independent and identically distributed across agents. Employment is Markov process with transition matrix denoted by $\pi(s'|s) = Prob(s_{t+1} = s'|s_t = s)$.

While households are hit by unemployment shocks, there are no aggregate shocks hitting the economy, hence given that at the steady-state the distribution of households are time-invariant and constant over asset and shock state pairs, the aggregate variables, such as total endowment, consumption; and prices, such as the price of bonds are time-invariant, and constant, as well.

Households do not have access to Arrow-Debreu securities to insure themselves against unemployment shocks. Instead, there is only a single type of non-contingent (risk-free) bond that pays 1 unit next period, which is traded this period at a price of q. Note that, this arrangement can be considered a bond that pays at a net real interest rate of r units, where the return satisfies $q = 1/(1 + r)$.

In order to curb unbounded capital accumulation motive of households, we assume that $\beta/q < 1$, which indented to generate sufficient impatience, and needs to be verified in equilibrium.

Let $a \in A$ denote the net asset holding position of a household. Then, the dynamic budget constraint of households could be written as:

$$c_t + qa_{t+1} = a_t + y(s_t)$$

While agents can go in debt, i.e. $a < 0$ is possible, it should be that their debt cannot grow unboundedly. For this purpose, there is a borrowing constraint $\underline{a} \in A$ such that $a \geq \underline{a}$ in any period t. Note that in this environment, the natural borrowing limit is $a^{NBL} = -\frac{y(u)}{1-q}$ which is endogenous with respect to the equilibrium bond price, q. We restrict the ad-hoc borrowing limit, $\underline{a}$ to be above the natural borrowing limit $\underline{a}^{NBL}$, i.e. $\underline{a} > \underline{a}^{NBL}$, which is to be verified in equilibrium, as well.

This property can easily be obtained by punishing negative consumption in utility function as in the previous problem-set.

It is common to write the above Bellman equation sans time subscripts through using the prime notation, where the variables with prime refer to next period variables. Then, for a given bond price q, the above Bellman equation to the household's problem can be written as follows:

$$V(a,s) = max_{a' \in A} u[y(s) + a - qa'] + \beta \sum_{s' \in S} \pi(s'|s) V(a', s') \qquad (1)$$

where a' satisfies

$$\underline{a} \leq a' \leq \frac{y(s) + a}{q} \qquad (2)$$

Note that while the former inequality $\underline{a} \leq a'$ is due the ad-hoc borrowing constraint, the latter inequality $a' \leq \frac{y(s)+a}{q}$ ensures consumption to be non-negative.

Let the solution to the above problem be $a' = g(a, s)$. Since households differ on their employment histories, they will in general differ in their amount of asset holdings, as well. This requires that in order to describe the economy at any point in time, we need to describe the amount of assets each agent has, as well as their employment status.

For this goal, the object we use to describe is a probability measure, μ, defined over all possible asset holdings and shocks, A x S. We can think of μ (A, S; q) as the fraction of population with shocks in the set S and asset holdings in the set A when the price is q.

In the absence of the aggregate shocks, the distribution of agents over the state pairs has to be time-invariant in economy's long-run equilibrium. This is satisfied by that the law of motion induced by the decision rule g(a, s) and the probability transition matrix of the stochastic employment process π generate a time-invariant distribution $\mu' = \mu$:

$$\mu(A, S; q) = \int_{AxS} \{ \int_{AxS} \chi_{a'=g(a,s;q)} \pi(s'|s) \mu(a, s; q) \} da' ds'$$

where $\chi_{a'=g(a,s;q)}$ is an indicator function that takes the value 1 if the statement is true, and zero otherwise. In other words, $\{a' = g(a, s; q)\}$ is the set of (a, s) which yield the true $a' \in A$.

Definition of the Economy

For this model economy, a steady-state equilibrium is defined as follows:

Definition: A steady-state equilibrium is a set of functions (v(a, s; q*), c(a, s; q*), $a'(a, s; q*)$); a price q* and an invariant distribution μ*(a, s) such that

1.For given q*, c(a, s; q*) and $a'(a, s; q*)$ solve household's problem in (1) and (2), and v(a, s; q*) is the resultant value function.

2. Given $\mu * (a, s; q*)$, goods and assets markets clear:

$$\int_{AxS} [c(a, s; q*) - y(s)] d\mu * (a, s; q*) = 0$$

$$\int_{AxS} g(a, s; q*) d\mu * (a, s; q) = 0$$

3. $\mu*(a,s;q*)$ is a stationary probability measure:

$$\mu(A,S;q*) = \int_{AxS}\{\int_{AxS}\chi_{a'=g(a,s;q*)}\pi(s'|s)\mu(a,s;q*)\}da'ds'$$

Numerical Method

To solve this model numerically, the algorithm we employ could be as follows:

1. Start with an initial guess for the bond price q^0 such that $\beta < q^0 < 1$.

2. Given q^0, solve households's problem using value function iteration to obtain $a' = g(a,s;q^0)$.

3. Using the obtained decision rule and the exogenous law of motion of the idiosyncratic shocks, derive the stationary distribution by iterative application. Define the fixed point as $\mu^0(a,s;q^0)$.

4. Compute $A^0 = \int_{AxS} ad\mu^0(a,s;q)$. If $A^0 = 0$ you are done, you found the equilibrium! If $A^0 \neq 0$, the price cannot clear the market, and we need to set a new price that clears the good and the asset markets. If $A^0 < 0$, there is not enough demand for the bond, and we need to raise the return so as to make the bond more attractive, which means we need to reduce its price, q0. If $A^0 > 0$, there is too much demand for the bond, so we need to decrease the return from the bond, or in other words, we need to increase its price, q^0. After updating the price, we return to step 2, and go over the same steps until we find the price at which markets clear.

Let the parameter values and functional forms be as follows: The discount factor $\beta = 0.96$, coefficient of relative risk aversion $\alpha = 1.5$ with a power utility function $u = \frac{c^{1-\alpha}}{1-\alpha}$, the set of possible earnings $S = \{e,u\}$ where y(e) = 1 and y(u) = 0.1 are interpreted as earnings when employed (normalized to unity) and unemployed respectively, the Markov-process for earnings $\pi(s'=e|s=e) = 0.80$ and $\pi(s'=u|s=u) = 0.5$, and the space of asset holdings be given by the compact set A = [-2,5].

Thus, asset grid is created as [-2:0.01:5], which includes 701 points. Tolerance value for economy-wide asset holdings, which should be zero in net supply in equilibrium, calculation is 0.0001.

As in the previous problem-set, negative consumption is punished with very high negative utility to prevent negative consumption selection. This is simply the ad-hoc borrowing constraint given in the problem.

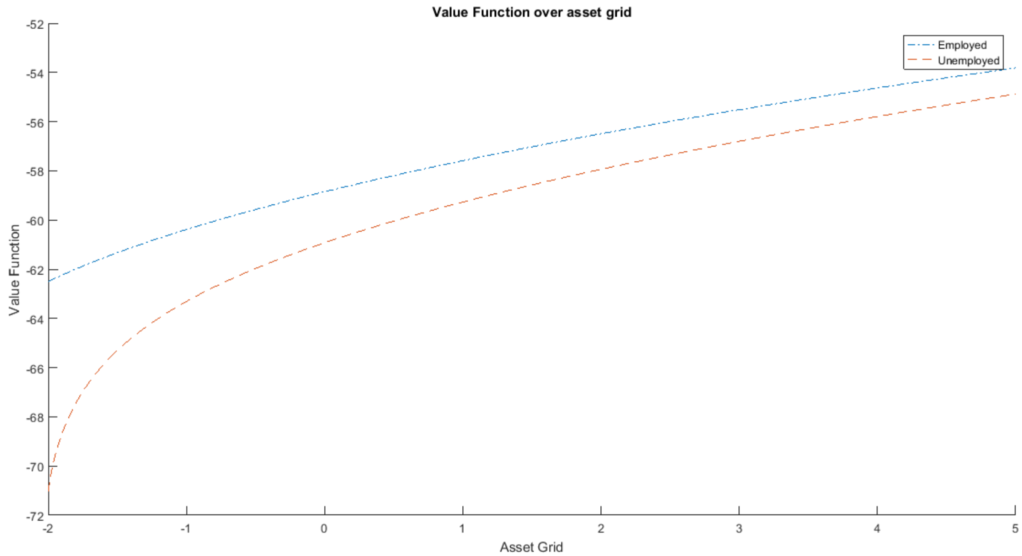
McQueen-Porteus algorithm is used to make calculation faster like in previous problem-set. However, the process still takes 130.95 seconds to complete.

Value Function and Decision Rule - Q1

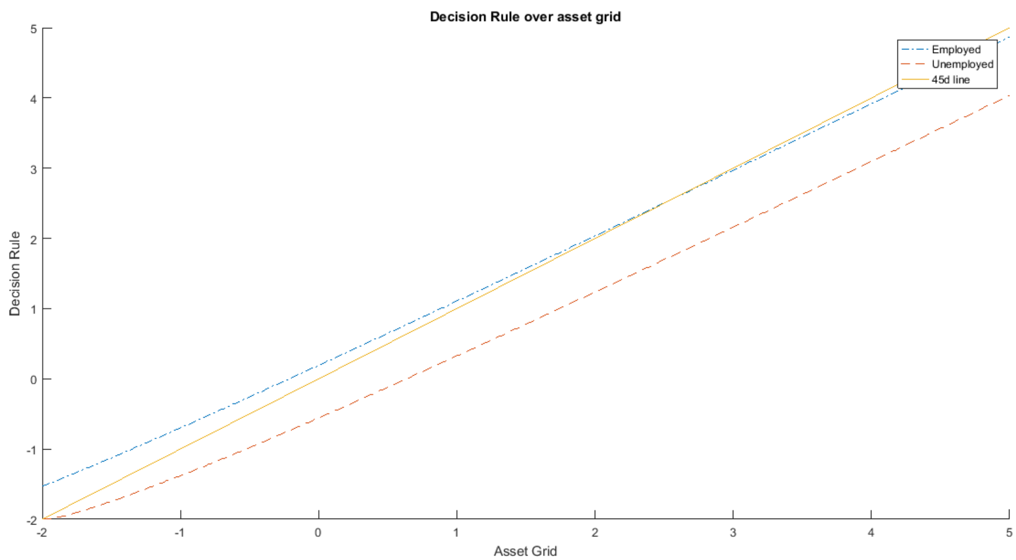
Value function is monotonously increasing with assed holdings as expected, however the difference between employed and unemployed case is changing with asset. For low asset values, the income of employed agent, which is 1, and the income of unemployed agent, which is 0.1, creates a big difference.

$$V(a,s) = max_{a'\in A}u[y(s) + a - qa'] + \beta\sum_{s'\in S}\pi(s'|s)V(a',s')$$

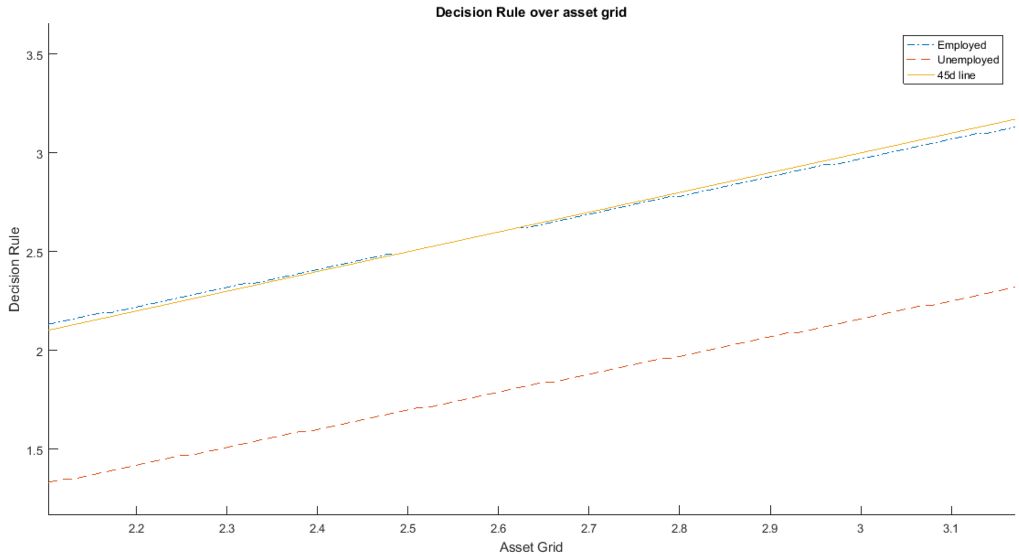
With increasing asset holding, the effect of difference in income is decreasing due to concave utility function.



For employed case the value function will always be higher than unemployed case for each asset point due to properties of utility function, however the difference is decreasing with increasing asset holdings due to above explained reasons.



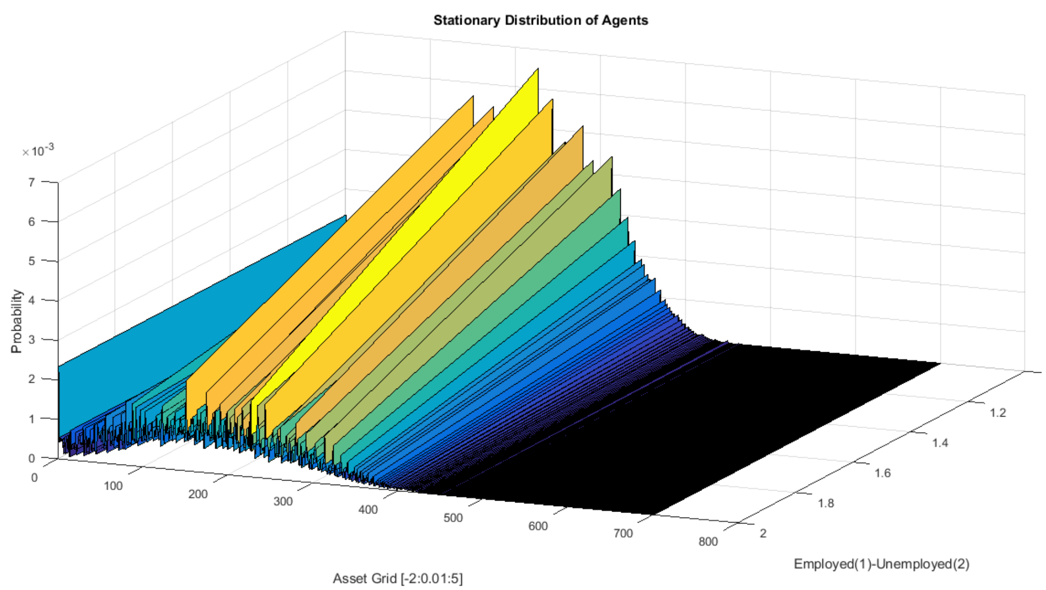
Decision rule is similar to Figure 1 of Huggett (1993) due to similar setup. For unemployed case the lower bound is binding, that is to say steady-state is at -2. But, this is the limit of borrowing, which means agent wants to borrow more optimally when unemployed, but the setup does not allow him to borrow more than 2. It can be said that steady-state for unemployed case is probably at point less than -2.
To find out steady-state for employed case, I zoomed in for clear description.

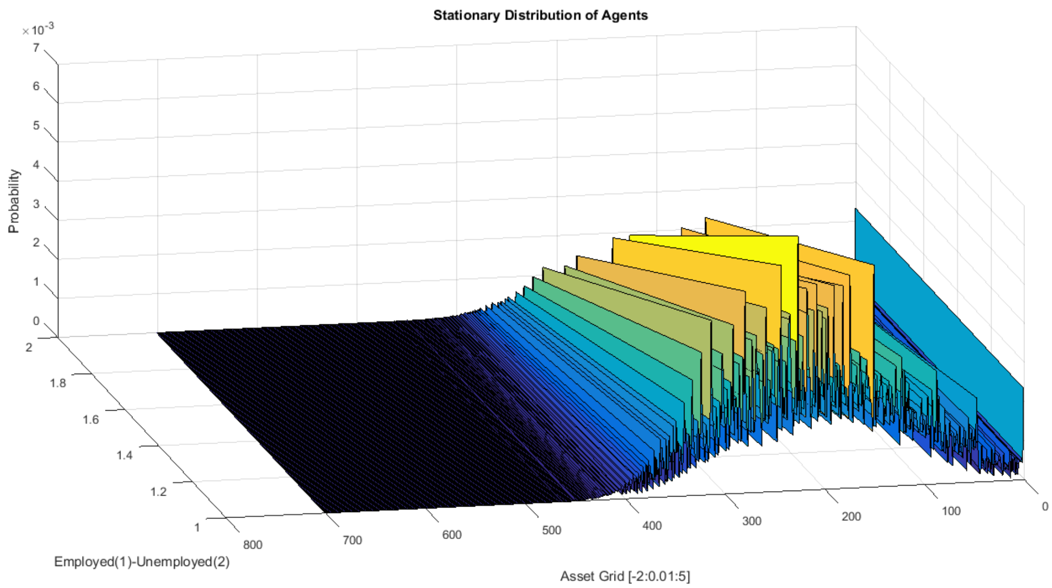


With a small MATLAB code, I find out that steady-state for employes case is at the interval of [2.5:0.01:2.63]. Ideally, I expect to see a single point for steady state due to being the intersection of decision rule and 45-degree line. However, iteration and tolerance choices are not small enough to provide that kind of precision. That is why steady-state is obtained in interval rather than a single point like in the previous problem-set.

Stationary Distribution μ - Q2

At equilibrium bond price q^*, which is found as 0.99154 represented in next section, following stationary distribution of agents $\mu(a, s; q^*)$ is obtained:





Actually, the employed-unemployed axis is binary, that is why the in-between values are meaningless. This figure shows us that probability of being employed at a certain asset point is higher than probability of being unemployed at that point. In general, for every asset point, the probability of being employed is greater than or equal to probability of being unemployed, due to property of Markov-chain probability matrix. For very high asset points, the probability of both states are 0, which means there is no people with that many assets, since it is not optimal to hold such high assets. So, in the long-run, there is nobody left in that part of asset grid.

Economy-wide Asset Holdings - Q3

To calculate stationary distribution of agent μ, the tolerance value is set to 0.00001 such that the distribution is no longer time-variant. Then, with these discrete probabilities asset holdings are multiplied to obtain sum of 0, conditional on the q^*. The MATLAB code gives the following result under above explained conditions:

q^0(Bond Price)	A^0(Total Asset)	Iteration	Elapsed Time
0.99154	9.1586e-05	20th	130.95

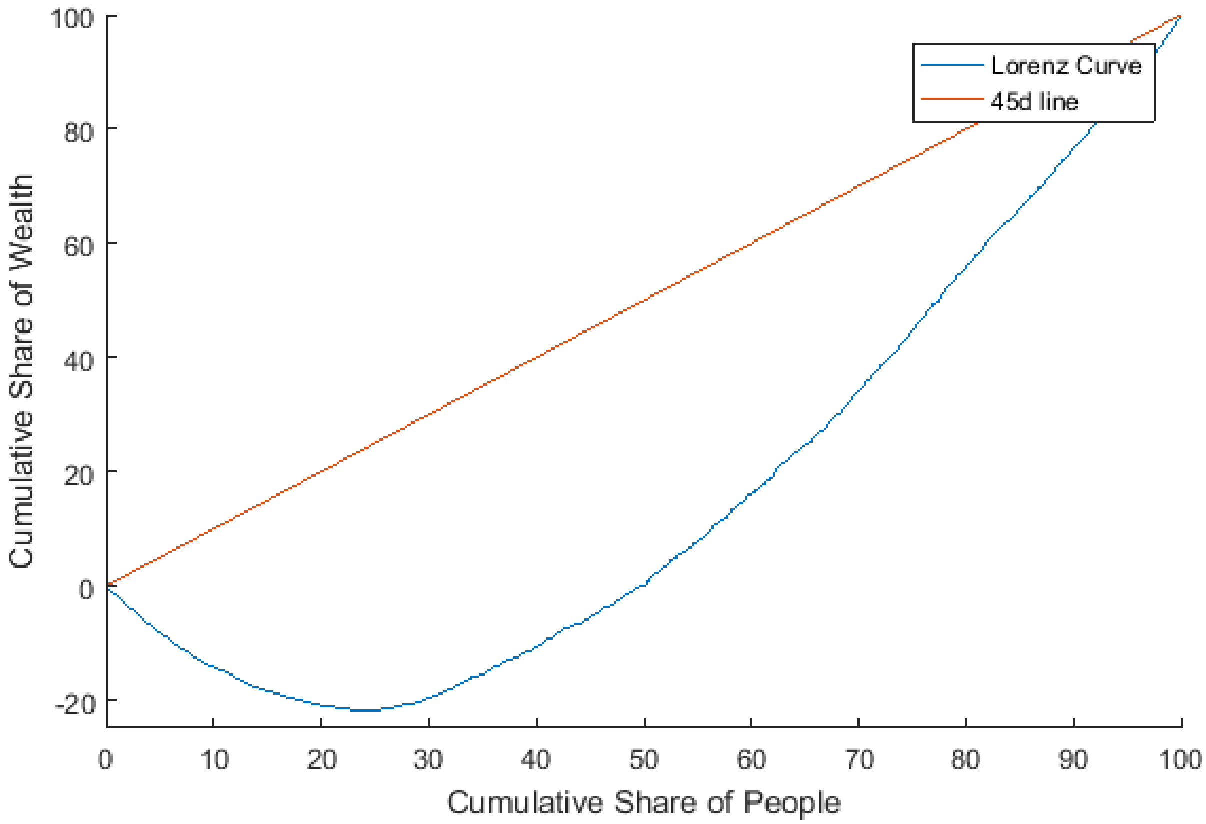
Since the tolerance for A^0 is set to 0.0001, resulted A^0 is a bit less than that value and q^0 is the bond price that corresponds to A^0. Ideally, we are looking for q^0, which makes level of economy-wide asset holdings equal to zero in net supply in equilibrium.
However, computationally it is not possible to get exact 0 for total asset holdings, that is why the bond price, which clears the bond market completely, is very close to our q^0, but not exactly is.

Lorenz Curve and Gini Index - Q4

Total resources is defined as $w = y(s) + a$. For employment state y(s) is 1 and for unemployment state y(s) is 0.1. The stationary, time-invariant distribution of agents and their assets are already known. That is why total resources can simply be calculated by sum of multiplied discrete probabilities and corresponding resources. The result is as follows:

Total Resources	Gini Coefficient
0.73713	0.67

The Gini coefficient is two times the area between Lorenz-Curve and the 45-degree line. For this model, Gini coefficient is quite high, because half of population have negative assets, which means they are in debt.



45-degree line represents the full equality in terms of resources, which makes the area 0 and correspondingly Gini coefficient 0. For our case, half of population are in debt, and the highest 22.5% owns the half of total wealth.

The source of heterogeneity is the idiosyncratic shocks that agents are subject to. However, the aggregate variables are not changing due to time-invariant distribution of agents and lack of population growth.

Welfare - Q5

Ex-ante welfare in incomplete market economy is simply defined as:

$$W^I = \sum_{(a,s)} v(a,s;q^*)\mu(a,s;q^*)$$

which is calculated as welfareInc=-59.7110 for our model.

Note that "full insurance" corresponds to the economy where a full set of contingent claims is traded. As for calculating the welfare in the full insurance world, note that one could first time invariant distribution over the two states by calculating $\widetilde{\pi}(s) = [\pi(s'|s)]^N$, where N is a sufficiently large number such as 100,000. This gives the following matrix:

$$\begin{bmatrix} 0.7143 & 0.2857 \\ 0.7143 & 0.2857 \end{bmatrix}$$

This means that the probability of being employed is 0.7143 and probability of being unemployed is 0.2857 in the long run. Then, the consumption level of the average person could be calculated as $\widetilde{c} = \widetilde{\pi}(e)$ x $y(e)$ + $\widetilde{\pi}(u)$ x $y(u)$, and as a result her deterministic full-insurance life-time utility would be as following:

$$W^F = \sum_{t=0}^{\infty} \beta^t u(\widetilde{c}) = \frac{\widetilde{c}^{1-\alpha}}{(1-\beta)(1-\alpha)}$$

which is calculated as welfareFull=-58.0119 for our model.
Thus, the welfare difference is 1.6991, which is positive as expected. For full-insurance case, agents have more life-time utility due to lack of uncertainty as a result of incompleteness.

Conclusion

Heterogeneous agent incomplete market model of Huggett (1993) is analyzed in detail by considering many aspects of it. Computational results do not give exactly the true results of analytical solution, however they are very good approximation and allow us to solve very complex economic models. With idiosyncratic shocks, the heterogeneity property of an economy is captured. The main downside is the aggregate variables do not change to be able to obtain time-invariant distribution of agents, which makes the calculation much easier. However, in general the findings are very intuitive and allow us to understand the working of an economy in a better way.