04_ Introduction to Constrained Systems

Invariance

Conceptions

::: definition
Invariance:

Region in which an **autonomous** system will satisfy the constraints
for all time
:::

::: definition
Positive Invariant Set:

A set \mathcal{O} is said to be a **positive invariant set** for the
autonomous system x_{i+1}=f\left(x_{i}\right) if
x_{i} \in \mathcal{O} \Rightarrow x_{i} \in \mathcal{O}, \forall i \in\{0,1, \ldots\}
:::

Notes: The invariant set provides a set of **initial states** from
which the trajectory will **never violate** the system constraints.

::: definition
Maximal Positive Invariant Set The set
\mathcal{O}_{\infty} \subset \mathbb{X} is the **maximal invariant**
set with respect to \mathbb{X} if 0 \in \mathcal{O}_{\infty}.

$\mathcal{O}_{\infty}$ is invariant and $\mathcal{O}_{\infty}$ contains all invariant sets that contain the origin.
:::

Notes: The maximal invariant set is the set of all states for which the system will remain feasible if it starts in $\mathcal{O}_{\infty}$.

::: definition
Preset: Given a set S and the dynamic system $x^{+}=f(x)$, the pre-set of S is the set of states that evolve into the target set S in one time step: $\operatorname{pre}(S):=\{x \mid f(x) \in S\}$
:::

Conditions of Invariant Set

A set $\mathcal{O}$ is a positive invariant set **if and only if** $\mathcal{O} \subset \text{pre}(\mathcal{O})$

Computation of Invariant Set

Notes: The algorithm generates the set sequence $\left\{\Omega_{i}\right\}$ satisfying $\Omega_{i+1} \subseteq \Omega_{i}$ for all $i \in \mathbb{N}$ and it terminates when $\Omega_{i+1}=\Omega_{i}$ so that Ω_{i} is the maximal positive invariant set $\mathcal{O}_{\infty}$ for $x^{+}=f(x)$.

Control Invariance

Conceptions

::: definition
Controlled Invariance:

Region for which there **exists** a controller so that the system satisfies the constraints for all time
:::

::: definition
Control Invariant Set: A set $\mathcal{C} \subseteq \mathbb{X}$ is said to be a control invariant set if
$x_{i} \in \mathcal{C} \quad \Rightarrow \quad \exists u_{i} \in \mathbb{U} \quad \text{such that } f\left(x_{i}, u_{i}\right) \in \mathcal{C} \quad \text{for all } i \in \mathbb{N}^{+}$
:::

::: definition
Maximal Control Invariant Set:

The set $\mathcal{C}_{\infty}$ is said to be the maximal control invariant set for the system $x^{+}=f(x, u)$ subject to the constraints $(x, u) \in \mathbb{X} \times \mathbb{U}$ if it is control invariant and contains all control invariant sets contained in $\mathbb{X}$.
:::

::: definition
Preset:

$\operatorname{pre}(S):=\{x \mid \exists u \in \mathbb{U} \text { s.t. } f(x, u) \in S\}$
:::

Conditions of Controlled Invariant Set

A set $\mathcal{C}$ is a positive invariant set **if and only if** $\mathcal{C} \subset \text{pre}(\mathcal{C})$

Computation of Controlled Invariant Set

Control Invariant Set and Control Law

Polytopes and Polytopic Computation

Conceptions

::: definition
Convex Hull For any subset S of $\mathbb{R}^{d}$, the convex hull conv (S) of S is the intersection of all convex sets containing S.
Since the intersection of two convex sets is convex, it is the smallest convex set containing S.
:::

::: theorem
Minkowski-Weyl Theorem For $P \subseteq \mathbb{R}^{d}$, the following statements are equivalent:

- **P is a polytope, i.e., P is bounded and there exist $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^{m}$ such that $P=\{x \mid A x \leq b\}$**
- **P is finitely generated, i.e., there exist a finite set of vectors $\left\{v_{i}\right\}$ such that $P=\operatorname{conv}\left(\left\{v_{1}, \ldots, v_{s}\right\}\right)$
:::**

::: definition
The intersection $I \subseteq \mathbb{R}^{n}$ of sets $S \subseteq \mathbb{R}^{n}$ and $T \subseteq \mathbb{R}^{n}$ is $I=S \cap T:=\{x \mid x \in S \text { and } x \in T\}$
:::

Notes: Intersection of polytopes in inequality form is easy:
$$\begin{aligned}
&S:=\{x \mid C x \leq c\} \\
&T:=\{x \mid D x \leq d\}
\end{aligned} \quad S \cap T=\left\{x \mid\left[\begin{array}{l}
C \\
D$$

\end{array}\right] x \leq\left[\begin{array}{l}
c \\
d
\end{array}\right]\right\}

::: definition
polytopic projection Given a polytope
P=\left\{(x, y) \in \mathbb{R}^{n} \times \mathbb{R}^{d} \mid C x+D y \leq b\right\},
find a matrix E and vector e, such that the polytope
P_{\pi}=\{x \mid E x \leq e\}=\{x \mid \exists y,(x, y) \in P\}
:::

Polytopes in MPC

Input Saturation

\begin{gathered}
u_{l b} \leq u \leq u^{u b} \\
\Downarrow \\
{\left[\begin{array}{c}
1 \\
-1
\end{array}\right] u \leq\left[\begin{array}{c}
u^{u b} \\
-u_{l b}
\end{array}\right]}
\end{gathered}

Magnitude Constraints

\begin{gathered}
\|C x\|_{\infty} \leq \alpha \\
\Downarrow \\
{\left[\begin{array}{c}
C \\
-C
\end{array}\right] x \leq 1 \alpha}
\end{gathered}

Rate Constraints

\begin{gathered}
\left\|x_{i}-x_{i+1}\right\|_{\infty} \leq \alpha \\
\Downarrow \\
{\left[\begin{array}{cc}
1 & -l \\
-1 & 1
\end{array}\right]\left(\begin{array}{c}
x_{i} \\
x_{i+1}

\end{array}\right) \leq \mathbf{1} \alpha}
\end{gathered}

Integral Constraints

$$
\begin{gathered}
\|x\|_{1} \leq \alpha \\
\Downarrow \\
x \in \operatorname{conv}\left(e_{i} \alpha\right)
\end{gathered}
$$

Computation of Pre-Set

Autonomous Systems

::: method
If $S:=\{x \mid F x \leq f\}$, then
$\operatorname{pre}(S)=\{x \mid F A x \leq f\}$
:::

Controlled Systems

Consider the system $x^{+}=A x+B u$ under the constraints $u \in \mathbb{U}:=\{u \mid G u \leq g\}$ and the set $S:=\{x \mid F x \leq f\}$.

$$
\begin{aligned}
\operatorname{pre}(S) &=\{x \mid \exists u \in \mathbb{U}, A x+B u \in S\} \\
&=\{x \mid \exists u \in \mathbb{U}, F A x+F B u \leq f\} \\
&=\left\{x \mid \exists u,\left[\begin{array}{rr}
F A & F B \\
0 & G
\end{array}\right]\left(\begin{array}{l}
x \\
u
\end{array}\right) \leq\left[\begin{array}{l}
f \\
g
\end{array}\right]\right\}
\end{aligned}
$$

Notes: this is actually a projection operation.

Equality Test

One important problem is how to check whether two set are the same? i.e. Is $P:=\{x \mid C x \leq c\}$ contained in $Q:=\{x \mid D x \leq d\}$?. The statement is true if $P \subset \{x|D_ix<d_i\}$ for each row D_i of D.

Define the support function of the set P:

$$
\begin{aligned}
h_{P}\left(D_{i}\right):=& \max _{x} D_{i} x \\
& \text { s.t. } C x \leq c
\end{aligned}
$$

if $h_p(D_1) \le d_1$, then it is true, if not , it is false.

Notes: Do not try to translate to straight line representation. It can be directly understood by the definition of Q, if the false case happen, it means at least one of the constrain in Q is violated, because we use Q as $\le$ format.

Convergence Discussion

Another problem is: Does the invariant set algorithm guarantees finite step termination?

In general, **no**! The boundary of the a maximal invariant set can be curvy, which needs infinite many half-space to define. In practice, to save memory and to ensure efficiency, the algorithm stop up to some **specific criteria or we use simpler convex set(i.e. box, ellipsoid)** to represent a smaller forward invariant set.

Ellipsoids and Invariance

Summary

This chapter introduce the content of invariant set and controlled invariant set and the method to compute them.