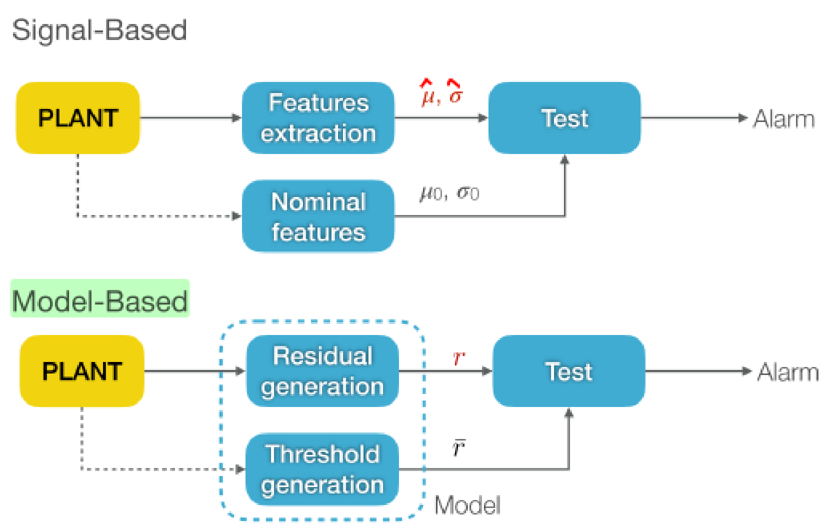
05_01_Model_Based_Fault_Detection

1. Overview

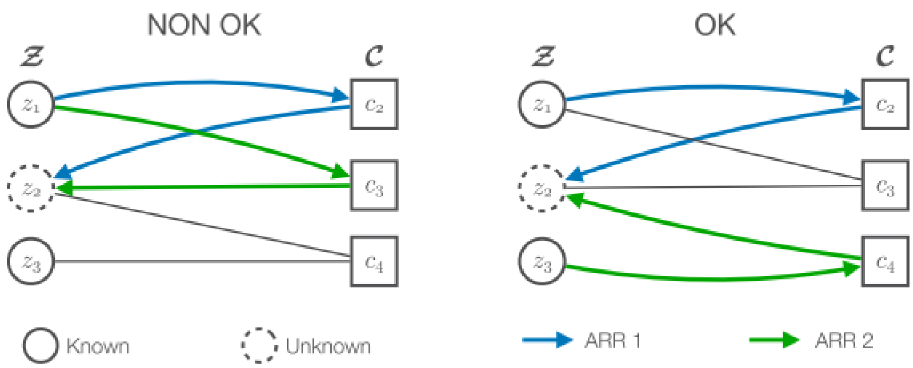
Signal-Based VS Model-Based



Definitions

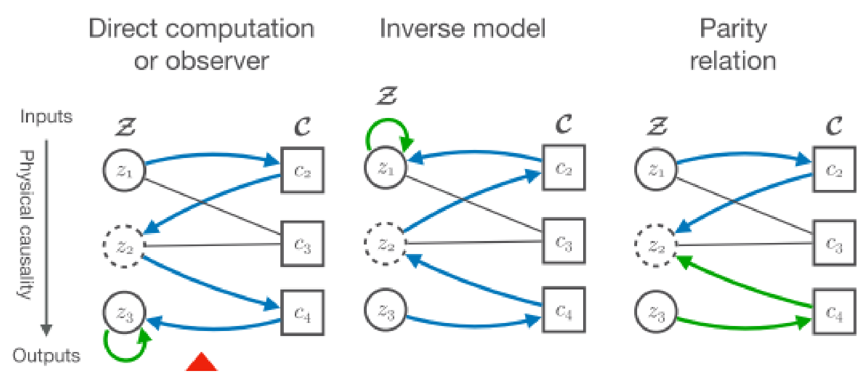
Definition: Residual r

Difference between the output of two **Analytical Redundancy Relations** producing the same variable and **depending on different sets of known variables**



Different ways to obtain a residual via ARRs

Direct Computation or Observer, Inverse Model, Parity Relation



- Direct Computation or Observer is preferred, because the observer can remove some noise
- Parity Relation are sometimes **badly affected by noise**

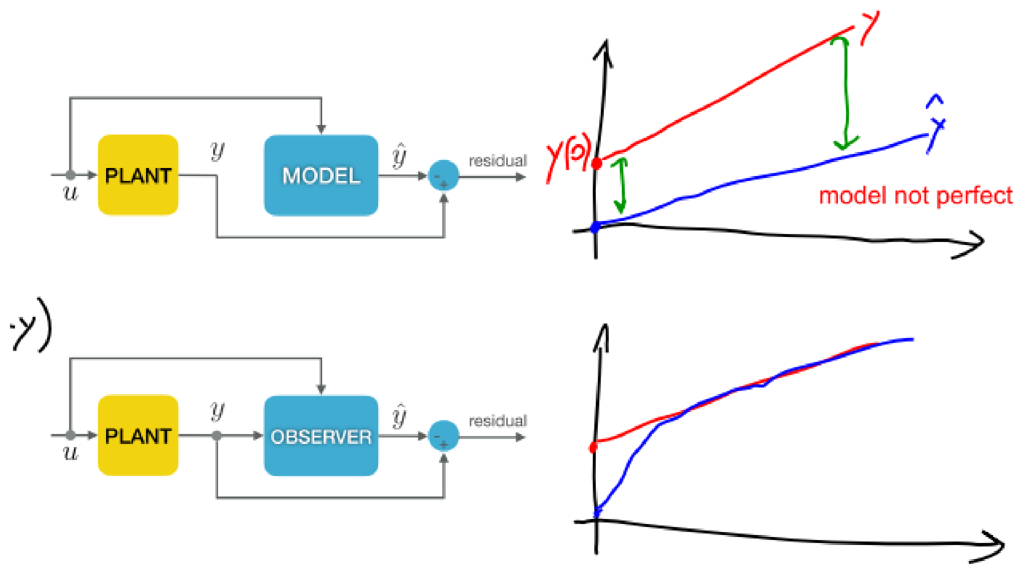
Definition: Threshold $\bar{r}$

Maximum expected value of (a function of) the residual under **healthy conditions**

Detection Observers

Why not Direct Computation?

Because our model of the system is not always perfect, then:



By using observer, we can somehow **guarantee convergence (may not totally same)**

3. Deterministic Threshold

Problem Formulation

<u>Nominal Model</u>

$$\dot{x} = f(x, u) \quad x(k+1) = f(x(k), u(k))$$

<u>Real Plant</u>

$$x(k+1) = f(x(k), u(k)) + \eta(x(k), u(k), k) + \beta\left(k - k_0\right) \phi(x(k), u(k), k)$$

- η: uncertainty
- β: fault profile
    - **Abrupt**

$$\beta\left(k - k_0\right) = \begin{cases} 0 & \text{if} \quad k < k_0 \\ 1 & \text{if} \quad k \geq k_0 \end{cases}$$

    - **Incipient**

$$\beta\left(k - k_0\right) = \begin{cases} 0 & \text{if} \quad k < k_0 \\ 1 - b^{-\left(k - k_0\right)} & \text{if} \quad k \geq k_0 \end{cases}$$

- ϕ: fault function

- Can be anything that changes the dynamics, e.g.

Actuator fault $\tilde{u} = (1-\theta)u$

Healthy $\dot{x} = f(x,u)$

Faulty $\dot{x} = f(x,(1-\theta)u)$

$\phi(k) = f(x,(1-\theta)u) - f(x,u)$

Full State Measurement Assumption

$$y(k) = x(k) + \xi(k)$$

- ξ: uncertainty

Assumptions

- **At time $k = 0$ no faults act on the system.** Moreover, the state variables $x(k)$ and control variables $u(k)$ remain **bounded** before and after the occurrence of a fault, i.e., there exist some stability regions $\mathcal{R} = \mathcal{R}^x \times \mathcal{R}^u \subset \mathbb{R}^n \times \mathbb{R}^m$, such that $(x(k), u(k)) \in \mathcal{R}^x \times \mathcal{R}^u, \forall k$.
- The time profile parameter b is **unknown but it is lower bounded** by a known constant $\bar{b}$
- η is an **unstructured and unknown** nonlinear function of x, u, and k, but it is **bounded** by a known positive function $\bar{\eta}$, i.e.,

$$|\eta_i(x(k), u(k), k)| \le \bar{\eta}_i(x(k), u(k), k), \quad \forall (x,u) \in \mathcal{R}, \forall k$$

- ξ is an **unknown** signal, but it is **bounded** by a known positive quantity $\bar{\xi}$, i.e., $|\xi_i(k)| \le \bar{\xi}_i, \forall k$.

Residual Generation and Residual Dynamics

Residual Generation

$$\begin{cases} \hat{x}(k+1) & = f(y(k), u(k)) + \Lambda[\hat{y}(k) - y(k)] \\ \hat{y}(k) & = \hat{x}(k) \end{cases}$$
$$r(k) \triangleq y(k) - \hat{y}(k)$$

- $y(k)$ is the measurement
- notice that in the nominal dynamics, we use measurement $y(k)$ instead of using $\hat{x}(k)$

Residual Dynamics (Without Fault)

$$
\begin{aligned}
&r(k+1) = y(k+1) - \hat{y}(k+1) = \\
&= x(k+1) + \xi(k+1) - \hat{x}(k+1) = \\
&= f(x(k), u(k)) + \eta(k) + \xi(k+1) - f(y(k), u(k)) - \Lambda(\hat{y}(k) - y(k)) \\
&= \Lambda r(k) + \underbrace{f(y(k) - \xi(k), u(k)) - f(y(k), u(k)) + \eta(k) + \xi(k+1)} \\
&r(k+1) = \Lambda r(k) + \delta(k)
\end{aligned}
$$

Threshold Design

Bounding the Residual Dynamics

$$
\begin{aligned}
&r(k+1) = \Lambda r(k) + \delta(k) \\
&\Lambda = \text{diag}(\{\lambda_i\}) \quad \lambda_i \in [0 \quad 1] \\
&|r_i(k+1)| \leqslant \lambda_i\,|r_i(k)| + |\delta_i(k)| \\
&\bar{\delta}_i(k) \geqslant |\delta_i(k)| \\
&|r_i(k+1)| \leqslant \lambda_i\,|r_i(k)| + \bar{\delta}_i(k) \\
&\bar{r}_i(k+1) = \lambda \bar{r}_i(k) + \bar{\delta}_i(k) \geqslant |r_i(k+1)|
\end{aligned}
$$

- This is an iterative process to compute $\bar{r_i}(k+1)$, it can also be written by:

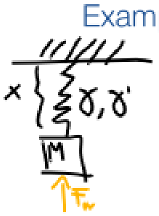
$$
\bar{r}(k+1) = \sum_{h=0}^{k-1} \Lambda^{k-1-h} \bar{\delta}(h) + \Lambda^k \bar{r}(0)
$$

- The calculation of $\bar{\delta}$ is very complex:

$$
\bar{\delta}_i(k) \triangleq \max_{\eta} \max_{\xi} |\delta_i(K)|
$$

it is always complex to compute that, it may be time variant, corresponding to x

Example



NOMINAL: $\ddot{x} = -1/M\,(\gamma x + \gamma' x^2)$

UNCERTAINTIES

- True $(\tilde{\gamma}, \tilde{\gamma}', \tilde{M}) \neq (\gamma, \gamma', M)$
- External unknown force F_w
- Unmodelled dynamics $-\gamma'' x^3$

TRUE DYNAMICS: $\ddot{x} = -1/\tilde{M}\,(\tilde{\gamma} x + \tilde{\gamma}' x^2 + \tilde{\gamma}'' x^3 - F_w)$

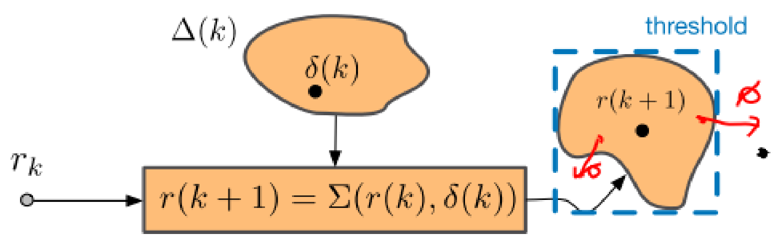
$$
\eta = \frac{1}{M}(\gamma x + \gamma' x^2) - \frac{1}{\tilde{M}}(\tilde{\gamma} x + \tilde{\gamma}' x^2 + \tilde{\gamma}'' x^3 - F_w)
$$

$$
= \frac{1}{\tilde{M} M}\left(-(M\tilde{\gamma} - \tilde{M}\gamma)x - (M\tilde{\gamma}' - \tilde{M}\gamma')x^2 - M\tilde{\gamma}'' x^3 + F_w\right)
$$

Robustness and Detectability

Robust Property

- the threshold is **robust against uncertainties** and leads to **zero FAR**



Residual Dynamics (With Fault)

$$r(k+1) = \Lambda r(k) + \delta(k) + \left(1 - b^{k-k_0}\right) \phi(k)$$

$$r(k) = \sum_{h=0}^{k-1} \Lambda^{k-1-h} \left[\delta(h) + \left(1 - b^{h-k_0}\right) \phi(h)\right] + \Lambda^k r(0)$$

Remember the expression of the threshold:

$$\bar{r}(k+1) = \sum_{h=0}^{k-1} \Lambda^{k-1-h} \bar{\delta}(h) + \Lambda^k \bar{r}(0)$$

Theorem of Detectability

If there exist two time indexes $k_2 > k_1 \geq k_0$ such that the fault ϕ fulfills the following inequality for at least one component $i \in \{1, \ldots, n\}$

$$\left| \sum_{h=k_1}^{k_2-1} \lambda^{k_2-1-h} \left(1 - b^{-(h-k_0)}\right) \phi_{(i)}(h) \right| > 2\bar{r}_{(i)}\left(k_2\right)$$

then it will be detected at k_2, that is $\left|r_{(i)}\left(k_2\right)\right| > \bar{r}_{(i)}\left(k_2\right)$.

Illustration:

It can be seen that, the δ **may compensate the effect of** ϕ. However, because we assume know that the nominal residue will be b**ounded by** $\bar{r}$, which means as long as the total effect of ϕ is larger than $2\bar{r}$, the sum will outside the threshold.

4. Probabilistic Threshold

- The deterministic threshold is quite **conservative**, we can get **high-MDR** in return for **zero-FAR**

Problem Formulation

Problem Formulation

$$
\begin{aligned}
x(k+1) &= f((x(k), u(k), \eta(k), \phi(k)) \\
y(k) &= x(k) + \xi(k)
\end{aligned}
$$

Assumptions

- No faults act on the system, that is $\phi(k) = 0$, for $0 \leq k < k_0$, with k_0 being the anomaly occurrence time. Moreover, the variables $x(k)$ **and** $u(k)$ **remain bounded** before and after the occurrence of an anomaly, i.e., there exist some stability regions $\mathcal{R} = \mathcal{R}^x \times \mathcal{R}^u \subset \mathbb{R}^n \times \mathbb{R}^m$, such that $(x(k), u(k)) \in \mathcal{R}, \forall k$.
- $\eta(k)$ and $\xi(k)$ are **random variables**, ar**e not correlated and are independent** from $x(k), u(k)$ and $\phi(k), \forall k$

Residual Generation and Residual Dynamics

Residual Generation

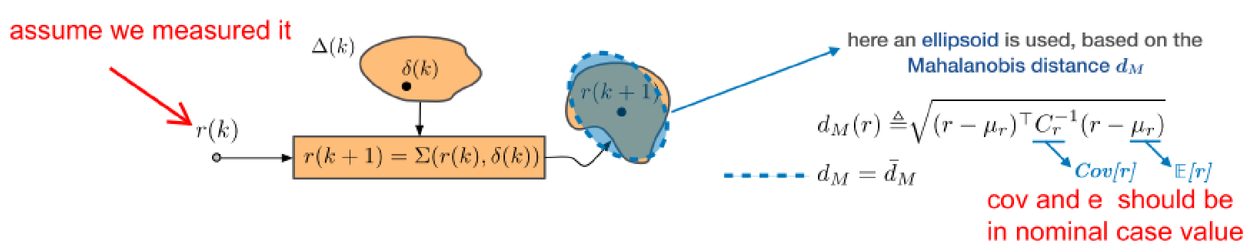
$$
\begin{cases}
\hat{x}(k+1) = f(y(k), u(k), 0, 0) + \Lambda[\hat{y}(k) - y(k)] \\
\qquad \hat{y}(k) = \hat{x}(k) \\
\qquad r(k) \triangleq y(k) - \hat{y}(k)
\end{cases}
$$

Residual Dynamics (without fault)

$$
\begin{aligned}
\delta(k) &\triangleq f(x(k), u(k), \eta(k), 0) - f(y(k), u(k), 0, 0) + \xi(k+1) \\
&= f(y(k) - \xi(k), u(k), \eta(k), 0) - f(y(k), u(k), 0, 0) + \xi(k+1)
\end{aligned}
$$

- **because we assume η and ξ are random variables, so $r(k+1)$ is also a random variables at time k**

Threshold Design



A probabilistic threshold based on Mahalanobis distance

$$\bar{d}_M \triangleq \frac{n}{\alpha} \Rightarrow \mathbb{P}\left[d_M^2\left(r_{k+1}\right) > \bar{d}_M\right] < \alpha \text{ in healthy conditions}$$

Summary

- Detection Observers: filter out noise and uncertainty
- Deterministic Threshold
  - **Residual Generation and Residual Dynamics**
  - Threshold Design
  - Robustness: zero FAR
  - **Detectability**: double magnitude
- Probabilistic Threshold
  - Residual Generation
  - Threshold Design: Chelbeshev