04_Extended Kalman Filter

1. Extended Kalman Filter

Linear Kalman Filter only consider scenarios that the system model is a linear model.

For nonlinear system, we can extend Kalman Filter by linearize the system.

Considering a nonlinear system:

$$
\begin{aligned}
\mathbf{x}_k &= \mathbf{f}_{k-1}\left(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1}\right) \\
\mathbf{y}_k &= \mathbf{h}_k\left(\mathbf{x}_k, \mathbf{v}_k\right)
\end{aligned}
$$

1.1. Linearization of Nonlinear System

We need first linearize the system to:

$$
\begin{aligned}
\mathbf{x}_k &= \mathbf{f}_{k-1}\left(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0}\right) + \mathbf{F}_{k-1}\left(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1}\right) + \mathbf{L}_{k-1}\mathbf{w}_{k-1} \\
\mathbf{y}_k &= \mathbf{h}_k\left(\check{\mathbf{x}}_k, \mathbf{0}\right) + \mathbf{H}_k\left(\mathbf{x}_k - \check{\mathbf{x}}_k\right) + \mathbf{M}_k\mathbf{v}_k
\end{aligned}
$$

1.2. Process

Prediction

$$
\begin{aligned}
\check{\mathbf{x}}_k &= \mathbf{f}_{k-1}\left(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0}\right) \\
\check{\mathbf{P}}_k &= \mathbf{F}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{F}_{k-1}^T + \mathbf{L}_{k-1}\mathbf{Q}_{k-1}\mathbf{L}_{k-1}^T
\end{aligned}
$$

Kalman Gain

$$
\mathbf{K}_k = \check{\mathbf{P}}_k\mathbf{H}_k^T\left(\mathbf{H}_k\check{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{M}_k\mathbf{R}_k\mathbf{M}_k^T\right)^{-1}
$$

Correction

$$
\begin{aligned}
\hat{\mathbf{x}}_k &= \check{\mathbf{x}}_k + \mathbf{K}_k\left(\mathbf{y}_k - \mathbf{h}_k\left(\check{\mathbf{x}}_k, \mathbf{0}\right)\right) \\
\hat{\mathbf{P}}_k &= \left(\mathbf{1} - \mathbf{K}_k\mathbf{H}_k\right)\check{\mathbf{P}}_k
\end{aligned}
$$

2. Error-state Extended Kalman Filter (ES-EKF)

Intuition

- The mistake made by EFK will accumulated when time passing by
- Compared to state of object, in a lot of reality scenarios, the error behaves much closer to a linear behavior
- Error is always smaller compared to nominal state so most times it will be more accurate when linearization

Based on that, ES-EKF tried to separates the state into a “large” nominal state and a “small” error state and uses local linearization to estimate the error state and uses it to correct the nominal state.

State Decomposition

$$\underbrace{\mathbf{x}_k - \mathbf{f}_{k-1}\left(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0}\right)}_{\delta\mathbf{x}_k} = \mathbf{F}_{k-1}\underbrace{\left(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1}\right)}_{\delta\mathbf{x}_{k-1}} + \mathbf{L}_{k-1}\mathbf{w}_{k-1}$$
$$\mathbf{y}_k = \mathbf{h}_k\left(\check{\mathbf{x}}_k, \mathbf{0}\right) + \mathbf{H}_k\underbrace{\left(\mathbf{x}_k - \check{\mathbf{x}}_k\right)}_{\delta\mathbf{x}_k} + \mathbf{M}_k\mathbf{v}_k$$

Procedure

Prediction

$$\check{\mathbf{x}}_k = \mathbf{f}_{k-1}\left(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0}\right)$$
$$\check{\mathbf{P}}_k = \mathbf{F}_{k-1}\mathbf{P}_{k-1}\mathbf{F}_{k-1}^T + \mathbf{L}_{k-1}\mathbf{Q}_{k-1}\mathbf{L}_{k-1}^T$$

- x_{k-1} can be predicted value or corrected value

Kalman Gain

$$\mathbf{K}_k = \check{\mathbf{P}}_k\mathbf{H}_k^T\left(\mathbf{H}_k\check{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{R}\right)^{-1}$$

Correction

$$\delta\hat{\mathbf{x}}_k = \mathbf{K}_k\left(\mathbf{y}_k - \mathbf{h}_k\left(\check{\mathbf{x}}_k, \mathbf{0}\right)\right)$$
$$\hat{\mathbf{x}}_k = \check{\mathbf{x}}_k + \delta\hat{\mathbf{x}}_k$$
$$\hat{\mathbf{P}}_k = \left(\mathbf{1} - \mathbf{K}_k\mathbf{H}_k\right)\check{\mathbf{P}}_k$$

3. Limitation of EKF

- The EKF uses analytical local linearization and, as a result, is sensitive to **linearization errors**
- For highly nonlinear systems, the EKF estimate can **diverge** and become unreliable
- Computing **complex Jacobian matrices** is an error-prone process and must be done with substantial care

Summary