DTMC

1. Basic Knowledge

Markov Chains

DTMC

- State Transition Matrix

① States: $\pi_k = (\pi_1, \pi_2, \pi_3)$

② Transition Matrix: → end state

$$\begin{array}{c|ccc} & P_1 & P_2 & P_3 \\ \hline P_1 & P_{11} & \cdots & P_{13} \\ P_2 & P_{21} & \cdots & P_{23} \\ P_3 & P_{31} & \cdots & P_{33} \end{array}$$

↓
Start State

③ Equation:

$$\pi_{k+1} = \pi_k \cdot P$$

- Stationary Distribution

two Definitions:

① $\lim_{n\to\infty} (P)^n$:

② $\pi \cdot P = \pi$ $\quad \pi \to$ Steady State probability

Properties:

① $\lim_{n\to\infty} P^n$: each row are equal

② $\pi_j = \lim_{n\to\infty} P_{ij}^n$

- Property of DTMC

① Accessibility: $j \leftarrow i$

State j is accessible from state i if $P_{ij}^{(n)} > 0$. for some $n \geq 0$

② Communicability: $j \leftrightarrow i$

State j is communicate if $j \leftarrow i$ & $i \leftarrow j$

③ Irreducibility: ☆ (property of a chain)

A Markov Chain is irreducible

$\iff$ all states belong to one class (all states communicate with each other)

· Lemma:

there exists some $n > 0$, for all i and j, $P_{ij}^{(n)} > 0$

$\Rightarrow$ all states communicate $\Rightarrow$ the Markov Chain irreducible

④ Periodicity: ☆ (property of a class)

☆ ≠ minimum n. ☆

Period of j: the greatest common divisor (GCD) of the set of integers, such that $P_{jj}^n > 0$

Aperiodic: A state has period 1.

Periodicity is a class property → All nodes in the same class has the same property

Lemma:

A MC with periodical state

$\Rightarrow$ there is no steady state

☆ Lemma 2:

A markov chain periodical + reducibility

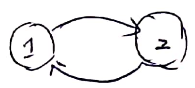
$\Rightarrow$ there is no steady state

① reducibility: $3 \not\to 1$

If start at 3 $\pi = (0,0,1)$

If start at 1 π will not be $(0,0,1)$

② Periodical:



$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

π will change in sequence: (start from $(1,0)$)

$(1,0) \to (0,1) \to (1,0) \to \cdots$

- Transient and Recurrent State (for state i)

① Transient:

there is a probability that the process will move to state j and never return to state i

class property

② Recurrent:

The process returns to state i with probability 1

☆ (State i no transient $\longrightarrow$ must be recurrent)

class property

③ Absorbing: if $P_{ii} = 1$

(that means, after arriving state i, we can never leave it)

☆ If i Absorbing $\to$ i recurrent

- Classes

Two states are in the same classes

$\triangleq$ two states communicate with each other

<1> A way to solve reducibility

$$M = (1-p) B + p \cdot A$$

where: $A \rightarrow$ original transition matrix

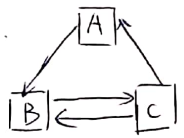
$B \rightarrow \frac{1}{n} \cdot \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{n \times n}$

that is:

① create virtual connections to all other nodes

② weight of new connections lower than original ones

e.g.



$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0 & 0.5 \end{bmatrix}$ take $p = 0.3$, $n = 3$

$\therefore M = 0.7 \cdot A + 0.3/3 \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

- Ergodicity

If All states in a MC: recurrent, aperiodic and communicate with each other

$\Leftrightarrow$ ergodicity

☆ If a chain is ergodicity

$\Rightarrow$ the steady state exists

2. Introduction

2.1. Random Variables and Random Process

- A **random variable** is a variable whose value is unknown or a function that assigns values to each of an experiment's outcomes.
- A **random process** is a **collection of random variables** usually indexed by time.

2.2. Definition

Consider the random process $X_n, n = 0, 1, 2, \cdots$, where $R_{X_i} = S \subset 0, 1, 2, \cdots$. We say that this process is a **Markov chain** if:

$$
\begin{aligned}
P(X_{m+1} = j | X_m = i, X_{m-1} = i_{m-1}, \cdots, X_0 = i_0) \\
= P(X_{m+1} = j | X_m = i)
\end{aligned}
$$

for all $m, j, i, i_0, i_1, \cdots i_{m-1}$. If the number of states is finite, e.g., $S = 0, 1, 2, \cdots, r$, we call it a **finite Markov chain**.

- $P(X_{m+1} = j | X_m = i)$ is called **transition probabilities**

2.3.State Transition Matrix

We often list the transition probabilities in a matrix. The matrix is called the **state transition matrix** or **transition probability matrix** and is usually shown by **P**. Assuming the states are **1, 2, ⋯, r**, then the state transition matrix is given by

$$
P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix}.
$$

Note that $p_{ij} \geq 0$ and for all i, we have

$$
\begin{aligned}
\sum_{k=1}^{r} p_{ik} = \sum_{k=1}^{r} P(X_{m+1} = k | X_m = i) \\
= 1.
\end{aligned}
$$

2.4. Classification of States

2.4.1. Accessible

We say that state j is accessible from state i, written as i → j, if $p_{ij}^{(n)} > 0$ for some n. We assume every state is accessible from itself since $p_{ii}^{(0)} = 1$.

2.4.2. Communicate

Two states i and j are said to **communicat**e, written as i ↔ j, if they are accessible from each other. In other words,

$$i \leftrightarrow j \text{ means } i \to j \text{ and } j \to i.$$

- Communication is an **equivalence relation**

2.4.3. Irreducible

A Markov chain is said to be **irreducible** if all states communicate with each other.

2.4.4. Transient and Recurrent and Absorbing

For any state i, we define

$$f_{ii} = P(X_n = i, \text{ for some } n \geq 1 | X_0 = i).$$

State i is **recurrent** if $f_{ii} = 1$, and it is **transient** if $f_{ii} < 1$.

- if a state is not recurrent, then it will be transient

A state i is called **absorbing** if there are no outgoing transitions from the state. that is $p_{ii} = 1$

- an absorbing state is a recurrent state

2.4.4. Periodicity

The **period** of a state i is the largest integer d satisfying the following property: $p_{ii}^{(n)} = 0$, whenever n is not divisible by d. The period of i is shown by $d(i)$. If $p_{ii}^{(n)} = 0$, for all $n > 0$, then we let $d(i) = \infty$.

- If $d(i) > 1$, we say that state i is **periodic**
- If $d(i) = 1$, we say that state i is **aperiodic**.

or we can say:

The **period** of state j is the **greatest common divisor (GCD)** of the set of integer, such that $P_{jj}^{n} > 0$

2.4.5. Ergodicity

If all states in a chain are recurrent, aperiodic, and communicate with each other, the chain is said to be ergodic

3. Stationary and Limiting Distributions

3.1. Limiting Distributions (Steady state distribution)

The probability distribution $\pi = [\pi_0, \pi_1, \pi_2, \cdots]$ is called the **limiting distribution** of the Markov chain X_n iff

$$\pi_j = \lim_{n\to\infty} P(X_n = j | X_0 = i)$$

for all $i, j \in S$, and we have

$$\sum_{j\in S} \pi_j = 1$$

when a limiting distribution exists, it does not depend on the initial state($X_0 = i$), so we can write

$$\pi_j = \lim_{n\to\infty} P(X_n = j), \text{ for all } j \in S.$$

3.2. Stationary distribution

$$\pi = \pi P, \sum_{j\in S} \pi_j = 1$$

3.3. Relations and Properties

- The limiting distribution of a Markov chain is a stationary distribution of the Markov chain.
- Limiting distribution is Unique (because it need suitable for all initial state)
- Stationary Distribution may not be unique (linear equation groups' solution)
- Not All Markov Chains Have a Stationary Distribution (linear equation group's solution)
- Not All Markov Chains Have Limiting Distribution
- if a limiting distribution exists, it is the unique stationary distribution

3.4. Theorem

- If a chain is **irreducible** and **aperiodic**, it will has a unique stationary distribution, and this stationary distribution will be its limiting distribution