Basic tabular methods: Temporal Difference SARSA and Q-learning

Reinforcement Learning

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Outline

1. Introduction
2. Learning Rule
3. Explore vs. Exploit
4. Long-term Reward
5. Temporal Difference Methods

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Solving an Environment

Problem: How can we solve the environment if we do not know the underlying model?

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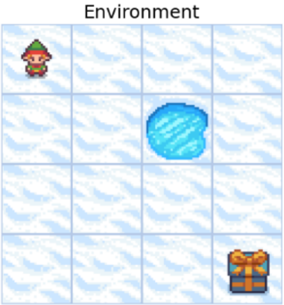

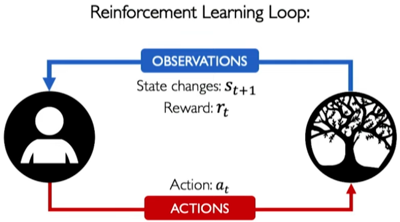

☞ The agent must learn to act based on its experience over an extended period of time.

Components of Reinforcement Learning

- Learning rule
- Explore vs. Exploit
- Long-term reward

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Learning as a correction of the utility estimator's error based on experience.

Components of Reinforcement Learning

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Balancing between exploiting current information and exploring to gather better information.

Components of Reinforcement Learning

- Learning rule
- Explore vs. Exploit
- Long-term reward

The agent must learn to maximize the total reward of the episode, not just the reward for the current action.

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Estimation

Each round, we want to estimate a natural number q based on imperfect observations $G_1, \ldots, G_T$, where G_i is the observation obtained in the i-th round. Suppose Q is our current estimate.

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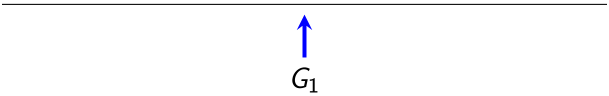

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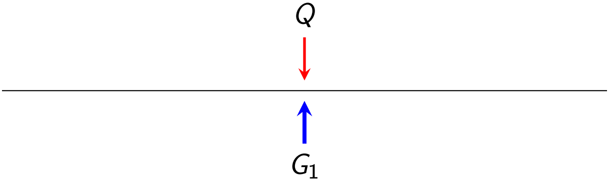

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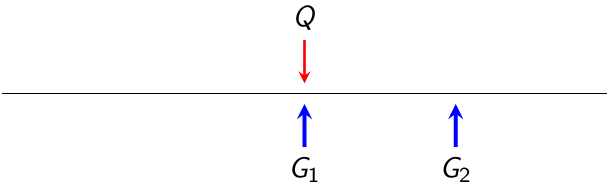

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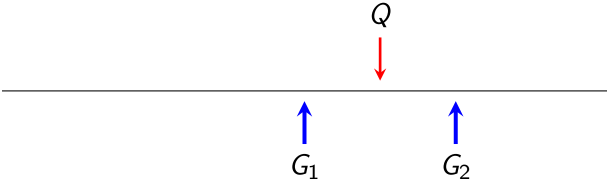

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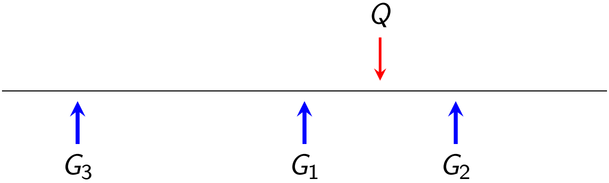

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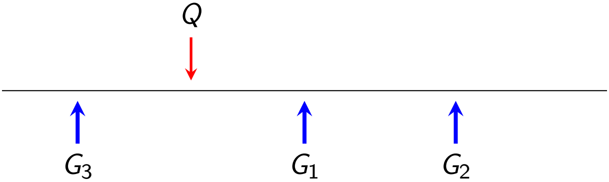

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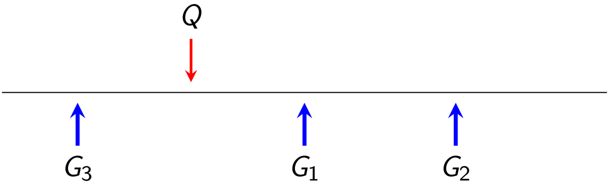
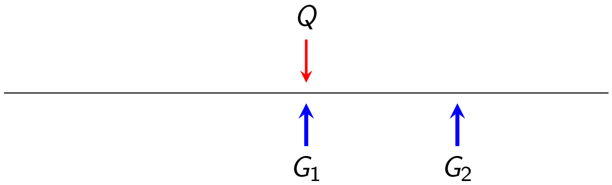


What formula can we use to measure the direction and magnitude of the change in the estimate?

Error Correction in Estimation

Each round, we observe a G_i and note the variation with respect to our current estimate q.



The estimation error is $\delta = G_2 - Q$.

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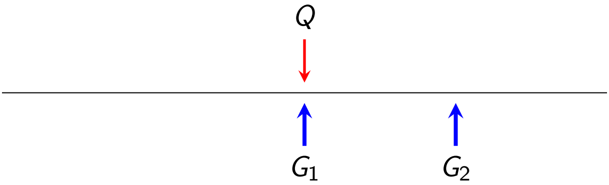

G_2 is not the actual data q, so we must weight the error by a learning rate α.

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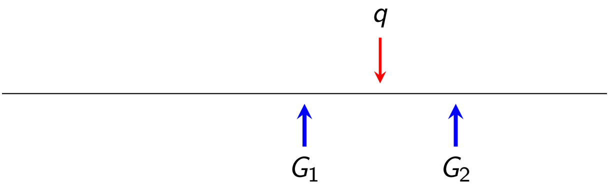

We update Q using the rule

$$Q \leftarrow Q + \alpha\delta$$

Error Correction in Estimation

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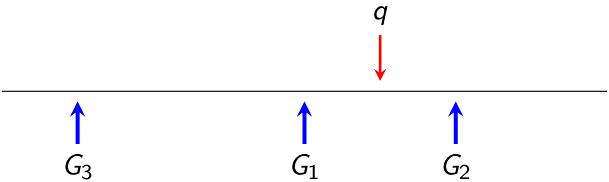
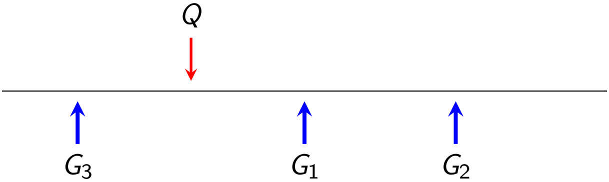


The estimation error is $\delta = G_3 - Q$.

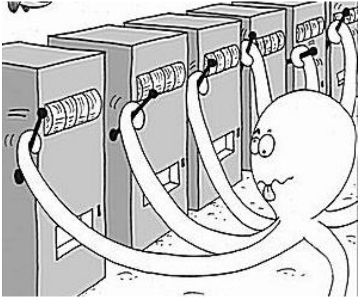
Error Correction in Estimation

Each round, we observe a G_i and note the variation with respect to our current estimate q.



$$Q \leftarrow Q + \alpha\big(G_3 - Q\big)$$

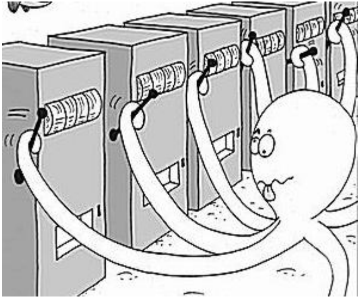
Multi-armed Bandits



Stochastic scheduling problem

- The agent pulls the lever of one of the machines and gets a reward of 0 or 1.

Multi-armed Bandits



Stochastic scheduling problem

- The success probability of each machine is different and initially unknown to the agent.

Multi-armed Bandit Problem

$r(1)$

$r(2)$

Problem: Which machine a offers success with the highest probability $Q(a)$?

Solution Plan

☞ Estimate the success probability $q(a)$ of each machine a.

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☞ Each round i, select a machine a and observe the reward r_i.

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☞ Each round i, select a machine a and observe the reward r_i.

☞ Maintain the estimators $Q_i(a)$ and update using the rule:

$$Q_i(a) = \begin{cases} Q_{i-1}(a) + \alpha\big(r_i - Q_{i-1}(a)\big), & \text{if } a \text{ is selected} \\ Q_{i-1}(a), & \text{otherwise} \end{cases}$$

Example

Suppose $\alpha = \frac{1}{2}$.

Machine 1

- *Initial state:* $Q_0(1) = 0$

Machine 2

- *Initial state:* $Q_0(2) = 0$

Both estimators start at 0.

Example

Suppose $\alpha = \frac{1}{2}$.

Machine 1

- *Initial state:* $Q_0(1) = 0$
- *Round 1:*
  $Q_1(1) = 0 + \alpha(1 - 0) = \frac{1}{2}$

Machine 2

- *Initial state:* $Q_0(2) = 0$
- *Round 1:* $Q_1(2) = 0$

We select 1 and achieve success ($r_1 = 1$).

Example

Suppose $\alpha = \frac{1}{2}$.

Machine 1

- *Initial state:* $Q_0(1) = 0$
- *Round 1:* $Q_1(1) = 0 + \alpha(1 - 0) = \frac{1}{2}$
- *Round 2:* $Q_2(1) = \frac{1}{2} + \alpha(0 - \frac{1}{2}) = \frac{1}{4}$

Machine 2

- *Initial state:* $Q_0(2) = 0$
- *Round 1:* $Q_1(2) = 0$
- *Round 2:* $Q_2(2) = 0$

We select 1 and do not achieve success ($r_2 = 0$).

Example

Suppose $\alpha = \frac{1}{2}$.

Machine 1

- *Initial state:* $Q_0(1) = 0$
- *Round 1:* $Q_1(1) = 0 + \alpha(1 - 0) = \frac{1}{2}$
- *Round 2:* $Q_2(1) = \frac{1}{2} + \alpha(0 - \frac{1}{2}) = \frac{1}{4}$
- *Round 3:* $Q_3(1) = \frac{1}{4}$

Machine 2

- *Initial state:* $Q_0(2) = 0$
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- *Round 3:* $Q_3(2) = 0 + \alpha(1 - 0) = \frac{1}{2}$

We select 2 and achieve success ($r_3 = 1$).

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Explore vs. Exploit (1/3)

Two extreme approaches:

- **Explore**: Sample both arms.
- **Exploit**: Select the arm that has provided the best rewards so far (greedy strategy).

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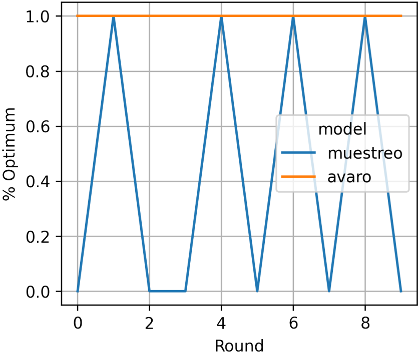

- The sampling strategy exhibits random behavior.

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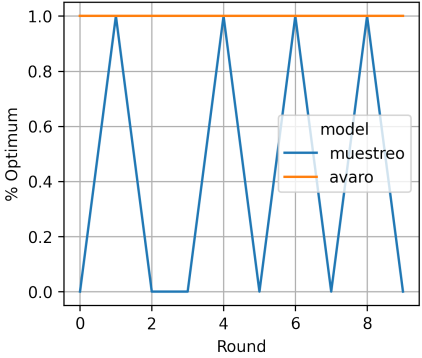

- The sampling strategy exhibits random behavior.
- The greedy strategy tried the optimal arm and succeeded.

Explore vs. Exploit (2/3)

Two extreme approaches:

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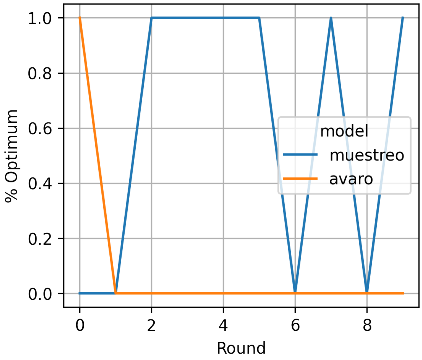

- The greedy strategy tried the optimal arm without success.
- Then, the greedy strategy succeeded by trying the arm that is NOT optimal.

Explore vs. Exploit (3/3)

Two extreme approaches:

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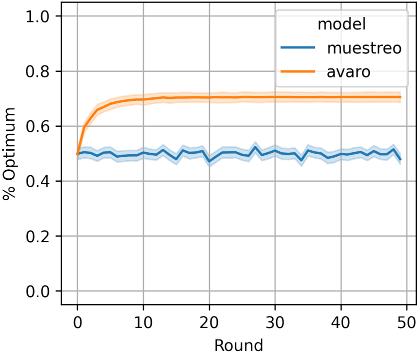


Average over 50 experiments of 50 trials each.

Possible Solutions

There are various ways to address the dilemma between explore and exploit:

- Optimistic initial greedy
- ϵ-greedy
- ϵ-greedy with annealing
- Upper Confidence Bound
- Softmax
- Etc.

☞ Here, we will only discuss the ϵ-greedy strategy.

ϵ-greedy (1/2)

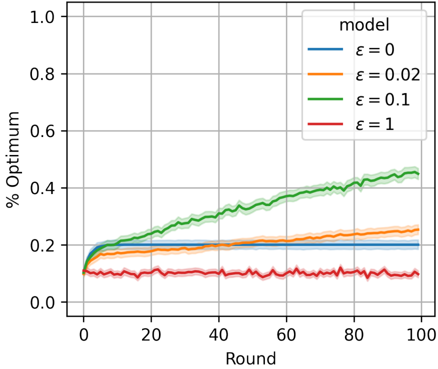
Balance between exploit (with probability $1 - \epsilon$) and explore (with probability ϵ).

Algorithm 1: ϵ-greedy bandit algorithm

Data: una probabilidad de exploración ϵ (donde $0 \leq \epsilon \leq 1$)
Result: índice del brazo seleccionado
$Q(a) \leftarrow 0$ para cada brazo a;
while *True* **do**
    if *probabilidad* $1 - \epsilon$ **then**
        $a \leftarrow \arg\max Q(a)$;
    else
        $a \leftarrow a$ aleatoria;
    end
    Presentar la acción a al entorno y obtener la recompensa r;
    $Q(a) \leftarrow Q(a) + \alpha\left[r - Q(a)\right]$
end

ϵ-greedy (2/2)

Results:



Protocol:

10 arms. 50 experiments of 50 trials.

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Definitions

- **Utility**:

$$G = r_1 + \gamma r_2 + \gamma^2 r_3 + \ldots = \sum_{t=0}^{\infty} \gamma^k r_{t+1}$$

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- **Action Value**: The expected utility of performing an action a in state s and then following π: $q_\pi(s, a) = \mathbb{E}[G|s, a]$

Transition Stochasticity

After the agent executes action a in state s, it transitions to state s_i with probability $p(s_i|s,a)$.

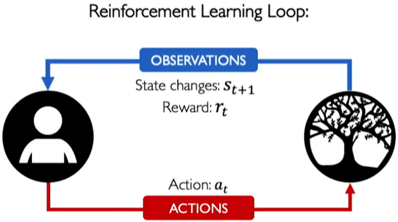
$$\{p(s_1|s,a); p(s_2|s,a); \ldots; p(s_n|s,a)\}$$

Markov Property

☞ Path Independence

$$p(s_{t+1}|s_0, a_0, s_1, a_1, \ldots, s_t, a_t) = p(s_{t+1}|s_t, a_t)$$

Components of MDPs



- Set of states
- Subset of terminal states
- Set of actions
- Transitions $p(s'|s, a)$
- Rewards $r(s, a, s')$

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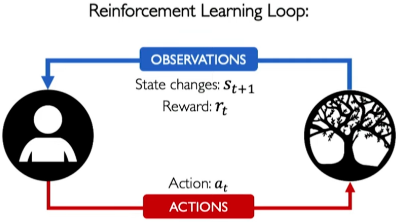

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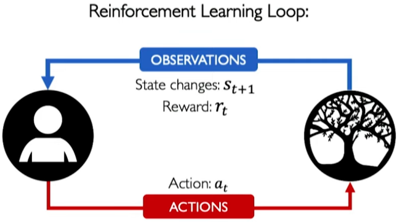
$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s'} \left(p(s'|s,a) \left[r(s,a,s') + \gamma v_\pi(s') \right] \right)$$

Components of MDPs



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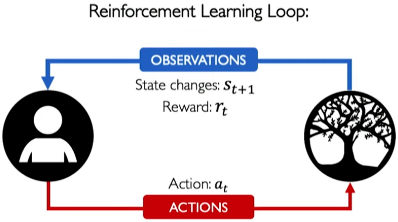
☞ Let us assume that we do not know the MDP model.

Components of MDPs



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☞ We aim to estimate v_* and q_* directly.

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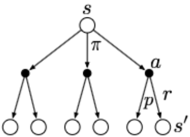
Using the Bellman Equation

Dynamic Programming:

$$V_{k+1}(s) \leftarrow \sum_a \pi(a|s) \sum_{s'} \Big(p(s'|s,a)\Big[r + \gamma V_k(s')\Big]\Big)$$



Backup diagram for v_π

Temporal Difference:

The backup diagram for TD(0)

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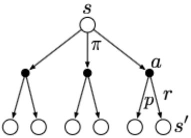

Backup diagram for v_π

Temporal Difference:

$$V_{k+1}(s) \leftarrow V_k(s) + \alpha \left(G - V_k(s) \right)$$


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Dynamic Programming:

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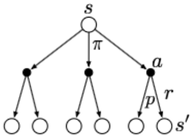

Backup diagram for v_π

Temporal Difference:

$$V_{k+1}(s) \leftarrow V_k(s) + \alpha \left(r + \gamma V_k(s') - V_k(s) \right)$$

The backup diagram for TD(0)

Learning Rule

$$V(s) \leftarrow \overbrace{V(s)}^{\text{previous estimate}} + \underbrace{\alpha}_{\text{step size}} \Big(\overbrace{r_1 + \gamma \underbrace{V(s_1)}_{\textit{bootstrap}}}^{\text{new data}} - \underbrace{V(s)}_{\text{previous estimate}} \Big)$$

Learning a Policy (SARSA)

Suppose a policy π.

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- Rule to update state-action values:

$$Q(s,a) \leftarrow Q(s,a) + \alpha\Big(r + \gamma Q(s',a') - Q(s,a)\Big)$$

  Where s' is the state reached after performing a in s,
  and $a' \leftarrow$ action by sampling $\pi(s')$.

Learning a Policy (SARSA)

Suppose a policy π.

- Rule to update state-action values:

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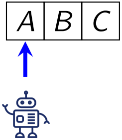
  Where s' is the state reached after performing a in s,
  and $a' \leftarrow$ action by sampling $\pi(s')$.
- Improve $\pi(s)$ with ϵ-greedy using Q for all s.

The Q Table

Consider the ABC environment presented two classes ago.



	Left	Right
A	0	0
B	0	0
C	0	0

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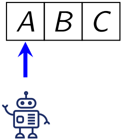

	Left	Right
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B	0	0
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Randomly, the agent selects the action Left.

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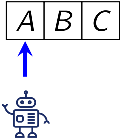

	Left	Right
A	0	0
B	0	0
C	0	0

The agent remains in state A.

$$s = A$$
$$a = \text{Left}$$
$$s' = A$$
$$a' \leftarrow \text{random action}$$

The Q Table

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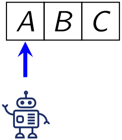

	Left	Right
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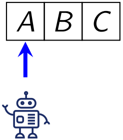
The agent remains in state A.

$$s = A$$
$$a = \text{Left}$$
$$s' = A$$
$$a' \leftarrow \text{Right}$$

The Q Table

Consider the ABC environment presented two classes ago.

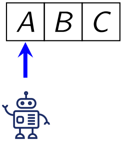


	Left	Right
A	0	0
B	0	0
C	0	0

$$q(A, \text{Left}){+} \;=\; \alpha\Big(-1 \;+\; \gamma q(A, \text{Right}) \;-\; q(A, \text{Left})\Big)$$

The Q Table

Consider the ABC environment presented two classes ago.



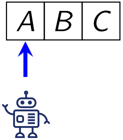
	Left	Right
A	0	0
B	0	0
C	0	0

$$q(A, \text{Left}) += 0.1\Big(-1 + 0.8 \times 0 - 0\Big)$$

Suppose $\alpha = 0.1$

The Q Table

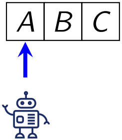
Consider the ABC environment presented two classes ago.



	Left	Right
A	-0.1	0
B	0	0
C	0	0

The Q Table

Consider the ABC environment presented two classes ago.

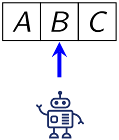


	Left	Right
A	-0.1	0
B	0	0
C	0	0

The agent selects the action with the highest q value, namely, Right (this occurs with probability $1 - \epsilon$).

The Q Table

Consider the ABC environment presented two classes ago.



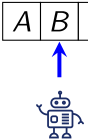
	Left	Right
A	-0.1	0
B	0	0
C	0	0

$$s = A$$
$$a = \text{Right}$$
$$s' = B$$
$$a' \leftarrow \text{Right}$$

Suppose the agent reaches B
(this occurs with probability 0.9).

The Q Table

Consider the ABC environment presented two classes ago.

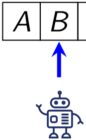


	Left	Right
A	-0.1	0
B	0	0
C	0	0

$$q(A, \text{Right}) += \alpha\Big(-1 + \gamma q(B, \text{Right}) - q(A, \text{Right})\Big)$$

The Q Table

Consider the ABC environment presented two classes ago.

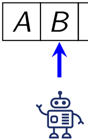


	Left	Right
A	-0.1	0
B	0	0
C	0	0

$$q(A, \text{Right}) += 0.1\Big(-1 + 0.8 \times 0 - 0\Big)$$

The Q Table

Consider the ABC environment presented two classes ago.



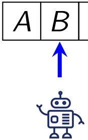
	Left	Right
A	-0.1	-0.1
B	0	0
C	0	0

Randomly, the agent selects the action Right.

$$s = B$$
$$a = \text{Right}$$
$$s' = C$$
$$a' \leftarrow \text{Right}$$

The Q Table

Consider the ABC environment presented two classes ago.

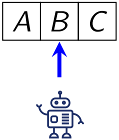


	Left	Right
A	-0.1	-0.1
B	0	0
C	0	0

The agent reaches C.

The Q Table

Consider the ABC environment presented two classes ago.

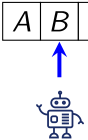


	Left	Right
A	-0.1	-0.1
B	0	0
C	0	0

$$q(B, \text{Right}) += \alpha\Big(10 + \gamma q(C, \text{Right}) - q(B, \text{Right})\Big)$$

The Q Table

Consider the ABC environment presented two classes ago.

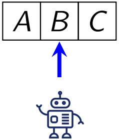


	Left	Right
A	-0.1	-0.1
B	0	0
C	0	0

$$q(B, \text{Right}) += 0.1\Big(10 + 0.8 \times 0 - 0\Big)$$

The Q Table

Consider the ABC environment presented two classes ago.



	Left	Right
A	-0.1	-0.1
B	0	1
C	0	0

SARSA Pseudocode

Algorithm 3: SARSA agent (update rule)

Data: Una acción a, un estado s' y una recompensa r

$Q(s, a) \leftarrow \text{self}.Q(s, a)$ (action-value para cada (s, a));

$\pi \leftarrow \text{self}.\pi$ (política ϵ-greedy sobre Q);

$s \leftarrow$ self.s (estado anterior);

$a' \leftarrow$ acción dada por π en s';

$\text{self}.Q(s, a) \leftarrow Q(s, a) + \alpha\Big(r + \gamma Q(s', a') - Q(s, a)\Big)$;

$\text{self}.\pi \leftarrow$ mejorar $\pi(s)$ con ϵ-greedy sobre Q;

$\text{self}.s \leftarrow s'$;

SARSA vs. Q-learning (1/2)

SARSA:

$$Q(s,a) \leftarrow Q(s,a) + \alpha\Big(r + \gamma Q(s',a') - Q(s,a)\Big)$$

Improve $\pi(s)$ with ϵ-greedy and Q

Q-learning:

$$Q(s,a) \leftarrow Q(s,a) + \alpha\Big(r + \gamma \max_{a'} Q(s',a') - Q(s,a)\Big)$$

Improve $\pi(s)$ with ϵ-greedy and Q

SARSA vs. Q-learning (2/2)

Algorithm 3: SARSA agent (update rule)

Data: Una acción a, un estado s' y una recompensa r
$Q(s,a) \leftarrow$ self.$Q(s,a)$ (action-value para cada (s,a));
$\pi \leftarrow$ self.π (política ϵ-greedy sobre Q);
$s \leftarrow$ self.s (estado anterior);
$a' \leftarrow$ acción dada por π en s';
self.$Q(s,a) \leftarrow Q(s,a) + \alpha\Big(r + \gamma Q(s',a') - Q(s,a)\Big)$;
self.$\pi \leftarrow$ mejorar $\pi(s)$ con ϵ-greedy sobre Q;
self.$s \leftarrow s'$;

Algorithm 5: Q-learning agent

Data: Una acción a, un estado s' y una recompensa r
$Q(s,a) \leftarrow$ self.$Q(s,a)$ (action-value para cada (s,a));
$\pi \leftarrow$ self.π (política ϵ-greedy sobre Q);
$s \leftarrow$ self.s (estado anterior);
self.$Q(s,a) \leftarrow Q(s,a) + \alpha\Big(r + \gamma \operatorname{máx}_{a'} Q(s',a') - Q(s,a)\Big)$;
self.$\pi \leftarrow$ actualizar $\pi(s)$ con ϵ-greedy sobre Q;
self.$s \leftarrow s'$;

The Cliff Problem

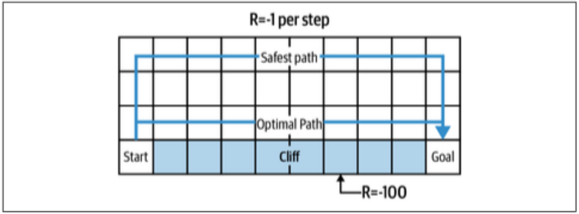


Figure 3-1. A depiction of the grid environment with a cliff along one side.[1]

Optimal Policy — SARSA vs. Q-learning

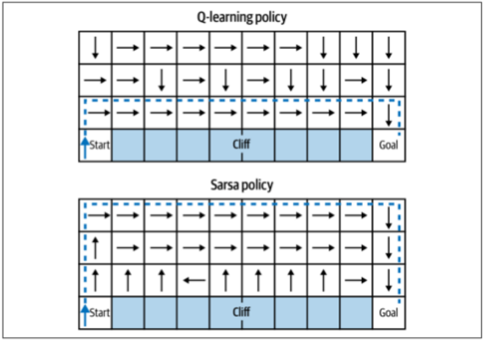


Figure 3-3. The policies derived by Q-learning and SARSA agents. Q-learning tends to prefer the optimal route. SARSA prefers the safe route.

Q Table

	Left	Right	Up	Down
20	-11	-10	-11	-100
21	-10	-9	-10	-100

What is the updated value of $q(20, \text{Right})$

if $a' = \text{Down}$, using:

- SARSA?
- Q-learning?

Utility — SARSA vs. Q-learning

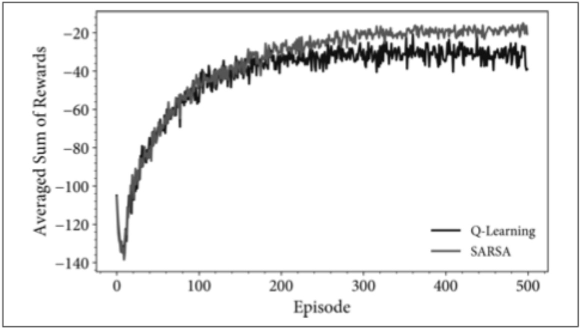


Figure 3-2. A comparison of Q-learning against SARSA for a simple grid problem. The agents were trained upon the environment in Figure 3-1. I used $\gamma \doteq 1.0$, $\epsilon \doteq 0.1$, and $\alpha \doteq 0.5$. The rewards for each episode were captured and averaged over 100 trials.

Takeaway

In this session, you learned:

- To analyze reinforcement learning as the combination of estimating long-term reward, correcting estimation error, and balancing exploit vs. explore.
- The temporal difference methods SARSA and Q-learning.