06_Optimization-Based Control

1. Optimal Control of a Class of Hybrid Systems

Example: Optimal Control of Hybrid Manufacturing Systems

Manufacturing System

Jobs move through network of work centers

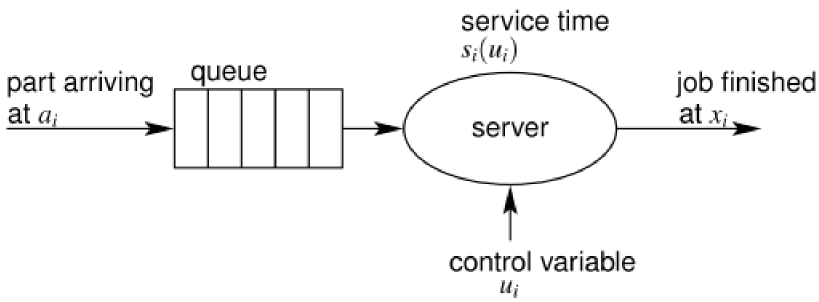
State

- **Temporal State (event-driven)**: waiting time, departure time, ...
- **Physical State (time-driven)**: temperature, size, weight, chemical composition

Trade-off:

- temporal requirements on job **completion times**
- physical requirements on **quality** of completed jobs

We always assume higher quality → longer processing times



- Single-stage, single-server queueing system
- N jobs (each job corresponds to mode)
- Buffer with capacity $> N$
- As job i is processed, physical state z_i evolves according to

$$\dot{z}_i = g_i(z_i, u_i, t) \qquad \text{with } z_i(\tau_i) = \zeta_i$$

with τ_i time instant at which processing begins

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- Control variable u_i is used to attain final desired physical state:
  If $s_i(u_i)$ is *service time* and $\Gamma_i(u_i)$ is target quality set, then

$$s_i(u_i) = \min\{t \geqslant 0 \mid z_i(\tau_i + t) \in \Gamma_i(u_i)\}$$

- *Temporal state* x_i represents time when job is completed:
  If a_i is arrival time of job i, then

$$x_i = \max(x_{i-1}, a_i) + s_i(u_i) \qquad \text{(Lindley equation)}$$

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Optimal control problem:

$$\min_{u_1,\ldots,u_N} J = \sum_{i=1}^{N} L_i(x_i, u_i)$$

subject to evolution equations for z_i and x_i

where $L(x_i, u_i)$ is cost function associated with job i

$\rightarrow$ classical discrete-time optimal control problems except for

- i does not count time steps
  $\rightarrow$ not really an issue
- max is non-differentiable for $a_i = x_{i-1}$
  $\rightarrow$ prevents use of standard gradient-based techniques
  $\rightarrow$ use non-differentiable calculus, generalized gradient

Example: heating/annealing manufacturing processes

See Lecture Slides

Optimality Condition

Augmented Cost

$$\bar{J}(x, \lambda, u)=\sum_{i=1}^N\left(L_i\left(x_i, u_i\right)+\lambda_i\left(\max \left(x_{i-1}, a_i\right)+s_i\left(u_i\right)-x_i\right)\right)$$

where λ is co-state

- Assume costs L_i and s_i are continuously differentiable
- First-Order Necessary Conditions of Optimality

$$\frac{\partial \bar{J}}{\partial u_i}=0, \quad \frac{\partial \bar{J}}{\partial \lambda_i}=0, \quad \frac{\partial \bar{J}}{\partial x_i}=0 \quad \text{for } i=1, \ldots, N$$

  Then we have:

  - **Stationarity condition**: $\frac{\partial L_i\left(x_i, u_i\right)}{\partial u_i}+\lambda_i \frac{d s_i\left(u_i\right)}{d u_i}=0$
  - **Temporal state equation**: $x_i=\max \left(x_{i-1}, a_i\right)+s_i\left(u_i\right)$ with $x_0=-\infty$
  - **Co-state equation**: $\lambda_i=\frac{\partial L_i\left(x_i, u_i\right)}{\partial x_i}+\lambda_{i+1} \frac{d \max \left(x_i, a_{i+1}\right)}{d x_i}$ with final boundary condition $\lambda_N=\frac{\partial L_N\left(x_N, u_N\right)}{\partial x_N}$
  - However, $\max$ function most time are not differentiable

Dealing with Non-differentiable: Generalized Gradient

- $\max$ is **Lipschitz continuous + differentiable** except for $x_i=a_{i+1}$

$$\frac{d \max \left(x_i, a_{i+1}\right)}{d x_i}= \begin{cases}0 & \text{if } x_i<a_{i+1} \\ 1 & \text{if } x_i>a_{i+1}\end{cases}$$

Generalized Gradient

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be **locally Lipschitz continuous**, and let $S(u)$ denote set of all sequences $\left\{u_m\right\}_{m=1}^{\infty}$ that satisfy

- $u_m \rightarrow u$ as $m \rightarrow \infty$
- gradient $\nabla f\left(u_m\right)$ exists for all m
- $\lim_{m \rightarrow \infty} \nabla f\left(u_m\right)=\phi$ exists

Then **generalized gradient** $\partial f(u)$ is defined as **convex hull** of all limits ϕ corresponding to some sequence $\left\{u_m\right\}_{m=1}^{\infty}$ in $S(u)$

Property

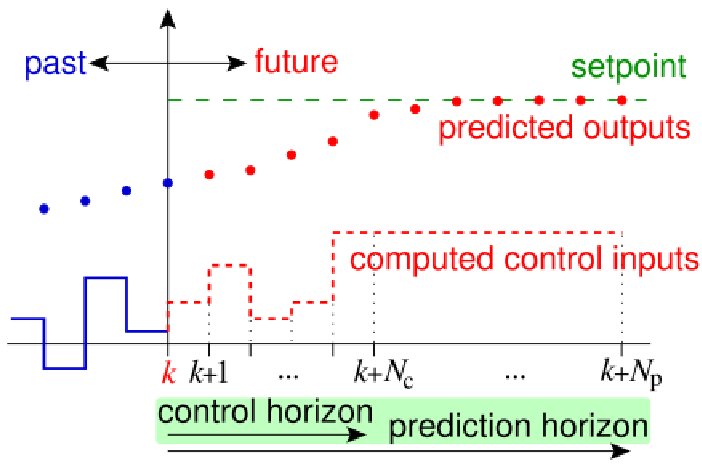
- if f is **continuously differentiable** in some open set containing u, then $\partial f(u) = \{\nabla f(u)\}$
- if u is **local minimum, then** $0 \in \partial f(u)$
  $\longrightarrow$ this becomes **first-order optimality condition in non-smooth optimization**

2. MPC for MLD and PWA Systems

Classical Settings of MPC

Classical MPC is quite familiar for me. Here I will not introduce MPC in detail, I will just mention some key-points.

One important thing is in this course, w have two **horizon: control horizon and prediction horizon**



Extra condition to reduce **computational complexity**: control horizon N_{c}

$$u(k+j) = u\left(k+N_{\mathrm{c}}-1\right) \quad \text{for } j = N_{\mathrm{c}}, \ldots, N_{\mathrm{p}}-1$$

$\longrightarrow$ smoother controller signal & stabilizing effect

MPC for MLD Systems

<u>**System Model**</u>

Consider MLD Systems

$$\begin{gathered}
x(k+1) = Ax(k) + B_1 u(k) + B_2 \delta(k) + B_3 z(k) \\
y(k) = Cx(k) + D_1 u(k) + D_2 \delta(k) + D_3 z(k) \\
E_1 x(k) + E_2 u(k) + E_3 \delta(k) + E_4 z(k) \leqslant g_5
\end{gathered}$$

- Consider **equilibrium state/input/output**: $(x_{\mathrm{eq}}, u_{\mathrm{eq}}, y_{\mathrm{eq}}) \rightarrow (\delta_{\mathrm{eq}}, z_{\mathrm{eq}})$
- $\hat{x}(k+j \mid k)$: estimate of x at sample step $k+j$ based on information available at sample step k

<u>**MPC Model**</u>

We tried to stabilize system to equilibrium state

$$
\begin{aligned}
J(k)=\sum_{j=1}^{N_{\mathrm{p}}} & \|\hat{x}(k+j \mid k)-x_{\mathrm{eq}}\|_{Qx}^2+\|u(k+j-1)-u_{\mathrm{eq}}\|_{Qu}^2+ \\
& \|\hat{y}(k+j \mid k)-y_{\mathrm{eq}}\|_{Qy}^2+\left\|\hat{\delta}(k+j-1 \mid k)-\delta_{\mathrm{eq}}\right\|_{Q\delta}^2+ \\
& \|\hat{z}(k+j-1 \mid k)-z_{\mathrm{eq}}\|_{Qz}^2
\end{aligned}
$$

with $Q.>0$

- End-point condition: $\hat{x}\left(k+N_{\mathrm{p}} \mid k\right)=x_{\mathrm{eq}}$
- Control horizon constraint:

  $u(k+j)=u\left(k+N_{\mathrm{c}}-1\right)$ for $j=N_{\mathrm{c}}, \ldots, N_{\mathrm{p}}-1$

Property

If feasible solution exists for $x(0)$, then MPC input stabilizes system, i.e.,

$$
\begin{array}{lll}
\lim_{k \rightarrow \infty} x(k)=x_{\mathrm{eq}} & \lim_{k \rightarrow \infty}\|y(k)-y_{\mathrm{eq}}\|_{Q_y}=0 & \lim_{k \rightarrow \infty}\|z(k)-z_{\mathrm{eq}}\|_{Q_z}=0 \\
\lim_{k \rightarrow \infty} u(k)=u_{\mathrm{eq}} & \lim_{k \rightarrow \infty}\|\delta(k)-\delta_{\mathrm{eq}}\|_{Q_\delta}=0 &
\end{array}
$$

Algorithms for MLD-MPC: mixed-integer quadratic programming (MIQP)

- **Successive substitution** of system equations: → $\hat{x}(k+j|k)$ is linear function of $x(k), \hat{u}, \hat{\delta}$ and $\hat{z}$ Also holds for $y(k \hat{+} j|k)$
  - When substitution, we assume u_k is known, we are now proving u and x to compute J, which is necessary for optimization steps
- Define $\tilde{V}(k)=\left[\begin{array}{lll}\tilde{u}^{\top}(k) & \tilde{\delta}^{\top}(k) & \tilde{z}^{\top}(k)\end{array}\right]^{\top}$
- Results in an **MIQP** Problem

$$
\begin{aligned}
& \min_{\tilde{V}(k)} \tilde{V}^{\top}(k) S_1 \tilde{V}(k)+2\left(S_2+x^{\top}(k) S_3\right) \tilde{V}(k) \\
& \text{subject to } F_1 \tilde{V}(k) \leqslant F_2+F_3 x(k),
\end{aligned}
$$

3. MPC for MMPS and Continuous PWA Systems

Note of PWA and MMPS Systems

- **Continuous** PWA model can be used as **approximation** of general nonlinear continuous state space model
- **General** PWA systems are equivalent to **constrained** MMPS systems
- **Continuous** PWA functions and **unconstrained** MMPS functions coincide.

  i.e., a continuous PWA function f can be rewritten as

$$f = \max_j \min_i \left(\alpha_i^T x + \beta_i\right)$$

Canonical Forms of MMPS Functions

Any MMPS function $f : \mathbb{R}^n \to \mathbb{R}$ can be rewritten into **min-max canonical form**

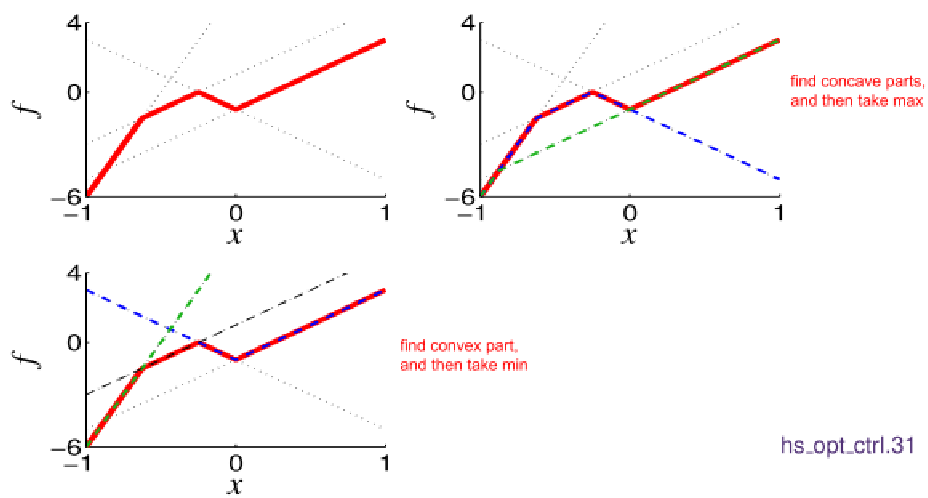
$$f = \min_i \max_j \left(\alpha_{(i,j)}^T x + \beta_{(i,j)}\right)$$

or into **max-min canonical form**

$$f = \max_i \min_j \left(\gamma_{(i,j)}^T x + \delta_{(i,j)}\right)$$

Example

$$\begin{aligned} f(x) &= \min(8x+6,1) - 2\max\left(\min(2x+1,1-2x),-2x\right) \\ &= \max\left(\min(12x+6,4x+1,-4x-1),\min(12x+6,4x-1)\right) \\ &= \min\left(\max(4x-1,-4x-1),12x+6,4x+1\right) \end{aligned}$$



<u>**Some Transformation Method: (From Lecture Notes)**</u>

- Max-Min and Min-Max
  - Minimization is distributive w.r.t. maximization, i.e., $\min(\alpha, \max(\beta, \gamma)) = \max(\min(\alpha, \beta),$ $\min(\alpha, \gamma))$, which results in:

    $$\min(\max(\alpha, \beta), \max(\gamma, \delta)) = \max(\min(\alpha, \gamma), \min(\alpha, \delta), \min(\beta, \gamma), \min(\beta, \delta)).$$

  - The max operation is distributive w.r.t. min. Hence,

    $$\max(\min(\alpha, \beta), \min(\gamma, \delta)) = \min(\max(\alpha, \gamma), \max(\alpha, \delta), \max(\beta, \gamma), \max(\beta, \delta)).$$

- We have

$$\min(\alpha, \beta) + \min(\gamma, \delta) = \min(\alpha + \gamma, \alpha + \delta, \beta + \gamma, \beta + \delta)$$
$$\max(\alpha, \beta) + \max(\gamma, \delta) = \max(\alpha + \gamma, \alpha + \delta, \beta + \gamma, \beta + \delta)$$

- The min and max operators are related as follows:

$$\max(\alpha, \beta) = -\min(-\alpha, -\beta)$$

- If $\rho \in \mathbb{R}$ is positive, then

$$\rho \max(\alpha, \beta) = \max(\rho\alpha, \rho\beta), \quad \rho \min(\alpha, \beta) = \min(\rho\alpha, \rho\beta)$$

MPC for MMPS Systems

MMPS Model

$$x(k) = \mathscr{M}_x(x(k-1), u(k))$$
$$y(k) = \mathscr{M}_y(x(k), u(k))$$

- Prediction Horizon: N_p
- Estimate $\hat{y}(k+j \mid k)$ of output at sample step $k+j$:

$$\hat{y}(k+j \mid k) = F_j(x(k-1), u(k), \ldots, u(k+j))$$

$\rightarrow F_j$ **is MMPS function**! (if we expand the formulation with higher-order x, we can find the combination is still MMPS equation)

MPC Model

- Some possible **cost functions** (need to meet MMPS format)

$$J_{\text{out },1}(k) = \|\tilde{y}(k) - \tilde{r}(k)\|_1 \quad J_{\text{out },\infty}(k) = \|\tilde{y}(k) - \tilde{r}(k)\|_\infty$$
$$J_{\text{in },1}(k) = \|\tilde{u}(k)\|_1 \quad J_{\text{in },\infty}(k) = \|\tilde{u}(k)\|_\infty$$

with

$$\tilde{u}(k) = \begin{bmatrix} u^T(k) & \ldots & u^T(k + N\mathrm{p} - 1) \end{bmatrix}^T$$
$$\tilde{y}(k) = \begin{bmatrix} \hat{y}^\top(k \mid k) & \ldots & \hat{y}^\top(k + N_\mathrm{p} - 1 \mid k) \end{bmatrix}^T$$
$$\tilde{r}(k) = \begin{bmatrix} r^T(k) & \ldots & r^T(k + N_\mathrm{p} - 1) \end{bmatrix}^T$$

Note: $|x| = \max(x, -x) \rightarrow$ cost functions are MMPS functions

- **Constraints**

$$C_\mathrm{c}(k, x(k-1), \tilde{u}(k), \tilde{y}(k)) \geqslant 0$$

Algorithms for Model Solution

3.4 Algorithms for MMPS-MPC

- Nonlinear optimization (SQP, ELCP):

  $\rightarrow$ local minima, excessive computation time

- MPC for mixed logical-dynamical (MLD) systems [Bemporad, Morari]:

  $\rightarrow$ mixed real-integer quadratic programming problems

- New approach based on canonical forms:

  $\rightarrow$ collection of linear programming problems

We will introduce the LP method in detail, we are familiar with LP and we will see how to **transform MMPS MPC to an LP** problem

Recall: $J(k)$ is MMPS function

$$
\begin{aligned}
\Rightarrow J(k) &= \max_i \left(\min_j \left(\gamma_{(i,j)}^T \tilde{u} + \delta_{(i,j)} \right) \right) \\
&= \min_i \left(\max_j \left(\alpha_{(i,j)}^T \tilde{u} + \beta_{(i,j)} \right) \right) \\
\Rightarrow \min_{\tilde{u}} J(k) &= \min_{\tilde{u}} \min_i \left(\max_j \left(\alpha_{(i,j)}^T \tilde{u} + \beta_{(i,j)} \right) \right) \\
&= \min_i \underbrace{\min_{\tilde{u}} \left(\max_j \left(\alpha_{(i,j)}^T \tilde{u} + \beta_{(i,j)} \right) \right)}_{\rightarrow \text{LP!}}
\end{aligned}
$$

Which means we will have i **LP problem** because the min-max part is equal to:

LP i :

$$
\begin{aligned}
&\min_{\tilde{u}} t \\
&\text{s.t.} \left\{ \begin{array}{ll} t \geqslant \alpha_{(i,j)}^T \tilde{u} + \beta_{(i,j)} & \text{for all } j \\ P\tilde{u} + q \geqslant 0 & \end{array} \right.
\end{aligned}
$$

4. Game-Theoretic Approach

We tried to use **Game-Theoretic Perspective** for **Safety-Critical Applications** such as collision avoidance in free flight or automated highways

- guarantee safety even in case intentions of other aircraft/vehicle are not known (**non-cooperative game**)
- if (partial) communication possible $\rightarrow$ **cooperative game**

Game-Theoretic Model

- **System Model:**

$$\dot{x} = f(x, u, d)$$

  - u control inputs (corresponding to 1st player)
  - d disturbance inputs (corresponding to 2nd player/adversary)

- **Safety Constraints** be represented by:

$$F = \{x \in X \mid S(x) \geqslant 0\}$$

- Cost Function
  - Let $t_0 \leqslant t_{\text{end}}$ and consider cost function

$$\begin{gathered} J : X \times \mathscr{U} \times \mathscr{D} \times [t_0, t_{\text{end}}] \rightarrow \mathbb{R} \\ i.e. \quad (x, u(\cdot), d(\cdot), t) \mapsto S\left(x\left(t_{\text{end}}\right)\right) \end{gathered}$$

  where $\mathscr{U}$ and $\mathscr{D}$ denote admissible control and disturbance functions

  J is cost associated with trajectory starting at x at time $t \in [t_0, t_{\text{end}}]$ with inputs $u(\cdot)$ and $d(\cdot)$, and ending at time $t = t_{\text{end}}$ at the final state $x\left(t_{\text{end}}\right)$

- **Optimization Target**

$$J^\star(x, t) = \max_{u \in \mathscr{U}} \min_{d \in \mathscr{D}} J(x, u, d, t)$$

Model Solution

- **Feasible Set(not officially)**

$$\left\{x \in X \mid \min_{\tau \in [t, t_{\text{end}}]} J^\star(x, \tau) \geqslant 0\right\}$$

  - contains all states for which system can be forced by control u to remain in safe set F for at least $|t_{\text{end}} - t|$ time units, irrespective 0 disturbance function d

- Value function $J^\star$ can be computed using **Hamilton-Jacobi equations**
  - Computation Tremendous Task
  - Provides systematic way to check safety properties for continuous-time system

Summary

5. Summary

- Optimal control of hybrid systems
- MPC for MLD and PWA systems
- MPC for MMPS and continuous PWA systems
- Game-theoretic approaches

- Optimal Control of Hybrid Manufacturing Systems
    - **temporal state (event-driven)**
    - physical state (time-driven)
    - event-driven part may cause **non-differentiable** when computing optimality condition of the cost function
- **Generalized Gradient** to deal with non-differentiable
- MPC for MLD and PWA
    - Control Horizon + Prediction Horizon
    - MLD: **mixed-integer quadratic programming (MIQP)**
    - PWA: transform to MLD
- MPC for MMPS and Continuous PWA
    - **Continuous PWA** $\leftrightarrow$ **Unconstrained MMPS**
    - Canonical form of MMPS function
    - MPC for MMPS: **canonical form + multiple LP problem**
- Gamte-Theoretical Approach
    - Safety-Critical
    - Hamilton-Jacobi equations