07_08_Robust MPC

1. Introduction of Robust Constrained Control

The real world: full of noise, parameters inaccurate, system structure not fully known.

Model: Uncertain Constrained Systems

$$x^+ = f(x, u, w; \theta) \quad (x, u) \in \mathbb{X}, \mathbb{U} \quad w \in \mathbb{W} \quad \theta \in \Theta$$

Different Uncertainty Model

Model 1: Measurement/Input Bias

θ unknown, but constant

$$g(x, u, w; \theta) = f(x, u) + \theta$$

Model 2: Linear Parameter Varying System

$$g(x,u,w;\theta)=\sum_{k=0}^{l}\theta_k A_k x+\sum_{k=0}^{\imath}\theta_k B_k u,\quad \mathbf{1}^\top\theta=1,\theta\geq 0$$

A_k, B_k known, θ_k unknown, but constant

Model 3: Polytopic Uncertainty

$$g(x,u,w;\theta)=\sum_{k=0}^{l}w_k A_k x+\sum_{k=0}w_k B_k u,\quad \mathbf{1}^\top w=1,w\geq 0$$

A_k, B_k known, w_k unknown and changing at each sample time

Model 4: Additive Stochastic Noise

$$g(x,u,w;\theta)=Ax+Bu+w$$

Distribution of w known

In this Chapter, we will mainly focus on **Additive Stochastic Noise** Model.

2. Impact of Bounded Additive Noise

Model: Additive Bounded Noise

$$g(x,u,w;\theta)=Ax+Bu+w,\quad w\in\mathbb{W}$$

A, B known, w unknown and changing with each sample

Property:

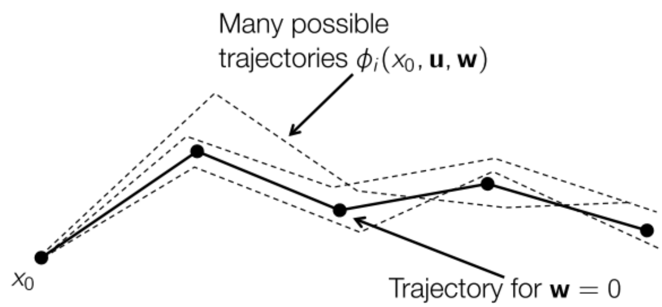
- Dynamics are linear, but impacted by **random, bounded noise at each time step**
- It is always a **conservative model**
- The noise is **persistent**, i.e. it does not converge to zero in the limit

Goal:

So the basic motivation is to design a control lay that will satisfy constraints and **stabilize the system for all possible disturbances**.

Uncertain State Evolution

Define $\phi_i\left(x_0,\mathbf{u},\mathbf{w}\right)$ as the state that the system will be in at time i if the state at time zero is x_0, we apply the input $\mathbf{u}:=\{u_0\ldots,u_{N-1}\}$ and we observe the disturbance $\mathbf{w}:=\{w_0,\ldots,w_{N-1}\}$.



Define $\phi_i(x_0, \mathbf{u}, \mathbf{w})$ as the state that the system will be in at time i if the state at time zero is x_0, we apply the input $\mathbf{u} := \{u_0, \ldots, u_{N-1}\}$ and we observe the disturbance $\mathbf{w} := \{w_0, \ldots, w_{N-1}\}$.

Then we can compare the nominal system and uncertain system:

For Nominal System: $x^+ = Ax + Bu$, so we will have:

$$
\begin{aligned}
x_1 &= Ax_0 + Bu_0 \\
x_2 &= A^2 x_0 + ABu_0 + Bu_1 \\
&\vdots \\
x_i &= A^i x_0 + \sum_{k=0}^{i-1} A^k B u_{i-k}
\end{aligned}
$$

For Uncertain System: $x^+ = Ax + Bu + w \quad w \in \mathcal{W}$, we will have:

$$
\begin{aligned}
\phi_1 &= Ax_0 + Bu_0 + w_0 \\
\phi_2 &= A^2 x_0 + ABu_0 + Bu_1 + Aw_0 + w_1 \\
&\vdots \\
\phi_i &= A^i x_0 + \sum_{k=0}^{i-1} A^k B u_{i-k} + \sum_{k=0}^{i-1} A^k w_{i-k} \\
\phi_i &= x_i + \sum_{k=0}^{i-1} A^k w_{i-k}
\end{aligned}
$$

Uncertain evolution is the nominal system + offset caused by the disturbance

Cost Function Definition

Another problem is how to define cost function in uncertain system. The cost is now a function of the disturbance, and therefore each possible trajectory has a different cost:

$$J\left(x_0, \mathbf{u}, \mathbf{w}\right) := \sum_{i=0}^{N-1} I\left(\phi_i\left(x_0, \mathbf{u}, \mathbf{w}\right), u_i\right) + V_f\left(\phi_N\left(x_0, \mathbf{u}, \mathbf{w}\right)\right)$$

Model: Minimize the Expected Value

$$V_N\left(x_0, \mathbf{u}\right) := \mathbf{E}\left[J\left(x_0, \mathbf{u}, \mathbf{w}\right)\right]$$

Model: Minimize the Variance

$$V_N\left(x_0, \mathbf{u}\right) := \operatorname{Var}\left(J\left(x_0, \mathbf{u}, \mathbf{w}\right)\right)$$

Model: Take the Worst-Case

$$V_N\left(x_0, \mathbf{u}\right) := \max_{\mathbf{w} \in \boldsymbol{W}^{N-1}} J\left(x_0, \mathbf{u}, \mathbf{w}\right)$$

Model: Take the Nominal-Case

$$V_N\left(x_0, \mathbf{u}\right) := J\left(x_0, \mathbf{u}, 0\right)$$

Robust Invariant Set

Robust Positive Invariant Set and Robust Pre Set

Definition: Robust Positive Invariant Set

A set $\mathcal{O}^W$ is said to be a robust positive invariant set for the autonomous system $x_{i+1} = f\left(x_i, w\right)$ if

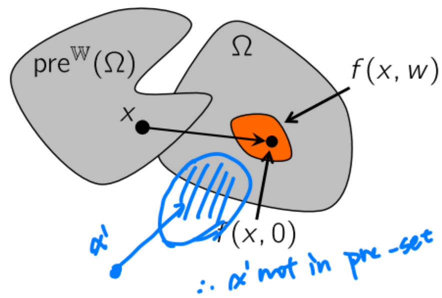
$$x \in \mathcal{O}^W \Rightarrow f(x, w) \in \mathcal{O}^W, \text{ for all } w \in \mathbb{W}$$

Definition: Robust Pre Set

Given a set Ω and the dynamic system $x^+ = f(x, w)$, the pre-set of Ω is the set of states that evolve into the target set Ω in one time step **for all** values of the disturbance $w \in \mathbb{W}$:

$$\operatorname{pre}^W(\Omega) := \{x \mid f(x, w) \in \Omega \textbf{ for all } w \in \mathbb{W}\}$$

These two definitions are illustrated in Figure:



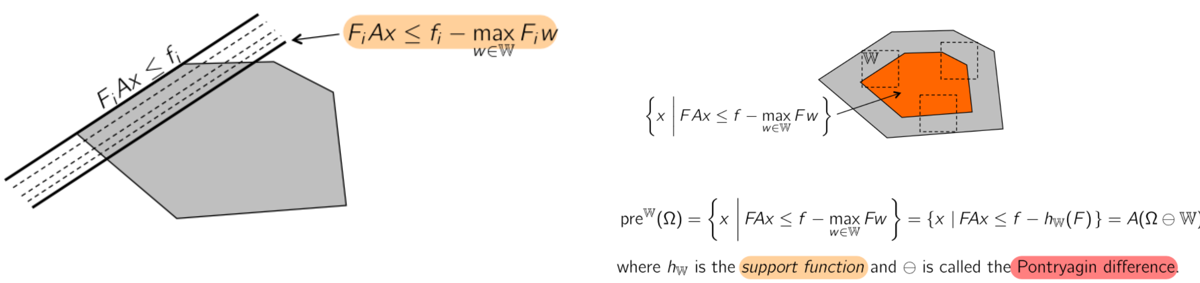
<u>Theorem: Geometric Condition for Robust Invariance</u>

A set $\mathcal{O}$ is a robust positive invariant set if and only if

$$\mathcal{O} \subseteq \mathrm{pre}^{\mathrm{W}}(\mathcal{O})$$

Computation of Robust Pre Set

The basic idea of compute the Robust Pre Set is shown in Figure



The robust pre set then can be computed by:

$$\mathrm{pre}^{\mathrm{W}}(\Omega) = \left\{ x \mid FAx \leq f - \max_{w \in W} F_W \right\} = \{x \mid FAx \leq f - h_W(F)\} = A(\Omega \ominus \mathbb{W})$$

Computing Robust Invariant Set

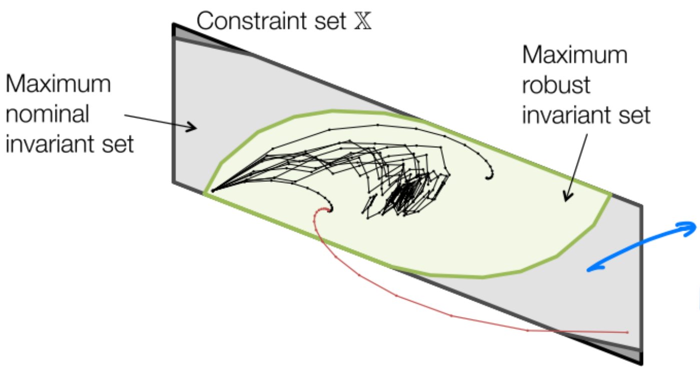
Conceptual Algorithm to Compute Robust Invariant Set

Input: f, $\mathbb{X}$, $\mathbb{W}$

Output: $\mathcal{O}_\infty^{\mathbb{W}}$

$\Omega_0 \leftarrow \mathbb{X}$
loop
  $\Omega_{i+1} \leftarrow \mathrm{pre}^{\mathbb{W}}(\Omega_i) \cap \Omega_i$
  if $\Omega_{i+1} = \Omega_i$ **then**
    return $\mathcal{O}_\infty = \Omega_i$
  end if
end loop

The following figure shows the relations between constraint set, maximum robust invariant set and maximum nominal invariant set.



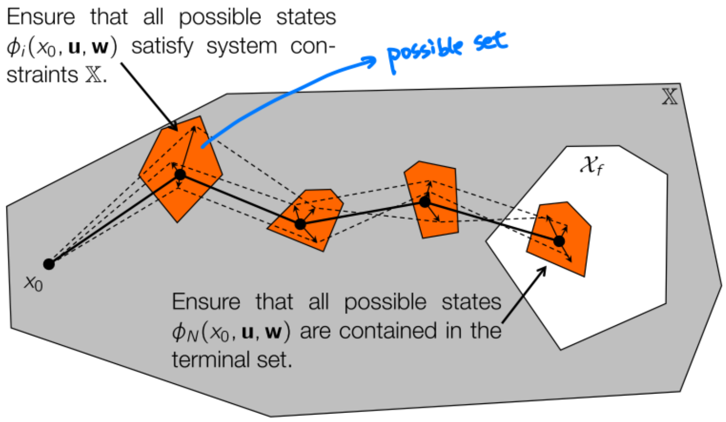
Does not mean every trajectory here will be malicious

One important thing to be noticed is that because in the robust invariant set, we considered **"for all possible disturbance",** it is quite conservative. So, in Figure, a state in the gray area does not indicate that every trajectory here will be malicious.

From Robust Invariant Set to Dealing with Uncertainty

Idea:

Compute a set of tighter constraints such that if the nominal system meets these constraints, then the uncertain system will too. We then do MPC on the nominal system.



The idea: Compute a set of tighter constraints such that if **the nominal system** meets these constraints, then the uncertain system will too. We then do MPC **on the nominal system.**

Robust Constraint Satisfaction

So the first problem is what should we do on the constraint satisfication. Remember the uncertain evaluation:

$$\phi_i(x_0, \mathbf{u}, \mathbf{w}) = \left\{ x_i + \sum_{k=0}^{i-1} A^k w_k \mid \mathbf{w} \in \mathbb{W}^i \right\} \subseteq \mathbb{X}$$

Assume that $\mathbb{X} = \{x \mid Fx \leq f\}$, then this is equivalent to

$$Fx_i + F\sum_{k=0}^{i-1} A^k w_k \leq f \quad \forall \mathbf{w} \in \mathbb{W}^i$$

We've seen this before while computing the robust pre-set:

$$Fx_i \leq f - \max_{w \in Wi} F \sum_{k=0}^{i-1} A^k w_k = f - h_{W,i}\left(F\sum_{k=0}^{i-1} A^k\right)$$

__**Definition: Support Function**__

let $A \subset \mathbb{R}^n$ be a non-empty convex set. The support function of A $h_A : \mathbb{R}^n \to (-\infty, \infty]$ is defined as: $h_A(u) := \sup\{\langle x, u\rangle : x \in A\}, u \in \mathbb{R}^n$.

We always use $h(A, .) := h_A$ to represent the support function of A

After tightening the constraints on the nominal system, we can guarantee that the system will meet constraints even with noise.

Robust Terminal State Constraints

Another problems is hot to deal with **terminal state constraints**:

$$\phi_N\left(x_0, \mathbf{u}, \mathbf{w}\right) \subseteq \mathcal{X}_f$$

Which can be done in a similar way.

3. Robust Open-Loop MPC

To be concluded, we can propose the **Robust Open-Loop MPC**.

Model: Robust Open-Loop MPC

$$\begin{aligned} \min_u \sum_{i=0}^{N-1} \quad & l\left(x_i, u_i\right)+V_f\left(x_N\right) \\ \text{s.t. } x_{i+1} \quad & =A x_i+B u_i \\ x_i \quad & \in \mathbb{X} \ominus \mathcal{A}_i \mathbb{W}^i \\ u_i \quad & \in \mathbb{U} \\ x_N \quad & \in \mathcal{X}_f \ominus \mathcal{A}_N \mathbb{W}^N \end{aligned}$$

where $\mathcal{A}_i:=\left[\begin{array}{llll}A^0 & A^1 & \ldots & A^i\end{array}\right]$ *and* $\tilde{\mathcal{X}}_f$ is a robust invariant set for the system $x^{+}=(A+B K) x$ for some stabilizing K.

- That is we do nominal MPC, but with tighter constraints.
- One important thing to be noticed that, in Robust Open-Loop MPC, we need the A to be a stable A. If not, the noisy part (the support function) will become larger and larger.

Property

If $\mathbf{u}^*(x)$ is the optimizer of the robust open-loop MPC problem, then the system $A x+B u_0^*(x)+w \in \mathbb{X}$ for all $w \in \mathbb{W}$.

However, robust open-loop MPC has a very small region of attraction. **This is a reason that we almost never use it in reality.**

4. Closed-Loop Prediction

Closed-Loop MPC

From the game perspective, when in MPC problem, what we want to do is to optimize over **a sequence of functions** $\left\{u_0, \mu_1(\cdot) \ldots \ldots, \mu_{N-1}(\cdot)\right\}$. where $\mu_i\left(x_i\right): \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a **control policy**. and maps the state at time i to an input at time i.

Notes:

- This is the same as making μ a **function** of the disturbances to time i, since the state is a function of the disturbances up to that point
- The first input u_0 is a **function of the current state**, which is known. Therefore it is not a function, but a single value.

It is hard to search for a "optimal solution" in the whole space. So firstly, we need to **assume the form of the function**. There are several popular forms:

1. **Pre-Stabilization**

$$\mu_i(x) = Kx + v_i$$

- Fixed K, such that $A + BK$ is stable
- Simple, often conservative
- v_i is optimized online

2. **Linear Feedback**

$$\mu_i(x) = K_i x + v_i$$

- Optimize over K_i and v_i
- Non-convex. Extremely difficult to solve...

3. **Disturbance Feedback**

$$\mu_i(x) = \sum_{j=0}^{i-1} M_{ij} w_j + v_i$$

- Optimize over M_{ij} and v_i
- Equivalent to linear feedback, but convex
- Can be very effective, but computationally intense

4. **Tube-MPC**

$$\mu_i(x) = v_i + K(x - \bar{x}_i)$$

- Fixed K, such that $A + BK$ is stable
- Optimize over $\bar{x}_i$ and v_i
- Simple, and can be effective

5. Tube-MPC

Model:

$$x^+ = Ax + Bu + w \quad (x, u) \in X \times \mathrm{U} \quad w \in W$$

Controller:

$$\mu_i(x) = v_i + K(x - \bar{x}_i)$$

Motivation:

The idea is to seperate the available control authority into two parts:

1. A portion that steers the noise-free system to the origin $z^+ = Az + Bv$
2. A portion that compensates for deviations from this system $e^+ = (A + BK)e + w$

System Decomposition

First, we define a **nominal noise-free subsystem**:

$$z_{i+1} = Az_i + Bv_i$$

The Kz_i part of the Tube-MPC controller is used to help the system stabilize this subsystem.

Then we can define a **error subsystem** $e_i = x_i - z_i$:

<u>Model: Error Subsystem:</u>

$$\begin{aligned} e_{i+1} &= x_{i+1} - z_{i+1} \\ &= Ax_i + Bu_i + w_i - Az_i - Bv_i \\ &= Ax_i + BK\left(x_i - z_j\right) + Bv_i + w_i - Az_i - Bv_i \\ &= \left(A + BK\right)\left(x_i - z_i\right) + w_i \\ &= (A + BK)e_i + w_i \end{aligned}$$

Dynamics $A + BK$ are stable, and the set W is bounded, so there is some set $\mathcal{E}$ that e will stay inside for all time.

We know that the real trajectory stays 'nearby' the nominal one: $x_i \in z_i \oplus \mathcal{E}$ because we plan to apply the controller $u_i = K\left(x_j - z_j\right) + v_i$ in the future (we won't actually do this, but it's a valid sub-optimal plan)

That is by setting the format of the controller, we shrink the possible set the trajectory and the nominal trajectory can be

Uncertain State Evolution

Consider the system $x^+ = Ax + w$ and assume that $x_0 = 0$

As sum goes to infinity, we arrive at the **minimum robust invariant set**

$$F_\infty = \bigoplus_{k=0}^{\infty} A^k W, \quad F_0 := \{0\}$$

If there exists an n such that $F_n = F_{n+1}$, then $F_n = F_\infty$

The minimal invariant set can be calculate following the progress in Figure

Minimal Invariant Set

Input: A
Output: F_∞
  $\Omega_0 \leftarrow \{0\}$
  loop
    $\Omega_{i+1} \leftarrow \Omega_i \oplus A^i \mathbb{W}$
    if $\Omega_{i+1} = \Omega_i$ **then**
      return $F_\infty = \Omega_i$
    end if
  end loop

<u>Note:</u>

- A finite n does not always exist, but a 'large' n is a good approximation.
- If we use controller $\mu_i(x) = v_i + K(x - \bar{x}_i)$, the input A actually should be $A + BK$

Method: Computing Minkowski Sums

Given $P := \{x \mid Tx \leq t\}$ and $Q := \{x \mid Rx \leq r\}$, the Minkowski sum is:

$$\begin{aligned} P \oplus Q &:= \{x + y \mid x \in P, y \in Q\} \\ &= \{z \mid \exists x, y z = x + y, Tx \leq t, Ry \leq r\} \\ &= \{z \mid \exists y \quad Tz - Ty \leq t, Ry \leq r\} \\ &= \left\{ z \middle| \exists y \begin{bmatrix} T & -T \\ 0 & R \end{bmatrix} \begin{pmatrix} z \\ y \end{pmatrix} \leq \begin{pmatrix} t \\ r \end{pmatrix} \right\} \end{aligned}$$

Note:

This is a **projection** of a polyhedron from (z, y) onto z

Constraint Tightening

Definition: Pontryagin Difference

Let A and B be subsets of $\mathbb{R}^n$. The Pontryagin Difference is

$$A \ominus B := \{x \mid x + e \in A \forall e \in B\}$$

Then the state constraints and the input constraints can be calculated by:

$$\begin{aligned} z_i \oplus \mathcal{E} \subseteq \mathbb{X} \quad &\Leftarrow \quad z_i \in \mathbb{X} \ominus \mathcal{E} \\ u_i \in K\mathcal{E} \oplus v_i \subset \mathbb{U} \quad &\Leftarrow \quad v_i \in \mathbb{U} \ominus K\mathcal{E} \end{aligned}$$

And the terminal constraint can be calculated by:

$$\operatorname{pre}\left(\mathcal{X}_f\right) \subseteq \mathcal{X}_f \text{ and } x_f \subseteq \mathbb{X} \in \mathcal{E} \text{ and } K\mathcal{X}_f \subseteq \mathbb{U} \in K\mathcal{E}$$

Tube-MPC Problem Formulation

$$\begin{aligned} &\text{Feasible set: } \mathcal{Z}\left(x_0\right) := \left\{ \begin{array}{c} \left\{ \begin{array}{cc} z_{i+1} = Az_i + Bv_i & i \in [0, N-1] \\ z_i \in X \oplus \mathcal{E} & i \in [0, N-1] \\ v_i \in U \oplus K\mathcal{E} & i \in [0, N-1] \\ z_N \in \mathcal{X}_f & x_0 \in z_0 \oplus \mathcal{E} \end{array} \right. \\ \text{Cost:} \quad V(\mathbf{z}, \mathbf{v}) := \sum_{i=0}^{N-1} I\left(z_i, v_i\right) + V_f\left(z_N\right) \end{array} \right\} \\ &\text{Opti. prob: } \left(\mathbf{v}^*\left(x_0\right), \mathbf{z}^*\left(x_0\right)\right) = \operatorname{argmin}_{v,z} \left\{V(\mathbf{z}, \mathbf{v}) \mid (\mathbf{z}, \mathbf{v}) \in \mathcal{Z}\left(x_0\right)\right\} \\ &\text{Control law; } \mu_{\text{tube}}\left(x\right) := K\left(x - z_0^+(x)\right) + v_0^*(x) \end{aligned}$$

Stability Analysis

Assumption

- The stage cost is a **positive definite** function.
- The terminal set is invariant for the nominal system under the local control law $K_f\,(z)$:
  $z^+ = Az + B\kappa_f\,(z) \in X_f \quad$ for all $z \in X_f$
  All tightened state and input constraints are satisfied in $\mathcal{X}_f$ *:*
  $\mathcal{X}_f \subseteq X \ominus \mathcal{E}, \kappa_f\,(z) \in$ U $\in \mathcal{E}$ for all $z \in \mathcal{X}_f$
- Terminal cost is a continuous Lyapunov function in the terminal set $\mathcal{X}_f$ *:*
  $V\,f\,(Az + B\kappa_f\,(z)) - V_f\,(z) \leq -l\,(z, \kappa_f\,(z))$ for all $z \in \mathcal{X}_f$

Then we have the following Theorem

Theorem

The state x of the system $x^+ = Ax + B\mu_{\text{tube}}\,(x) + w$ converges in the limit to the set $\mathcal{E}$.

That is the state x does not converge to zero.

The Overall Procedure of Tube-MPC

Offline Part

- Choose a stabilizing controller K so that $\|A + BK\| < 1$
- Compute the minimal robust invariant set $\mathcal{E} = F_\infty$ for the system $x^+ = (A + BK)x + w, w \in W^1$
- Compute the tightened constraints $\tilde{X} := X \ominus \mathcal{E},\ U_U\ = U \ominus \mathcal{E}$
- Choose terminal weight function V_f and constraint $\mathcal{X}_f$ satisfying assumptions

Online Part

- Measure / estimate state x
- Solve the problem $\left(\mathbf{v}^*(x), \mathbf{z}^*(x)\right) = \operatorname{argmin}_{v,z} \{v(\mathbf{z}, v) \mid (\mathbf{z}, \mathbf{v}) \in \mathcal{Z}(x)\}\backslash$
- Set the input to $u = K\,(x - z_0^*(x)) + v_0^*(x)$

6. Nominal MPC with Noise

In this section, we will discuss what will happen if we just ignore the noise and design controller without considering the noise.

One important fact is: For some states, the system will sometimes work fine.

Stability

Lyapunov Stability Analysis

Assume the optimal cost V^* is continuous

$$
\begin{aligned}
&| \; V^* \left(Ax + Bu^*(x) + w\right) - V^* \left(Ax + Bu^*(x)\right) \; | \\
&\qquad\qquad\qquad\qquad \leq \gamma \left\| Ax + Bu^*(x) + w - \left(Ax + Bu^\star(x)\right) \right\| = \gamma \|w\|
\end{aligned}
$$

Then the Lyapunov decrease con be calculated:

$$
\begin{aligned}
&V^* \left(Ax + Bu^*(x) + w\right) - V^*(x) \\
&= V^\star \left(Ax + Bu^\star(x) + w\right) - V^\star(x) - V^\star \left(Ax + Bu^\star(x)\right) + V^\star \left(Ax + Bu^\star\right) \\
&\leq V^* \left(Ax + Bu^\star(x)\right) - V^*(x) + \gamma \|w\| \\
&\leq -l \left(x, u^\star(x)\right) + \gamma \|w\|
\end{aligned}
$$

l is a decrease part and $||w||$ is a increase/constant part, that is:

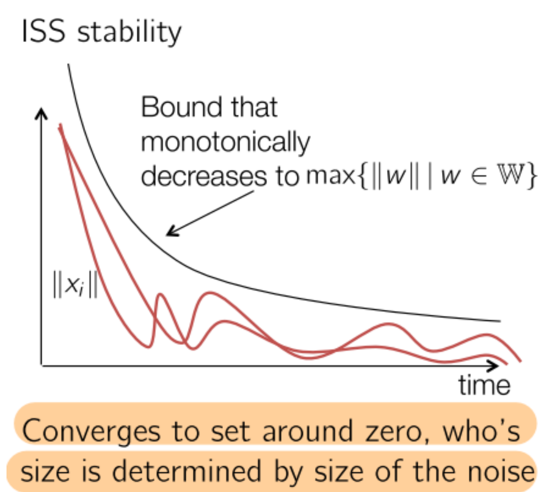
- Amount of decrease grows with $||x||$
- Amount of increase is upper bounded by $\max\{\|w\| \mid w \in \mathbb{W}\}$

Therefore, the system will move towards the origin until there is a balance between the size of x and the size of w

Input-To-State Stability (ISS Stability)

Roughly speaking, a control system is **ISS** if it is globally asymptotically stable in the absence of external inputs and if its trajectories are bounded by a function of the size of the input for all sufficiently large times.

Based on the Lyapunov analysis, we knows that the system ignoring noise is ISS stable.



Design of Nominal MPC

So in reality, we can just ignore the noise and hope it will work.

The analysis of Nominal MPC also tells as, for robust MPC, most of times we need to know the largest noise W, but even if we take do not know that, and we take W with **a smaller boundary assumption**, it will at least ISS.

Summary

- MPC Relies on model, but most of times, the models are not perfect and we may have **disturbance** based on some distribution.
- In order to solve this problem, we introduce the concept of **robust invariant set**, which means a set, the system keep feasible under all possible disturbance.
- By this conception, we can:
    - **Narrow the constraint** set so that when our controller meet the new constraints, no matter what disturbance happen, the system will always meet the constraint(conservative)
    - Use **robust invariant** modify the previous final terminal set
- Now we will get a new open-loop MPC problem that can be solved and is robust. But, it is always **too conservative** that it has a very small region of attraction, which means it is not good to use.
- **Tube MPC,** the Tube MPC defines some **explicit form of controller**: **feedback based on state error + an extra part**. Because of the feedback part, we can limit the effect of the disturbance.
- we discuss what if we just directly **ignore the disturbance** and design controller. The result shows that the system will be **ISS stable**.