Trace Operator

- 1. Definition
- 2. Properties
- 3. Derivatives

1. Definition

Definition:

$$ext{tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$$

• the trace of a matrix is the **sum of its (complex) eigenvalues** (counted with multiplicities)

2. Properties

Basic Properties

$$egin{aligned} \operatorname{tr}(\mathbf{A}+\mathbf{B}) &= \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B}) \ \operatorname{tr}(c\mathbf{A}) &= c\operatorname{tr}(\mathbf{A}) \ \operatorname{tr}(\mathbf{A}) &= \operatorname{tr}\left(\mathbf{A}^{ op}
ight) \ \operatorname{tr}(\mathbf{A}\mathbf{B}) &= \operatorname{tr}(\mathbf{B}\mathbf{A}) \ \operatorname{tr}\left(\mathbf{P}^{-1}(\mathbf{A}\mathbf{P})\right) &= \operatorname{tr}\left((\mathbf{A}\mathbf{P})\mathbf{P}^{-1}\right) &= \operatorname{tr}(\mathbf{A}) \end{aligned}$$

Trace of a Product

if A and B are two m imes n matrix

$$\mathrm{tr}\left(\mathbf{A}^{ op}\mathbf{B}
ight) = \mathrm{tr}\left(\mathbf{A}\mathbf{B}^{ op}
ight) = \mathrm{tr}\left(\mathbf{B}^{ op}\mathbf{A}
ight) = \mathrm{tr}\left(\mathbf{B}\mathbf{A}^{ op}
ight) = \sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}b_{ij}$$

Cyclic Property

$$\operatorname{tr}(\mathbf{ABCD}) = \operatorname{tr}(\mathbf{BCDA}) = \operatorname{tr}(\mathbf{CDAB}) = \operatorname{tr}(\mathbf{DABC})$$

Trace of Kronecker Product

$$\operatorname{tr}(\mathbf{A} \otimes \mathbf{B}) = \operatorname{tr}(\mathbf{A}) \operatorname{tr}(\mathbf{B})$$

3. Derivatives

$$egin{aligned} &rac{d}{dm{A}}(ext{tr}(m{A}m{B})) = m{B}^T \ &rac{d}{dm{A}}\left(ext{tr}\left(m{A}m{B}m{A}^T
ight)
ight) = 2m{A}m{B} \end{aligned}$$