01_02_Nonlinear Models

1. Standard System Model

Definition: Compact Notion of Standard Systems

$$\dot{x} = f(t, x, u), \quad x(t_0) = x_0, \quad \text{where } x = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T, \text{ etc.}$$

2. System Properties

Linear Systems

Definition: Linear

The systems $\dot{x} = f(t, x, u)$ is linear if it can be written in the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

where $A(t)$ and $B(t)$ are real matrices for all t

Remark:

$A(t)$ and $B(t)$ have to be independent of $x(t)$ and $u(t)$

Theorem:

The system $\dot{x} = f(t, x, u)$ is linear if and only if

1. $f(t, \alpha x_a + \beta x_b, 0) = \alpha f(t, x_a, 0) + \beta f(t, x_b, 0)$, The influence of the state is additive
2. $f(t, 0, \alpha u_a + \beta u_b) = \alpha f(t, 0, u_a) + \beta f(t, 0, u_b)$, The influence of the input is additive

3. $f\left(t, \alpha x_a, \beta u_b\right)=\alpha f\left(t, x_a, 0\right)+\beta f\left(t, 0, u_b\right)$, The influence of the input and the state are independent

for all real vectors x_a, x_b, u_a, u_b, scalars α, β and times $t \geq t_0$.

Note:

1. ***How to show that a system is linear***

Find matrices $A(t)$ and $B(t)$ such that $\dot{x}=A x+B u$.

2. ***Hot to show that a system is nonlinear***

Verify that one of the conditions on the previous slide is violated. (It is not sufficient to write " $f(x, t)$ can not be written as $A x+B u$ ")

Time-invariant and Autonomous Systems

Definition: Time Invariant

The system $\dot{x}=f(t, x, u)$ is called time invariant if

$$\dot{x}=f(x, u)$$

i.e., f does not depend on t. Otherwise, it is called time varying.

Definition: Autonomous

A ***time invariant*** system without inputs is called **autonomous**, i.e.,

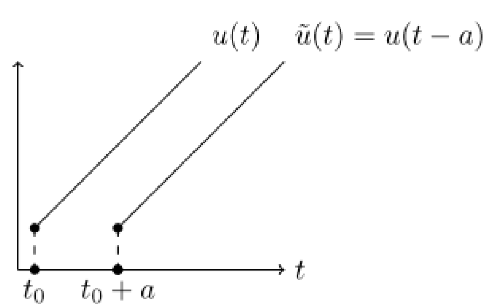
$$\dot{x}=f(x) .$$

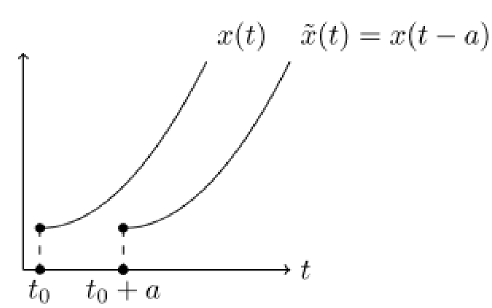
Theorem:

Let $x\left(t ; t_0, x_0, u\right)$ denote the state of the a time-invariant system that results from the initial condition $x\left(t_0\right)=x_0$ and the input $u(t)$. Then,

$$x\left(t ; t_0, x_0, u(t)\right)=x\left(t+a ; t_0+a, x_0, u(t-a)\right)$$

for all initial states x_0, inputs $u(t)$ and $a \in \mathbb{R}$.





Note:

1. ***How to show that a system is time invariant***

Show that $f(t, x, u)$ is independent of t for all x and u.

2. ***How to show that a system is time varying***

Show that $f(t, x, u)$ changes with t for some specific values of x and u. (It is not sufficient to write "t.v. since there is a t in $f(t, x) = \ldots$"!)

Equilibrium Points

<u>Definition: Unforced System</u>

A system without inputs, i.e., $\dot{x} = f(t, x)$, is called unforced.

<u>Definition: Equilibrium and Isolated Equilibrium</u>

An unforced system $\dot{x} = f(t, x)$ has an **<u>equilibrium</u>** at $x \in \mathbb{R}^n$ if $x(t) = x^*$ is a solution. That is,

$$0 = f\left(t, x^*\right)$$

for all $t \geq t_0$.

An equilibrium is **<u>isolated</u>** if there is an $\epsilon > 0$ such that every other equilibrium $x^* \neq x^*$ satisfies $\|x^* - x^*\| > \epsilon$.

<u>Property:</u>

linear systems cannot have multiple isolated equilibrium points. This is a unique nonlinear phenomenon:

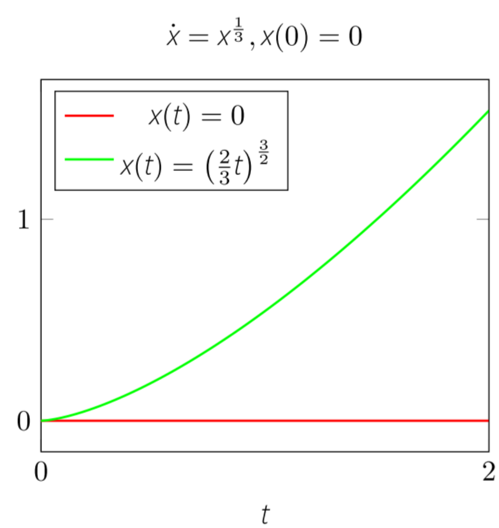
Illustration:

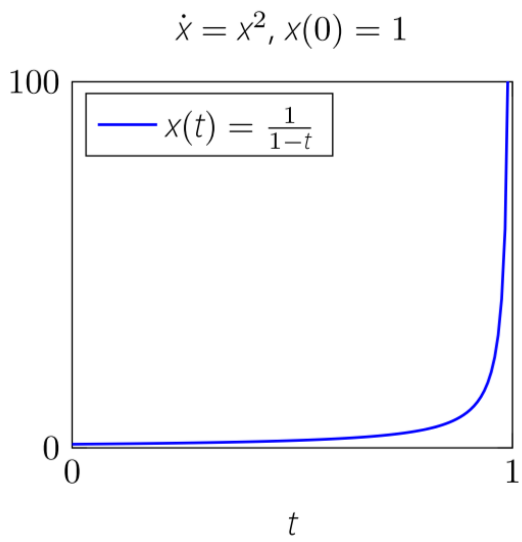
make a line between to isolated equilibrium points, and based on linear property, the line will all be equilibrium points

3. Solvable Conditions

Problem with Nonlinear System

- The solution may only exists on a finite time interval
- The system may have more than one solution





Piecewise Continuity

Definition: Piecewise Continuous

The function

$$f : J \times D \to \mathbb{R}^n, \quad J = \text{ interval }, D = \text{ domain in } \mathbb{R}^n,$$

is **piecewise continuous** in t if, for any fixed $\bar{x} \in D$, the function

$$g : J \to \mathbb{R}^n, \quad g(t) := f(t, \bar{x}),$$

satisfies the following two conditions:

- at any $t = t_0 \in J$, the function $g(t)$ is either **continuous or makes a jump of finite height**
- the number of jumps that $g(t)$ makes **in any finite interval**

$$J_0 = [a, b], \quad a < b, \quad a, b \in J,$$

is finite

Note:

A domain is a open and connected set:

- Open: a point in the domain has a $\epsilon > 0$ neighborhood in the domain
- Connected: Two points, can be connected by an arc in the set

Lipschitz Continuity

Definition: Lipschitz (Continuity)

The function

$$f : J \times D \to \mathbb{R}^n, \quad J = \text{ interval }, D = \text{ domain in } \mathbb{R}^n,$$

is **Lipschitz (continuous)** if there exists a $L > 0$ s.t.

$$\|f(t,y) - f(t,x)\| \le L\|y - x\| \quad \text{for all } x, y \in D \text{ and } t \in J.$$

Definition: Globally Lipschitz Continuity

It is **Globally Lipschitz Continuous** if furthermore $D = \mathbb{R}^n$

Definition: Locally Lipschitz Continuity

It is **Locally Lipschitz Continuous in** x_0 if there exists an $\epsilon > 0$ s.t.

$$f_0 : J \times \{x \in D : \|x - x_0\| < \epsilon\} \to \mathbb{R}^n, \quad f_0(t,x) = f(t,x)$$

is a Lipschitz Continuous Function

Theorem: Lipschitz continuity via partial derivatives

- Locally Lipschitz:

  Let $f : J \times D \to \mathbb{R}^n, J =$ interval, $D =$ domain in $\mathbb{R}^n$, and

$$\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

  both be **continuous.** Then, f is locally Lipschitz in x_0 for all $x_0 \in D$.

- Lipschitz

  Furthermore, f is Lipschitz if and only if $[\partial f / \partial x]$ is **uniformly bounded**, i.e, there exists a $B > 0$ such that

$$\left\|\left[\frac{\partial f(t,x)}{\partial x}\right]\right\| \le B \quad \text{for all } t \in J, x \in D$$

  Any $B > 0$ that fulfills this condition is a **Lipschitz constant**,

$$\|f(t,y) - f(t,x)\| \le B\|y - x\| \quad \text{for all } t \in J, x \in D$$

Definition: Frobenius Norm

- For a vector $x \in \mathbb{R}^n$, we consider the/usual **Eucledian norm** $\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}$.

- For a matrix $A \in \mathbb{R}^{n \times n}$, $\|A\| := \max_{x \in \mathbb{R}^n, \|x\|=1} \|Ax\|$ is the **spectral norm (**It corresponds to the largest singular value, which is not always easy to compute)
- The **Frobenius norm**

$$\|A\|_F := \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}^2}$$

is often a practical alternative when bounding $\|[\partial f / \partial x]\|$ since

$$\frac{1}{n} \|A\|_F \leq \|A\| \leq \|A\|_F$$

Solvability Conditions

Solvability over $[t_0, t_1]$

Theorem

Let the function

$$f : J \times D \to \mathbb{R}^n, \quad J = [t_0, t_1], D = \mathbb{R}^n,$$

be **piecewise continuous in** t and **globally Lipschitz continuous**. Then, the unforced system

$$\dot{x} = f(t, x), \quad x(t_0) = x_0,$$

has a unique solution over $[t_0, t_1]$ for any x_0.

Solvability over $[t_0, t_0 + \delta]$

Let the function

$$f : J \times D \to \mathbb{R}^n, \quad J = [t_0, t_1], D = \text{ domain in } \mathbb{R}^n,$$

be **piecewise continuous in** t and **locally Lipschitz continuous** at x_0. Then, there exists a $\delta > 0$ such that the unforced system

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

has a unique solution over $[t_0, t_0 + \delta]$.

Solvability over $[t_0, \infty)$

Let the function

$$f : J \times D \to \mathbb{R}^n, \quad J = [t_0, \infty), D = \text{ domain in } \mathbb{R}^n,$$

be **piecewise continuous in t** and **locally Lipschitz continuous in x**. Furthermore, let W **be a compact (closed and bounded) subset** of D such that $x_0 \in W$. If any solution $x(t)$ of

$$\dot{x} = f(t, x), \quad x(t_0) = x_0,$$

stays in W as long as it exists, then it has a unique solution on $[t_0, \infty)$.

Summary

- System Properties:
  - linear
  - time-varying
  - autonomous (first should be time-invariant)
- Piece-wise Continuous
- Lipschitz Continuity:
  - definition
  - sufficient condition: derivative continuous $\Rightarrow$ locally Lipschitz
  - n+s condition: bounded derivative $\Rightarrow$ globally Lipchitz
- Solvable: f, J, D
  - $[t_0, t_1]$: piecewise continuity (t) + global Lipchitz (x)
    - can be extended to $[t_0, \infty]$
  - $[t_0, t_0 + \delta]$: piecewise continuity (t) + locally Lipchitz (x)
  - $[t_0, \infty]$: piecewise continuity (t) + compact $W \subset D$ + locally lipcitz (x)