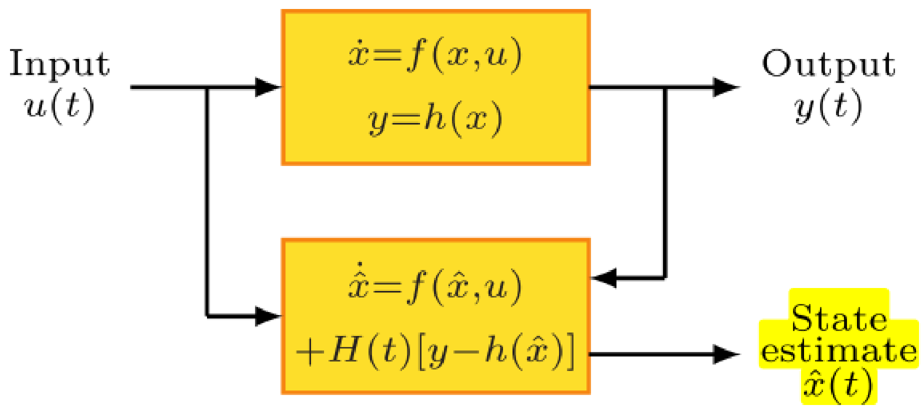
# 09_Nonlinear_Observers



## 1. Introduction




State is often not directly available, controllers need estimates system state.

**Goal:**

Choose an observer gain $H(t)$ such that the observer's state $\hat{x}(t)$ converges towards the state of the observed system, $x(t)$.

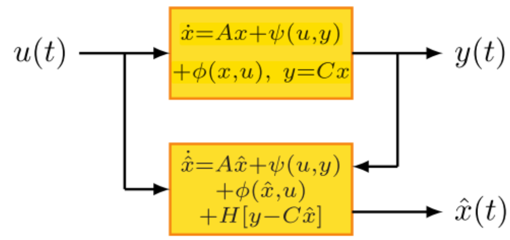
That is, the state reconstruction error $\tilde{x}(t)$ should vanish over time:

$$\tilde{x}(t) := x(t) - \hat{x}(t) \to 0, \quad t \to \infty.$$

## 2. Global Observer

**Target:**

Choose a **constant observer gain** $H(t) \equiv H$ such that
$\tilde{x}(t) := x(t) - \hat{x}(t) \rightarrow 0, \quad t \rightarrow \infty$




## Error Dynamics

$$\begin{aligned}\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} &= [Ax + \psi(u,y) + \phi(x,u)] - [A\hat{x} + \psi(u,y) + \phi(\hat{x},u) + H(y - C\hat{x})] \\ &= (A - HC)\tilde{x} + \phi(x,u) - \phi(\hat{x},u).\end{aligned}$$

## Theorem For Global Observers

**Theorem For Global Observers:**

We consider the system $\dot{x} = Ax + \psi(u,y) + \phi(x,u), y = Cx$ with

$$\psi : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n, \quad \phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p \quad \text{locally Lipschitz}$$

Let $x(t)$ and $u(t)$ exist at all times. Assume that
1. there exists a Lipschitz constant $L > 0$ such that (depend on system dynamics)

$$\|\phi(x,u) - \phi(z,u)\| \leq L\|x - z\|, \quad \forall x, z, u$$

2. the positive definite solution $P$ of the Lyapunov equation (depend on designer)

$$P(A - HC) + (A - HC)^T P = -I$$

satisfies $L < \frac{1}{2\|P\|}$.

Then there exist $k, \lambda > 0$, independent of $x(t)$ and $u(t)$, such that

$$\|\tilde{x}(t)\| \leq k \left\|\tilde{x}\left(t_0\right)\right\| e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0, \tilde{x}\left(t_0\right) \in \mathbb{R}^n$$

**Note:**

The $V(z) = z^T P z$ is a candidate Lyapunov function

# 3. High Gain Observers

**Model:**

- **System Dynamics:** $w$ is state/parameter we do not care about or we have uncertainty on it

$$
\begin{cases}
\dot{w} = f_0(w, x, u) \\
\dot{x}_i = x_{i+1} + \psi_i\left(x_1, \ldots, x_i, u\right), \quad i \in \{1, \ldots, \rho - 1\} \\
\dot{x}_\rho = \phi(w, x, u) \\
y = x_1
\end{cases}
$$

- **Observer Dynamics:**

$$
\begin{cases}
\dot{\hat{x}}_i = \hat{x}_{i+1} + \psi_i\left(\hat{x}_1, \ldots, \hat{x}_i, u\right) + \dfrac{\alpha_i}{\varepsilon^i}\left(y - \hat{x}_1\right) \\
\dot{\hat{x}}_\rho = \phi_0(\hat{x}, u) + \dfrac{\alpha_\rho}{\varepsilon^\rho}\left(y - \hat{x}_1\right)
\end{cases}
$$

**Target:**

Choose the constants $\alpha_1, \cdots \alpha_\rho$ and $\epsilon > 0$ s.t. $\tilde{x}(t) \to 0$

# Error Dynamics

$$
\begin{aligned}
\dot{\tilde{x}}_i = \dot{x}_i - \dot{\hat{x}}_i &= \left[x_{i+1} + \psi_i\left(x_1, \ldots, x_i, u\right)\right] - \left[\hat{x}_{i+1} + \psi_i\left(\hat{x}_1, \ldots, \hat{x}_i, u\right) + \frac{\alpha_i}{\varepsilon^i}\left(y - \hat{x}_1\right)\right] \\
&= \tilde{x}_{i+1} + \left[\psi_i\left(x_1, \ldots, x_i, u\right) - \psi_i\left(\hat{x}_1, \ldots, \hat{x}_i, u\right)\right] - \frac{\alpha_i}{\varepsilon^i}\tilde{x}_1
\end{aligned}
$$

It can be transformed to:

$$
\underbrace{\varepsilon \frac{\dot{\tilde{x}}_i}{\varepsilon^{\rho-i}}}_{=\dot{\eta}_i} = \underbrace{\frac{\tilde{x}_{i+1}}{\varepsilon^{\rho-(i+1)}}}_{=\eta_{i+1}} + \underbrace{\varepsilon \frac{\psi_i\left(x_1, \ldots, x_i, u\right) - \psi_i\left(\hat{x}_1, \ldots, \hat{x}_i, u\right)}{\varepsilon^{\rho-i}}}_{=:\delta_i(x, \tilde{x}, u)} - \underbrace{\frac{\alpha_i}{\varepsilon^i \varepsilon^{\rho-(i+1)}}\tilde{x}_1}_{=\alpha_i \eta_1}.
$$

Then it can be wrote to a perturbed linear system

$$
\dot{\eta} = \underbrace{\frac{1}{\varepsilon}\begin{bmatrix}
-\alpha_1 & 1 & & & \\
-\alpha_2 & 0 & 1 & & \\
\vdots & & 0 & \ddots & \\
-\alpha_{\rho-1} & & & \ddots & 1 \\
-\alpha_\rho & & & & 0
\end{bmatrix}}_{=:F}\eta + \delta(w, x, \tilde{x}, u), \quad \delta_\rho := \phi(w, x, u) - \phi_0(\hat{x}, u).
$$

Then the matrix $F$ can be made Hurwitz.

- When the gain $\frac{1}{\epsilon}$ is high, the perturbation $\delta$ is negligible

# Theorem for High Gain Observers

**Theorem for High Gain Observers:**

Let there be **compact sets** $W, X, U$ **such that** $w(t) \in W, x(t) \in X$ **and** $u(t) \in U$ for all times and initial states / inputs of interest. Assume that

1. Over the domain of interest, the functions $f_0, \psi_1, \ldots, \psi_{\rho-1}$ and $\phi$ are locally Lipschitz in their arguments;
2. The functions $\phi_0, \psi_1, \ldots, \psi_{\rho-1}$ are Lipschitz in $x$, uniformly in $u$ :

$$\exists L_0 > 0 \text{ s.t. } |\phi_0(x,u) - \phi_0(z,u)| \leq L_0 \sum_{k=1}^{\rho} |x_k - z_k|$$

$$\exists L_i > 0 \text{ s.t. } |\psi_i\left(x_1, \ldots, x_i, u\right) - \psi_i\left(z_1, \ldots, z_i, u\right)| \leq L_i \sum_{k=1}^{i} |x_k - z_k|$$

3. the matrix $F$ is Hurwitz (see the previous slide);
4. there are constants $L, M > 0$ such that

$$\|\phi(w,x,u) - \phi_0(z,u)\| \leq L\|x-z\| + M, \quad \forall w \in W, x,z \in X, u \in U.$$

Then, there exist constants $a, b, c, \varepsilon^* > 0$ such that

$$0 < \varepsilon \leq \varepsilon \Rightarrow \|\tilde{x}i(t)\| \leq \max\left\{\frac{b}{\varepsilon^{i-1}} e^{-at/\varepsilon}, \varepsilon^{\rho+1-i} cM\right\}, \forall t \geq t0.$$

**Pros**

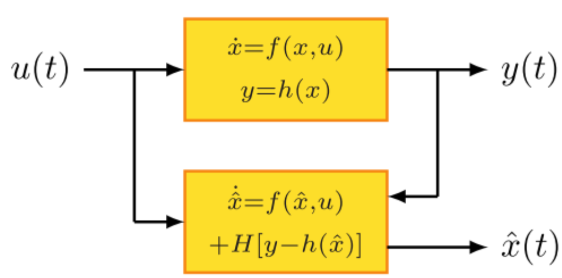
We do not have to know the exact system dynamics in order to design an observer. The function $\phi_0$ only has to approximate $\phi$.

**Cons:**

- can suffer from strong initial peaks in the estimation error
- sensitive to measurement noise

# 4. Local Observer

**Goal:** Choose a **constant observer gain** $H(t) \equiv H$ such that $\tilde{x}(t) := x(t) - \hat{x}(t) \to 0, \quad t \to \infty$




## Linearization

- Linearize around a constant state $x_{ss}$ and a constant input $u_{ss}$ to get $A$ and $C$
- Assume that $(A, C)$ is detectable, then we can find a $H \in \mathbb{R}^{n \times p}$ that makes $A - HC$ Hurwitz

## Theorem

We consider the system $\dot{x}=f(x,u), y=h(x)$ with $f: D \times U \rightarrow \mathbb{R}^n, \quad h: D \rightarrow \mathbb{R}^p, \quad D \subset \mathbb{R}^n, U \subset \mathbb{R}^m$ domains, with $f, h$ **twice continuously differentiable** in $(x, u)$. Assume that

1. $x_{ss} \in D$ is an **equilibrium** for a constant input $u(t) \equiv u_{ss} \in U$ with output zero. That is, $f\left(x_{ss}, u_{ss}\right)=0$ and $h\left(x_{ss}\right)=0$;
2. for any $\varepsilon>0$ there exist constants $\delta_1, \delta_2>0$ such that if

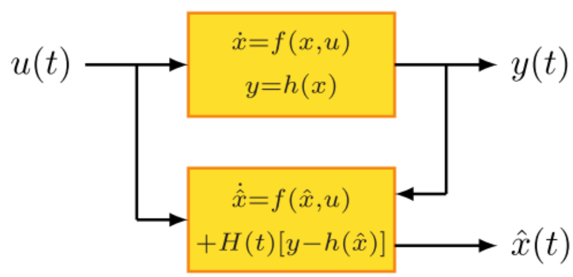
$$\left\|x\left(t_0\right)-x_{ss}\right\| \leq \delta_1,\left\|u(t)-u_{ss}\right\| \leq \delta_2, \forall t \geq t_0$$

   then $\left\|x(t)-x_{ss}\right\| \leq \varepsilon$ for all $t \geq t_0$;
3. the matrix $H \in \mathbb{R}^{n \times p}$ makes $A-HC$ Hurwitz.

Then $\lim_{t \rightarrow \infty} \tilde{x}(t)=0$ whenever $\|\tilde{x}\left(t0\right)\|$ , $\left\|x\left(t0\right)-x_{ss}\right\|$ and $\sup_{t \geq t0}\left\|u(t)-u_{ss}\right\|$ are small enough.

# 5. Extended Kalman Filter

**Goal:** Choose a **time-varying observer gain** $H(t)$ such that $\tilde{x}(t):=x(t)-\hat{x}(t) \rightarrow 0, \quad t \rightarrow \infty$




## Linearization

- We **linearize** around the current estimated state $\hat{x}(t)$ and input $u(t)$ :

$$A(t)=\left[\frac{\partial f(\hat{x}(t), u(t))}{\partial x}\right] \text{ and } C(t)=\left[\frac{\partial h(\hat{x}(t))}{\partial x}\right] .$$

- The **Kalman gain** is $H(t)=P(t) C^{\top}(t) R^{-1}$, where $P(t)$ is given by

$$\dot{P}=A P+P A^T+Q-P C^T R^{-1} C P, \quad P\left(t_0\right)=P_0,$$

  and the matrices $P_0, Q$ and $R$ are symmetric and positive definite.

## Theorem for the Extended Kalman Filter

We consider the system $\dot{x}=f(x, u), y=h(x)$ with

$$f: D \times U \rightarrow \mathbb{R}^n, \quad h: D \rightarrow \mathbb{R}^p, \quad D \subset \mathbb{R}^n, U \subset \mathbb{R}^m \text{bounded}$$

Assume that

1. $f, h$ are twice continuously differentiable in all arguments;
2. The solution of the Riccati equation exists for all $t \geq t_0$, and there are constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 l \leq P(t) \leq \alpha_2 l, \quad \forall t \geq t_0$$

Then there exist constants $c, k, \lambda > 0$ such that

$$\|\tilde{x}\left(t_0\right)\| \leq c \quad \Rightarrow \quad \|\tilde{x}(t)\| \leq k e^{-\lambda(t-t_0)}, \forall t \geq t_0 \geq 0.$$

# Summary

- Observers reconstruct the **unknown state** of a system from **inputs and output**
- Observers for **specific classes** of nonlinear systems
  - Global Observers (for systems with linear state updates)
  - High-gain Observers (for systems in a normal form)
- Generic Observers based on **linearization**
  - Local Observers (constant observer gain)
  - Extended Kalman Filter (time-varying observer gain)

There are a lot of theorem in this chapter to tell us the convergence and stability of the observer. These theorem has similar structures:

- They should try to give a bounded to uncertainty/nonlinear part (such as nonlinear part in dynamic and approximation of nonlinear part)
- Part to guarantee the stability and convergence in ideal part of the system