05_Unscented Kalman Filter

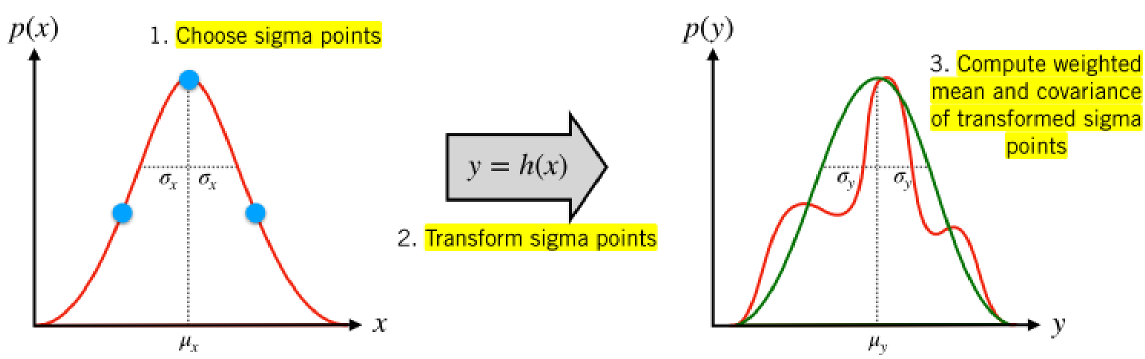
1. The Unscented Transform

It is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function.

Unscented Transformation is a method that can use very few sampling data to calculate mean and covariance after transformation.

The Unscented Transform consist of 3 steps:

1. Choose Sigma-Points
2. Transform Sigma Points
3. Compute weighted mean and covariance of transformed sigma points



Process

Choose Sigma-Points

For N-dimensional PDF $N(\mu_x, \Sigma_{xx})$ we ned $2N+1$ sigma points

1. Compute the Cholesky Decomposition of the covariance matrix

$$\mathbf{L}\mathbf{L}^T = \mathbf{\Sigma}_{xx} (\text{L lower triangular})$$

2. Calculate the sigma points:

$$
\begin{aligned}
\mathbf{x}_0 &= \boldsymbol{\mu}_x \\
\mathbf{x}_i &= \boldsymbol{\mu}_x + \sqrt{N+\kappa}\,\mathrm{col}_i\,\mathbf{L} \quad i=1,\ldots,N \\
\mathbf{x}_{i+N} &= \boldsymbol{\mu}_x - \sqrt{N+\kappa}\,\mathrm{col}_i\,\mathbf{L} \quad i=1,\ldots,N
\end{aligned}
$$

where $\kappa = 3 - N$

Transforming

$$
y_i = h(x_i) \quad i = 1\cdots,2N
$$

Recombining

- Mean:

$$
\boldsymbol{\mu} y = \sum_{i=0}^{2N} \alpha_i \mathbf{y}_i
$$

- Covariance:

$$
\boldsymbol{\Sigma}_{yy} = \sum_{i=0}^{2N} \alpha_i \left(\mathbf{y}_i - \boldsymbol{\mu}_y\right)\left(\mathbf{y}_i - \boldsymbol{\mu}_y\right)^T
$$

- Weights:

$$
\alpha_i = \begin{cases} \frac{\kappa}{N+\kappa} & i=0 \\ \frac{1}{2}\frac{1}{N+\kappa} & \text{otherwise} \end{cases}
$$

2. Unscented Kalman Filter

- In Kalman Filter, we need to calculate P and use P to calculate K
- For nonlinear system it is very hard to update P because of nonlinear relations. There are two ways to deal with this problem
    - First use **Monte-Carlo Method** simulate a lot of data points, use this points to do nonlinear transformation and then estimate mean and covariance. This method is time-consuming
    - Use Unscented Transform

Process

System Model

$$
\begin{aligned}
\mathbf{x}_k &= \mathbf{f}_{k-1}\left(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1}\right) \\
\mathbf{y}_k &= \mathbf{h}_k\left(\mathbf{x}_k, \mathbf{v}_k\right) \\
\mathbf{w}_k &\sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_k\right) \\
\mathbf{v}_k &\sim \mathcal{N}\left(\mathbf{0}, \mathbf{R}_k\right)
\end{aligned}
$$

Prediction

- Compute sigma-points

$$
\begin{aligned}
\hat{\mathbf{L}}_{k-1}\hat{\mathbf{L}}_{k-1}^T &= \hat{\mathbf{P}}_{k-1} \\
\hat{\mathbf{x}}_{k-1}^{(0)} &= \hat{\mathbf{x}}_{k-1} \\
\hat{\mathbf{x}}_{k-1}^{(i)} &= \hat{\mathbf{x}}_{k-1} + \sqrt{N+\kappa}\,\mathrm{col}_i\,\hat{\mathbf{L}}_{k-1} \quad i = 1\ldots N \\
\hat{\mathbf{x}}_{k-1}^{(i+N)} &= \hat{\mathbf{x}}_{k-1} - \sqrt{N+\kappa}\,\mathrm{col}_i\,\hat{\mathbf{L}}_{k-1} \quad i = 1\ldots N
\end{aligned}
$$

- Propagate sigma-points

$$
\check{\mathbf{x}}_k^{(i)} = \mathbf{f}_{k-1}\left(\hat{\mathbf{x}}_{k-1}^{(i)}, \mathbf{u}_{k-1}, \mathbf{0}\right) \quad i = 0\ldots 2N
$$

- Compute predicted mean and covrariance

$$
\begin{aligned}
\alpha^{(i)} &= \begin{cases} \frac{\kappa}{N+\kappa} & i = 0 \\ \frac{1}{2}\frac{1}{N+\kappa} & \text{otherwise} \end{cases} \\
\check{\mathbf{x}}_k &= \sum_{i=0}^{2N} \alpha^{(i)} \check{\mathbf{x}}_k^{(i)} \\
\check{\mathbf{P}}_k &= \sum_{i=0}^{2N} \alpha^{(i)} \left(\check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k\right)\left(\check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k\right)^T + \mathbf{Q}_{k-1}
\end{aligned}
$$

Correction Step

- Predict measurement from propagated sigma-points

$$
\hat{\mathbf{y}}_k^{(i)} = \mathbf{h}_k\left(\check{\mathbf{x}}_k^{(i)}, \mathbf{0}\right) \quad i = 0, \ldots, 2N
$$

- Estimate mean and covariance of predicted measurements

$$
\begin{aligned}
\hat{\mathbf{y}}_k &= \sum_{i=0}^{2N} \alpha^{(i)} \hat{\mathbf{y}}_k^{(i)} \\
\mathbf{P}_y &= \sum_{i=0}^{2N} \alpha^{(i)} \left(\hat{\mathbf{y}}_k^{(i)} - \hat{\mathbf{y}}_k\right)\left(\hat{\mathbf{y}}_k^{(i)} - \hat{\mathbf{y}}_k\right)^T + \mathbf{R}_k
\end{aligned}
$$

- Compute cross-covariance and Kalman gain

$$
\begin{aligned}
\mathbf{P}_{xy} &= \sum_{i=0}^{2N} \alpha^{(i)} \left(\check{\mathbf{x}}_k^{(i)} - \check{\mathbf{x}}_k\right)\left(\hat{\mathbf{y}}_k^{(i)} - \hat{\mathbf{y}}_k\right)^T \\
\mathbf{K}_k &= \mathbf{P}_{xy}\mathbf{P}_y^{-1}
\end{aligned}
$$

- Compute corrected mean and covariance

$$\hat{\mathbf{x}}_k = \check{\mathbf{x}}_k + \mathbf{K}_k \left(\mathbf{y}_k - \hat{\mathbf{y}}_k\right)$$
$$\hat{\mathbf{P}}_k = \check{\mathbf{P}}_k - \mathbf{K}_k \mathbf{P}_y \mathbf{K}_k^T$$

Summary

	EKF	ES-EKF	UKF
Operating Principle	Linearization (Full State)	Linearization (Error State)	Unscented Transform
Accuracy	Good	Better	Best
Jacobians	Required	Required	Not required
Speed	Slightly faster	Slightly faster	Slightly slower