Traditional Petri Net

3. Analysis of Petri net

3.1. Meaningful properties in Petri Net

Boundedness:

is the number of reachable markings bounded

Place Boundedness:

is there a bound on the number of tokens that can be created in a place

Semi-liveness:

is there a reachable marking from which a given transition can fire

Reachablity:

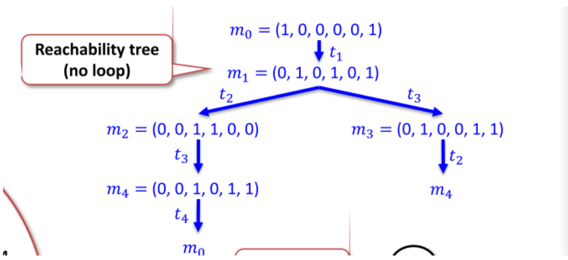
can a certain marking be reached, when we start from an initial marking

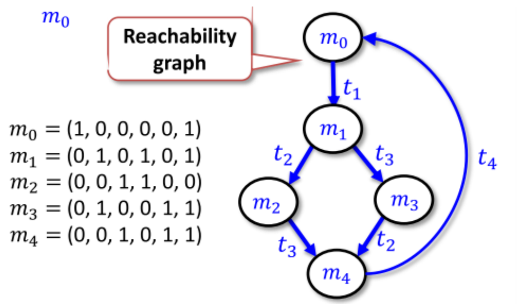
3.2.Reachability

A marking m is said to be **Reachabl**e from the initial marking m_0 (denoted by $m \in R(m_0)$), if and only if there exists a firing sequence $< t_1, t_2, t_3, \cdots, t_k >$ such that

$$m = m_0 \cdot t_1 \cdot t_2 \cdots t_k$$

3.2.1. Reachability Tree & reachability graph





<u>Drawbacks</u>

The tree maybe very long

3.3. Coverability Tree

The coverability tree allows us to analyze **unbounded Petri nets**

- It can be used to find out whether the reachability graph is infinite
- It is always **finite**, and its **construction always terminates**

3.3.1. Omega: the key to analyze unbounded Petri nets

Omega: the key to analyze unbounded Petri nets

- Coverability graph uses a concept called "Omega", denoted ω.
- ω represents "arbitrarily many tokens".
- We extend the arithmetic on natural numbers with ω as follows. For all $n \in \mathbb{N}$:

  - $n + \omega = \omega + n = \omega$
  - $\omega - n = \omega$
  - $\omega + \omega = \omega$
  - $0 \cdot \omega = 0$
  - $\omega \cdot \omega = \omega$
  - $n \geq 1 \Rightarrow n \cdot \omega = \omega \cdot n = \omega$
  - $n < \omega$ and $\omega \leq \omega$

  Note: $\omega - \omega$ remains undefined, but we will not need it.

- ω-markings extends the notion of markings.
- In an ω-marking, each place p will either have $n \in \mathbb{N}$ tokens or ω tokens (arbitrarily many).

Q: what does this mean (in terms of tokens in the marking)?

Example:

$m = (1, \omega, 0)$

- 1 token in the first place
- Any number of tokens in the second place
- No token in the third place

Firing rules with ω

1. If a transition has an input place with ω tokens, that place is considered to have **sufficient tokens** for the transition to fire, regardless of the arc weight.

2. If a place contains an ω-marking, then firing any transition connected with an arc to that place **will not change its marking**.

Covering Markings:

Marking m is **coverable** if exists m', reachable from m_0, such that

$$\forall p \in P, m'(p) \geq m(p)$$

Q: Assume that m' is reachable from m_0. Does m' cover m? (3 points)

(a)
$m = (1, 4, 1, 0)$
$m' = (1, 4, 2, 1)$
yes

(b)
$m = (1, 4, 1, 0)$
$m' = (2, 4, 0, 4)$
no

(c)
$m = (1, \omega, 0)$
$m' = (1, \omega - 10, 100)$
yes

(d)
$m = (7, \omega, 0)$
$m' = (7, \omega, \omega)$
yes

3.3.2. Algorithm to Construct Coverability tree

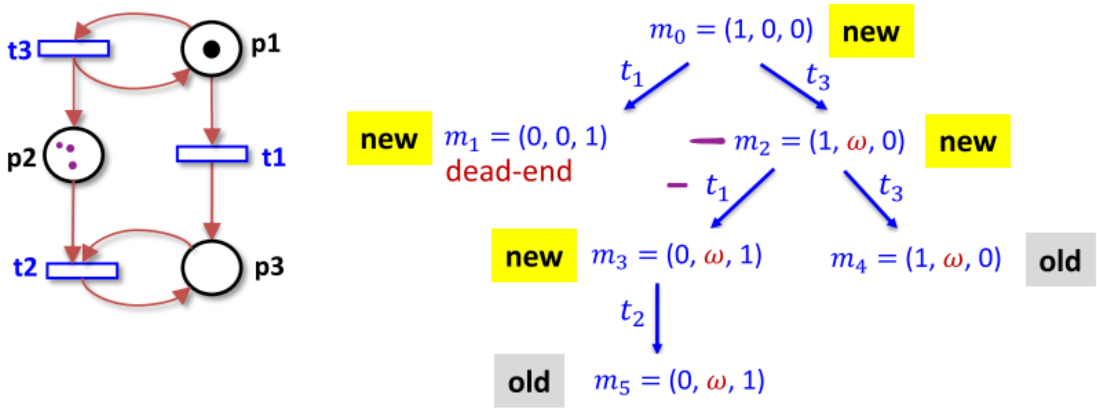
1. label the initial marking m_0 as root and tag it as "**new**"
2. **while** new markings exist, pick one, say m
   2.1. If m is identical to a marking on the way from the root to m,
      2.1.1. mark it as "**old**";
      2.1.2. **continue**;

   2.2. If no transitions are enabled at m,
      2.2.1. tag it as "**deadend**";
      2.2.2. **continue**;

   2.3. **For each** enabled transition t at m do
      2.3.1. obtain the new marking, denoted by m'

      2.3.2. if there exists a marking m'' on the path from m_0 to m such that $m' \geq m''$ (i.e., m' covers m'') and $m' \neq m''$,
         2.3.2.1. replace $m'(p)$ with ω for any $p \in P$ where $m'(p) > m''(p)$;

      2.3.3. introduce m' as a **new node** and connect it to m with a label t



3.3.3. Other Information from the voerability tree

Bounded:

The PN is **bounded** iff ω does not appear in any node label

Safe:

The PN is **safe** iff only ‘0’ and ‘1’ appear in the node labels

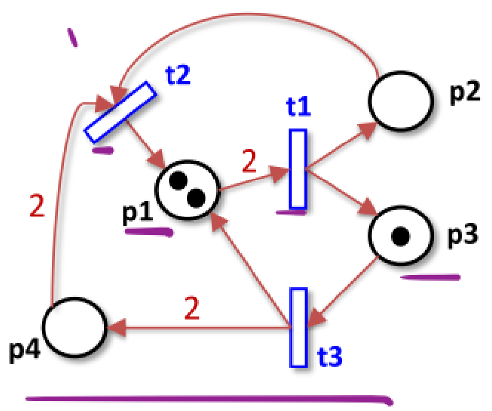
Dead:

A transition t is **dead** iff it does not appear as an arc

3.4. Incidence Matrix

Describe a Petri net through equations

- $A_{i,j}$: gain of tokens at node i when transition j fires
- A **marking** m is denoted by a $|P| \times 1$ column vector
- The **firing unit vector** u_i describes the firing of transition i
  - It consists of all ‘0’, except for the i-th position, where it has a ‘1’.
- A transition t from m_k to m_{k+1} is written as $m_{k+1} = m_k + A \cdot u_i$



$$A = \overbrace{\begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -2 & 2 \end{bmatrix}}^{\text{transitions}} \left.\vphantom{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}\right\} \text{places}$$

$$m_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Example: m_1 is obtained from m_0 by firing t_3

$$m_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Represents firing of t1 — Represents firing of t2 — Represents firing of t3

Neccessary Condition of Rreachable

A marking m is said to be **Reachabl**e from the initial marking m_0 (denoted by $m \in R(m_0)$), if and only if there exists a firing sequence $< t_1, t_2, t_3, \cdots, t_k >$ such that

$$m = m_0 \cdot t_1 \cdot t_2 \cdots t_k$$

Can be expressed as:

$$m = m_0 + A \cdot \sum_{i=1}^{k} u_i$$

or as

$$m - m_0 = \Delta = A \cdot \vec{x}$$

so, if m is reachable from m_0, the above equation must have a solution where all components of $\vec{x}$ are positive integers

- this is a **neccessary** conditions, not sufficient, because it ignore the sequence of transition
- it can used to prove a marking can not be arrived

3.5. Boundedness and Safety

3.5.1. K-Boundedness

A Petri net is said to be **K-bounded** if the number of tokens in **every place** doesn't exceed a finite number ***K***.

3.5.2. Safety

1-Boundedness: Every place holds at most 1 token at any time.

3.6. Liveness

Liveness focus on analysis particular transition

A transition t in a Petri Net is:

3.6.1. Dead

iff t cannot be fired in any firing sequence

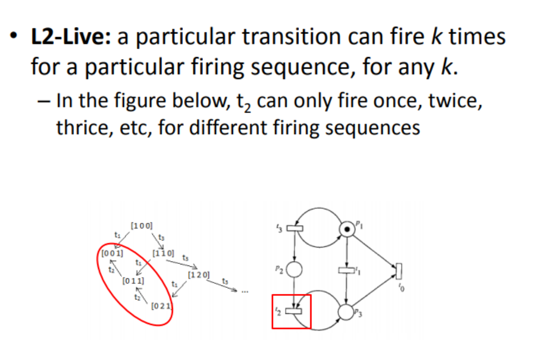
3.6.2. L1-live

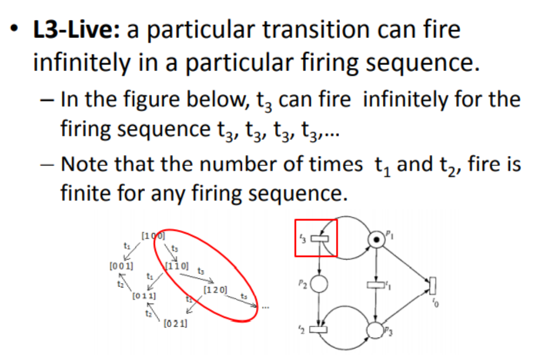
iff t can be fired **at least once** in some firing sequence

3.6.3. L2-live

iff $\forall k \in \mathbb{N}+$, t can be fired **at least *k* times** in some firing sequence(always means we can find a way to construct a sequence in which t can be fired at least k times but the sequence is not infinitely)

- In each sequence, there always some upper-bound or limitation for the firing times of this action





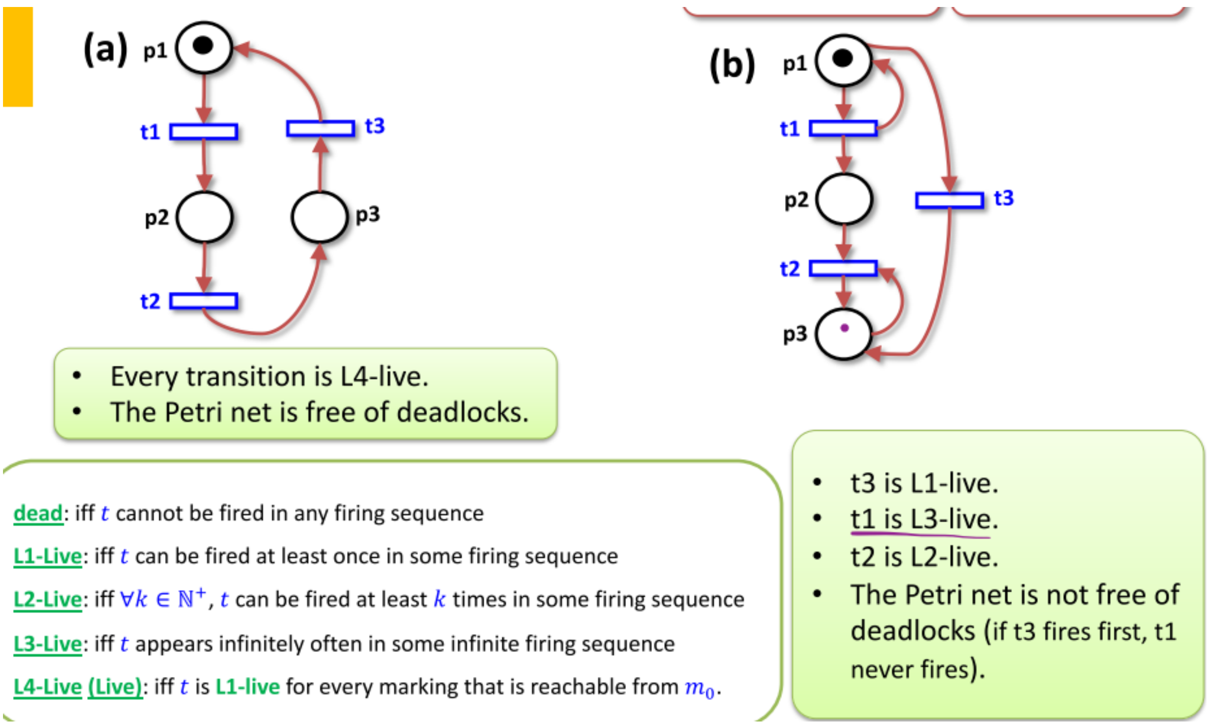
3.6.4. L3-live

iff t appears **infinitely often** in some infinite firing sequence

3.6.5. L4-live(live)

iff t is **L1-live** for **every marking that is reachable** from m_0

- A Petri net (N, M_0) is said to be Lk-live if every transition in the net is Lk-live, k = 0,1,2,3,4.
- L4 $\Rightarrow$ L3 $\Rightarrow$ L2 $\Rightarrow$ L1



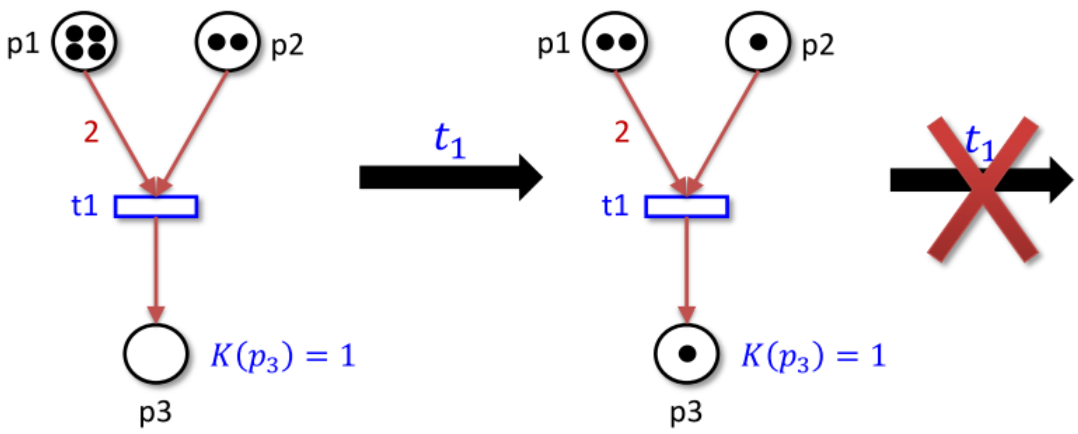
3.6.6. Free of Deadlock

A Petri net is **free of deadlocks** iff there is no reachable marking from $\boldsymbol{m0}$ in which all transitions are dead

4. Petri Net Flavors

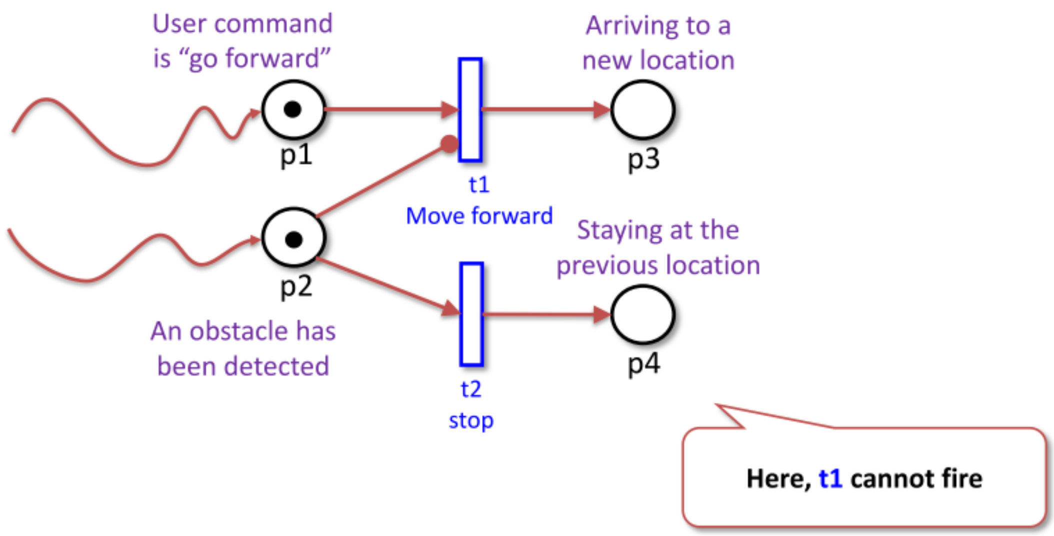
4.1. Finite capacity Petri net

- In a Petri net with **finite place capacity**, each place p can hold **at most $K(p)$** tokens
- A transition t is enabled only if all output places p_i of t will not exceed their capacity limit $K(p_i)$ after firing t



4.2. Inhibitor arcs

- An inhibitor arc is drawn from a place *p* to a transition *t* and means that *t* is disabled when *p* is marked with at least one token.
- This simplifies modeling "**absence of a condition**".



4.3. Timed Petri Nets

- Transitions can **take time to finish (ideally)**
- Time can be described as a **constant (deterministic)** or distribution (always, we differ it from Timed Petri Nets as Stochastic Petri Nets)
- In timed petri nets, transitions "**take time" to fire**. (Presentation of the first item in Petri Nets)
- Time **does not pass anywhere** else than on transitions.

4.4. Colored Petri Nets

Background

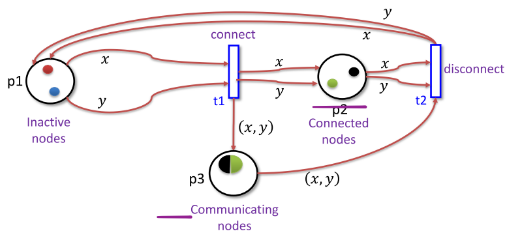
- In standard PN, tokens are indistinguishable entities.
- The semantics of the model does not allow to follow the behavior of an individual token through the PN

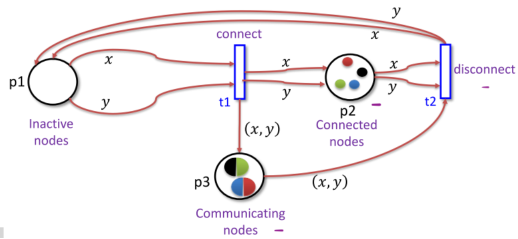
Idea

- add colors to the tokens

- Allow to distinguish between different types of tokens
- The colors can model data carried by the processes
- Transitions are aware of the colors

- Places, arcs and transitions can have functions and guards depending on the colors





4.5. Stochastic Petri Nets

- Transitions can **take time to finish (ideally)**
- Time can be described as a constant (deterministic) or **distribution** (always, we differ it from Timed Petri Nets as Stochastic Petri Nets)
- In timed petri nets, transitions "**take time" to fire**. (Presentation of the first item in Petri Nets)
- Time **does not pass anywhere** else than on transitions.