01_Modeling and Analysis of NCS

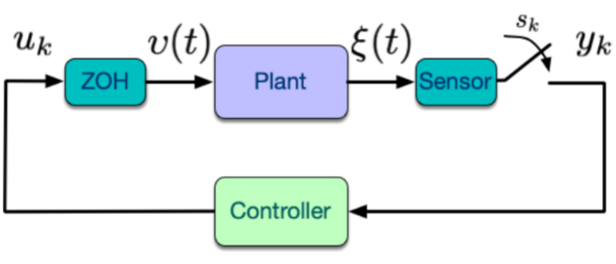
Summary

For a constant sample interval sampled-data control system, we can obtain its model:

$$\begin{aligned} x_{k+1} &= e^{Ah}x_k + \int_0^h e^{As} B ds u_k \\ &=: F(h)x_k + G(h)u_k \end{aligned}$$

Because of the property that **the inter-sample gain is bounded** we can use the GES theorem of discrete-time model: **A discrete-time system is stable iff** $\rho(\bar{A}) < 1$. The stability of a sampled-data system depend on the sample interval.

1. Sampled-data Control Systems



1.1. Assumption

- constant sampling interval h
- full state measurement: $y(k) = x(k) := \epsilon(kh)$
- ZOH mechanism

1.2. Model

$$\begin{aligned} \dot{\xi}(t) &= A\xi(t) + Bv(t) \quad && \forall t \in \mathbb{R}^+ \\ v(t) &= u_k && \text{for } t \in [s_k, s_{k+1}) \end{aligned}$$

where ϵ is trajectory, x_k is points

so we have:

$$\xi(t) = e^{A(t-t_0)}\mathcal{E}(t_0) + \int_{t_0}^{t} e^{A(t-s)}Bv(s)ds$$

and

$$x_{k+1} := \xi(s_{k+1}) = e^{Ah}\xi(s_k) + \int_{s_k}^{s_{k+1}} e^{A(s_{k+1}-s)}Bu_k ds$$

It can be denoted as:

$$\begin{aligned} x_{k+1} &= e^{Ah}x_k + \int_0^h e^{As}Bdsu_k \\ &=: F(h)x_k + G(h)u_k \end{aligned}$$

If A is invertible

$$\int_0^h e^{As}ds = \left(e^{Ah} - 1\right)A^{-1}$$

If A is not invertible, we can use Jordan's Decomposition

1.3. Globally Exponentially Stable (GES) of Discrete Linear Systems

1.3.1 Definition

The **origin** of the linear discrete-time system

$$x_{k+1} = \bar{A} x_k$$

is said to be a **globally exponentially stable (GES)** fixed point if there exist $c > 0$ and $\rho \in (0,1)$such that

$$\|x_k\| \leq c\rho^{k-k_0} \|x_0\|$$

That means:

> *For each start point, this system will get close to the origin with exponentially speed*

1.3.2. Theorem

The origin of the discrete-time system $x_{k+1} = \bar{A} x_k$ is <u>**GES**</u> **if and only if**

$$\rho(\bar{A}) = \max\left\{ \left|\lambda_1(\bar{A})\right|, \left|\lambda_2(\bar{A})\right|, \ldots, \left|\lambda_n(\bar{A})\right| \right\} < 1$$

that is: **the spectral radius of $\bar{A}$ is smaller than one**

1.4. Sampled Data System's Stability

For system

$$x_{k+1} = F(h) x_k + G(h) u_k$$

we consider a static state-feedback control law $u_k = -\bar{K} x_k$, then we will have

$$\begin{aligned} x_{k+1} &= (F(h) - G(h)\bar{K}) x_k \\ \text{with} \quad & (F(h) - G(h)\bar{K}) = e^{Ah} - \int_0^h e^{As} B\bar{K} ds \end{aligned}$$

The origin is GES iff:

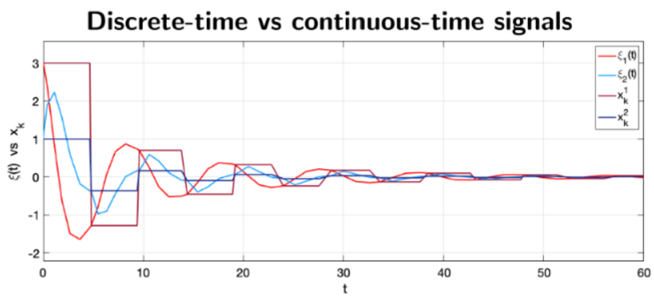
$$\rho(F(h) - G(h)\bar{K}) < 1$$

That means

> *For a sampled-data system, its stability depend on the sample period h*

1.5. Sampled Data Behavior

Up to now, we ignore the inter-sample behavior. However, these behaviors is important



During each interval $t \in [s_k, s_{k+1})$

$$\xi(t) = \xi(s_k + \Delta t) = \left(e^{A\Delta t} - \int_0^{\Delta t} e^{As} B ds \bar{K}\right) x_k =: \tilde{F}_{cl}(\Delta t) x_k,$$

$$\tilde{F}_{cl}(\Delta t) := \left(e^{A\Delta t} - \int_0^{\Delta t} e^{As} B ds \bar{K}\right)$$

where $\Delta t \in [0, h)$

Relation between discrete-time model stable and sampled-data system stable

stability of exact discrete-time model $\Rightarrow$ stability of the sampled-data system

Proof

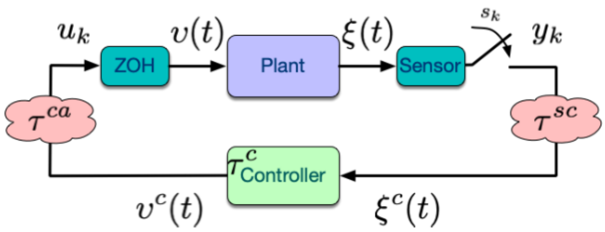
Because $e^{A\Delta t}$, $e^{As}B$ is bounded, so we have

$$\Delta t < \infty \Rightarrow \left\|\tilde{F}_{cl}(\Delta t)\right\| \leq M < \infty$$

That is

> *the inter-sample gain is bounded*

2. Networked Control Systems with Delays



2.1. Assumptions

- Sensor-to-controller delay τ^{sc}

$$\xi^c(t) = \xi(s_k), \text{ for } t \in [s_k + \tau^{sc}, s_{k+1} + \tau^{sc})$$

- Computational Delay τ^c

$$v^c(t) = \kappa(\xi^c(t - \tau^c))$$

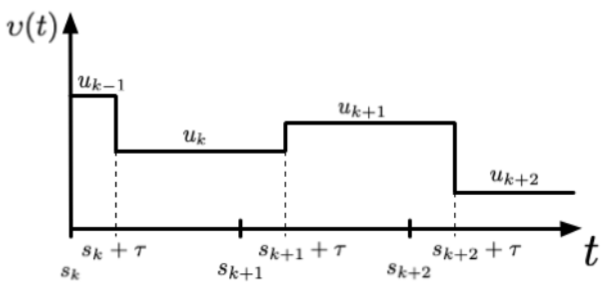
- Controller-to-actuator delay τ^{ac}

$$v(t) = v^c(t - \tau^{ca})$$

- **Constant and Small total** delay:

$$\tau = \tau^{SC} + \tau^{C} + \tau^{Ca}$$
$$0 \leq \tau \leq h$$

2.2. Modelling



For time interval $t \in [s_k, s_{k+1})$

$$\xi\left(s_{k+1}\right)=e^{A(h)} \xi\left(s_k\right)+\int_{s_k}^{s_{k+1}} e^{A(t-s)} B v(s) d s$$
$$v(t)=u_{k-1} \quad \text{for} \quad t \in\left[s_k, s_k+\tau\right)$$
$$v(t)=u_k \quad \text{for} \quad t \in\left[s_k+\tau, s_{k+1}\right)$$

We can rewrite it as:

$$x_{k+1}=e^{A(h)} x_k+\int_{h-\tau}^h e^{A s} B d s u_{k-1}+\int_0^{h-\tau} e^{A s} B d s u_k$$

then rewrite as:

$$x_{k+1}=F_x(h) x_k+F_u(h, \tau) u_{k-1}+G_1(h, \tau) u_k$$

where

$$F_x(h):=e^{A(h)}, F_u(h, \tau):=\int_{h-\tau}^h e^{A s} B d s, \text{ and } G_1(h, \tau):=\int_0^{h-\tau} e^{A s} B d s$$

2.3. State Augmentation Modelling

with **extended state vector** $x_k^e=\left[\begin{array}{ll}x_k^T & u_{k-1}^T\end{array}\right]$

$$x_{k+1}^e=F(h, \tau) x_k^e+G(h, \tau) u_k$$

where

$$F(h, \tau):=\left[\begin{array}{cc}F_x(h) & F_u(h, \tau) \\ 0 & 0\end{array}\right], G(h, \tau):=\left[\begin{array}{c}G_1(h, \tau) \\ I\end{array}\right]$$

2.4. Stability Analysis

For a given system we have fixed h and τ, so the rule for Discrete-Time Stability is still available.

We consider a **static state-feedback control law**

$$u_k=-K x_k^e=-\bar{K} x_k-K_u u_{k-1} .$$

Then we have

$$x_{k+1}^e=(F(h, \tau)-G(h, \tau) K) x_k^e$$

The **origin** of the discrete-time closed-loop is a **GES** fixed point **iff**

$$\rho(F(h,\tau) - G(h,\tau)K) < 1$$

> *Stability depends on h and τ*

The controller $K = [\bar{K} \quad K_u]$ is truly "static" iff $K_u = 0$.

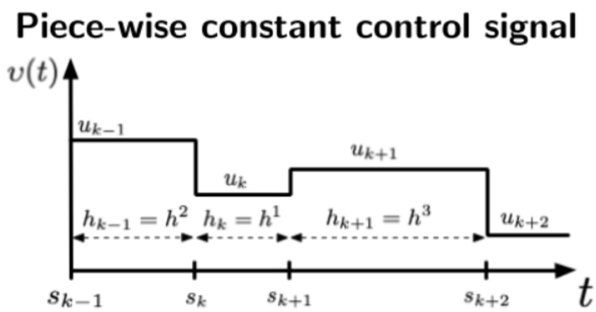
3. NCS with varying sampling intervals

3.1. Assumptions

- Time varying sampling intervals: $s_{k+1} = \sum_{i=0}^{k} h_i$
- Non-constant sampling interval: $h_k = s_{k+1} - s_k$
- $h_k \in \{h^1, \ldots, h'\}, \forall k$
- ZOH

> *Piece-wise constant control signal*

3.2. Modelling



Different from previous models, now we have a **Linear Time-Varing Discrete Time Systems**

$$x_{k+1} = e^{Ah_k}x_k + \int_0^{h_k} e^{As}Bdsu_k$$
$$=: F(h_k)x_k + G(h_k)u_k$$

> *Exact discrete-time system matrices depend on varying sampling interval h_k*

Time-varying vs Uncertain but constant

- **Time-varying**: intervals vary but belongs to a certain set
- **Uncertain but constant**: many choice of interval but we do not know, however, after start, each interval are constant

3.3. Stability Analysis

Uncertain but Constant Sampling Interval

$$\rho\left(F\left(h^i\right)-G\left(h^i\right) \bar{K}\right)<1, \forall h^i \in \mathcal{H} \Rightarrow \text { origin GES. }$$

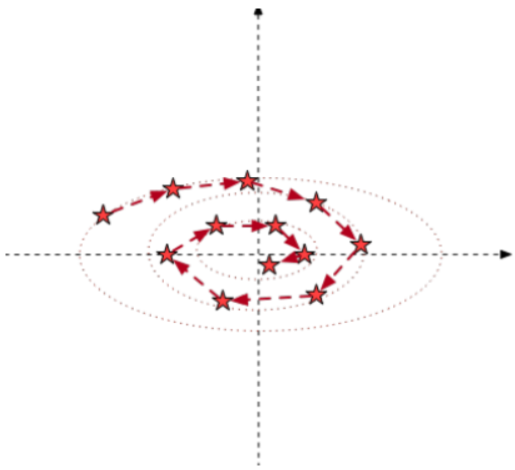
Time Varying interval

$$\rho\left(F\left(h_k\right)-G\left(h_k\right) \bar{K}\right)<1, \forall k$$

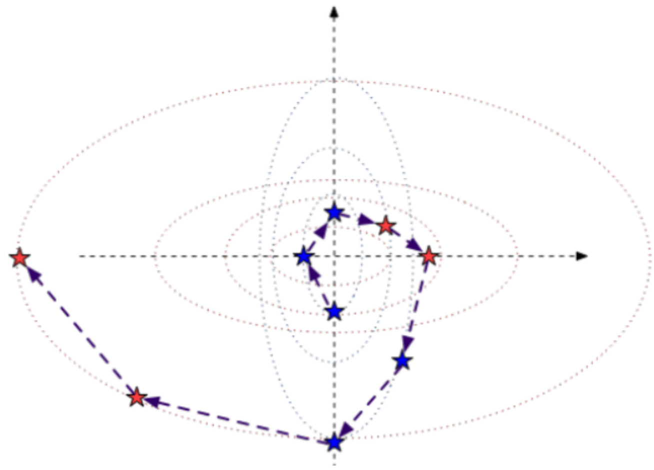
is **Neither Sufficient Nor Necessary**

<u>Example</u>

For a system, if with uncertain but constant sampling interval, it can be:



However, after change it to a Switching System



4. Stability Analysis for non-linear discrete time systems

4.1. Framework

a Discrete-time non-linear time-varying system:

$$x(k+1) = f(x(k), k), \forall k$$

with $x_k \in R^n$ and the origin is a fixed point, i.e.

$$0 = f(0, k), \quad \forall k$$

4.2. Stability Definitions

Stability

The origin of system is said to be **stable** if for every $\epsilon > 0$ and $k_0 \geq 0$ there exists a function δ of ϵ and k_0, i.e. $\delta = \bar{\delta}(\epsilon, k_0)$, such that:

$$\|x_{k0}\| < \bar{\delta}\,(\epsilon, k_0) \Rightarrow \|x_k\| < \epsilon \quad \forall k \geq k_0$$

> *For a given standard (ϵ) and a given state, we can always find a value $\bar{\delta}(\epsilon, k_0)$ that if the given state is bounded by the value, the following trajectory is bounded by the standard*
>
> *a system pass some $\delta - ball$ around the equilibrium at time step k_0 will not leave a $\epsilon - ball$ around the equilibrium*

Uniform Stability

The origin of system is said to be **uniformly stable** if for every $\epsilon > 0$ and $k_0 \geq 0$ there exists a $\delta = \bar{\delta}(\epsilon)$ such that:

$$\|x_{k0}\| < \bar{\delta}(\epsilon) \Rightarrow \|x_k\| < \epsilon \quad \forall k \geq k_0$$

Uniformity $\Rightarrow$ δ is independent of the initial time k_0

> *For a given standard (ϵ), we can always find a value $\bar{\delta}(\epsilon)$ that* ***no matter from which point, as long as this point is bounded by the value****, the following trajectory is bounded by the standard*
>
> *a system starting in some $\delta - ball$ around the equilibrium will not leave a $\epsilon - ball$ around the equilibrium*

Global Exponential Stability

The origin of system (1) is said to be said **globally uniformly asymptotically** stable if there exists an exponential function $\beta \in \mathcal{KL}$, i.e. $\beta(\|x\|, k) = c\rho^k\|x\|$ with $c > 0, \rho \in (0, 1)$, such that:

$$\|x_k\| \leq \beta(\|x_{k0}\|, k), \forall k \geq k_0$$

4.3. Attractivity Definitions

Attractivity

The origin of system is said to be **attractive** if for every $k_0 \geq 0$ there exists a scalar η function of k_0, i.e. $\eta = \bar{\eta}(k_0) > 0$, such that

$$\|x_{k0}\| < \bar{\eta}(k_0) \Rightarrow \|x_k\| \rightarrow 0; \text{ as } k \rightarrow \infty$$

> *For each start point, we can find a bound that is the point meet the bound, the trajectory will be attracted by the origin*

Uniform Attractivity

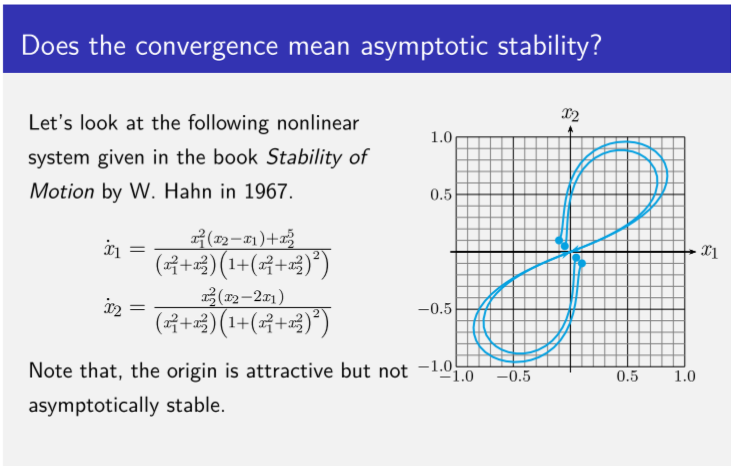
The origin of system is said to be **uniformly attractive** if for every $k_0 \geq 0$ there exists a scalar $\eta > 0$, such that

$$\|x_{k0}\| < \eta \Rightarrow \|x_k\| \to 0; \text{ as } k \to \infty, \text{ uniformly in } k_0$$

uniformity $\Rightarrow$ independent with k_0

> *For all points, we can find a uniform bound that if the points meet this bound, the trajectory will be attracted by the origin*

Attractive but not stable



4.4. Asymptotic Stability

Local Asymptotic Stability

The origin of system is said to be said **locally asymptotically stable** if it is **both stable and attractive**

Local Uniform Asymptotic Stability

The origin of system is said to be said **locally asymptotically uniformly stable** if it is **both uniformly stable and uniformly attractive.**

Global Uniform Asymptotic Stability

The origin of system is said to be said **globally uniformly asymptotically stable** if there exists a function $\beta \in \mathcal{KL}$ such that:

$$\|x_k\| \leq \beta\left(\|x_{k0}\|, k\right), \forall k \geq k_0$$

> - *The trajectory should always below a $\mathcal{KL}$ function*

4.5. Comparison Functions

$\mathcal{K}$

A function $\gamma : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, is of class $\mathcal{K}$ if

- it is continuous,
- strictly increasing
- $\gamma(0) = 0$.

$\mathcal{K}_\infty$

A function $\gamma \in \mathcal{K}$ is of class $\mathcal{K}_\infty$ if additionally it satisfies:

- $\gamma(s) \rightarrow \infty$ as $s \rightarrow \infty$

A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is of class $\mathcal{KL}$ if

- $\beta(\cdot, \tau)$ is of class $\mathcal{K}$ for each $\tau \geq 0$
- $\beta(s, \cdot)$ is monotonically decreasing to zero for each $s \geq 0 : \beta(s, \tau) \rightarrow 0$ as $\tau \rightarrow \infty$.

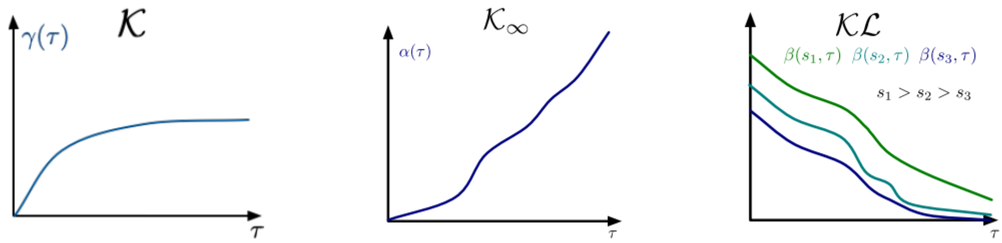
> *For fixed τ, when s increase, β increase*

> *For fixed s, when τ increase, β decrease*

Exponential

A class $\mathcal{KL}$ function β is called **exponential** if

$$\beta(s,\tau) \le \sigma s e^{-c\tau} \text{where} \sigma > 0, c > 0$$



4.6. Stability Characterization (For discrete-time systems)

Theorem for Uniform Asymptotic Stability

The origin of system is **local uniformly asymptotically stable** if **there exists** a function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_0^+$, a scalar $r > 0$, and $\mathcal{K}$ functions α_1, α_2, and α_3 such that:

$$\begin{aligned}
&\alpha_1(\|x\|) \le V(x,k) \le \alpha_2(\|x\|), \forall k \ge k_0, \forall x \in B_r \\
&V(x_{k+1}, k+1) - V(x_k, k) \le -\alpha_3(\|x_k\|), \forall k \ge k_0, \forall x \in B_r \\
&\text{ where } B_r = \{x \in \mathbb{R}^n \mid \|x\| \le r\}
\end{aligned}$$

> - *Can be regarded as energy decrease*
> - *bounded by two $\mathcal{K}$ function*
> - *decrease and decrease more and more rapidly*

Theorem for Global Uniform Asymptotic Stability

The origin of system is globally uniformly asymptotically stable if there exists a function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_0^+$, and $\mathcal{K}$ functions α_1, α_2, and
α_3 such that:

$$
\begin{aligned}
&\alpha_1(\|x\|) \leq V(x,k) \leq \alpha_2(\|x\|), \forall k \geq k_0, \forall x \in \mathbb{R}^n \\
&V\left(x_{k+1}, k+1\right) - V\left(x_k, k\right) \leq -\alpha_3\left(\left\|x_k\right\|\right), \forall k \geq k_0, \forall x \in R^n \\
&V(x,k) \text{is radially unbounded, i.e. } V(x,r) \rightarrow \infty \text{ as } \|x\| \rightarrow \infty \text{ uniformly in } k.
\end{aligned}
$$

Theorem for Global Exponential Stability

The origin of system (1) is globally exponentially stable if there exists a function $V: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}_0^+$, constants $\alpha_1, \alpha_2, \alpha_3 > 0$, and $p > 1$ such that:

$$
\begin{aligned}
&\alpha_1\|x\|^p \leq V(x,k) \leq \alpha_2\|x\|^p, \forall k \geq k_0, \forall x \in \mathbb{R}^n \\
&V\left(x_{k+1}, k+1\right) - V\left(x_k, k\right) \leq -\alpha_3\left\|x_k\right\|^p, \forall k \geq k_0, \forall x \in R^n
\end{aligned}
$$

4.7. LTI Systems Stability

Consider now a Linear Time-Invariant (LTI) system:

$$
x_{k+1} = A x_k
$$

and $V(x) = x^T P x$, with $P = P^T > 0$ as a Lyapunov candidate function. Observe:

$$
\begin{aligned}
&\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2 \\
&V\left(x_{k+1}\right) - V\left(x_k\right) = x_k^T\left(A^T P A - P\right) x_k \\
&\left(A^T P A - P\right) \leq -Q, Q > 0 \Rightarrow V\left(x_{k+1}\right) - V\left(x_k\right) \leq -x_k^T Q x_k \leq -\lambda_{\min}(Q)\|x\|^2, \\
&\text{i.e. GES with } p = 2.
\end{aligned}
$$

> *Above first equation:*
>
> - *we can always find a Unit Orthogonal Array to transform P to a diagonal format, and because orthogonal transformation does not change the norm of the vector, so it can be easily proved.*

And we can prove that:

A Schur [1] $\Leftrightarrow$ for each $Q > 0 \ \exists! P > 0$ such that $A^T P A - P = -Q$

> *Spectral and Lyapunov stability tests equivalence for LTI systems*

5. NCS with varying sampling intervals. LMIs for Stability

The origin of the system is **GES** if there **exist** a matrix $P = P^T > 0$ such that the following LMIs hold for all $h \in \mathcal{H}$:

$$(F(h) - G(h)\bar{K})^T P(F(h) - G(h)\bar{K}) - P \leq -Q$$

for some $Q = Q^T > 0$.

This result is based on finding a common Lyapunov function.