04_Observer, Feedback Controller, PID Controller and LQR Controller

1. Observer

Luenberger Observer Design

State Estimation Problem Target

Find $\hat{x} \in \mathbb{R}^n$ and associated linear model (observer):

$$\frac{d\hat{x}}{dt} = F\hat{x} + Gu + Hy$$

such that $\hat{x}(t) \to x(t)$ as $t \to \infty$

Theorem: Observer design by eigen value assignment

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)$$

with characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

If the system is **observable**, then there **exists a observer**

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

that gives a closed loop system with the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1}s + p_n$$

The feedback gain is given by

$$L = T^{-1}\tilde{L} = W_o^{-1}\tilde{W}o \begin{bmatrix} p1 - a_1 & p_2 - a_2 & \ldots & p_n - a_n \end{bmatrix}^T$$

Some Mathematical Details

$$\begin{aligned}
&\tilde{A} - \tilde{L}\tilde{C} = \\
&= \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ -a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \\
&= \begin{bmatrix} -p_1 & 1 & \cdots & 0 \\ -p_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -p_n & 0 & \cdots & 0 \end{bmatrix}
\end{aligned}$$

Some Discussion (self understanding)

One interesting thing of the observer is:

Consider a real-life system $< A, B, C, D >$ and we have the model $< \hat{A}, \hat{B}, \hat{C}, \hat{D} >$.

If we write out the error dynamics, we can see that, because of the “feedback” part, the system will still gradually get close to the true value, perhaps with some bounded variation (like ISS stability).

Kalman Filter (Continuous Time Case)

Consider

- Continuous-time Random Process
- Estimate the state of a system **in the presence of noisy measurements**

Model

$$\begin{aligned}
\dot{x} &= Ax + Bu + Fv, & E\left\{v(s)v^T(t)\right\} &= R_v\delta(t-s) \\
y &= Cx + w & E\left\{w(s)w^T(t)\right\} &= R_w\delta(t-s)
\end{aligned}$$

Disturbance v and noise w are zero-mean and Gaussian

Target

$$\min \quad E\left\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\right\}$$

Solution

If system is **observable,** then the solution is given by Kalman observer gain

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$L = PC^T R_w^{-1}$$

where $P \in \mathbb{R}^{n\times n}, P > 0$, is the solution of A.R.E.:

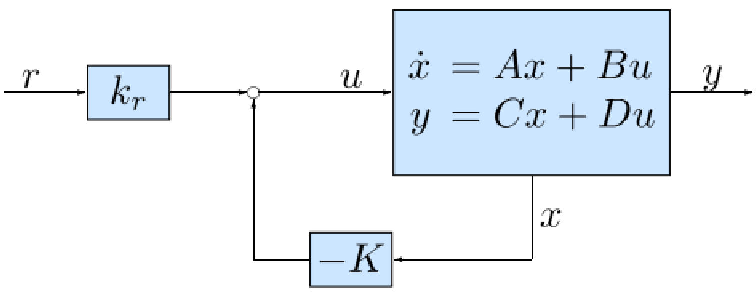
$$AP + PA^T - PC^T R_w^{-1} CP + FR_v F^T = 0$$

A.R.E. can be solved numerically in MATLAB with command `lqe`

2. Feedback Control System Design

Structure 1: $u = -Kx + k_r r$

Generally, can set $u = -Kx + k_r r$, where r is a (constant) reference signal



- Closed-Loop System: $\dot{x} = (A - BK)x + Bk_r r$
- We want to select K so that closed loop has assigned, desired characteristic polynomial

State Feedback Design: Eigenvalue assignment by state feedback

For a **controllable** system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

Original Characteristic Polynomial:

$$a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

Expected Characteristic Polynomial:

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1}s + p_n$$

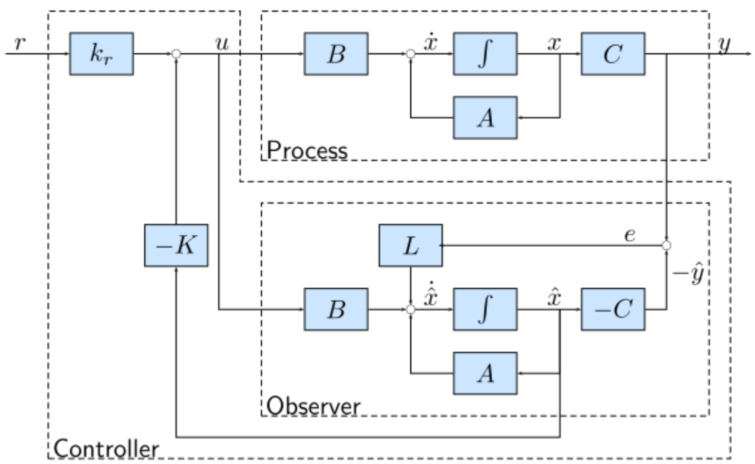
The feedback gain then should be:

$$K=\tilde{K}T=\left[\begin{array}{llll} p_1-a_1 & p_2-a_2 & \ldots & p_n-a_n \end{array}\right]\tilde{W}_r W_r^{-1}$$

Some Mathematical Details

$$\begin{aligned}\tilde{A}-\tilde{B}\tilde{K}&=\left[\begin{array}{cccc} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{array}\right]-\left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array}\right]\left[\begin{array}{llll} p_1-a_1 & p_2-a_2 & \ldots & p_n-a_n \end{array}\right] \\ &=\left[\begin{array}{cccc} -p_1 & -p_2 & \cdots & -p_n \\ 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{array}\right]\end{aligned}$$

Output Feedback(Estimated State) Controller Design: Separation Principle



Assume that not all the variables are observed, that is we need the **observer** to observe some states. That is we need: **stabilization by output feedback.**

Then the controller has structure: $u=-K\hat{x}+k_r r$. We assume there is **no noise** in the system

System Model

We can illustrate new **augmented state**

$$x_{\text{new}}=\left[\begin{array}{c} x \\ \tilde{x} \end{array}\right]=\left[\begin{array}{c} x \\ \hat{x}-x \end{array}\right]$$

Then the **closed-loop system** will be:

$$\frac{d}{dt}\left[\begin{array}{c} x \\ \tilde{x} \end{array}\right]=\left[\begin{array}{cc} A-BK & BK \\ 0 & A-LC \end{array}\right]\left[\begin{array}{c} x \\ \tilde{x} \end{array}\right]+\left[\begin{array}{c} Bk_r \\ 0 \end{array}\right]r$$

Then we should try to **select** K so that closed loop has assigned characteristic polynomial

Property

Consider characteristic polynomial of closed-loop system:

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC)$$

That means that the characteristic polynomial can be **decoupled**

The characteristic polynomial can be **assigned arbitrary roots** if

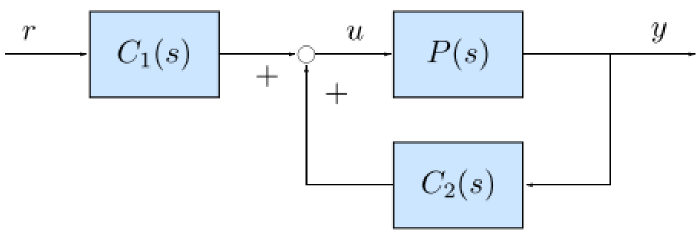
- (A, B) is controllable
- (A, C) is observable

Separation Principle

Eigenvalue assignment for output feedback can be neatly **split into two separate problems**

1. eigenvalue assignment for state feedback
2. eigenvalue assignment for observer

Controller Structure Analysis



$$C_1(s) = -K(sI - A + BK + LC)^{-1}Bk_r + k_r$$
$$C_2(s) = -K(sI - A + BK + LC)^{-1}L$$

Integral Feedback

Evaluation of $u = -Kx + k_r r$ Controller Structure

State feedback is very effective when:

1. model is perfectly known
2. no disturbance is present

Objective of Integral Feedback

1. provide zero steady-state error
2. achieve robustness

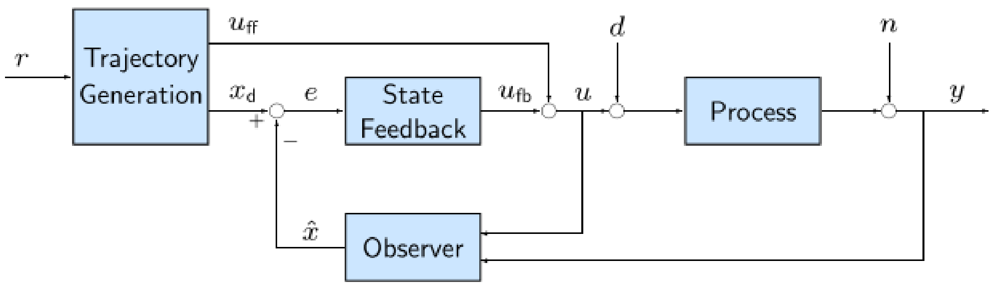
Model

- Introduce **new state**, a signal $z(t) = \int_0^t (y(\tau) - r) d\tau$
- Design $u = -Kx + k_r r - k_i z$ based on $\dot{z} = y - r = Cx - r$
- New state-space model:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A - BK & -Bk_i \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} Bk_r \\ -1 \end{bmatrix} r$$

A General Controller Structure

Based on what is discussed above, a general controller structure can then be established as follows:

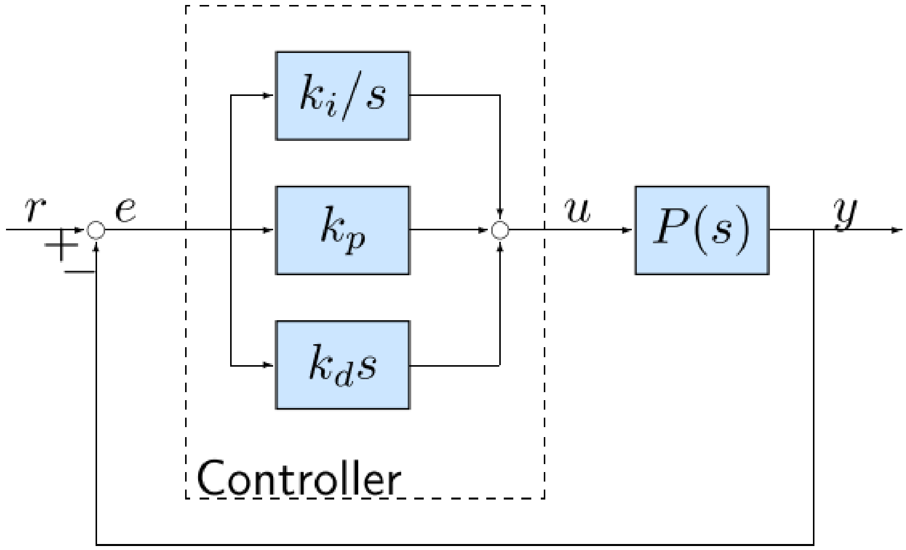


It comprises of 3 parts:

1. Trajectory Generator
   a. Computes input signal u_{ff} such that $y \approx r$
   b. computes corresponding **state** x_d
2. Observer
   a. Built observer to **reconstruct** state $\hat{x}$
3. State Feedback
   a. Built state feedback to **handle disturbance** and **model mismatch**

3. PID Control

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d \frac{de(t)}{dt}$$



P-Controller

- Will have steady-state error
- Will have oscillation

I-Controller

- Can overcome steady-state error
- Will take in oscillation

PID-Controller

- Can Overcome oscillation

3. Linear Quadratic Regulator

$$\int_0^\infty \left(x^T Q_x x + u^T Q_u u\right) dt, \quad Q_x \geq 0, Q_u > 0$$

- Trade off closed-loop performance with input effort
- Trade off convergence rate with cost of control input

Solution:

$$u = -Q_u^{-1} B^T P x$$

where $P \in \mathbb{R}^{n \times n}, P > 0$, is the solution of the **Algebraic Riccati Equation (ARE)**:

$$PA + A^T P - PBQ_u^{-1}B^T P + Q_x = 0$$

- A.R.E. can be solved numerically in MATLAB with command `lqr`
- Solution depends on choice of Q_x, Q_u (often taken as diagonal matrices), **different pairs means different limitation on input and state trajectory**

Summary

- Observer:
  - Normal Observer: regard error as a state
  - Kalman Filter
- Feedback Control
  - State Feedback Design, assuming state x is totally known
  - Output Feedback Design, introduce **observer**
  - Integral Feedback, introduce **integral error** part
  - General Structure: 3 parts
- PID controller
- LQR controller: an optimization-based controller