02_Dynamic Behavior of System and Linear System

1. Solution to Linear ODE and Matrix Exponential

Autonomous Case

$$\frac{d}{dt}x(t) = Ax(t), \quad x(0) = x_0$$

has solution

$$x(t) = e^{At}x_0$$

where

$$e^{At} := I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \dots$$

General Case

For system $\dot{x} = Ax + Bu$, the solution is

$$x(t) = e^{At}\left[x_0 + \int_0^t e^{-At}Bu(\tau)d\tau\right]$$

Matrix Exponential

A is diagonal

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \Rightarrow e^A = \begin{bmatrix} e^{a_1} & 0 & \cdots & 0 \\ 0 & e^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_n} \end{bmatrix}$$

A is diagonalizable

<u>Case 1: If A has n distinct eiganvalues</u>

If $A = T^{-1}DT$, then $e^A = T^{-1}e^DT$

A is not diagonalizable

Things may become more complex, we just choose one case: A has eigenvalues with multiplicity

- Jordan Transformation

$$TAT^{-1} = J = \begin{bmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & J_{k-1} & 0 \\ 0 & 0 & \cdots & 0 & J_k \end{bmatrix}$$

where $J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix}$

- Exponential

$$
e^{Jt} = \begin{bmatrix} e^{J_1 t} & 0 & \cdots & 0 & 0 \\ 0 & e^{J_2 t} & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & e^{J_{k-1} t} & 0 \\ 0 & 0 & \cdots & 0 & e^{J_k t} \end{bmatrix}
$$

where $e^{J_i t} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & & \frac{t^{n-2}}{(n-2)!} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & t \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} e^{\lambda_i t}$

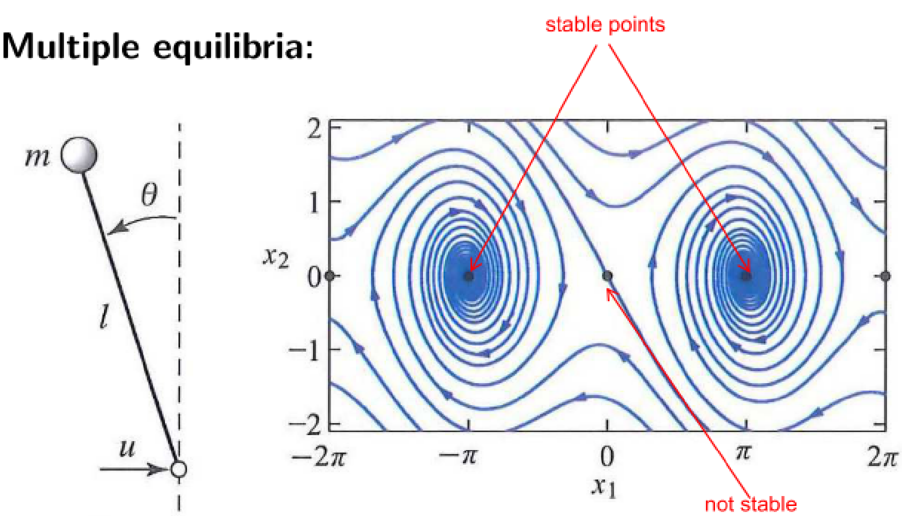
- $e^{At} = T^{-1} e^{Jt} T$

2. Qualitative Analysis of System

Phase Portraits

- Mainly for 2-dimensions systems, consider planar case, $x \in R^2$
- Plot with the two states on the axis

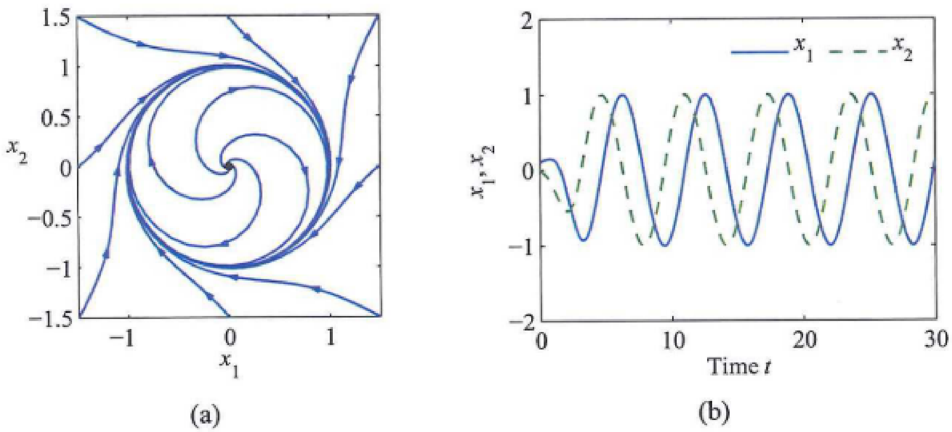
Multiple equilibria:



$$
Pendulum: \quad \dot{x} = \begin{bmatrix} x_2 \\ \sin x_1 - c\, x_2 + u \cos x_1 \end{bmatrix}, \quad x_1 = \theta, \quad x_2 = \dot{\theta}
$$

Limit Cycle

- The solutions in the phase plane **converge to a circular trajectory**
- In the time domain this corresponds to an **oscillatory solution**



$$Example:\ \dot{x} = y + x(1 - x^2 - y^2) \quad , \quad \dot{y} = -x + y(1 - x^2 - y^2)$$

3. Lyapunov Stability Analysis

Stability Definitions

Equilibrium Point

x_e is an **equilibrium point** for $\dot{x} = f(x)$ if $f(x_e) = 0$

- Note that for linear ODE the origin is an equilibrium point

Lyapunov stable

An equilibrium point x_e is **Lyapunov stable** if

$$\forall \epsilon > 0, \exists \delta > 0 : \|x_0 - x_e\| < \delta \rightarrow \|x(t, x_0) - x_e\| < \epsilon, \forall t \geq 0$$

Asymptotically Stable

An equilibrium point x_e is **asymptotically stable** if

1. it is Lyapunov stable
2. $x(t, x_0) \rightarrow x_e, \text{ as } t \rightarrow \infty$

Local Stable (Local Asymptotically Stable)

A solution to be **Locally Stable** if it is stable for all initial condition $x \in B_r(a)$, where

$$B_r(a) = x : ||x - a|| < r$$

Global Stable

A system is **globally stable** if it is stable for all $r > 0$

Lyapunov Stability Analysis

Consider system $\dot{x}(t) = f(x(t)), x(0) = x_0$,$V(x)$ be scalar function having continuous first derivatives, satisfying

1. $V(x)$ is positive definite
2. $\dot{V}(x) = \frac{dV(x)}{dx}\dot{x}$ is negative definite

Then the system is **asymptotically stable**

- V(x) is called *Lyapunov function*
- V(x) can be a **measure for the total energy** in the system

For Linear Systems

If $A^T P + PA < 0$ the function $V(x)$ is a Lyapunov Function

Region of Attraction

Set of **all initial conditions** that converge to a **given** asymptotically stable equilibrium point

Bifurcation

Definition: Bifurcation

Consider the nonlinear system

$$\dot{x} = F(\mu, x)$$

where μ is a set of parameters that describe the family of equations

Bifurcation at $\mu = \mu^*$ if the behavior of the system changes qualitatively at μ^*

Example

Example (Bicyle)

$\phi =$ roll angle

$\delta =$ steer angle

$v_0 =$ velocity

Model:

$$M \begin{bmatrix} \ddot{\phi} \\ \ddot{\delta} \end{bmatrix} + C\, v_0 \begin{bmatrix} \dot{\phi} \\ \dot{\delta} \end{bmatrix} + (K_0 + K_2 v_0^2) \begin{bmatrix} \phi \\ \delta \end{bmatrix}$$

For state $x = \begin{bmatrix} \phi & \delta & \dot{\phi} & \dot{\delta} \end{bmatrix}^T$ the model becomes:

$$\dot{x} = \begin{bmatrix} 0 & I \\ -M^{-1}(K_0 + K_2 v_0^2) & -M^{-1} C v_0 \end{bmatrix} x = A(v_0)\, x$$

eigenvalue change with v

4. Frequency Domain Analysis

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Time-Domain Solution

$$x(t) = e^{At} \left[x_0 + \int_0^t e^{-At} B u(\tau) d\tau \right]$$

If $u(t) = e^{st},\ s \neq \lambda(A) \Rightarrow x(t) = e^{At} x_0 + e^{At} (sI - A)^{-1} \left(e^{(sI-A)t} - I \right) B$

From State-Space Model to Frequency Domain Analysis

$$y(t) = G(s) u(t),\ \text{for}\ u(t) = e^{st}$$

where

$$\begin{aligned} G(s) &= C(sI - A)^{-1} B + D \\ &= \frac{C\,\mathrm{adj}(sI - A) B + D \det(sI - A)}{\det(sI - A)} \end{aligned}$$

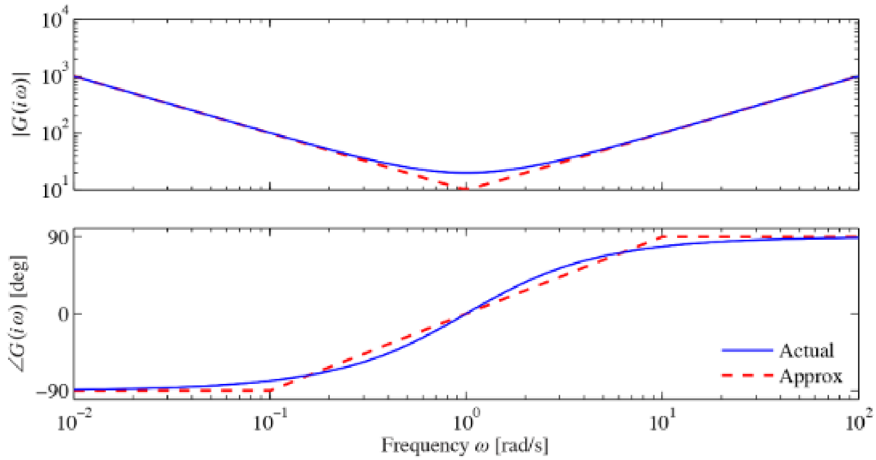
Then we define *Gain* and *Phase*

$$G(s) = M(s) e^{j\phi(s)}$$

Gain: $M(s) = |G(s)|$
Phase: $\phi(s) = \angle G(s)$

Bode Plots

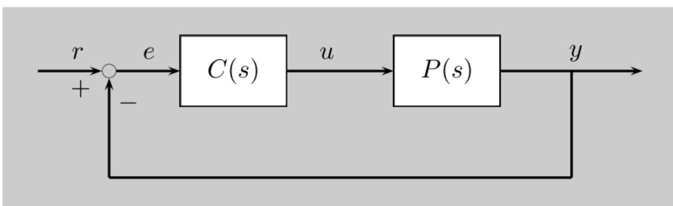
The graph of **Gain** and **Phase** regarding to **input frequency**



Nyquist Graph

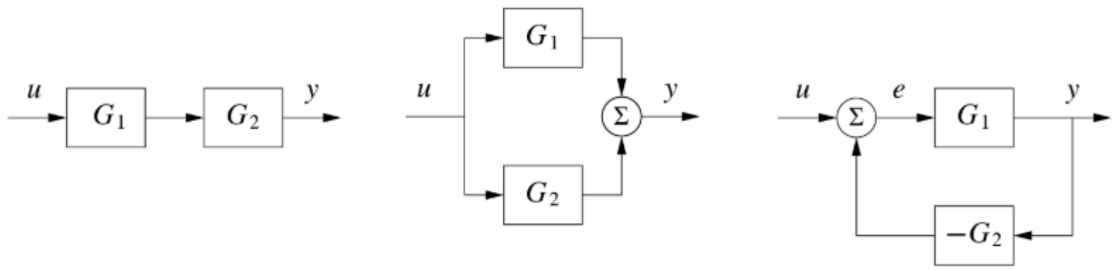
Loop Transfer Function

Loop transfer function is obtained by **breaking the feedback loop**



$$L(s) = P(s)C(s)$$

From Block Diagram to Loop Transfer Function

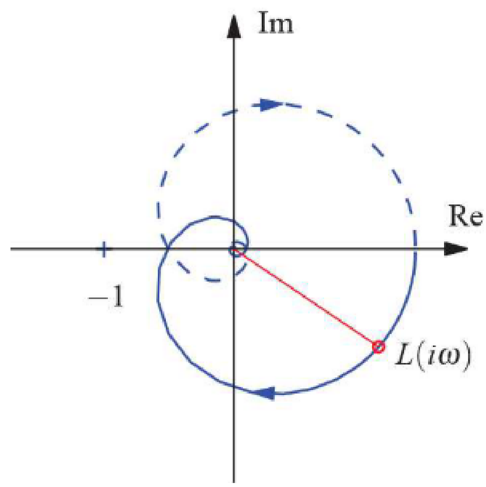


(a) $G_{yu} = G_2 G_1$ (b) $G_{yu} = G_1 + G_2$ (c) $G_{yu} = \dfrac{G_1}{1 + G_1 G_2}$

Nyquist Graph

Nyquist Graph is the curve $L(j\omega)$ in the complex plane parameterized by ω

$$L(s) = \frac{1.4e^{-s}}{(s+1)^2}$$



Nyquist Stability Theorem (Simplified Version)

Assume that L has **no poles in the closed right half-plan**e except for **single** poles on the imaginary axis

The closed loop system is stable **if and only if** the closed contour $L(j\omega)$ **does not encircle** the point **−1** in the **clockwise direction**.

5. Stability of Linear ODE: State-Space Model

For LTI system $\dot{x} = Ax, x \in \mathbb{R}^n$, stability of equilibrium is related to the **eigenvalues** of state matrix A

- **Stable** if $Re(\lambda_i) \leq 0$
- **Asymptotically Stable** if $Re(\lambda_i) \leq 0$
- State Coordinate Change does not change the eigenvalues

6. Skills in Linear Systems Analysis

Linearization

Consider the system $\dot{x} = f(x, u)$. For a steady-state or equilibrium point (x_0, u_0, y_0) there holds

$$\begin{gathered} f\left(x_0, u_0\right) = 0 \\ y_0 = g(x_o, u_0) \end{gathered}$$

Look at **small variation**s $\tilde{x}, \tilde{u}$, and $\tilde{y}$ about the equilibrium (x_0, u_0, y_0):

$$\begin{gathered} x(t) = x_0 + \tilde{x}(t) \\ u(t) = u_0 + \tilde{u}(t) \\ y(t) = y_0 + \tilde{y}(t) \end{gathered}$$

Then, we have

$$\begin{gathered} \dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t) \\ \tilde{y}(t) = C\tilde{x}(t) + D\tilde{u}(t) \end{gathered}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x \equiv x_0 \\ u = u_0}}, B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x \equiv x_0 \\ u = u_0}} \quad C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x \equiv x_0 \\ u = u_0}}, D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x \equiv x_0 \\ u = u_0}}$$

Proof Procedure:

First of all note that $\dot{\tilde{x}}(t) = \dot{x} - \dot{x}_0 = \dot{x}$, and so

$$\begin{gathered} \dot{\tilde{x}}(t) = f\left(x_0 + \tilde{x}(t), u_0 + \tilde{u}(t)\right) \\ y_0 + \tilde{y}(t) = g\left(x_0 + \tilde{x}(t), u_0 + \tilde{u}(t)\right) \end{gathered}$$

Use **Taylor expansion** to describe nonlinear equations in terms of $\tilde{x}$ and $\tilde{u}$:

$$\begin{gathered} \dot{\tilde{x}}(t) = f\left(x_0, u_0\right) + A\tilde{x}(t) + B\tilde{u}(t) \\ \tilde{y}(t) = g\left(x_0, u_0\right) + C\tilde{x}(t) + D\tilde{u}(t) - y_0 \end{gathered}$$

and at the same time, we have $f\left(x_0, u_0\right) = 0$ and $y_0 = g\left(x_0, u_0\right)$

State Coordinate Change

Consider the original system

$$\begin{gathered} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{gathered}$$

with $z(t) = Tx(t)$ or $x(t) = T^{-1}z(t)$ with T is non-singular matrix, we will have

$$\dot{z}(t) = A'z(t) + B'u(t)$$
$$y(t) = C'z(t) + D'u(t)$$

with

$$\begin{aligned} A' &= TAT^{-1} \quad & B' &= TB \\ C' &= CT^{-1} \quad & D' &= D \end{aligned}$$

- The **transfer function keeps unchanged** before and after the state coordinate change

$$Y(s) = \left(C(sI - A)^{-1}B + D\right)U(s)$$

Summary

- Solution to Autonomous ODE and Matrix Exponential
  - Solution for Autonomous system
  - Matrix Exponential
    - Series Definition
    - Diagonal and Diagonalizable
    - Not Diagonalizable: Multiplicity Eigenvalue Case

- **Qualitative** Analysis
  - Phase Portraits
  - Limit Cycle
  - Oscillation
- **Lyapunov Stability** Analysis
  - Definitions: Stable, Asymptotically, Exponentially, Globally ...
  - Lyapunov Stability Analysis: Lyapunov Function
  - Region of Attraction for initial conditions
  - Bifurcation
- **Frequency Domain** Analysis
  - State-Space $\Rightarrow$ Frequency Domain
  - Bode Plots
  - Loop transfer function, Nyquist Graph and Nyquist Stability Criterion
- State Space Analysis

- For nonlinear system: **linearization**
- For linear system: state coordinate change: **transfer function keeps unchanged**