04_Observer, Feedback Controller, PID Controller and LQR Controller

1. Observer

Luenberger Observer Design

State Estimation Problem Target

Find $\hat{x} \in \mathbb{R}^n$ and associated linear model (observer):

$$\frac{d\hat{x}}{dt} = F\hat{x} + Gu + Hy$$

such that $\hat{x}(t) \to x(t)$ as $t \to \infty$

Theorem: Observer design by eigen value assignment

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)$$

with characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

If the system is **observable**, then there **exists a observer**

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

that gives a closed loop system with the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1}s + p_n$$

The feedback gain is given by

$$L = T^{-1}\tilde{L} = W_o^{-1}\tilde{W}o \begin{bmatrix} p1 - a_1 & p_2 - a_2 & \ldots & p_n - a_n \end{bmatrix}^T$$

Some Mathematical Details

$$\begin{aligned}
&\tilde{A} - \tilde{L}\tilde{C} = \\
&= \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ -a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \\
&= \begin{bmatrix} -p_1 & 1 & \cdots & 0 \\ -p_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -p_n & 0 & \cdots & 0 \end{bmatrix}
\end{aligned}$$

Kalman Filter (Continuous Time Case)

Consider

- Continuous-time Random Process
- Estimate the state of a system **in the presence of noisy measurements**

Model

$$\begin{aligned}
\dot{x} &= Ax + Bu + Fv, & E\left\{v(s)v^T(t)\right\} &= R_v\delta(t-s) \\
y &= Cx + w & E\left\{w(s)w^T(t)\right\} &= R_w\delta(t-s)
\end{aligned}$$

Disturbance v and noise w are zero-mean and Gaussian

Target

$$\min \quad E\left\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\right\}$$

Solution

If system is **observable,** then the solution is given by Kalman observer gain

$$\begin{aligned}
&\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \\
&L = PC^T R_w^{-1}
\end{aligned}$$

where $P \in \mathbb{R}^{n\times n}, P > 0$, is the solution of A.R.E.:

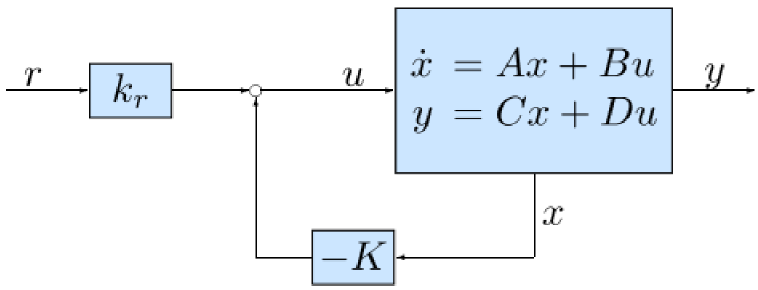
$$AP + PA^T - PC^T R_w^{-1} CP + FR_v F^T = 0$$

A.R.E. can be solved numerically in MATLAB with command `lqe`

2. Feedback Control System Design

Structure 1: $u = -Kx + k_r r$

Generally, can set $u = -Kx + k_r r$, where r is a (constant) reference signal



- Closed-Loop System: $\dot{x} = (A - BK)x + Bk_r r$
- We want to select K so that closed loop has assigned, desired characteristic polynomial

State Feedback Design: Eigenvalue assignment by state feedback

For a **controllable** system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

Original Characteristic Polynomial:

$$a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

Expected Characteristic Polynomial:

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1} s + p_n$$

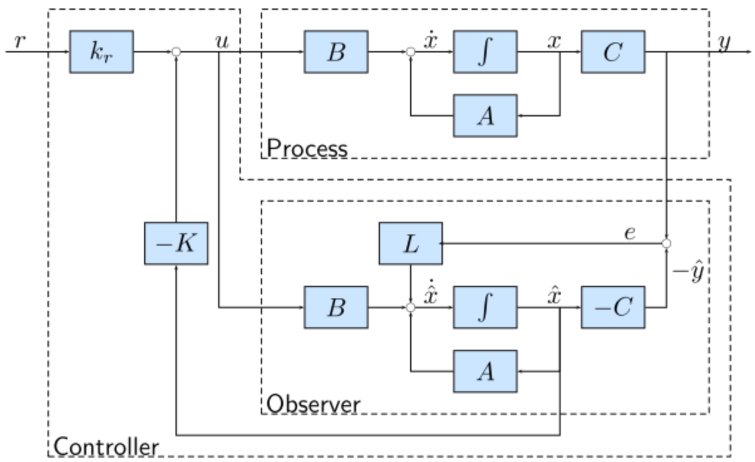
The feedback gain then should be:

$$K = \tilde{K}T = \begin{bmatrix} p_1 - a_1 & p_2 - a_2 & \ldots & p_n - a_n \end{bmatrix} \tilde{W}_r W_r^{-1}$$

<u>Some Mathematical Details</u>

$$
\begin{aligned}
\tilde{A}-\tilde{B}\tilde{K} &= \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} p_1-a_1 & p_2-a_2 & \ldots & p_n-a_n \end{bmatrix} \\
&= \begin{bmatrix} -p_1 & -p_2 & \cdots & -p_n \\ 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}
\end{aligned}
$$

Output Feedback(Estimated State) Controller Design: Separation Principle



Assume that not all the variables are observed, that is we need the **observer** to observe some states. That is we need: **stabilization by output feedback.**

Then the controller has structure: $u = -K\hat{x} + k_r r$. We assume there is **no noise** in the system

System Model

We can illustrate new **augmented state**

$$
x_{\text{new}} = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} x \\ \hat{x} - x \end{bmatrix}
$$

Then the **closed-loop system** will be:

$$
\frac{d}{dt}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r
$$

Then we should try to **select** K so that closed loop has assigned characteristic polynomial

Property

Consider characteristic polynomial of closed-loop system:

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC)$$

That means that the characteristic polynomial can be **decoupled**

The characteristic polynomial can be **assigned arbitrary roots** if

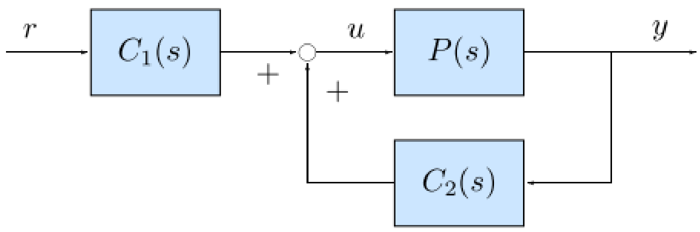
- (A, B) is controllable
- (A, C) is observable

Separation Principle

Eigenvalue assignment for output feedback can be neatly **split into two separate problems**

1. eigenvalue assignment for state feedback
2. eigenvalue assignment for observer

Controller Structure Analysis



$$C_1(s) = -K(sI - A + BK + LC)^{-1}Bk_r + k_r$$
$$C_2(s) = -K(sI - A + BK + LC)^{-1}L$$

Integral Feedback

Evaluation of $u = -Kx + k_r r$ Controller Structure

State feedback is very effective when:

1. model is perfectly known
2. no disturbance is present

Objective of Integral Feedback

1. provide zero steady-state error
2. achieve robustness

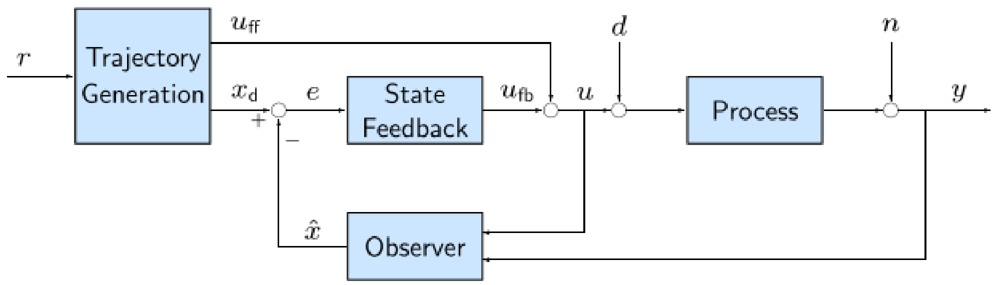
Model

- Introduce **new state**, a signal $z(t) = \int_0^t (y(\tau) - r)d\tau$
- Design $u = -Kx + k_r r - k_i z$ based on $\dot{z} = y - r = Cx - r$
- New state-space model:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A - BK & -Bk_i \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} Bk_r \\ -1 \end{bmatrix} r$$

A General Controller Structure

Based on what is discussed above, a general controller structure can then be established as follows:

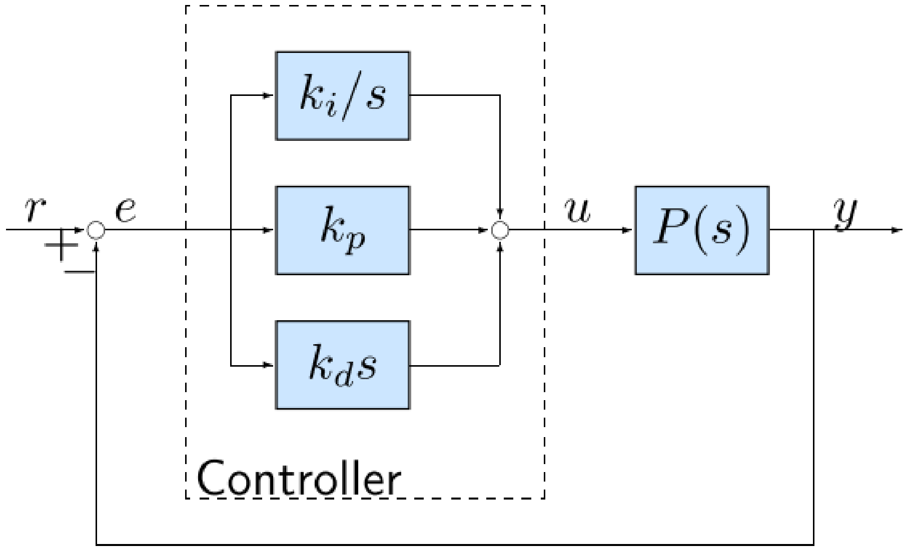


It comprises of 3 parts:

1. Trajectory Generator
    a. Computes input signal u_{ff} such that $y \approx r$
    b. computes corresponding **state** x_d
2. Observer
    a. Built observer to **reconstruct** state $\hat{x}$
3. State Feedback
    a. Built state feedback to **handle disturbance** and **model mismatch**

3. PID Control

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d \frac{de(t)}{dt}$$



P-Controller

- Will have steady-state error
- Will have oscillation

I-Controller

- Can overcome steady-state error
- Will take in oscillation

PID-Controller

- Can Overcome oscillation

3. Linear Quadratic Regulator

$$\int_0^{\infty} \left(x^T Q_x x + u^T Q_u u\right) dt, \quad Q_x \geq 0, Q_u > 0$$

- Trade off closed-loop performance with input effort
- Trade off convergence rate with cost of control input

Solution:

$$u = -Q_u^{-1} B^T P x$$

where $P \in \mathbb{R}^{n \times n}, P > 0$, is the solution of the **Algebraic Riccati Equation (ARE)**:

$$PA + A^T P - PBQ_u^{-1}B^T P + Q_x = 0$$

- A.R.E. can be solved numerically in MATLAB with command `lqr`
- Solution depends on choice of Q_x, Q_u (often taken as diagonal matrices), **different pairs means different limitation on input and state trajectory**

Summary

- Observer:
  - Normal Observer: regard error as a state
  - Kalman Filter
- Feedback Control
  - State Feedback Design, assuming state x is totally known
  - Output Feedback Design, introduce **observer**
  - Integral Feedback, introduce **integral error** part
  - General Structure: 3 parts
- PID controller
- LQR controller: an optimization-based controller