

# 01\_01\_Mathematical Structure of Fuzzy Logic

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## 01\_01\_Mathematical Structure of Fuzzy Logic

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## 1. Definition of Fuzzy Set

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A fuzzy set  $A$  on universe (domain)  $X$  is a **set** defined by the **membership function**  $\mu_A(x)$  which is a mapping from the universe  $X$  into the unit interval:

$$\mu_A(x) : X \rightarrow [0, 1]$$

$\mathcal{F}(X)$  denotes the set of all fuzzy sets on  $X$

$$\mu_A(x) \begin{cases} = 1 & x \text{ is a full member of } A \\ \in (0, 1) & x \text{ is a partial member of } A \\ = 0 & x \text{ is not member of } A \end{cases}$$

## 2. Properties of Fuzzy Set

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### Complement

Let  $A$  be a fuzzy set in  $X$ . The complement of  $A$  is a fuzzy set, denoted  $\bar{A}$ , such that for each  $x \in X$

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

$\lambda$ -complement

$$\mu_{\bar{A}}(x) = \frac{1 - \mu_A(x)}{1 + \lambda\mu_A(x)}$$

### Intersection of Fuzzy Sets

Let  $A$  and  $B$  be two fuzzy sets in  $X$ . The intersection of  $A$  and  $B$  is a fuzzy set  $C$ , denoted  $C = A \cap B$ , such that for each  $x \in X$

$$\mu_C(x) = \min(\mu_A(x), \mu_B(x))$$

The minimum operator is also denoted by  $\wedge$ , i.e.,  $\mu_C(x) = \mu_A(x) \wedge \mu_B(x)$

### Union of Fuzzy Sets

Let  $A$  and  $B$  be two fuzzy sets in  $X$ . The union of  $A$  and  $B$  is a fuzzy set  $C$ , denoted  $C = A \cup B$ , such that for each  $x \in X$

$$\mu_C(x) = \max(\mu_A(x), \mu_B(x))$$

The maximum operator is also denoted by  $\vee$ , i.e.,  $\mu_C(x) = \mu_A(x) \vee \mu_B(x)$

### T-norms and T-conorms

## T-Norm (Intersection)

A **t-norm**  $T$  is a **binary operation** on the unit interval that satisfies at least the following axioms for all  $a, b, c \in [0, 1]$  :

$$\begin{aligned}T(a, 1) &= a \quad (\text{boundary condition}) \\b \leq c \quad &\text{implies} \quad T(a, b) \leq T(a, c) \quad (\text{monotonicity}) \\T(a, b) &= T(b, a) \quad (\text{commutativity}), \\T(a, T(b, c)) &= T(T(a, b), c) \quad (\text{associativity}).\end{aligned}$$

Some frequently used  $t$ -norms are:

|   |                                |
|---|--------------------------------|
| standard (Zadeh) intersection:                  | $T(a, b) = \min(a, b)$         |
| algebraic product (probabilistic intersection): | $T(a, b) = ab$                 |
| Łukasiewicz (bold) intersection:                | $T(a, b) = \max(0, a + b - 1)$ |

- **minimum** is the **largest t-norm** (intersection operator)
- means the membership functions of fuzzy intersections  $A \cap B$  obtained with other t-norms are all **below** the bold membership function

## T-Conorm(Union)

A **t-conorm**  $S$  is a binary operation on the unit interval that satisfies at least the following axioms for all  $a, b, c \in [0, 1]$  (Klir and Yuan, 1995):

$$\begin{aligned}S(a, 0) &= a \quad (\text{boundary condition}) \\b \leq c \quad &\text{implies} \quad S(a, b) \leq S(a, c) \quad (\text{monotonicity}), \\S(a, b) &= S(b, a) \quad (\text{commutativity}) \\S(a, S(b, c)) &= S(S(a, b), c) \quad (\text{associativity}).\end{aligned}$$

Some frequently used  $t$ -conorms are:

|                                      |                              |
|--------------------------------------|------------------------------|
| standard (Zadeh) union:              | $S(a, b) = \max(a, b),$      |
| algebraic sum (probabilistic union): | $S(a, b) = a + b - ab,$      |
| Łukasiewicz (bold) union:            | $S(a, b) = \min(1, a + b) .$ |

- The **maximum** is the **smallest t-conorm** (union operator)
- the membership functions of fuzzy unions  $A \cup B$  obtained with other t-conorms are all **above** the bold membership function

# Projection and Cylindrical Extention

**Projection** reduces a fuzzy set defined in a multi-dimensional domain

**Cylindrical extension** extend of a fuzzy set defined in low-dimensional domain into a higher-dimensional domain

## Projection

Let  $U \in U_1 \times U_2$  be a subset of a Cartesian product space, where  $U_1$  and  $U_2$  can themselves be Cartesian products of lower dimensional domains. The projection of fuzzy set  $A$  defined in  $U$  onto  $U_1$  is the mapping  $proj_{U_1} : \mathcal{F}(U) \rightarrow \mathcal{F}(U_1)$  defined by:

$$proj_{U_1}(A) = \left\{ \sup_{U_2} \mu_A(u) / u_1 \mid u_1 \in U_1 \right\}$$

where *sup* is the supremum operation

The **supremum** (abbreviated *sup*; plural *suprema*) of a subset  $S$  of a partially ordered set  $T$  is the least element in  $T$  that is greater than or equal to all elements of  $S$ , if such an element exists. Consequently, the supremum is also referred to as the **least upper bound (or LUB)**.

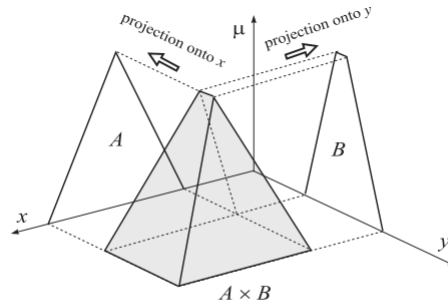


Figure 2.9.: Example of projection from  $\mathbb{R}^2$  to  $\mathbb{R}$ .

**Example 2.4 (Projection)** Assume a fuzzy set  $A$  defined in  $U \subset X \times Y \times Z$  with  $X = \{x_1, x_2\}$ ,  $Y = \{y_1, y_2\}$  and  $Z = \{z_1, z_2\}$ , as follows:

$$A = \{ \mu_1 / (x_1, y_1, z_1), \mu_2 / (x_1, y_2, z_1), \mu_3 / (x_2, y_1, z_1), \mu_4 / (x_2, y_2, z_1), \mu_5 / (x_2, y_2, z_2) \} \quad (2.31)$$

Let us compute the projections of  $A$  onto  $X$ ,  $Y$  and  $X \times Y$ :

$$proj_X(A) = \{ \max(\mu_1, \mu_2) / x_1, \max(\mu_3, \mu_4, \mu_5) / x_2 \}, \quad (2.32)$$

$$proj_Y(A) = \{ \max(\mu_1, \mu_3) / y_1, \max(\mu_2, \mu_4, \mu_5) / y_2 \}, \quad (2.33)$$

$$proj_{X \times Y}(A) = \{ \mu_1 / (x_1, y_1), \mu_2 / (x_1, y_2), \mu_3 / (x_2, y_1), \max(\mu_4, \mu_5) / (x_2, y_2) \}. \quad (2.34)$$

## Cylindrical Extension

Let  $U \subseteq U_1 \times U_2$  be a subset of a Cartesian product space, where  $U_1$  and  $U_2$  can themselves be Cartesian products of lower-dimensional domains. The cylindrical extension of fuzzy set  $A$  defined in  $U_1$  onto  $U$  is the mapping  $ext_U : \mathcal{F}(U_1) \rightarrow \mathcal{F}(U)$  defined by

$$ext_U(A) = \{\mu_A(u_1)/u \mid u \in U\}$$

- Cylindrical extension thus **simply replicates** the membership degrees from the existing dimensions into the new dimensions.

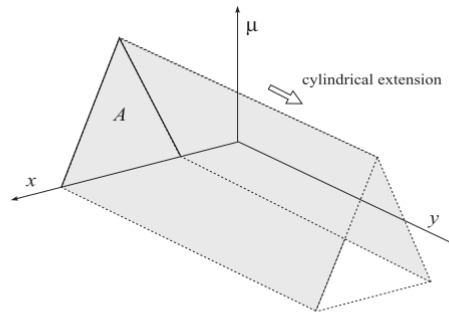


Figure 2.10.: Example of cylindrical extension from  $\mathbb{R}$  to  $\mathbb{R}^2$ .

## Properties

- **projection** leads to a **loss of information**, so we have

$$A = \text{proj}_{X^n} (\text{ext}_{X^m}(A))$$
$$A \neq \text{ext}_{X^m} (\text{proj}_{X^n}(A))$$

## Operations on Cartesian Product Domains

Set-theoretic operations such as the **union or intersection** applied to fuzzy sets **defined in different domains** result in a **multi-dimensional fuzzy set** in the Cartesian product of those domains.

- first **extending** the original fuzzy sets into the Cartesian product domain
- then **computing the operation** on those multi-dimensional sets.

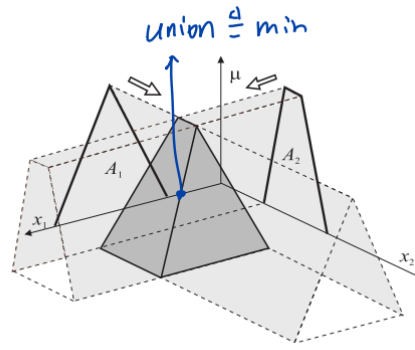
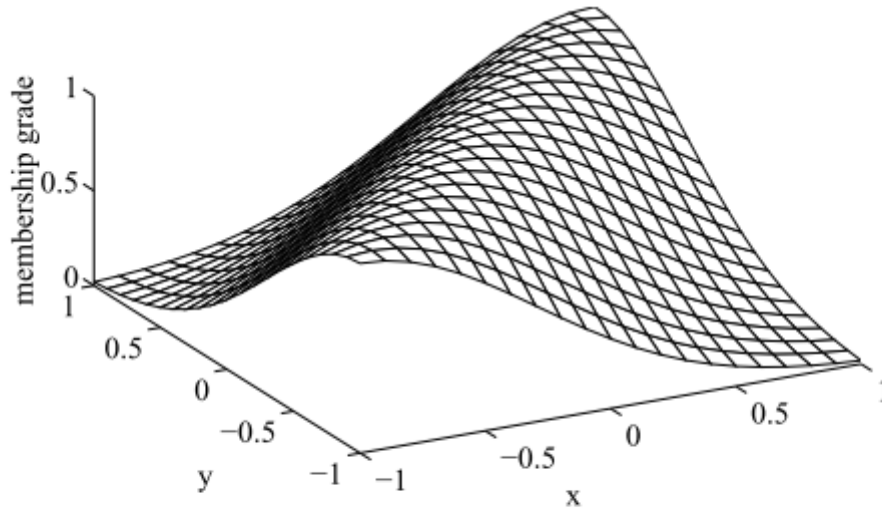


Figure 2.11.: Cartesian-product intersection.

Figure 2.13.: Fuzzy relation  $\mu_R(x, y) = e^{-(x-y)^2}$ .

**Example 2.5 (Cartesian-Product Intersection)** Consider two fuzzy sets  $A_1$  and  $A_2$  defined in domains  $X_1$  and  $X_2$ , respectively. The intersection  $A_1 \cap A_2$ , also denoted by  $A_1 \times A_2$  is given by:

$$A_1 \times A_2 = \text{ext}_{X_2}(A_1) \cap \text{ext}_{X_1}(A_2). \quad (2.38)$$

This **cylindrical extension is usually considered implicitly** and it is not stated in the notation:

$$\mu_{A_1 \times A_2}(x_1, x_2) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2). \quad (2.39)$$

Figure 2.11 gives a graphical illustration of this operation.

## 4. Fuzzy Relations

An n-ary **fuzzy relation** is a **mapping**

$$R : X_1 \times X_2 \times \cdots \times X_n \rightarrow [0, 1]$$

which assigns membership grades to all n-tuples  $(x_1, x_2, \dots, x_n)$  from the Cartesian product  $X_1 \times X_2 \times \cdots \times X_n$ .

- For **computer implementations**,  $R$  is conveniently represented as an **n-dimensional array**:  $R = [r_{i_1, i_2, \dots, i_n}]$
- A **fuzzy relation** is a **fuzzy set** in the Cartesian product  $X_1 \times X_2 \times \dots \times X_n$ . The membership grades represent the degree of association (correlation) among the elements of the different domains  $X_i$ .

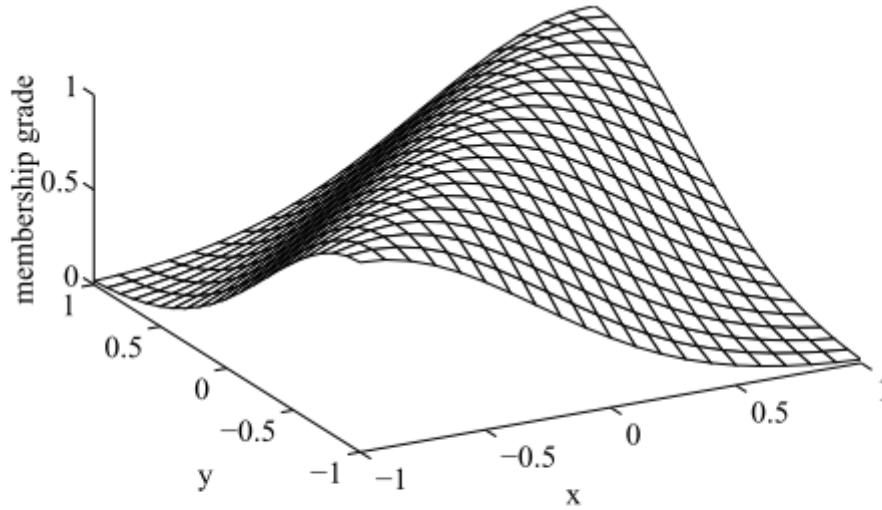


Figure 2.13.: Fuzzy relation  $\mu_R(x, y) = e^{-(x-y)^2}$ .

## 5. Relational Composition

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The **composition** is defined as follows : suppose there exists a fuzzy relation  $R$  in  $X \times Y$  and  $A$  is a fuzzy set in  $X$ . Then, fuzzy subset  $B$  of  $Y$  can be **induced** by  $A$  through the composition of  $A$  and  $R$ :

$$B = A \circ R$$

And is defined by:

$$B = \text{proj}_Y (R \cap \text{ext}_{X \times Y}(A))$$

### Understanding

The composition can be regarded in two phases:

- combination(intersection)
- projection

## sup-min composition

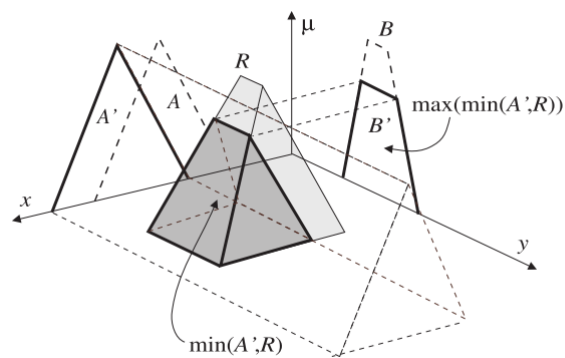
$$\mu_B(y) = \sup_x (\min(\mu_A(x), \mu_R(x, y)))$$

"given y, find x, in which make the min() part max"

More general form

$$\mu_B(y) = \sup_x (T(\mu_A(x), \mu_R(x, y)))$$

the  $T$  means a t-norm  $T$



## Example



**Example 2.8 (Relational Composition)** Consider a fuzzy relation  $R$  which represents the relationship “ $x$  is *approximately equal* to  $y$ ”:

$$\mu_R(x, y) = \max(1 - 0.5 \cdot |x - y|, 0). \quad (2.45)$$

Further, consider a fuzzy set  $A$  “*approximately 5*”:

$$\mu_A(x) = \max(1 - 0.5 \cdot |x - 5|, 0). \quad (2.46)$$

Suppose that  $R$  and  $A$  are discretized with  $x, y = 0, 1, 2, \dots$ , in  $[0, 10]$ . **For a discrete set**

***max* is equivalent to *sup*.** Then, the composition is:

$$\begin{aligned} \mu_B(y) &= \overbrace{\left( 0 \ 0 \ 0 \ 0 \ 0 \ \frac{1}{2} \ 1 \ \frac{1}{2} \ 0 \ 0 \ 0 \ 0 \right)}^{\mu_A(x)} \circ \overbrace{\begin{pmatrix} \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}}^{\mu_R(x, y)} = \\ &= \max_x \overbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}^{\min(\mu_A(x), \mu_R(x, y))} = \\ &= \overbrace{\left( 0 \ 0 \ 0 \ 0 \ \frac{1}{2} \ \frac{1}{2} \ 1 \ \frac{1}{2} \ \frac{1}{2} \ 0 \ 0 \ 0 \right)}^{\max_x \left( \min(\mu_A(x), \mu_R(x, y)) \right)} \end{aligned}$$

In this graph, different column means different x

This resulting fuzzy set, defined in Y **can be interpreted as “approximately 5”** . Note, however, that it is **broadier (more uncertain)** than the set from which it was induced. This is because the combination of uncertainty in input fuzzy and relation