# 02_Modelling Framework of Hybrid Systems



# 1. Frameworks

## Piecewise Affine Systems (PWA)

> region partition and piece-wise affine constraint

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + f_i \\ y(k) &= C_i x(k) + D_i u(k) + g_i \end{aligned} \text{ for } \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \Omega_i, i = 1, \ldots, N$$

with

$\Omega_1, \cdots, \Omega_N$: **convex** polyhedra (i.e., given by finite number of linear inequalities) in input/state space, **non-overlapping** interiors

- PWA can be used as **approximation of nonlinear model:** e.g. using least square criterion

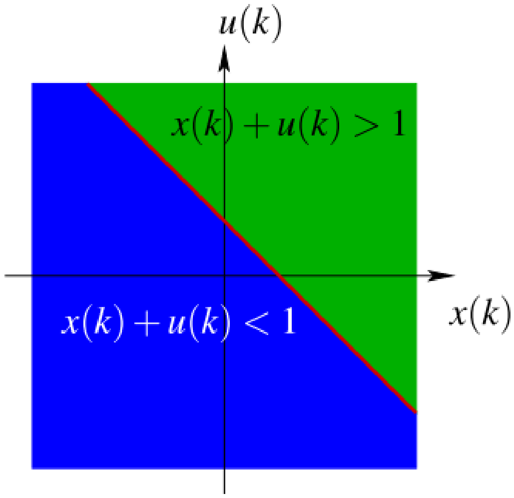
**Example**

Integrator with upper saturation:

$$x(k+1)=\begin{cases}x(k)+u(k) & \text{if } x(k)+u(k)\leqslant 1\\ 1 & \text{if } x(k)+u(k)\geqslant 1\end{cases}$$

$$y(k)=x(k)$$

deterministic?
* if in blue,  no problem
* if in green, no problem
* if in red line, output is equal to 1
so, if from input-output behavior:
deterministic
if from mode perspective,
cannot deterministic




hs_mod.5

# Mixed Logical Dynamical (MLD) Systems

> boolean variable + linear equality constraint

$$\begin{aligned} x(k+1) &= Ax(k)+B_1u(k)+B_2\delta(k)+B_3z(k)\\ y(k) &= Cx(k)+D_1u(k)+D_2\delta(k)+D_3z(k)\\ E_1x(k)+E_2u(k)+E_3\delta(k)+E_4z(k) &\leqslant g_5, \end{aligned}$$

- $x(k)=\left[x_{\mathrm{r}}^{\top}(k) x_{\mathrm{b}}^{\top}(k)\right]^{\top}$ with $x_{\mathrm{r}}(k)$ real-valued, $x_{\mathrm{b}}(k)$ boolean
- $z(k)$: real-valued auxiliary variables
- $\delta(k)$: boolean auxiliary variables

**Transformation of Logical part**

- Associate with literal $X_i$ logical variable $\delta_i\in\{0,1\}$:

  $\delta_i=1$ iff $X_i=\mathrm{T}, \delta_i=0$ iff $X_i=\mathrm{F}$
- Other Logical Operations

$$\begin{aligned} &X_1\wedge X_2 \text{ equivalent to } \delta_1=\delta_2=1\\ &X_1\vee X_2 \text{ equivalent to } \delta_1+\delta_2\geqslant 1\\ &\sim X_1 \text{ equivalent to } \delta_1=0\\ &X_1\Rightarrow X_2 \text{ equivalent to } \delta_1-\delta_2\leqslant 0\\ &X_1\Leftrightarrow X_2 \text{ equivalent to } \delta_1-\delta_2=0\\ &X_1\oplus X_2 \text{ equivalent to } \delta_1+\delta_2=1 \end{aligned}$$

**Transformation of Real-Value Function**

- For $f:\mathbb{R}^n \to \mathbb{R}$ and $x \in \mathscr{X} with \mathscr{X}$ bounded, define
- $M \stackrel{\text{def}}{=} \max_{x\in\mathscr{X}} f(x) \quad m \stackrel{\text{def}}{=} \min_{x\in\mathscr{X}} f(x)$
- Mixture Logic of real-value function and boolean variables

$$
\begin{array}{l}
[f(x) \leqslant 0] \wedge [\delta = 1] \text{ true iff } f(x) - \delta \leqslant -1 + m(1-\delta) \\
[f(x) \leqslant 0] \vee [\delta = 1] \text{ true iff } f(x) \leqslant M\delta \\
\sim [f(x) \leqslant 0] \text{ true iff } f(x) \geqslant \varepsilon \quad \text{(with } \varepsilon \text{ machine precision)} \\
[f(x) \leqslant 0] \Rightarrow [\delta = 1] \text{ true iff } f(x) \geqslant \varepsilon + (m-\varepsilon)\delta \\
[f(x) \leqslant 0] \Leftrightarrow [\delta = 1] \text{ true iff } \begin{cases} f(x) \leqslant M(1-\delta) \\ f(x) \geqslant \varepsilon + (m-\varepsilon)\delta \end{cases}
\end{array}
$$

**Transformation of Product of Logic Variable**

Product $\delta_1\delta_2$ can be replaced by auxiliary variable $\delta_3 = \delta_1\delta_2$

$$
\delta_3 = \delta_1\delta_2 \quad \text{is equivalent to} \begin{cases} -\delta_1 + \delta_3 \leqslant 0 \\ -\delta_2 + \delta_3 \leqslant 0 \\ \delta_1 + \delta_2 - \delta_3 \leqslant 1 \end{cases}
$$

**Transformation of Product of Logic Variable and Real-value Function**

$\delta f(x)$ can be replaced by auxiliary real variable $y = \delta f(x)$

$$
y = \delta f(x) \quad \text{is equivalent to} \begin{cases} y \leqslant M\delta \\ y \geqslant m\delta \\ y \leqslant f(x) - m(1-\delta) \\ y \geqslant f(x) - M(1-\delta) \end{cases}
$$

**One example of transformation from PWA to MLD**

- Consider PWA system:

$$x(k+1) = \begin{cases} 0.8x(k) + u(k) & \text{if } x(k) \geqslant 0 \\ -0.8x(k) + u(k) & \text{if } x(k) < 0 \end{cases}$$

where $x(k) \in [-10, 10]$ and $u(k) \in [-1, 1]$

- Associate binary variable $\delta(k)$ to condition $x(k) \geqslant 0$ such that $[\delta(k) = 1] \Leftrightarrow [x(k) \geqslant 0]$ or

$$\begin{aligned} -m\delta(k) &\leqslant x(k) - m \\ -(M + \varepsilon)\delta(k) &\leqslant -x(k) - \varepsilon \end{aligned}$$

where $M = -m = 10$, and $\varepsilon$ is machine precision

- PWA system can be rewritten as

$$x(k+1) = 1.6\,\delta(k)x(k) - 0.8\,x(k) + u(k)$$

hs_mod.11

- $x(k+1) = 1.6\,\delta(k)x(k) - 0.8\,x(k) + u(k)$
- Define new variable $z(k) = \delta(k)x(k)$ or

$$\begin{aligned} z(k) &\leqslant M\delta(k) \\ z(k) &\geqslant m\delta(k) \\ z(k) &\leqslant x(k) - m(1 - \delta(k)) \\ z(k) &\geqslant x(k) - M(1 - \delta(k)) \end{aligned}$$

- PWA system now becomes

$$x(k+1) = 1.6z(k) - 0.8x(k) + u(k)$$

subject to linear constraints above $\rightarrow$ MLD

## Linear Complementarity (LC) systems

> complementarity condition + linear equality constraint

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1u(k) + B_2w(k) \\
y(k) &= Cx(k) + D_1u(k) + D_2w(k) \\
v(k) &= E_1x(k) + E_2u(k) + E_3w(k) + e_4 \\
0 \leqslant v(k) &\perp w(k) \geqslant 0
\end{aligned}
$$

$v(k), w(k)$: “complementarity variables” (real-valued)

- LC systems do not have $\delta$ and inequality constraints
- LC systems only have equation and complementary conditions

## Extended Linear Complementarity (ELC) systems

> group complementarity condition + linear equality constraint

$$
\begin{aligned}
&x(k+1) = Ax(k) + B_1u(k) + B_2d(k) \\
&y(k) = Cx(k) + D_1u(k) + D_2d(k) \\
&E_1x(k) + E_2u(k) + E_3d(k) \leqslant e_4 \\
&\textstyle\sum_{i=1}^{p} \prod_{j\in\phi_i} \left(e_4 - E_1x(k) - E_2u(k) - E_3d(k)\right)_j = 0
\end{aligned}
$$

- $d(k)$: real-valued auxiliary variable
- The 4th condition is equal to $\prod_{j\in\phi_i} \left(e_4 - E_1x(k) - E_2u(k) - E_3d(k)\right)_j = 0 \quad \text{for each } i \in \{1, \ldots, p\}$
    - system of linear inequalities with $p$ groups, in each group **at least one inequality should hold with equality**

## Max-Min-Plus-Scaling (MMPS) systems

**Max-min-plus-scaling expression :**

$$
f := x_i |\alpha| \max\left(f_k, f_l\right) |\min\left(f_k, f_l\right)| \, f_k + f_l \mid \beta f_k
$$

with $\alpha, \beta \in \mathbb{R}$ and $f_k, f_l$ again **MMPS expressions.**

**MMPS Systems**

$$
\begin{aligned}
&x(k+1) = \mathscr{M}_x(x(k), u(k), d(k)) \\
&y(k) = \mathscr{M}_y(x(k), u(k), d(k)) \\
&\mathscr{M}_c(x(k), u(k), d(k)) \leqslant c \\
&\text{ with } \mathscr{M}_x, \mathscr{M}_y, \mathscr{M}_\text{c} \text{ MMPS expressions}
\end{aligned}
$$

$d(k)$: real-valued auxiliary variables

- It is good for systems with soft & hard synchronization constraints

**Some Illustrated Example**

- **max**: one of two product is ready is okay
- **min**: both of two product are needed

**Example**

- Integrator with upper saturation:

$$x(k+1) = \begin{cases} x(k)+u(k) & \text{if } x(k)+u(k) \leqslant 1 \\ 1 & \text{if } x(k)+u(k) \geqslant 1 \end{cases}$$
$$y(k) = x(k)$$

can be recast as

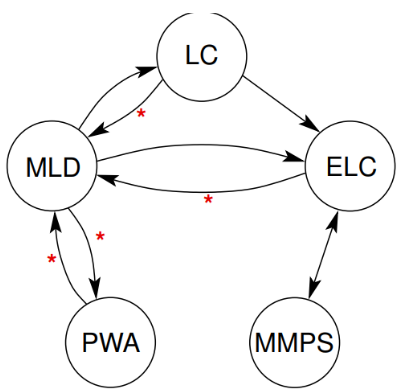
$$x(k+1) = \min(x(k)+u(k), 1)$$
$$y(k) = x(k)$$

# 2. Equivalence of MLD, LC, ELC, PWA, and MMPS systems

## Definition of Equivalence

Equivalence between model classes $\mathscr{A}$ and $\mathscr{B}$ :
for each model $\in \mathscr{A}$ there exists model $\in \mathscr{B}$ with same i**nput/output behavior** (+ vice versa )

## Equivalence Relations




- Red Star means this transformation need some constraints

## Different Advantages

Each subclass has own advantages:

- **stability criteria** for PWA
- control and **verification** techniques for MLD
- **control** techniques for MMPS

- conditions of **existence and uniqueness** of solutions for LC
- **transfer techniques** from one class to other

Based on the transformation, we can transform among them for different usage.

# 3. Transformation Among Different Models

## MLD and LC systems

**Theorem**

Every MLD system can be written as LC system

**Method**

- $\delta_i(k) \in \{0,1\}$ is equivalent to $0 \leqslant \delta_i(k) \perp 1-\delta_i(k) \geqslant 0$
  $\to$ introduce auxiliary variable $p(k) = [1 \;\; 1 \; \dots \; 1]^\top - \delta(k)$ with

$$0 \leqslant \delta(k) \perp p(k) \geqslant 0$$

- For constraint $E_1x(k)+E_2u(k)+E_3\delta(k)+E_4z(k) \leqslant g_5$, introduce auxiliary variables $q(k) = g_5 - E_1x(k) - E_2u(k) - E_3\delta(k) - E_4z(k) \geqslant 0$ and $r(k) \geqslant 0$ with

$$0 \leqslant q(k) \perp r(k) \geqslant 0$$

  Complementarity condition will always hold (select $r(k) = 0$)

- For LC: all variables $\geqslant 0$
  $\to$ split real-valued variable $z(k)$ in "positive" and "negative part":
  $z(k) = z^+(k) - z^-(k)$ with $z^+(k) = \max(0, z(k))$, $z^-(k) = \max(0, -z(k))$
  or $0 \leqslant z^+(k) \perp z^-(k) \geqslant 0$

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1u(k) + \begin{bmatrix} B_2 & 0 & B_3 & -B_3 \end{bmatrix} w(k) \\
y(k) &= Cx(k) + D_1u(k) + \begin{bmatrix} D_2 & 0 & D_3 & -D_3 \end{bmatrix} w(k) \\
\underbrace{\begin{pmatrix} p(k) \\ q(k) \\ s(k) \\ t(k) \end{pmatrix}}_{=:v(k)} &= \begin{pmatrix} e \\ g_5 - E_1x(k) - E_2u(k) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -I & 0 & 0 & 0 \\ -E_3 & 0 & -E_4 & E_4 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \delta(k) \\ r(k) \\ z^+(k) \\ z^-(k) \end{pmatrix}}_{=:w(k)} \\
0 &\leqslant v(k) \perp w(k) \geqslant 0
\end{aligned}
$$

**Theorem**

Every LC system can be written as MLD provided that $w(k)$ and $v(k)$ are **bounded**

**Method**

- Transform $v(k)$ and $w(k)$
  - LC complementarity condition $0 \leqslant v(k) \perp w(k) \geqslant 0$ implies that for each $i$ we have $v_i(k) = 0, w_i(k) \geqslant 0$ or $v_i(k) \geqslant 0, w_i(k) = 0$
  - **Introduce boolean vector** $\delta(k)$ such that

$$\begin{aligned} v_i(k) = 0, w_i(k) \geqslant 0 \leftrightarrow \delta_i(k) = 1 \\ v_i(k) \geqslant 0, w_i(k) = 0 \leftrightarrow \delta_i(k) = 0 \end{aligned}$$

  - Can be achieved by **introducing constraints**

$$\begin{aligned} & w(k) \leqslant M_w \delta(k) \\ & v(k) \leqslant M_v \left( [11 \ldots 1]^\top - \delta(k) \right) \\ & w(k), v(k) \geqslant 0 \end{aligned}$$

    with $M_w, M_v$ diagonal matrices containing **upper bounds** on $w(k), v(k)$(in practice, the upper bounds usually known due to physical reasons/insight)

  - use $z(k)$ to repesent $w(k)$
  - replacing linear equality constraints: Replacing $v(k)$ by $E_1 x(k) + E_2 u(k) + E_3 w(k) + e_4$ in inequalities finally results in MLD model

$$\begin{gathered} x(k+1) = Ax(k) + B_1 u(k) + B_2 z(k) \\ y(k) = Cx(k) + D_1 u(k) + D_2 z(k) \\ \begin{bmatrix} 0 \\ E_1 \\ 0 \\ -E_1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ E_2 \\ 0 \\ -E_2 \end{bmatrix} u(k) + \begin{bmatrix} -M_w \\ M_v \\ 0 \\ 0 \end{bmatrix} \delta(k) + \begin{bmatrix} I \\ E_3 \\ -I \\ -E_3 \end{bmatrix} z(k) \leqslant \begin{bmatrix} 0 \\ M_v e - e_4 \\ 0 \\ e_4 \end{bmatrix} \end{gathered}$$

    with $e = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^\top, z(k) = w(k)$

## LC and ELC systems

**Theorem**

Every LC system can be written as ELC system

$$v(k) \perp w(k) \text{ is equivalent to } \sum v_i(k) w_i(k) = 0$$

## PWA and MLD systems

**Theorem**

**Well-posed** PWA system can be rewritten as MLD system assuming that set of **feasible states and inputs is bounded**

See the example

<u>**Theorem**</u>

**Completely well-posed** MLD can be rewritten as PWA

<u>**well-posed**</u>: given $x(k), u(k)$, then $x(k+1), u(k+1)$ are unique

<u>**completely well-posed**</u>: given $x(k), u(k)$, then $x(k+1), u(k+1), \delta(k), z(k)$ are unique. In MLD it means the inequalities can only yield one possible solution

## MMPS and ELC

<u>**Theorem**</u>

The classes of MMPS and ELC systems **coincide**

**MMPS $\subseteq$ ELC**

- Expressions of form $f = x_i, f = \alpha, f = f_k + f_l, f = \beta f_k$ result in linear equations
- $f = \max(f_k, f_l) = -\min(-f_k, -f_l)$ can be rewritten as

$$f - f_k \geqslant 0, \quad f - f_l \geqslant 0, \quad (f - f_k)(f - f_l) = 0$$

- Two or more ELC systems can be combined into one large ELC

**ELC $\subseteq$ MMPS**

- Linear equations are MMPS expressions
- Complementarity condition can be transformed
  - $\forall i, \exists j \in \phi_i \text{ such that } \underbrace{(e_4 - E_1 x(k) - E_2 u(k) - E_3 d(k))_j}_{\geqslant 0} = 0$
  - Then: $\min_{j \in \phi_i} (e_4 - E_1 x(k) - E_2 u(k) - E_3 d(k))_j = 0 \quad \text{for each } i$

## MLD and ELC systems

<u>**Theorem**</u>

**Every MLD system can be rewritten as ELC system**

- boolean variables:

$$\begin{aligned} & -\delta_i(k) \leqslant 0 \\ & \delta_i(k) \leqslant 1 \\ & \delta_i(k)(1 - \delta_i(k)) = 0 \end{aligned}$$

- Note, if want to direct transform MLD to MMPS:

$$\begin{aligned} & \max(-\delta_i(k), \delta_i(k) - 1) = 0 \\ & \min(\delta_i(k), 1 - \delta_i(k)) = 0 \end{aligned}$$

<u>**Theorem**</u>

Every ELC system can be written as MLD system, provided that $e_4 - E_1 x(k) - E_2 u(k) - E_3 d(k)$ is **bounded**

- Transform Complementary Condition

$$\begin{aligned} &(e_4)_j - (E_1 x(k) + E_2 u(k) + E_3 d(k))_j \leqslant M_j \delta_j(k) \quad \text{for each } j \in \phi_i \\ &\sum_{j \in \phi_i} \delta_j(k) \leqslant \#\phi_i - 1 \end{aligned}$$

  with $\delta_j(k) \in \{0, 1\}$ auxiliary variables, and $M_j$ upper bound for $(e_4 - E_1 x(k) - E_2 u(k) - E_3 d(k))_j$

## An example

See Lecture

# 4. Timed Automata

**Definition: Rectangular Sets**

Subset of $\mathbb{R}^n$ set is called **rectangular** if it can be written as finite boolean combination of constraints of form

$$x_i \leqslant a, \quad x_i < b, \quad x_i = c, \quad x_i \geqslant d, \quad x_i > e$$

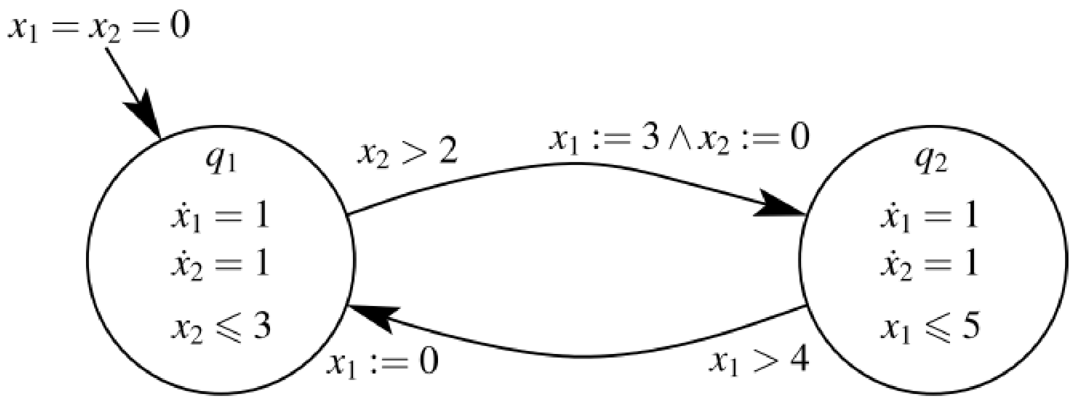
- Rectangular sets are “rectangles” or “boxes” in $\mathbb{R}^n$ whose sides are **aligned with the axes**, or **unions** of such rectangles/boxes(including empty set)

## Timed Automaton

Timed automaton is hybrid automaton with following characteristics:

- automaton involves **differential equations of form** $\dot{x}_i = 1$ continuous variables governed by this differential equation are called **"clocks" or "timers", all differential equations should be equa to 1**
- sets involved in definition of initial states, guards, and invariants are **rectangular sets**
- reset maps involve either **rectangular set**, or may **leave certain states unchanged**

## Properties




Timed automata involve simple **continuous dynamics**

- all differential equations of form $\dot{x} = 1$ (**clock dynamics**)
- all invariants, guards, etc. involve comparison of real-valued states **with constants**
- reset maps involve either **rectangular set,** or may leave certain states **unchanged**

Timed automata are **limited for modeling physical system**s and very well suited for **encoding timing constraints**

# 5. Timed Petri Nets

- **Transition enabled** if all input places $(\bullet_t)$ contain at least 1 token

Compared to Untimed Petri Nets, Timed Petri Nets has **two more variables**:

- **discrete state variables** (markings, $m_\theta(p)$)
- **continuous state variables**(arrival times $M_\theta(p)$)

$M_\theta(p) := \{\theta_1, \ldots, \theta_{m_\theta(p)}\}$, with arrival times $\theta_1 \leqslant \theta_2 \leqslant \ldots \leqslant \theta_{m_\theta(p)}$ of $m_\theta(p)$ token in place $p$

And we also have **interval** $[L(t), U(t)]$: time interval for specified token during which it must be transmitted

## Time Analysis

Transition $t$ becomes enabled at

$$\max_{p \in \bullet} \min M_\theta(p)$$

Then transition $t$ may fire at some time

$$\theta \in \left[\max_{p \in {}^\bullet t} \min M_\theta(p) + L(t), \max_{p \in {}^\bullet t} \min M_\theta(p) + U(t)\right]$$

- If enabling condition is still **valid at final time** of firing interval, then transition is **forced to fire**
- However, many problems are **undecidable or NP-hard**

# Summary

- Frameworks
    - PWA
    - MLD
    - LC
    - ELC
    - MMPS
- Equivalence of these systems
- Transformation among these models
- Timed automata and Timed Petri Nets