03_04_05_Two_Dimensional_Systems

1. Graphical Representations of Two-Dimensional Systems

Diagrams

Definition: Vector Field Diagram

Consider the autonomous two-dimensional system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

A **vector field diagram** is obtained when, for each x on a grid in the $x_1 - x_2$ plane, an **arrow is drawn from x to** $x + cf(x)$, where $c > 0$ is a scale factor such that the diagram is easy to interpret.

- Sometimes, the length of the arrows is also set to a fixed value

Definition: Trajectories

Consider the autonomous two-dimensional system

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2), \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = x_0$$

The curve that results from plotting $x_2(t)$ over $x_1(t)$ for $t \geq 0$ is called **trajectory (or orbit)** from x_0.

Definition: Phase Portrait

The **phase portrait** of an autonomous two-dimensional system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

is obtained if the **trajectories for many different initial points x_0 are shown in a single plot**. Sketching a phase portait becomes easier when it is drawn on top of a vector field diagram.

Definition: Isoclines

The **isocline** of an autonomous two-dimensional system

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2),$$

with slope $s \in \mathbb{R}$ is the curve of all $x \in \mathbb{R}^2$ that satisfy

$$f_2(x_1, x_2) = s f_1(x_1, x_2).$$

The **nullclines** are the two curves given by $f_1 = 0$ and $f_2 = 0$.

Note:

- Isocline means **equal slope**
- The nullclines divide the phase plane into regions in which the signs of $f_1$and f_2 are constant (if f is continuous). In these regions, the arrows on an isocline all point into the same direction
- **The intersections of the nullclines are the equilibria of the system**

Method: Method of Drawing Isoclines

1. **Sketch the nullclines** $f_1 = 0$ and $f_2 = 0$. Indicate the slope of f on the nullclines using equal-length arrows
2. **Sketch a few isoclines** $f_2 = s f_1$ for different slopes s. Indicate the slope of f on the isoclines using equal-length arrows
3. **Sketch trajectories and add equilibria.**

2. Linear Time Invariant Systems in Modal Coordinates

If the **eigenvalues of A are distinct and different from zero**, there exists a real invertible matrix M such that $M^{-1}AM = J_r,$ where

$$J_r = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

if the eigenvalues λ_1, λ_2 are both real. Otherwise, $\lambda_1 = \alpha + \beta j . \lambda_2 = \alpha - \beta j$

$$J_r = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

with α, β real. (This is the **real Jordan form**.)

Change of Coordinates

$$\dot{z}(t) = J_r z(t), \quad z(t) := M^{-1}\chi(t).$$

Then we will have

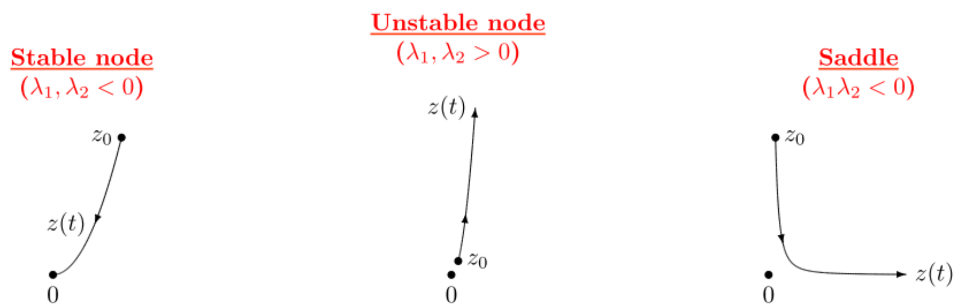
$$\dot{x}(t) = Ax(t) = MJ_r M^{-1}x(t)$$

3. Qualitative Behavior of Linear Time Invariant Systems

Two Real Eigenvalues

$$\dot{Z}(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} z(t) \Rightarrow \dot{Z}_1 = \lambda_1 Z_1, \dot{z}_2 = \lambda_2 Z_2$$

$$z_1(t) = z_1^0 e^{\lambda_1 t}, \quad z_2(t) = z_2^0 e^{\lambda_2 t}$$

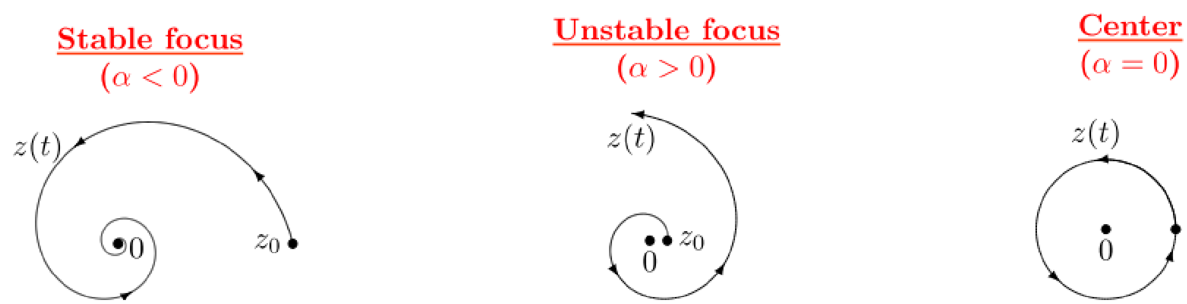


Complex Eigenvalues

We consider it in polar representation

$$\dot{r} = \alpha r, \quad \dot{\theta} = \beta.$$

$$z(t) = r_0 e^{\alpha t} \begin{bmatrix} \cos(\theta_0 + \beta t) \\ \sin(\theta_0 + \beta t) \end{bmatrix}$$



4. Classification of Nonlinear Equilibrium Points

Concepts

Definition: Hyperbolic Equilibrium Points for Linear Systems

The equilibrium point $x^* = 0$ of the linear system $\dot{x}(t) = Ax(t)$ is called **Hyperbolic** if no eigenvalue of A are zero real part.

- Hyperbolic equilibrium points do not change their type in the perturbed system $\dot{x} = (A + \Delta A)x$ if ΔA is small enough
- They are thus said to be **structurally stable**
- Nodes, focuses and saddles are hyperbolic, centers are not

Definition: Hyperbolic Equilibrium Points for Nonlinear Systems

The equilibrium point x^* is called **hyperbolic** if the linearized system has a hyperbolic equilibrium point at zero.

Note:

- If an equilibrium point of $x^* = f(x)$ is hyperbolic, the system will behave locally like its linearization around that point
- This is not always true for non-hyperbolic equilibrium points
- If the linearization has a center, nonlinear analysis is required

Theorem

An equilibrium point x^* of the nonlinear system $\dot{x} = f(x)$ is called **(un)stable node / (un)stable focus / saddle** if the linearized system

$$\dot{y} = Ay, \quad A = \left[\frac{\partial f(x)}{\partial x}\right]$$

has a (un)stable node / (un)stable focus / saddle at zero.

5. Periodic Orbits and Limit Cycles

Concepts

Definition: Oscillates and Periodic Orbits

A system **oscillates** if it has a nonconstant periodic solution

$$x(t) = x(t+T) \quad \forall t \geq t_0, \quad x(t) \not\equiv \text{ const.}$$

These are usually called **periodic or closed orbits**

Definition: Harmonic Oscillators

system has a equilibrium point as a center $\Rightarrow$ all trajectories with $z(t_0) \neq 0$ are periodic orbits

Note:

A harmonic oscillator is useless in practice: even very small pertubations can turn it into a focus

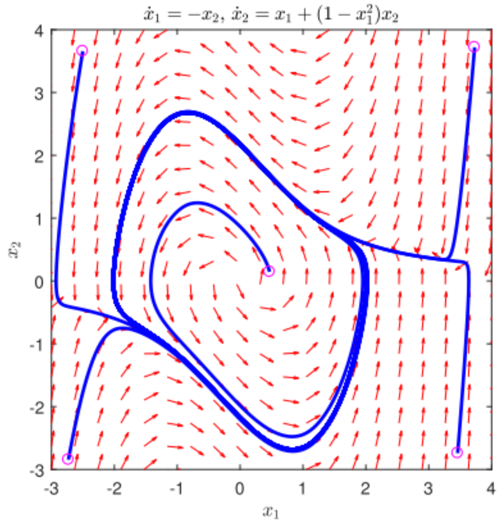
Definition: Limit Cycle

An isolated closed orbit is called a **limit cycle**

- In the case where all the neighboring trajectories approach the limit cycle as time approaches infinity, it is called a **stable or attractive limit cycle** (ω-limit cycle).
- If instead, all neighboring trajectories approach it as time approaches negative infinity, then it is an **unstable limit cycle** (α-limit cycle).
- If there is a neighboring trajectory which spirals into the limit cycle as time approaches infinity, and another one which spirals into it as time approaches negative infinity, then it is a **semi-stable limit cycle.**

Example: Van der Pol Oscillator

$$\dot{x}_1 = x_2, \dot{x}_2 = -x_1 + \epsilon\left(1 - x_1^2\right) x_2, \quad \epsilon > 0$$



$\epsilon = 1$

Existence of Periodic Orbits

Lemma: Trapping Trajectories

We consider the system

$$\dot{x} = f(x), \quad f : \mathbb{R}^2 \to \mathbb{R}^2 \text{ continuously differentiable.}$$

Let $V : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that

$$f(x) \cdot \nabla V(x) \leq 0 \quad \text{whenever} \quad V(x) = c$$

for some constant $c > 0$. Then, any trajectory starting in

$$M := \left\{x \in \mathbb{R}^2 : V(x) \leq c\right\},$$

will stay in M. That is, $x\left(t_0\right) \in M$ implies $x(t) \in M$ for all $t \geq t_0$.

Note:

$$f(x) \cdot \nabla V(x) \leq 0 \quad \text{whenever} \quad V(x) = c$$

Guarantees that $f(x)$ is pushed to the opposite direction of $V(x)$ increasing

- The lemma still holds if the condition is replaced with $f(x) \cdot \nabla V(x) \geq 0$ and the set is changed to $M' := \left\{x \in \mathbb{R}^2 : V(x) \geq c\right\}$.

> The condition means x change opposite to the $V(x)$ change

Theorem: Poincare-Bendixson Criterion

We consider the autonomous second-order system

$$\dot{x}_1 = f_1\left(x_1, x_2\right), \quad \dot{x}_2 = f_2\left(x_1, x_2\right), \quad x\left(t_0\right) = x_0,$$

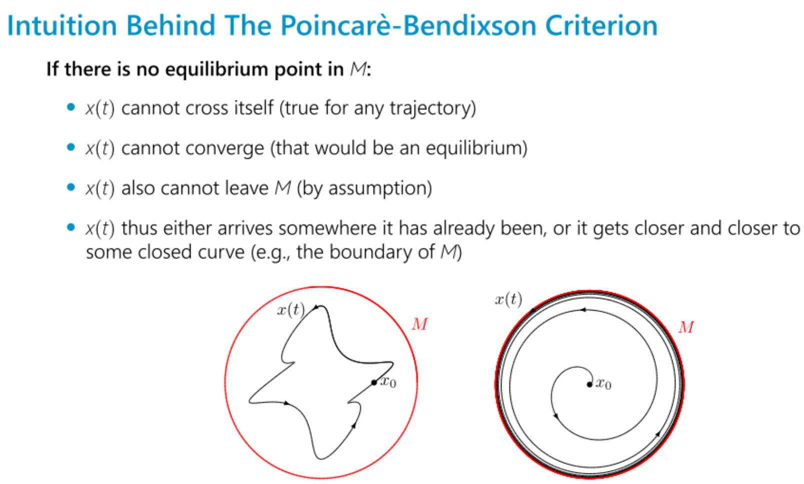
where f is continuously differentiable. Let M be a closed bounded subset of the plane such that

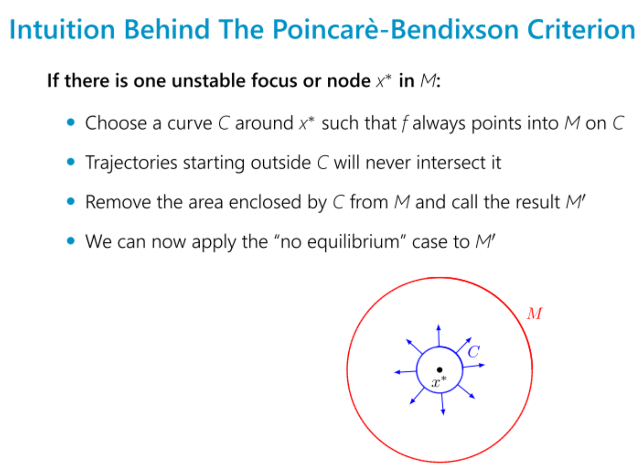
- M contains **at most one equilibrium point** that is either an unstable focus or an unstable node
- Every solution with $x_0 \in M$ satisfies $x(t) \in M$ for all $t \geq t_0$

Then, there exists an initial condition $x_0 \in M$ and a period $T > 0$ such that the solution of the system is not constant and satisfies

$$x(t+T) = x(t) \quad \forall t \geq t_0.$$

That is, M contains at least one periodic orbit





Absence of Periodic Orbits

Theorem: Bendixson Criterion

We consider the system

$$\dot{x} = f(x), \quad f: \mathbb{R}^2 \to \mathbb{R}^2 \text{ continuously differentiable.}$$

Let $D \subseteq \mathbb{R}^2$ be a simply connected domain. If either

$$\nabla \cdot f(x) > 0 \quad \forall x \in D \quad \text{or} \quad \nabla \cdot f(x) < 0 \quad \forall x \in D,$$

then there are no periodic orbits that lie entirely in D.

> The condition means, we cannot find a trajectory in which $f(x)$ is periodic, that is increasing of $f(x)$ along the closed trajectory is zero

We use a proof by contradiction. Assume that a periodic orbit $\gamma := \{x(t) : t \geq t_0\}$ would exist. Since

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \dot{x} = f(x) \quad \Longrightarrow \quad \frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2}{f_1} \quad \Longrightarrow \quad f_1 dx_2 = f_2 dx_1,$$

we find that

$$\oint_\gamma f_2(x_1, x_2)dx_1 - f_1(x_1, x_2)dx_2 = 0.$$

Let S denote the interior of γ. By Green's theorem, we would thus have

$$\iint_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 = 0.$$

However, our assumptions imply that either

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} > 0 \quad \forall x \in D \qquad \text{or} \qquad \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0 \quad \forall x \in D$$

everywhere in S. Hence, it must be

$$\iint_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 \neq 0,$$

and there can be no periodic orbit in D.

Green's Theorem

Let C **be a positively oriented**, piecewise smooth, **simple closed curve** in a plane, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and having continuous partial derivatives there, then

$$\oint_C (L dx + M dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx dy$$

where the path of integration along C is **anticlockwise**

5. Bifurcations

For bifurcations, we consider parameterized system model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mu\right), \quad \mu \in \mathbb{R}, f \text{ continuously differentiable}$$

We care interested in the **influence of small changes in the parameter μ on changes in system behavior**

Conceptions

Definition: Bifurcations

The system $\dot{x} = f(x, \mu)$ **bifurcates** at the point $\mu = \mu_0$ if the phase portrait is **qualitatively different** for $\mu < \mu_0$ and $\mu > \mu_0$.

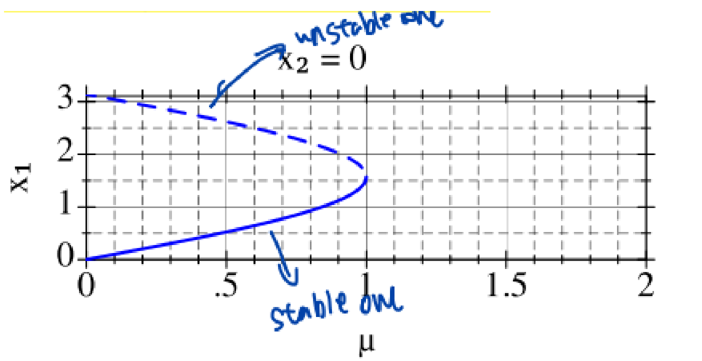
Bifurcation Diagram

A plot with the axes μ, x_1 and x_2 that shows the locations of the involved equilibrium points/limit cycles.

- Straight lines indicate stable equilibria (i.e., stable nodes, focuses or centers) or limit cycles;
- Dashed lines indicate unstable equilibria (i.e., unstable nodes, focuses or saddles) or limit cycles.

Note:

If all equilibrium points satisfy either $x_1 = 0$ or $x_2 = 0$, the bifurcation diagram becomes two dimensional



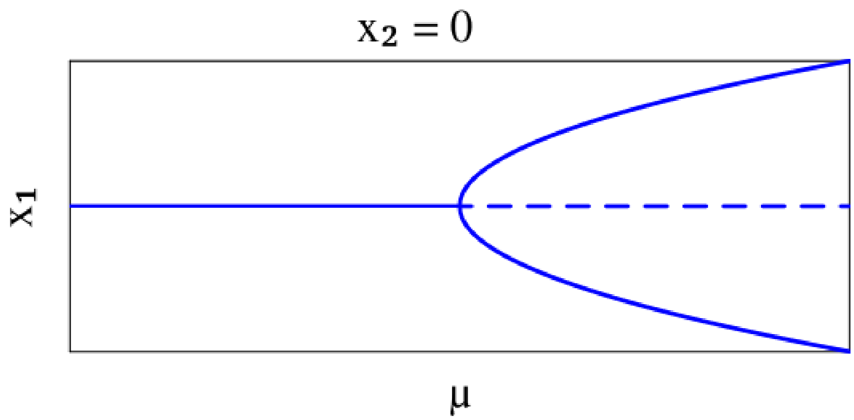
Type 1: Saddle-node Bifurcation

An unstable and a stable equilibrium collide and cancel

Pitchfork Bifurcations

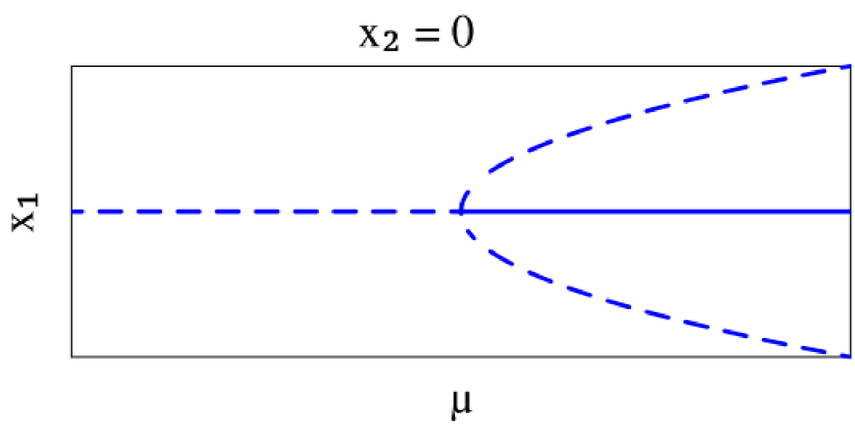
Type 2: Supercritical Pitchfork Bifurcation

A stable equilibrium becomes unstable at the bifurcation point and spawns two new stable equilibria



<u>Type 3: Subcritical Pitchfork Bifurcation</u>

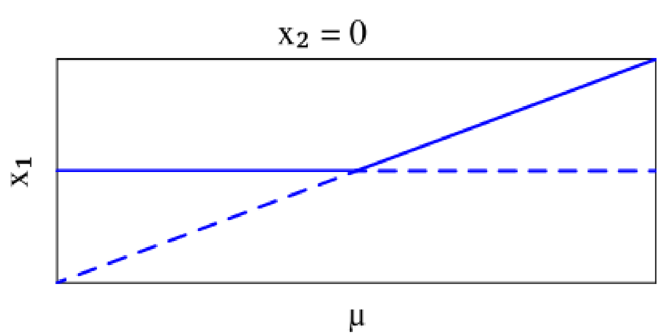
An unstable equilibrium point becomes stable at the bifurcation point and spawns two new unstable equilibria



Transcritical and Hopf Bifurcations

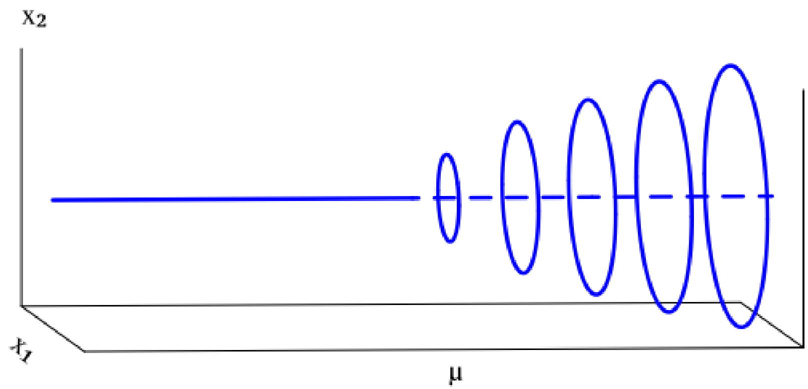
<u>Type 4: Transcritical Bifurcation</u>

A stable equilibrium and a non-stable equilibrium collide at the bifurcation point and exchange stability properties.



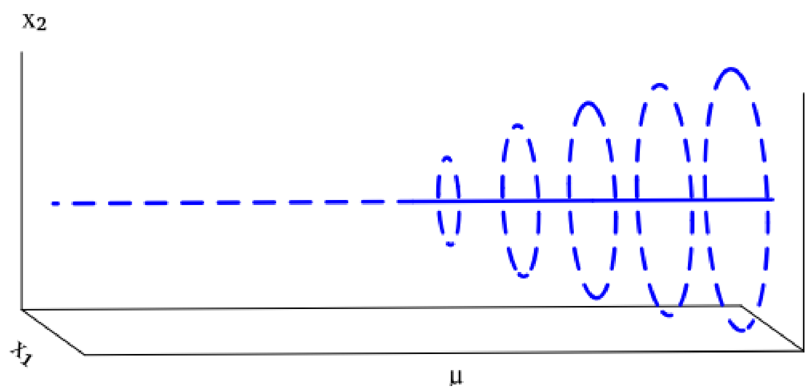
<u>Type 5: Supercritical Hopf bifurcation</u>

A stable focus turns into an unstable focus and spawns a stable limit cycle.



Type 6: Subcritical Hopf bifurcation

An unstable focus turns into a stable focus and spawns an unstable limit cycle



Finding Bifurcation Points

Assume $f(x, \mu)$ is continuously differentiable

Let $\bar{x}$ be the equilibrium of the system $\dot{x} = f(x, \mu)$ at the bifurcation point $\mu = \mu^*$. Then $\bar{x}$ **is not hyperbolic**. In particular:

- the equilibrium is a center for Hopf bifurcations
- at least one of the eigenvalues of the linearization is zero for the other bifurcation types discussed today.

Useful Result From Linear Systems Theory

Theorem: Routh-Hurwitz Stability Criterion for 2×2 Matrix

The eigenvalues of a matrix $A \in \mathbb{R}^{2\times 2}$ are all in the open left half-plane $\mathrm{s(s)} < 0$ if and only if

$$\mathrm{Tr}(A) < 0 \text{ and } \det(A) > 0.$$

Summary

- Investigated two-dimensional systems $\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$
- Graphical representations: Vector field diagrams, phase portraits
- Method of isoclines for sketching phase portraits
- Investigated the equilibrium point $x^* = 0$ for a large class of linear time invariant systems (distinct non-zero eigenvalues)
- The equilibrium point can have on of the following types:
  - Stable / unstable node, saddle
  - Stable / unstable focus, center

- An equilibrium point x^* of the nonlinear system $\dot{x} = f(x)$ is hyperbolic of the same type if its linearization is hyperbolic at zero
- If the linearization has non-hyperbolic equilibrium at the origin, a nonlinear analysis has to be carried out
- Limit cycle = isolated periodic orbit
- Poincaré-Bendixson criterion $\longrightarrow$ existence of periodic orbits
- Bendixson criterion $\longrightarrow$ non-existence of periodic orbits

- Considered parameter-dependent systems $\dot{x} = f(x, \mu)$
- At a bifurcation point $\mu = \mu_0$, the phase portrait changes qualitatively
- The corresponding equilibrium is not hyperbolic at μ_0
- Different types:
  - pitchfork
  - transcritical
  - Hopf