Iterative Control

- Super-Vector Notation

Original Model:

$$x_j^{k+1} = A(k)\cdot x_j^k + B(k)\cdot u_j^k \quad \leftarrow \text{timestep}$$

$$y_j^{k+1} = C(k+1)\, x_j^{k+1} + D(k)\cdot u_j^{k+1} \quad \leftarrow \text{iteration}$$

Expand:

$$y_j^{k+1} = \left[C(k+1)\cdot\left(\prod_{i=0}^{k} A(i)\right)\cdot x_j^0\right] \quad \leftarrow \text{IID. } x_j^0 = x^0$$

$$+ \sum_{i=0}^{k}\left[C(k+1)\cdot\left(\prod_{\kappa=i+1}^{k} A(\kappa)\right)\cdot B(i)\right]\cdot u_j^i$$

$$+ D(k)\cdot u_j^{k+1}$$

Super-Vector:

$$\begin{array}{l} k=0 \\ k=1 \\ k=2 \\ \vdots \\ k=N-1 \end{array}
\underbrace{\begin{bmatrix} y_j^1 \\ y_j^2 \\ y_j^3 \\ \vdots \\ y_j^N \end{bmatrix}}_{Y_j \in R^{MN}}
=
\underbrace{\begin{bmatrix} C(1)\cdot A(0)\cdot x^0 \\ C(2)\cdot A(1)\cdot A(0)\cdot x^0 \\ \vdots \\ \vdots \end{bmatrix}}_{H \in R^{MN}}
+
\underbrace{\begin{bmatrix} C(1)B(0), & D(0), & 0 & \cdots & 0 \\ C(2)A(1)B(0), & C(2)B(1), & D(1) & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots \\ & & & & \end{bmatrix}}_{P}
\cdot \underbrace{\vec{u}}_{u}$$

independ to j

$$Y_j = H + P\cdot u_j$$

For time-invariant A, B, C, D.

$$H = \left[CAx_0,\ CA^2x_0,\ \cdots\ CA^N x_0\right]^T$$

$$P = \begin{bmatrix} CB & D & & & \\ CAB & CB & D & & \\ CA^2B & & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots \\ CA^{N-1}B & & & & \ddots \end{bmatrix}$$

- **Linear Learning Rule**

$$U_{j+1} = Q(U_j + L\cdot E_j)$$

$\bar{Y} - Y_j$

Expand Using Super-Vector Dynamic Model:

$$U_{j+1} = Q(U_j + L\cdot(\bar{Y} - H - P\cdot U_j))$$

$$= Q(I - LP)\cdot U_j + QL(\bar{Y} - H)$$

Asymptotical Stable iff: $\rho(Q(I-LP)) < 1$

The Attractor: $U_j \rightarrow U_\infty$

$$(I - Q(I-LP))\,U_\infty = QL(\bar{Y} - H)$$

$$\Rightarrow \quad U_\infty = (I - Q(I-LP))^{-1}\cdot QL(\bar{Y} - H)$$

$$\therefore \quad Y_\infty = H + P\cdot(I - Q(I-LP))^{-1}\,QL(\bar{Y} - H)$$

$$\Rightarrow \quad \underbrace{\bar{Y} - Y_\infty}_{E_\infty} = (I - P(I - Q(I-LP))^{-1}QL)\,(\bar{Y} - H)$$

$\therefore$ If $Q = I$

$$E_\infty = (I - \overbrace{P(LP)^{-1}L}^{I})(\bar{Y} - H) = 0$$

- **DeadBeat Learning Rule** $(Q = I)$

If $L = P^{-1}$ $\quad E_1 = E_\infty = 0$ $\leftarrow$ learn in one step

- **P Learning Rule** $(Q = I)$

Same format as Linear

$$u_{j+1}^{k-1} = u_j^{k-1} + l(\bar{y}^k - y_j^k)$$

A.S. $\rho(I - LP) < 1 \Rightarrow \rho(I - CBl) < 1$

Then, use $l = (CB)^{-1}$ if $D = 0$, time independent A, B, C, D

$$\begin{bmatrix} CB & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ \cdots & \cdots & \cdots & CB \end{bmatrix}$$

- PD-Type Learning Rate

nothing with u_j^k

$$u_{j+1}^{k-1} = u_j^{k-1} + L e_j^k + d\,(e_j^k - e_j^{k-1})$$

A.S. $\rho(I - LP) < 1 \Rightarrow \rho(I - CB(L+d)) < 1$

We can use $CB(L+d) = I$

- LQ-Type Learning Rule (with Linear Learning model)

• Cost function: $J_{j+1}(u_{j+1}) = E_{j+1}^T Q_{LQ} E_{j+1} + (u_{j+1} - u_j)^T S_{LQ} (u_{j+1} - u_j)$

Solution: $L^* = (P^T Q_{LQ} P + S_{LQ})^{-1} P^T Q_{LQ}$

• If $\det(P) \neq 0$, $S_{LQ} = 0$

Then $L^* = P^{-1}$