Mathematical Structure of Fuzzy Logic

1. Definition of Fuzzy Set

A fuzzy set A on universe (domain) X is a **set** defined by the **membership function** $\mu_A(x)$ which is a mapping from the universe X into the unit interval:

$$\mu_A(x) : X \to [0, 1] \tag{1}$$

$\mathcal{F}(X)$ denotes the set of all fuzzy sets on X

$$\mu_A(x) \begin{cases} = 1 & x \text{ is a full member of } A \\ \in (0, 1) & x \text{ is a partial member of } A \\ = 0 & x \text{ is not member of } A \end{cases} \tag{2}$$

2. Properties of Fuzzy Set

Complement

Let A be a fuzzy set in X. The complement of A is a fuzzy set, denoted $\overline{A}$, such that for each $x \in X$

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x) \tag{3}$$

λ-complement

$$\mu_{\bar{A}}(x) = \frac{1 - \mu_A(x)}{1 + \lambda\mu_A(x)} \tag{4}$$

Intersection of Fuzzy Sets

Let A and B be two fuzzy sets in X. The intersection of A and B is a fuzzy set C, denoted C = $A \cap B$, such that for each $x \in X$

$$\mu_C(x) = \min\left(\mu_A(x), \mu_B(x)\right) \tag{5}$$

The minimum operator is also denoted by $\wedge$, i.e., $\mu_C(x) = \mu_A(x) \wedge \mu_B(x)$

Union of Fuzzy Sets

Let A and B be two fuzzy sets in X. The union of A and B is a fuzzy set C, denoted $C = A \cup B$, such that for each $x \in X$

$$\mu_C(x) = max(\mu_A(x), \mu_B(x)) \tag{6}$$

The maximum operator is also denoted by $\vee$, i.e., $\mu_C(x) = \mu_A(x) \vee \mu_B(x)$

3.2. T-norms and T-conorms

T-Norm (Intersection)

A **t-norm** T is a **binary operation** on the unit interval that satisfies at least the following axioms for all $a, b, c \in [0,1]$:

$$\begin{aligned} T(a,1) = a \quad &(\text{ boundary condition}) \\ b \leq c \quad \textit{implies} \quad &T(a,b) \leq T(a,c)(\text{monotonicity}) \\ T(a,b) = &T(b,a)(\text{commutativity}), \\ T(a,T(b,c)) = &T(T(a,b),c) \quad (\text{associativity}). \end{aligned} \tag{7}$$

Some frequently used t-norms are:

standard (Zadeh) intersection:	$T(a,b) = \min(a,b)$
algebraic product (probabilistic intersection):	$T(a,b) = ab$
Łukasiewicz (bold) intersection:	$T(a,b) = \max(0, a+b-1)$

- **minimum** is the **largest t-norm** (intersection operator)
- means the membership functions of fuzzy intersections $A \cap B$ obtained with other t-norms are all **below** the bold membership function

T-Conorm(Union)

A **t-conorm** S is a binary operation on the unit interval that satisfies at least the following axioms for all a, b, c ∈ [0, 1] (Klir and Yuan, 1995):

$$\begin{aligned} S(a,0) = a \quad &(\text{ boundary condition }) \\ b \leq c \quad \textit{implies} \quad &S(a,b) \leq S(a,c) \quad (\text{monotonicity}), \\ S(a,b) = &S(b,a) \quad (\text{ commutativity }) \\ S(a,S(b,c)) = &S(S(a,b),c)(\text{associativity}). \end{aligned} \tag{8}$$

Some frequently used t-conorms are:

standard (Zadeh) union:	$S(a,b) = \max(a,b)$,
algebraic sum (probabilistic union):	$S(a,b) = a + b - ab$,
Łukasiewicz (bold) union:	$S(a,b) = \min(1, a+b)$.

- The **maximum** is the **smallest** t-conorm (union operator)
- the membership functions of fuzzy unions A ∪ B obtained with other t-conorms are all **above** the bold membership function

3.3. Projection and Cylindrical Extenstion

Projection reduces a fuzzy set defined in a multi-dimensional domain

Cylindrical extension extend of a fuzzy set defined in low-dimensional domain into a higher-dimensional domain

Projection

Let $U \in U_1 \times U_2$ be a subset of a Cartesian product space, where U_1 and U_2 can themselves be Cartesian products of lower dimensional domains. The projection of fuzzy set A defined in U onto U_1 is the mapping $proj_{U_1} : \mathcal{F}(U) \to \mathcal{F}(U_1)$ defined by:

$$\operatorname{proj}_{U_1}(A) = \left\{ \sup_{U_2} \mu_A(u)/u_1 \mid u_1 \in U_1 \right\} \tag{9}$$

where sup is the supremum operation

> *The **supremum** (abbreviated sup; plural suprema) of a subset S of a partially ordered set T is the least element in T that is greater than or equal to all elements of S, if such an element exists. Consequently, the supremum is also referred to as the **least upper bound (or LUB)**.*

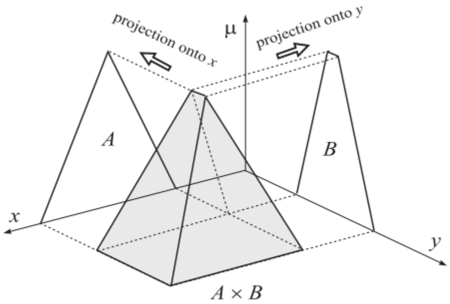


Figure 2.9.: Example of projection from $\mathbb{R}^2$ to $\mathbb{R}$.

Example 2.4 (Projection) Assume a fuzzy set A defined in $U \subset X \times Y \times Z$ with $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$, as follows:

$$\begin{aligned} A = \{&\mu_1/(x_1, y_1, z_1),\ \mu_2/(x_1, y_2, z_1),\ \mu_3/(x_2, y_1, z_1), \\ &\mu_4/(x_2, y_2, z_1),\ \mu_5/(x_2, y_2, z_2)\} \end{aligned} \tag{2.31}$$

Let us compute the projections of A onto X, Y and $X \times Y$:

$$\operatorname{proj}_X(A) = \{\max(\mu_1, \mu_2)/x_1,\ \max(\mu_3, \mu_4, \mu_5)/x_2\}, \tag{2.32}$$

$$\operatorname{proj}_Y(A) = \{\max(\mu_1, \mu_3)/y_1,\ \max(\mu_2, \mu_4, \mu_5)/y_2\}, \tag{2.33}$$

$$\begin{aligned} \operatorname{proj}_{X\times Y}(A) = \{&\mu_1/(x_1, y_1),\ \mu_2/(x_1, y_2), \\ &\mu_3/(x_2, y_1),\ \max(\mu_4, \mu_5)/(x_2, y_2)\}. \end{aligned} \tag{2.34}$$

Cylindrical Extension

Let $U \subseteq U_1 \times U_2$ be a subset of a Cartesian product space, where U_1 and U_2 can themselves be Cartesian products of lower-dimensional domains. The cylindrical extension of fuzzy set A defined in U1 onto U is the mapping $ext_U : \mathcal{F}(U_1) \to \mathcal{F}(U)$ defined by

$$\mathrm{ext}_U(A) = \{\mu_A(u_1)/u \mid u \in U\} \tag{10}$$

- Cylindrical extension thus **simply replicates** the membership degrees from the existing dimensions into the new dimensions.

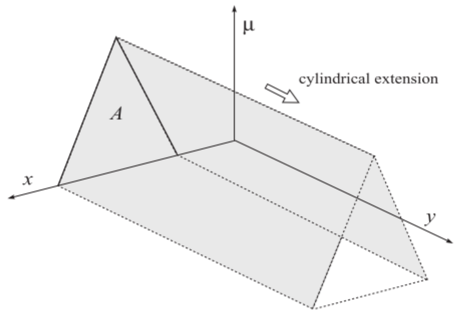


Figure 2.10.: Example of cylindrical extension from $\mathbb{R}$ to $\mathbb{R}^2$.

Properties

- **projection** leads to **a loss of information** , so we have

$$\begin{aligned} A &= \mathrm{proj}_{X^n}(\mathrm{ext}_{X^m}(A)) \\ A &\neq \mathrm{ext}_{X^m}(\mathrm{proj}_{X^n}(A)) \end{aligned} \tag{11}$$

3.4. Operations on Cartesian Product Domains

Set-theoretic operations such as the **union or intersection** applied to fuzzy sets **defined in different domains** result in a **multi-dimensional fuzzy set** in the Cartesian product of those domains.

- first **extending** the original fuzzy sets into the Cartesian product domain
- then **computing the operation** on those multi-dimensional sets.

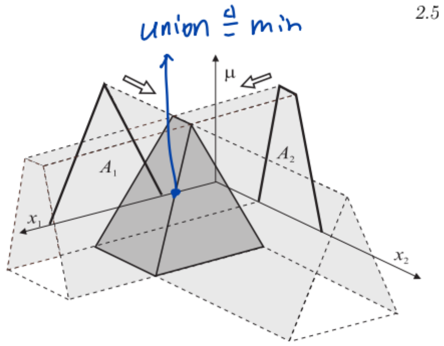


Figure 2.11.: Cartesian-product intersection.

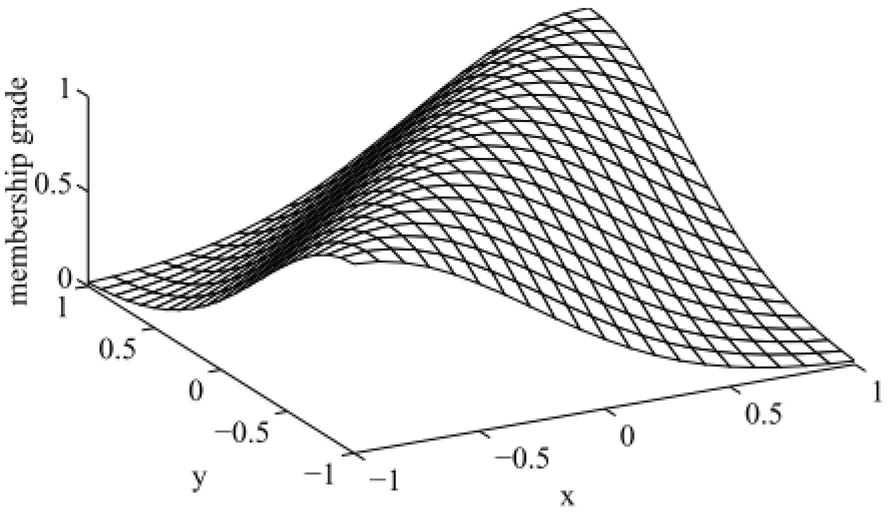


Figure 2.13.: Fuzzy relation $\mu_R(x, y) = e^{-(x-y)^2}$.

Example 2.5 (Cartesian-Product Intersection) Consider two fuzzy sets A_1 and A_2 defined in domains X_1 and X_2, respectively. The intersection $A_1 \cap A_2$, also denoted by $A_1 \times A_2$ is given by:

$$A_1 \times A_2 = \text{ext}_{X_2}(A_1) \cap \text{ext}_{X_1}(A_2)\,. \tag{2.38}$$

This cylindrical extension is usually considered implicitly and it is not stated in the notation:

$$\mu_{A_1 \times A_2}(x_1, x_2) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2)\,. \tag{2.39}$$

Figure 2.11 gives a graphical illustration of this operation.

4. Fuzzy Relations

An n-ary **fuzzy relation** is a **mapping**

$$R : X_1 \times X_2 \times \cdots \times X_n \rightarrow [0, 1] \tag{12}$$

which assigns membership grades to all n-tuples $(x_1, x_2, \ldots, x_n)$ from the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$.

- For **computer implementations** , R is conveniently represented as an **n-dimensional array:** $R = [r_{i_1, i_2, \ldots, i_n}]$
- A **fuzzy relation** is a **fuzzy set** in the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$. The membership grades represent the degree of association (correlation) among the elements of the different domains X_i.

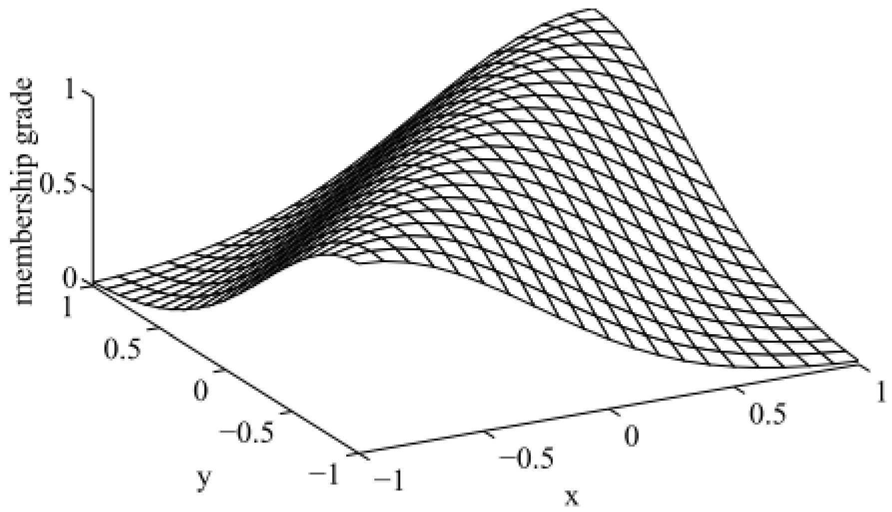


Figure 2.13.: Fuzzy relation $\mu_R(x, y) = e^{-(x-y)^2}$.

5. Relational Composition

The **composition** is defined as follows : suppose there exists a fuzzy relation R in $X \times Y$ and A is a fuzzy set in X. Then, fuzzy subset B of Y can be **induced** by A through the composition of A and R:

$$B = A \circ R \tag{13}$$

And is definied by:

$$B = \text{proj}_Y(R \cap \text{ext}_{X \times Y}(A)) \tag{14}$$

Understanding

The composition can be regarded in two phases:

- combination(intersection)
- projection

sup-min composition

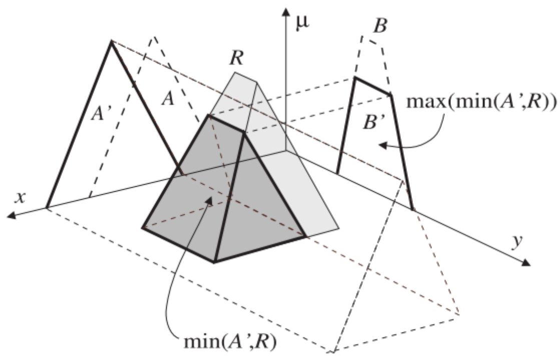
$$\mu_B(y) = \sup_x \left(\min\left(\mu_A(x), \mu_R(x, y)\right)\right) \tag{15}$$

"given y, find x, in which make the min() part max"

More general form

$$\mu_B(y) = \sup_x \left(\ \mathrm{T}\left(\mu_A(x), \mu_R(x, y)\right)\right) \tag{16}$$

the T means a t-norm T



Example

Example 2.8 (Relational Composition) Consider a fuzzy relation R which represents the relationship "x is *approximately equal* to y":

$$\mu_R(x, y) = \max(1 - 0.5 \cdot |x - y|, 0)\,. \tag{2.45}$$

Further, consider a fuzzy set A *"approximately 5"*:

$$\mu_A(x) = \max(1 - 0.5 \cdot |x - 5|, 0)\,. \tag{2.46}$$

Suppose that R and A are discretized with $x, y = 0, 1, 2, \ldots$, in $[0, 10]$. For a discrete set *max* is equivalent to *sup*. Then, the composition is:

$$\mu_B(y) = \overbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}}^{\mu_A(x)} \circ \overbrace{\begin{pmatrix}
1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1
\end{pmatrix}}^{\mu_R(x,y)} =$$

$$= \max_x \overbrace{\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}}^{\min\left(\mu_A(x), \mu_R(x,y)\right)} =$$

$$= \overbrace{\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}}^{\max_x\left(\min\left(\mu_A(x), \mu_R(x,y)\right)\right)}$$

In this graph, different column means different x

This resulting fuzzy set, defined in Y **can be interpreted as "approximately 5"** . Note, however, that it is **broader (more uncertain)** than the set from which it was induced。 This is because the combination of uncertainty in input fuzzy and relation