04_Introduction to Constrained Systems

1. Invariance & Control Invariance

Conceptions

Definition (Invariance)

Region in which an **autonomous** system will satisfy the constraints **for all time**

Definition (Positive Invariant Set)

A set $\mathcal{O}$ is said to be a **positive invariant set** for the autonomous system $x_{i+1} = f(x_i)$ if

$$x_i \in \mathcal{O} \Rightarrow x_i \in \mathcal{O}, \forall i \in \{0, 1, \ldots\}$$

Notes

The invariant set provides a set of **initial states** from which the trajectory will **never violate** the system constraints.

Definition (Maximal Positive Invariant Set)

The set $\mathcal{O}_\infty \subset \mathbb{X}$ is the **maximal invariant set** with respect to $\mathbb{X}$ if $0 \in \mathcal{O}_\infty,\ \mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all invariant sets that contain the origin.

Notes

The maximal invariant set is the set of all states for which the system will remain feasible if it starts in $\mathcal{O}_\infty$

Definition (Preset)

Given a set S and the dynamic system $x^+ = f(x)$, the pre-set of S is the set of states that evolve into the target set S in one time step:

$$\text{pre}(S) := \{x \mid f(x) \in S\}$$

Condition of Invariant Set

Theorem

A set $\mathcal{O}$ is a positive invariant set **if and only if**

$$\mathcal{O} \subset \text{pre}(\mathcal{O})$$

Computation

Conceptual Algorithm to Compute Invariant Set

maximal positive invariant set

Input: f, $\mathbb{X}$
Output: $\mathcal{O}_\infty$

```
Ω_0 ← X
loop
    Ω_{i+1} ← pre(Ω_i) ∩ Ω_i
    if Ω_{i+1} = Ω_i then
        return  O_∞ = Ω_i
    end if
end loop
```

The algorithm generates the set sequence $\{\Omega_i\}$ satisfying $\Omega_{i+1} \subseteq \Omega_i$ for all $i \in \mathbb{N}$ and it terminates when $\Omega_{i+1} = \Omega_i$ so that Ω_i is the maximal positive invariant set $\mathcal{O}_\infty$ for $x^+ = f(x)$.

Notes:

The algorithm generates the set sequence $\{\Omega_i\}$ satisfying $\Omega_{i+1} \subseteq \Omega_i$ for all $i \in \mathbb{N}$ and it terminates when $\Omega_{i+1} = \Omega_i$ so that Ω_i is the maximal positive invariant set $\mathcal{O}_\infty$ for $x^+ = f(x)$.

Control Invariance

Definition (Controlled Invariance)

Region for which there **exists** a controller so that the system satisfies the constraints for all time

Definition (Control Invariant Set)

A set $\mathcal{C} \subseteq \mathbb{X}$ is said to be a control invariant set if

$$x_i \in \mathcal{C} \quad \Rightarrow \quad \exists u_i \in \mathbb{U} \quad \text{such that } f(x_i, u_i) \in \mathcal{C} \quad \text{for all } i \in \mathbb{N}^+$$

Definition (Maximal Control Invariant Set)

The set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the system $x^+ = f(x, u)$ subject to the constraints $(x, u) \in \mathbb{X} \times \mathbb{U}$ if it is control invariant and contains all control invariant sets contained in $\mathbb{X}$.

Definition (Control Preset)

Condition of Control Invariant Set

Theorem

A set $\mathcal{C}$ is a positive invariant set **if and only if**

$$\mathcal{C} \subset \text{pre}(\mathcal{C})$$

Computation of Control Invariant Set

$\Omega_0 \leftarrow \mathbb{X}$
loop
  $\Omega_{i+1} \leftarrow \text{pre}(\Omega_i) \cap \Omega_i$
  if $\Omega_{i+1} = \Omega_i$ **then**
    return $\mathcal{C}_\infty = \Omega_i$
  end if
end loop

Control Invariant Set & Control Law

We can use this fact to **synthesize** a control law from a control invariant set by solving an optimization problem:

$$\kappa(x) := \text{argmin}\,\{g(x, u) \mid f(x, u) \in C\}$$

where g is any function (including $g(x, u) = 0$).

This doesn't ensure that the system will converge, but it will satisfy constraints.

2. Polytopes and Polytopic Computation

Conceptions

Definition (Polytope and Polyhedron)

A **polyhedron** is the intersection of a finite number of halfspaces.

$$P := \left\{x \mid a_i^T x \leq b_i, i = 1, \ldots, n\right\}$$

A **polytope** is a bounded polyhedron.

Definition (Convex Hull)

For any subset S of $\mathbb{R}^d$, the convex hull conv (S) of S is the intersection of all convex sets containing S. Since the intersection of two convex sets is convex, it is the **smallest convex set** containing S

Theorem (Minkowski-Weyl Theorem)

For $P \subseteq \mathbb{R}^d$, the following statements are equivalent:

- P is a polytope, i.e., P is bounded and there exist $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$ such that $P = \{x \mid Ax \leq b\}$
- P is finitely generated, i.e., there exist a finite set of vectors $\{v_i\}$ such that $P = \text{conv}\left(\{v_1, \ldots, v_s\}\right)$

Definition (Intersection)

The intersection $I \subseteq \mathbb{R}^n$ of sets $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^n$ is

$$I = S \cap T := \{x \mid x \in S \text{ and } x \in T\}$$

Notes:

Intersection of polytopes in inequality form is easy:

$$\begin{aligned} S &:= \{x \mid Cx \leq c\} \\ T &:= \{x \mid Dx \leq d\} \end{aligned} \qquad S \cap T = \left\{x \mid \begin{bmatrix} C \\ D \end{bmatrix} x \leq \begin{bmatrix} c \\ d \end{bmatrix}\right\}$$

Definition (Polytopic Projection)

Given a polytope $P = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^d \mid Cx + Dy \leq b\right\}$, find a matrix E and vector e, such that the polytope

$$P_\pi = \{x \mid Ex \leq e\} = \{x \mid \exists y, (x, y) \in P\}$$

PolyTopes in MPC

Input saturation

$$u_{lb} \leq u \leq u^{ub}$$
$$\Downarrow$$
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} u \leq \begin{bmatrix} u^{ub} \\ -u_{lb} \end{bmatrix}$$

Magnitude constraints

$$\|Cx\|_\infty \leq \alpha$$
$$\Downarrow$$
$$\begin{bmatrix} C \\ -C \end{bmatrix} x \leq \mathbf{1}\alpha$$

Rate constraints

$$\|x_i - x_{i+1}\|_\infty \leq \alpha$$
$$\Downarrow$$
$$\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \begin{pmatrix} x_i \\ x_{i+1} \end{pmatrix} \leq \mathbf{1}\alpha$$

Integral constraints

$$\|x\|_1 \leq \alpha$$
$$\Downarrow$$
$$x \in \text{conv}\,(e_i \alpha)$$

Polytopes in MPC are commonly described as a set of **inequalities**.
This is a standing assumption in the following.

Computation of Pre-Set

Autonomous System

Method

If $S := \{x \mid Fx \leq f\}$, then $\operatorname{pre}(S) = \{x \mid FAx \leq f\}$

Controlled Systems

Method

Consider the system $x^+ = Ax + Bu$ under the constraints $u \in \mathbb{U} := \{u \mid Gu \leq g\}$ and the set $S := \{x \mid Fx \leq f\}$.

$$\begin{aligned} \operatorname{pre}(S) &= \{x \mid \exists u \in \mathbb{U}, Ax + Bu \in S\} \\ &= \{x \mid \exists u \in \mathbb{U}, FAx + FBu \leq f\} \\ &= \left\{x \mid \exists u, \begin{bmatrix} FA & FB \\ 0 & G \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leq \begin{bmatrix} f \\ g \end{bmatrix}\right\} \end{aligned}$$

Note:

This is actually a projection operation.

Equality Test

One important problem is how to check whether two set are the same. i.e. Is $P := \{x \mid Cx \leq c\}$ contained in $Q := \{x \mid Dx \leq d\}$. The statement is true if $P \subset \{x|D_i x < d_i\}$ for each row D_i of D.

Method

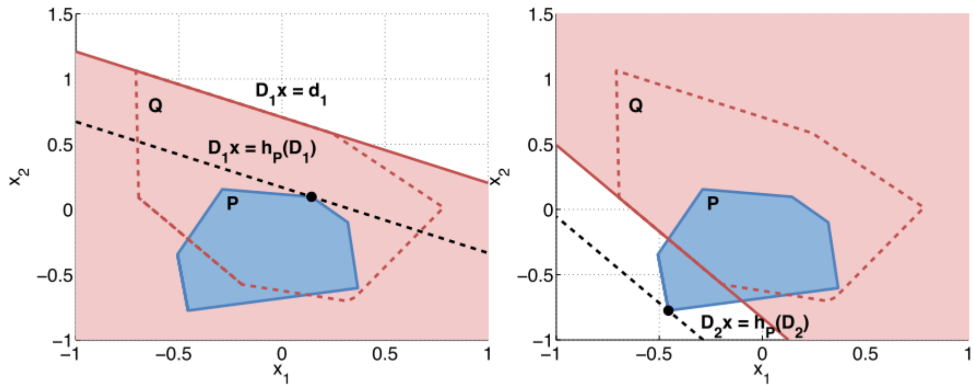
Define the support function of the set P :

$$\begin{aligned} h_P\left(D_i\right) := &\max_x D_i x \\ &\text{s.t. } Cx \leq c \end{aligned}$$

if $h_p(D_1) \leq d_1$, then it is true, if not , it is false.

Define the **support function** of the set P:

$$h_P(D_i) := \max_x \; D_i x$$
$$\text{s.t.} \quad Cx \leq c$$



$h_P(D_1) \leq d_1$ ✔ $\qquad$ $h_P(D_2) > d_2$ ✖

Note:

Do not try to translate to straight line representation. It can be directly understood by the definition of Q, if the false case happen, it means at least one of the constrain in Q is violated, because we use Q as $\leq$ format.

Convergence Discussion

Another problem is: Does the invariant set algorithm guarantees finite step termination?

In general, **no** The boundary of the a maximal invariant set can be curvy, which needs infinite many half-space to define. In practice, to save memory and to ensure efficiency, the algorithm stop up to some **specific criteria or we use simpler convex set(i.e. box, ellipsoid)** to represent a smaller forward invariant set.

3. Ellipsoids and Invariance (not on exam)

Ellipsoids

Ellipse

Let $P \succeq 0$ by a symmetric and positive-definite matrix in $\mathbb{R}^{n \times n}$ and $x_c \in \mathbb{R}^n$. The set

$$E := \{x \mid (x - x_c)^T P (x - x_c) \leq 1\}$$

is an **ellipse**.

Ellipsoids are useful because **the complexity of evaluating containment is always quadratic in the dimension,** whereas polyhedra can be arbitrarily complexy

Invariant Sets from Lyapunov Functions

> Lemma: Invariant set from Lyapunov function
>
> If $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the system $x^+ = f(x)$, then
>
> $$Y := \{x \mid V(x) \leq \alpha\}$$
>
> is an invariant set for all $\alpha \geq 0$.

The property of Lyapunov Function $V(f(x)) - V(x) < 0$ implies that once $V(x_i) \leq \alpha$, $V(x_j)$ will be less than α for all $j \geq i$

- We want to find the **largest invariant set contained** in our constraints

$$Y_\alpha := \{x \mid V(x) \leq \alpha\} \subseteq \mathbb{X}$$

Lyapunov Functions 1:

Use following construction method of Lyapunov Function

$$A^T PA - P \prec 0$$

Our goal is to find the largest α such that the invariant set Y_α is contained in the system constraints $\mathbb{X}$:

$$\boxed{Y_\alpha := \left\{x \;\middle|\; x^T Px \leq \alpha^2 \right\} \subset \mathbb{X} := \{x \mid Fx \leq f\}}$$

Equivalently, we want to solve the problem:

$$\boxed{\begin{aligned} &\max_\alpha \ \alpha \\ &\text{s.t. } \ h_{Y_\alpha}(F_i) \leq f_i \text{ for all } i \in \{1, \ldots, n\} \end{aligned}}$$

Use support function, then we can derive:

Largest ellipse in a polytope is now a one-dimensional optimization problem:

$$\begin{aligned} \alpha^\star &= \max_\alpha \ \alpha \quad \text{s.t.} \ \ \|P^{-1/2}F_i^T\|\alpha \leq f_i \text{ for all } i \in \{1, \ldots, n\} \\ &= \min_{i \in \{1,\ldots,n\}} \frac{f_i}{\|P^{-1/2}F_i^T\|} \end{aligned}$$

More General Lyapunov Function

The function $V(x) = x^T P x$ is only one of many possible Lyapunov functions for the system $x^+ = (A + BK)x$. Can we find one that will give a larger ellipse?

The function $V(x) = x^T P x$ is a Lyapunov function for the system $x^+ = (A + BK)x$ if it satisfies the Lyapunov equation

$$A^T P A - P = -Q$$

Note that this is equivalent to the condition

$$A^T P^{-1} A - P^{-1} \preceq 0$$

We can now pose a convex optimization problem, which returns the largest invariant ellipse within a polytope $\mathbb{X} = \{x \mid Fx \leq f\}$ (where we define $\tilde{P} := P^{-1}$)

$$\boxed{\begin{aligned} \max_{\tilde{P}} \quad & -\log\det\tilde{P} \quad \text{: volume of an ellipse} \\ \text{s.t.} \quad & A^T \tilde{P} A - \tilde{P} \preceq 0 \\ & F_i \tilde{P} F_i^T \leq f_i^2 \quad \text{for all} \, i = 1 \ldots n \end{aligned}}$$

Notes:

- The volume of an ellipse is $\log\det P^{-1}$
- $\|P^{-1/2} F_i^T\|^2 = F_i P^{-1} F_i^T$

The largest volume ellipse centered at zero within the polytope $\mathbb{X}$ is then

$$\mathcal{E} = \left\{x \;\middle|\; x^T \tilde{P}^{-1} x \leq 1\right\} \subset \mathbb{X}$$

Summary

- Invariant set
- Controlled invariant set
- The method to compute them.