# 03_Dynamics and Well-Posedness



# 1. Well-Posedness for Smooth Systems

For smooth system, we consider smooth system presented by differential equations.

**Definition: Solution**

A function $x:[t_0,t_1]\to\mathbb{R}^n$ is called a **solution** to the differential equation with initial state $x_0$ at time $t_0$, if

- $x$ is **continuous**,
- $x$ is **differentiable** on $(t_0,t_1)$ (i.e., $\dot{x}(t)$ exists for all $t\in(t_0,t_1)$),
- $\dot{x}(t)=f(t,x(t))$ for all $t\in(t_0,t_1)$ and $x\left(t_0\right)=x_0$

**Well-Posedness Problem:**

given initial conditions, does there **exist** a solution and is it **unique**?

## Well-Posedness

**Theorem for local existence and uniqueness of solutions**

Let $f(t,x)$ be piece-wise continuous in $t$ and satisfy the following Lipschitz condition: there exist $L>0$ and $r>0$ such that

$$\|f(t,x)-f(t,y)\|\leqslant L\|x-y\|$$

for all $x$ and $y$ in neighborhood $B:=\{x\in\mathbb{R}^n \mid \|x-x_0\|<r\}$ of $x_0$ and for all $t\in[t_0,t_1]$.

Then there exists $\delta>0$ such that unique solution exists on $[t_0,t_0+\delta]$ starting in $x_0$ at $t_0$.

### Global Well-Posedness

<u>**Theorem: Global Lipschitz Condition**</u>

Suppose $f(t,x)$ is piece-wise continuous in $t$ and satisfies

$$\|f(t,x)-f(t,y)\| \leqslant L\|x-y\|$$

**for all** $x, y$ in $\mathbb{R}^n$ and for all $t \in [t_0, t_1]$.
Then unique solution exists on $[t_0, t_1]$ for any initial state $x_0$ at $t_0$.

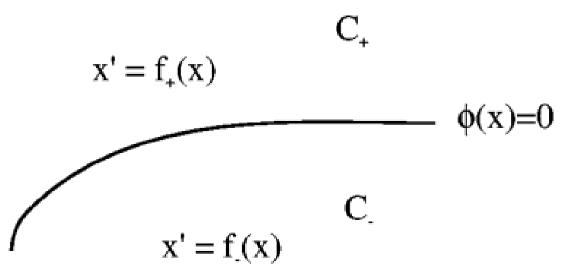
**Note:**

It is a **sufficient** condition, not an necessary condition

## 2. Solution Concept and Well-Posedness for Switched Systems

For switched system, we assume it has **discontinuous** differential equations.

For example:




$$\dot{x} = \begin{cases} f_+(x) & \text{if } x \in C_+ := \{x \mid \phi(x) > 0\} \\ f_-(x) & \text{if } x \in C_- := \{x \mid \phi(x) < 0\} \end{cases}$$

- if $x$ in interior of $C_-$ or $C_+$: just follow!
- if $f_-(x)$ and $f_+(x)$ point in same direction: just follow!
- if $f_+(x)$ points towards $C_+$ and $f_-(x)$ points towards $C_-$: At least two trajectories

$f_+(x)$ points towards $C_-$and $f_-(x)$ points towards $C_+$

$\rightarrow$ no classical solution

If one would **allow that the state evolves only according to one of the dynamics (as the initial definition of solution)**, then in the third class, there will be two solutions, and in the first case, there will be no solutions.

So, we need **generalization of the solution concept.**

### Classical Generalization

- **Relaxation:** spatial (hysteresis) $\Delta$, time delay $\tau$, smoothing $\epsilon$ (use a continuous function to approximate the ‘gap’)
- **Chattering/Infinitely Fast Switching**

## Sliding Mode and Differential Inclusion

**Filippov’s Convex Definition**

**Convex combination** of both dynamics

$$\dot{x} = \lambda f_+(x) + (1-\lambda) f_-(x) \text{ with } 0 \leqslant \lambda \leqslant 1$$

such that $x$ moves ("slides") along surface $\phi(x) = 0$

**Differential Inclusion**

$\dot{x} \in F(x)$ with set-valued

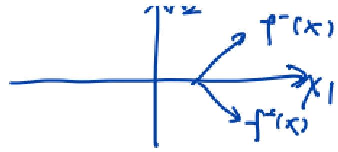
For example

$$\begin{aligned}
F(x) &= \{f_+(x)\} && (\phi(x) > 0) \\
F(x) &= \{f_-(x)\} && (\phi(x) < 0), \\
F(x) &= \{\lambda f_+(x) + (1-\lambda) f_-(x) \mid \lambda \in [0,1]\} && (\phi(x) = 0),
\end{aligned}$$

**Generalization of Solution Concept:**

Function $x : [a,b] \to \mathbb{R}^n$ is **solution** of $\dot{x} \in F(x)$ if $x$ is **absolutely continuous** and **satisfies** $\dot{x}(t) \in F(x(t))$ for almost all $t \in [a,b]$

**Example**




$$\phi(x) = x_2, f_+(x) = (x_1^2, -x_1 + \tfrac{1}{2}x_1^2)^\mathsf{T},\ f_-(x) = (1, x_1^2)^\mathsf{T}$$

Sliding for $x_0 = (1,0)^\mathsf{T}$ as $f_+(x_0) = (1, -\frac{1}{2})^\mathsf{T}$ and $f_-(x_0) = (1,1)^\mathsf{T}$

Sliding behavior: find convex combination such that $\phi(x) = 0$

$$\frac{d\phi}{dt}(x(t)) = \frac{d\phi}{dx}\dot{x}(t) = \dot{x}_2(t) = \lambda(-x_1 + \tfrac{1}{2}x_1^2) + (1-\lambda)x_1^2 = 0 \quad \rightarrow$$

$\frac{d\phi}{dx_1} \cdot \frac{dx_1}{dt}$ . $\frac{d\phi}{dx_2} \cdot \frac{dx_2}{dt}$

$$\lambda(x) = \frac{x_1}{\frac{1}{2}x_1 + 1}$$

Sliding mode is valid as long as $\lambda(x) \in [0,1]$, "invariant"

$$\dot{x}_1 = \lambda x_1^2 + (1-\lambda) = \frac{2x_1^3 - x_1 + 2}{x_1 + 2}$$

as long as $0 \leqslant x_1 \leqslant 2$ $\rightarrow$ $f_+(x) = [4, 0]$

$f_-(x) = [1, 4]$

hs_dyn.10

### Well-posedness Result for Sliding Mode

**Theorem: A well-posedness result for sliding mode**

Assume

- $f_-$ and $f_+$ are continuously differentiable $(C^1)$
- $\phi$ is $C^2$, discontinuity vector $h(x) := f_+(x) - f_-(x)$ is $C^1$

If for each $x$ with $\phi(x) = 0$ at least one of the conditions

- $f_+(x)$ points towards $C_-$ or
- $f_-(x)$ points towards $C_+$

holds (where for different points $x$ a different condition may hold), then the **Filippov solutions exist and are unique**

## 3. Event-Times Criterion

### Conceptions

**Definition: Admissible Event Times Set**

Set $\mathscr{E} \subset \mathbb{R}_+$ is **admissible event times set**, if it is **closed and countable**, and $0 \in \mathscr{E}$ (0: initial time)

**Definition: Accumulation Points**

- **left accumulation point:** $t \in \mathscr{E}$ is said to be **left accumulation point** of $\mathscr{E}$, if for all $t' > t$, $(t, t') \cap \mathscr{E}$ is not empty: e.g. bouncing ball
- **right accumulation point:** $t \in \mathscr{E}$ is said to be right **accumulation point** of $\mathscr{E}$, if for all $t' < t$, $(t', t) \cap \mathscr{E}$ is not empty:

**Definition: Zeno Free**

Admissible event times set $\mathscr{E}$ (or the corresponding solution) is said to be **left (right) Zeno free**, if it **does not contain** any left (right) accumulation points

## Solution Concept Relate to Event-Times

- If solution concept **left Zeno free**: only one solution from origin (Filippov's example)
- If solution concept **right Zeno free**: only local existence (bouncing ball)
- If solution concept **allows Zeno**, then multiple solutions from origin (Filippov's example) and global solutions for bouncing ball

# 4. Well-Posedness for Hybrid Automata

**Definition: Hybrid Time Trajectory**

**Hybrid time trajectory** $\tau = \{I_i\}_{i=0}^{N}$ is finite $(N < \infty)$ or infinite $(N = \infty)$ sequence of intervals of real line, such that

- $I_i = [\tau_i, \tau_i']$ with $\tau_i \leqslant \tau_i' = \tau_{i+1}$ for $0 \leqslant i < N$;
- if $N < \infty$, either $I_N = [\tau_N, \tau_N']$ with $\tau_N \leqslant \tau_N' \neq \infty$ or $I_N = [\tau_N, \tau_N')$ with $\tau_N \leqslant \tau_N' \leqslant \infty$.

**Note:**

No left accumulations of event times!

## Well-Posedness for Hybrid Automata

**Definition: Initial Well-Posedness**

If hybrid automaton is **non-blocking + deterministic**, that is:

- no dead-lock
- no splitting of trajectories

**Note:**

- There exits theoretical condition for the initial well-posedness, but it is not easy to check

- Compared to well-posedness, IWP. do not need to consider the existence interval of the solution. The IWP. makes sure that there is a solution exists and time interval $[0, 0^+]$

**<u>Dilemma of Statement about Hyrbid Automata</u>**

**No statements** by hybrid automata theory on existence,
absence, or continuation

- beyond live-lock: an infinite number of jumps at one time instant, so no solution on $[0, \epsilon)$ for some $\epsilon > 0$
- for left accumulations of event times $\rightarrow$ prevent uniqueness
- for right accumulations of event times $\rightarrow$ prevent global existence

# 5. Well-Posedness for Complementarity Systems

$$\begin{aligned} x(k+1) &= Ax(k) + Bz(k) + Eu(k) \\ w(k) &= Cx(k) + Dz(k) + Fu(k) \\ 0 \leqslant w(k) &\perp z(k) \geqslant 0 \end{aligned}$$

**<u>Well-Posedness</u>**

Given $x(k), u(k) \rightarrow x(k+1), z(k), w(k)$ uniquely determined

## Theorem for Well-Posedness for LCS

Here, we regard the $w(k)$ as $w(k) = Mz(k) + q$

**<u>Linear Complementarity Problem LCP(q,M)</u>**

Given vector $q \in \mathbb{R}^m$ and matrix $M \in \mathbb{R}^{m \times m}$ find $z \in \mathbb{R}^m$ such that

$$0 \leqslant (q + Mz) \perp z \geqslant 0$$

$M \in \mathbb{R}^{m \times m}$ is **<u>P-matrix</u>**, if $\det M_{II} > 0$ for all $I \subseteq \{1, \ldots, m\}$ (that is all subset of the set)

**<u>Theorem</u>**

Discrete-time LCS is well-posed if $D$ is a P-matrix

## Theorem for Initial Well-Posedness for LCS

Consider LCS:

$$\dot{x}(t) = Ax(t) + Bz(t), \quad w(t) = Cx(t) + Dz(t), \quad 0 \leqslant z(t) \perp w(t) \geqslant 0$$

Define

$$G(s) := C(s\mathscr{I} - A)^{-1}B + D \quad Q(s) = C(s\mathscr{I} - A)^{-1}$$

**Theorem:**

LCS is i**nitially well-posed if and only if** for all $x_0$ $LCP(Q(\sigma)x_0, G(\sigma))$ is uniquely solvable for sufficiently large $\sigma \in \mathbb{R}$

- "sufficiently large" means we just need to find one $\sigma$
- dynamical properties can now be **linked to static results on LCPs** which are abundant in literature!
- $G(\sigma)$ being $P$-matrix for sufficiently large $\sigma$ is s**ufficient condition** for initial well-posedness

# Summary

- Solution concepts for smooth and switched systems:
  - well-posedness
  - sliding modes
  - Filippov solutions
- Event times
- Well-posedness for hybrid automata
- Well-posedness for complementarity systems

- Well-Posedness for Smooth System
  - Lipschitz condition
- Solution Concept  and Well-Posedness for Switched Systems
  - Traditional: only allow one dynamics
  - classical generations
  - Fillip's convex definition + sliding mode + differential inclusion + generalized solution concept
  - well-posedeness theorem
- Event-Time Criterion: countable + closed
  - Accumulation Points
  - Zeno Free
  - Solution Concepts
- Well-Posedeness for Hyrbid Automata
  - IWP.: non-blocking + deterministic

- Well-Posedeness for Comlementarity System
    - LCP(q,M), P-matrix
        - D is P-matrix → well-posed
    - IWP. $LCP(Q(\sigma)x_0, G(\sigma))$