

# 03\_Reachability, Observability

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## 1. Reachability and Observability

### **Definition: Reachability (Controllability)**

System is **reachable** if for any  $x_0 \in \mathbb{R}^n, x_f \in \mathbb{R}^n$ , there exists input  $u(\tau)$ , with  $\tau \in [0, t], 0 < t < \infty$ , such that  $x(t) = x_f$

- That is, **We can control system state using our input**
- $\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) = \mathbf{H}_1(s)\mathbf{U}(s)$

### **Definition: Observability**

System is **observable** if any  $x_0 \in \mathbb{R}^n$  can be derived from observation  $y(\tau)$  within the interval  $\tau \in [0, t], t > 0$

- $\mathbf{y} = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$
- **We can derive system state based on output**

## 2. Judgment of Reachability and Observability

### **Kalman Rank Condition**

#### **Reachability**

#### **Reachability Matrix**

$$W_r = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

### Kalman Rank Condition

$(A, B)$  reachable **if and only if** rank

$$W_r = \text{rank} \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n (= \text{number of rows})$$

- In MATLAB, use command `ctrb` over `ss` structure to obtain Reachability matrix

## Observability

### Observability Matrix

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

### Kalman Rank Condition

$(A, C)$  observable **if and only if**

$$\text{rank} W_o = n (= \text{number of columns})$$

- In MATLAB, use `obsv` over `ss` structure to obtain obs. matrix

## Hautus Condition

### Reachability

$(A, B)$  is reachable **iff**

$$\text{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix} = n, \text{ for all } \lambda \in \mathbb{C}$$

### Observability

$(A, C)$  is observable **iff**

$$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n, \text{ for all } \lambda \in \mathbb{C}$$

## Duality

$(A, B)$  reachable  $\Leftrightarrow (A^T, B^T)$  observable

$(A, C)$  observable  $\Leftrightarrow (A^T, C^T)$  reachable

- State feedback and observer design problems are closely related
- Both Reachability and observability are **invariant under state transformations**

## Transfer Function Perspective

### Theorem

- If there are **no zero-pole cancellations** in the transfer function of a **single-input single-output** system, then the system is both **controllable and observable**.
- If a **zero-pole cancellation** occurs, then the system is either **uncontrollable or unobservable** or both **uncontrollable and unobservable**.

### Minimal realization

In control theory, given any transfer function, any state-space model that is **both controllable and observable** and has the **same input-output behaviour** as the transfer function is said to be a minimal realization of the transfer function.

### Illustration

If there exists **zero-pole cancellations**, which means, there are **some information hidden** in the transfer function.

### **Note:**

Use transfer function we can easily figure out the system **has some problem** on reachability and observability. Whether it is unreachable or unobservable still needs to be tested.

For example, based on:

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) = \mathbf{H}_1(s)\mathbf{U}(s)$$

If  $[s\mathbf{I} - \mathbf{A}]^{-1}$  has **zero rows** or **linear independent rows**, which means uncontrollable.

Similar methods can be also used for observability.

## 3. Lack of Reachability/Observability

**UC 1** - physical unreachability - uncoupled variables are not affected by input

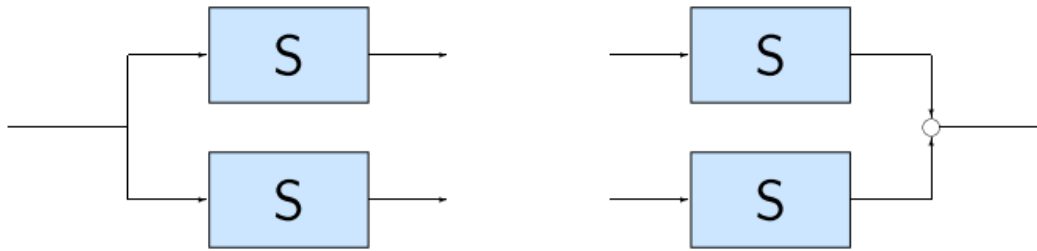
**UC 2** - parallel interconnection controlled by single input

**UO 1** - directly unmeasured variables are not fed back to measured ones

**UO 2** - single variables cannot be extracted from global observation function

### Example 1

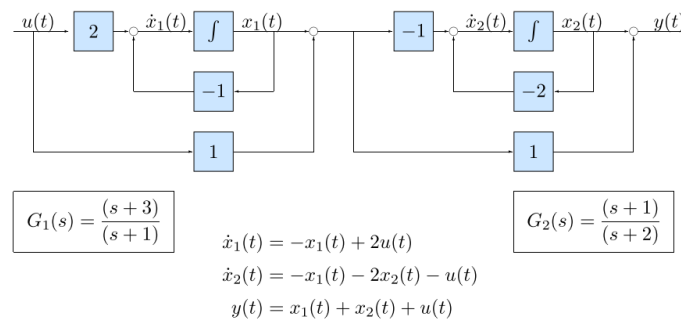
Consider identical systems in parallel:



two system with same State-Space Model

The first is not reachable, whereas the second is not observable

### Example 2: Reachable but not observable

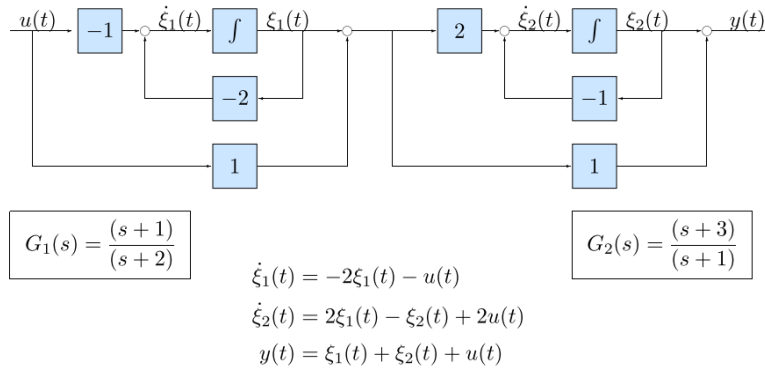


Compute transfer function

$$\begin{aligned}
 G(s) &= C(sI - A)^{-1}B + D \\
 &= \frac{s^2 + 4s + 3}{s^2 + 3s + 2} \\
 &= \frac{(s+3)(s+1)}{(s+2)(s+1)}
 \end{aligned}$$

**Pole-zero cancellation** for  $s = -1$ .

### Example 3: Observable but not reachable



$$G(s) = C(sI - A)^{-1}B + D$$

$$\begin{aligned}
 &= \frac{s^2 + 4s + 3}{s^2 + 3s + 2} \\
 &= \frac{(s+3)(s+1)}{(s+2)(s+1)}
 \end{aligned}$$

**Pole-zero cancellation** for  $s = -1$ .

## 4. Canonical Form

### Unchangeable Property

Given state transformation  $z(t) = Tx(t)$

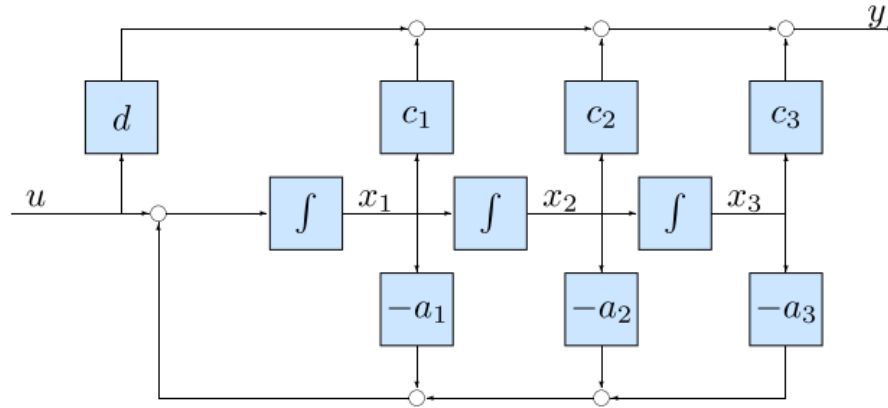
$$\begin{aligned}
 A' &= TAT^{-1} & B' &= TB \\
 C' &= CT^{-1} & D' &= D
 \end{aligned}$$

- If  $(A, B)$  is **reachable**  $\Rightarrow (A', B')$  is **reachable**
- If  $(A, C)$  is **observable**  $\Rightarrow (A', C')$  is **observable**

### Reachable Canonical Form

Transform SISO model coordinates as  $z = Tx$  to obtain the following:

$$\begin{aligned}
 A_r &= \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & 1 & 0 \end{bmatrix}, & B_r &= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \\
 C_r &= [c_1 \quad c_2 \quad c_3 \quad \cdots \quad c_n], & D_r &= [d],
 \end{aligned}$$



- The characteristic polynomial is **the same as that of original model**:

$$\lambda(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$$

- $(A_r, B_r)$  in reachable canonical form is reachable
- state transformations do not affect reachability
- If  $(A, B)$  is **reachable**, Then there **exists** a transformation  $T$  that **yields the reachable canonical form**
- $W_r$  = reachability matrix of original system.  
 $\tilde{W}_r$  = reachability matrix of system in reachable canonical form.

$$T = \tilde{W}_r W_r^{-1}$$

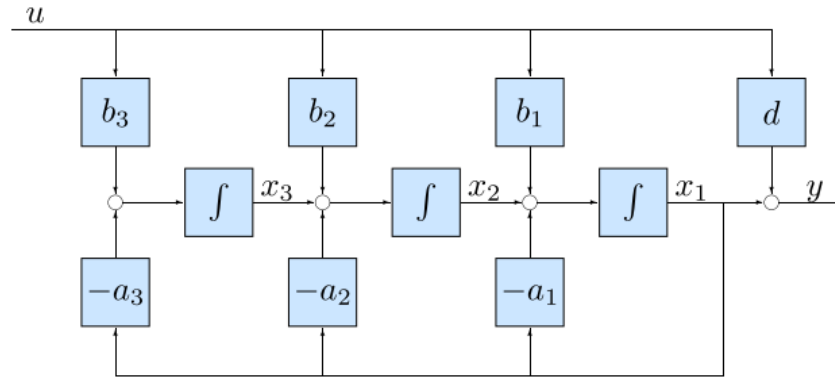
This conclusion can be easily proved by Kalman Rank Condition

## Observable Canonical Form

Transform SISO model coordinates as  $x = T\bar{x}$  to obtain

$$A_o = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix},$$

$$C_o = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad D_o = [d]$$



- The characteristic polynomial is **the same as that of original model**:

$$\lambda(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$$

- $(A_o, C_o)$  in observable canonical form is observable
- If  $(A, C)$  is **observable**, Then there **exists a transformation**  $T$  that yields the observable canonical form
- $W_o$  = observability matrix of original system.  
 $\tilde{W}_o$  = observability matrix of system in observable canonical form.

$$T = \tilde{W}_o^{-1} W_o$$

## Summary

- Reachability and Observability
  - Reachability: input  $\rightarrow$  state
  - Observability: output  $\rightarrow$  state
- Judgement
  - Kalman Rank Condition
  - Hautus Condition
  - Transfer Function Judgement
- Property
  - Duality
  - Unchangeable under similarity transformation
- Canonical Form
  - Reachable Canonical Form: can be used for controller design
  - Observable Canonical Form: can be used for observer design