Model Predictive Control : Exercise 3

Prob 1 | Compute invariant sets

Consider the discrete-time linear time-invariant system defined by

$$x^+ = Ax$$

with

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\beta \qquad \alpha = \pi/6 \qquad \beta = 0.8$$

and state constraint set

$$X = \{x \mid Hx \le h\} \qquad H = \begin{bmatrix} \cos(\pi/3) & \sin(\pi/3) \\ -\cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \\ -\sin(\pi/3) & \cos(\pi/3) \end{bmatrix} \qquad h = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 5 \end{bmatrix}$$

Tasks:

- Compute the largest invariant set $\mathcal{O}_\infty$ of the constrained system such that $\mathcal{O}_\infty \subseteq X$
- Plot state trajectories of the system for various $x_0 \in X$
- Plot the maximum invariant set $\mathcal{O}_\infty$
- Plot a trajectory where $x_0 \in X \backslash \mathcal{O}_\infty$ and there exists an $x_i \notin X$
- Plot several trajectories starting from various states within $\mathcal{O}_\infty$, demonstrating that the entire trajectory $\{x_i\}$ remains within $\mathcal{O}_\infty$

Prob 2 | Compute Controlled Invariant Sets

Consider the discrete-time LTI system defined by

$$x^+ = Ax + Bu$$

with

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\beta \qquad \alpha = \pi/6 \qquad \beta = 0.8 \qquad B = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

and state and input constraints $(x, u) \in X \times U$

$$X = \{x \,|\, Hx \leq h\} \qquad H = \begin{bmatrix} \cos(\pi/3) & \sin(\pi/3) \\ -\cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \\ -\sin(\pi/3) & \cos(\pi/3) \end{bmatrix} \qquad h = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 5 \end{bmatrix}$$

$$U = \{u \,|\, -0.5 \leq u \leq 0.5\}$$

Tasks:

- Compute the maximum controlled invariant set $\mathcal{C}_\infty$ of the constrained system such that $\mathcal{C}_\infty \subseteq X$
- Compute the optimal LQR controller K for $Q = I$, $R = 1$. Define the stable system $x^+ = (A + BK)x$, with constraints $x \in X \cap KU$, and compute its maximum invariant set $\mathcal{O}_\infty$.
- Plot the maximum controlled invariant set $\mathcal{C}_\infty$
- Plot the maximum invariant set $\mathcal{O}_\infty$ for the closed-loop system $(A + BK)x$
- Compare $\mathcal{O}_\infty$ to $\mathcal{C}_\infty$. Which would you expect to be bigger? Why?

Hints In both exercises, your goal will be to implement the following algorithm: (The only difference between the two exercises is the pre operator.)

Require: f, X
Ensure: $\mathcal{O}_\infty$

```
Ω_0 ← X
loop
   Ω_{i+1} ← pre Ω_i ∩ Ω_i
   if Ω_{i+1} = Ω_i then
      return O_∞ = Ω_i
   end if
end loop
```

Some matlab hints:

- The function $P = Polyhedron(H, h)$ creates the polytope $\{x \mid Hx \leq h\}$
- Given two polytopes $P1$ and $P2$, the operator $P1 == P2$ returns true if the polytopes are the same, and false otherwise
- Given a polytope P the function $A = P.A; b = P.b$; returns A and b such that $P = \{x \mid Ax \leq b\}$
- To plot a polytope P use $plot(P)$ or $P.plot$
- You can plot several polytopes using $plot([P1 P2 P3])$
- The function projection computes the projection of a polytope:

  $$P = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid H \begin{bmatrix} x \\ y \end{bmatrix} \leq b\right\}$$

  The projection of P onto x is

  $$P_x = \{x \mid \exists y \, (x, y) \in P\} = \{x \mid Ex \leq e\}$$

  You can compute P_x with the matlab command: $Px = projection(P, 1:n);$
- The matlab command $dlqr$ defines the feedback matrix K to be $-K$ as in the notes. i.e., $A - BK$ is a stable matrix