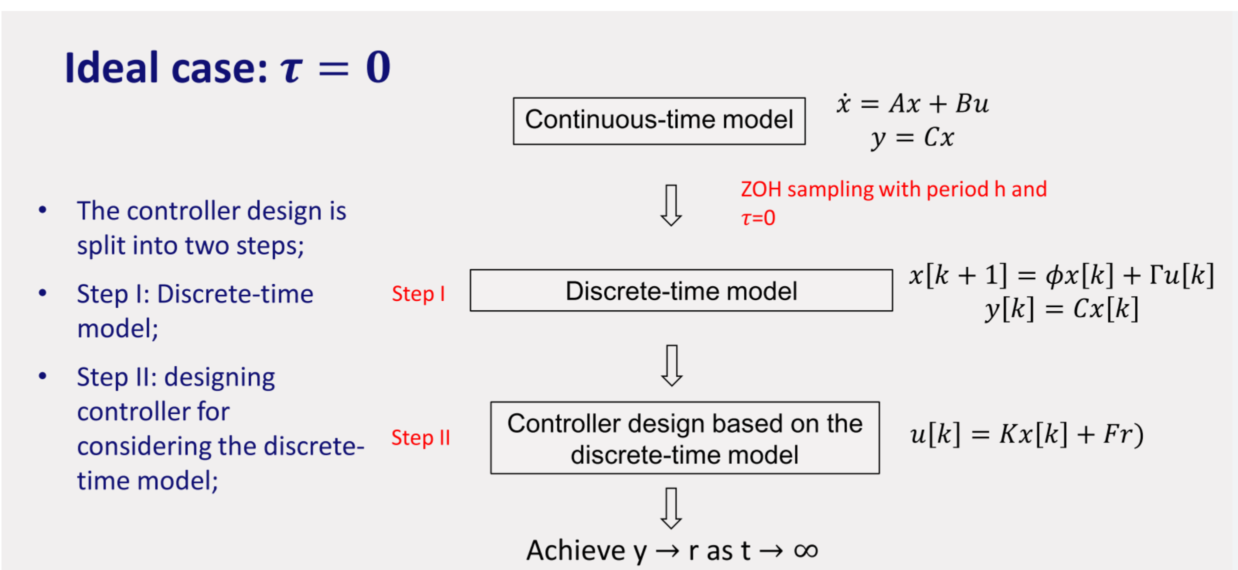
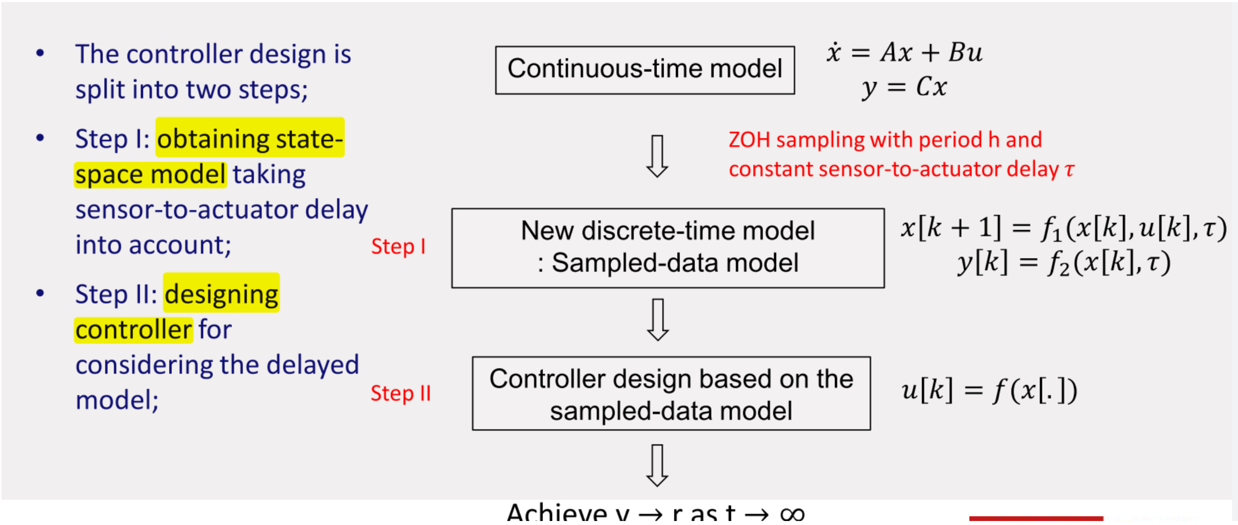
1_3_Controller Design With Small Delay

1. Basic Design Flow

Ideal case with τ=0

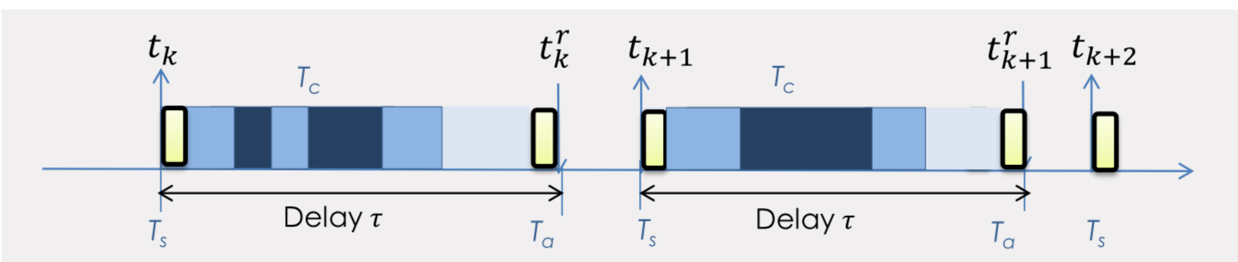


Non-ideal case



2. New discrete-time model: Sampled-data Model

Signals model



<u>**Constant sensor-to-actuator delay τ**</u>

$$t_k^r = t_k + \tau$$
$$t_{k+1}^r = t_{k+1} + \tau$$

<u>**Sampling period h**</u>

$$t_{k+1} = t_k + h$$
$$t_{k+2} = t_{k+1} + h$$

Signals

- Measurement is done in every sampling instant. So, we can assume **state unchanged** between measurements:

$$x(t) = x(t_k) = x[k], t_k \leq t \leq t_{k+1}$$

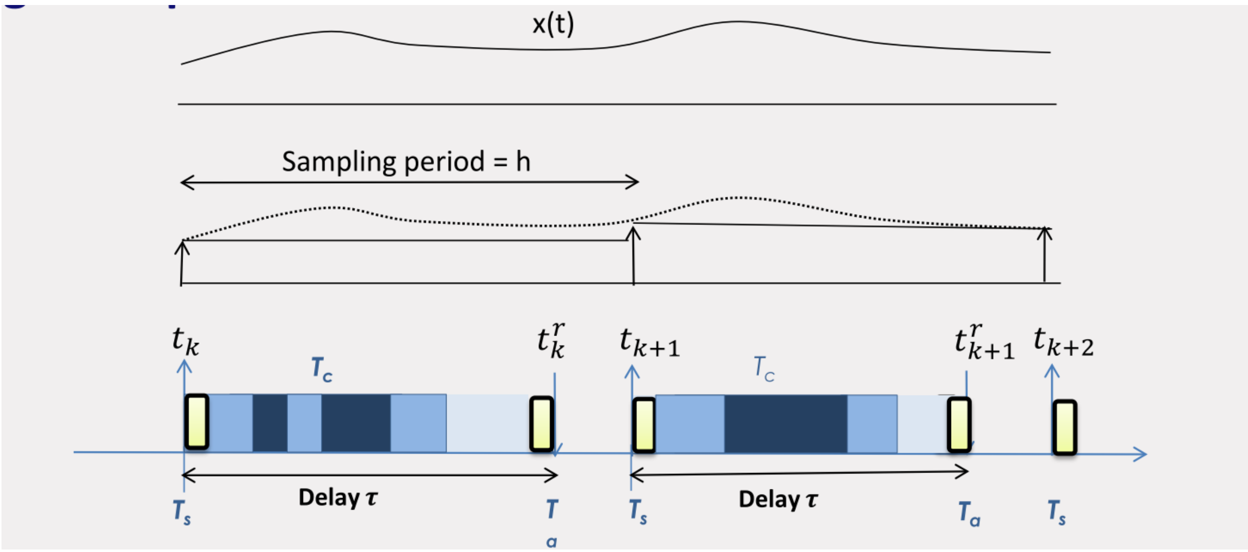
- The **input signal** is held **constant** for one sampling interval

$$u(t) = u(t_k) = u[k], t_k^r \leq t \leq t_{k+1}^r$$
$$u(t) = u(t_{k+1}) = u[k+1], t_{k+1}^r \leq t \leq t_{k+2}^r$$

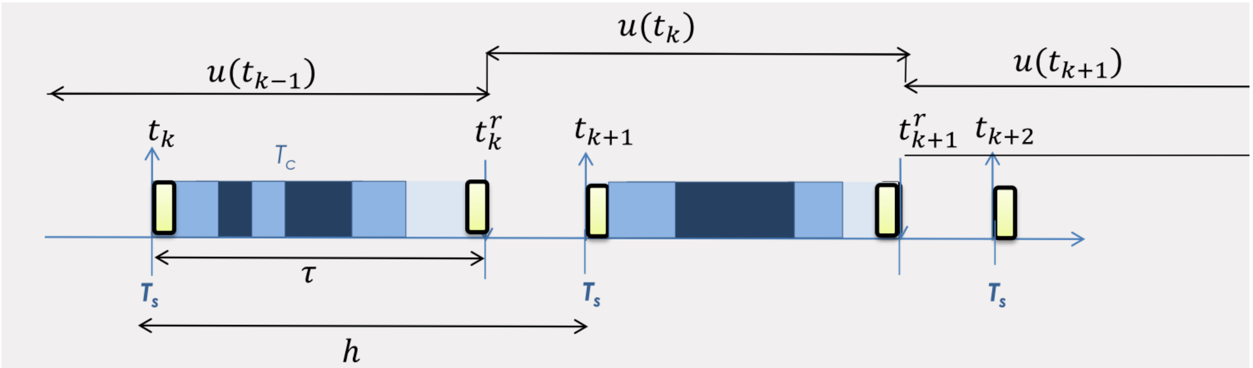
- Control input **updated once** in every sampling interval

$$t_{k+1}^r - t_k^r = h$$

States



Input Signals

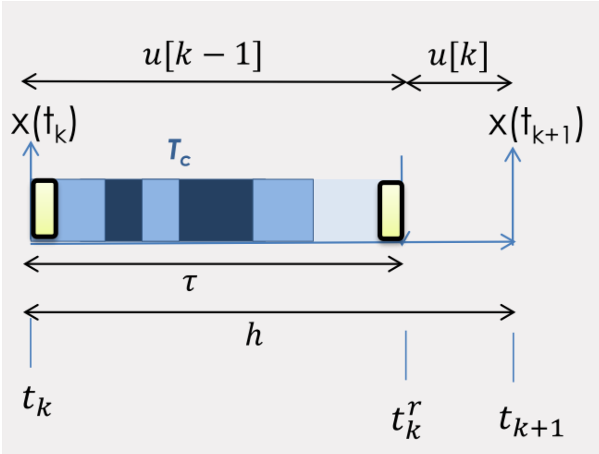


- The $u(t_k)$ is computed based on the latest measurement $x(t_k)$, i.e., $u(t_k) = f(x(t_k))$;
- $u(t_k)$ is applied at $t = (t_k + \tau) = t_k^r$
- Between $t_{k-1}^r \leq t \leq t_k^r$, the previous control input is held, i.e. $u(t) = u(t_{k-1}) = u[k-1]$

Sampling Period Analysis

For $t_k \leq t \leq t_k + \tau, u(t) = u[k-1]$

For $t_k + \tau < t \leq t_{k+1}, u(t) = u[k]$



(Sampled-data) State-space model with $\tau < h$

$$x\left(t_{k+1}\right)=e^{A\left(t_{k+1}-t_k\right)} x\left(t_k\right)+\int_{t_k}^{t_{k+1}} e^{A\left(t_{k+1}-\tau\right)} B u(\tau) d \tau$$

Considering Signals Characteristics

$$\begin{gathered}x[k+1]=e^{A h} x[k]+\int_{t_k}^{t_k^r} e^{A\left(t_{k+1}-\tau\right)} B d \tau \cdot u[k-1]+ \\ +\int_{t_k^r}^{t_k+1} e^{A\left(t_{k+1}-\tau\right)} B d \tau \cdot u[k]\end{gathered}$$

It can be written as:

$$\begin{gathered}x[k+1]=\phi x[k]+\Gamma_1(\tau) u[k-1]+\Gamma_0(\tau) u[k] \\ \phi=e^{A h} \\ \Gamma_1(\tau)=\int_{h-\tau}^h e^{A s} B d s \\ \Gamma_0(\tau)=\int_0^{h-\tau} e^{A s} B d s\end{gathered}$$

Approximation

when τ **is very “short”** compared to h and A matrix needs to be invertible:

$$x[k+1] \approx \phi x[k]+\tau B u[k-1]+(h-\tau) B u[k]$$

Proof

$$
\begin{aligned}
\phi = e^{Ah} &\approx I + Aq \quad (h \text{ Taylor, } h \text{ when is small })\\
\Gamma_1(\tau) = \int_{h-\tau}^{h} e^{As} B ds &= A^{-1}\left(e^{Ah} - e^{A(h-\tau)}\right) B\\
&\approx A^{-1}(I + Ah - I - A(h-\tau))B = \tau B\\
\Gamma_0(\tau) &= (h-\tau)B
\end{aligned}
$$

Property

$$
\begin{gathered}
\text{when } \tau << h\\
\Gamma_1(\tau) \approx 0\\
\Gamma_0 \approx \Gamma
\end{gathered}
$$

When τ is not very short, We are not suitable to only maintain two component in Taylor Series

State Augmentation

This model has the term with old input $u[k-1]$, the state augmentation can help us refine it;

We define

$$
z[k] = \begin{bmatrix} x[k] \\ u[k-1] \end{bmatrix}
$$

Then we have

$$
\begin{aligned}
z[k+1] &= \begin{bmatrix} \phi & \Gamma_1(\tau) \\ 0 & 0 \end{bmatrix} z[k] + \begin{bmatrix} \Gamma_0(\tau) \\ 1 \end{bmatrix} u[k]\\
y[k] &= \begin{bmatrix} C & 0 \end{bmatrix} z[k]
\end{aligned}
$$

We can denote:

$$
\phi_{aug} = \begin{bmatrix} \phi & \Gamma_1(\tau) \\ 0 & 0 \end{bmatrix}, \Gamma_{aug} = \begin{bmatrix} \Gamma_0(\tau) \\ 1 \end{bmatrix}, C_{aug} = \begin{bmatrix} C & 0 \end{bmatrix}
$$

3. Controller design with $\tau < h$

Model

We have

$$
z[k+1] = \phi_{aug} z[k] + \Gamma_{aug} u[k]; y[k] = C_{aug} z[k]
$$

And we aim to design

$$
u[k] = K z[k] + F r
$$

Our goal is:

$$y[k] \to r \text{ as } k \to \infty$$

Process

1. **Check for controllability** of ϕ_{aug}, Γ_{aug}; if they are controllable, then design K & F in the following way
2. **K** is designed using **pole placement** or any other advanced technique
3. **F** is designed using $F = \frac{1}{C_{aug}(I - \phi_{aug} - \Gamma_{aug} K)^{-1} \Gamma_{aug}}$;

4. Alternative Design

4.1. Linear Quadratic Regulator (LQR) Design

Consider system $z[k+1] = \phi_{aug} z[k] + \Gamma_{aug} u[k]$

LQR should minimize:

$$J = \sum_{k=0}^{\infty} \left(z^T[k] Q z[k] + u^T[k] R u[k] + 2 z^T[k] N u[k] \right)$$

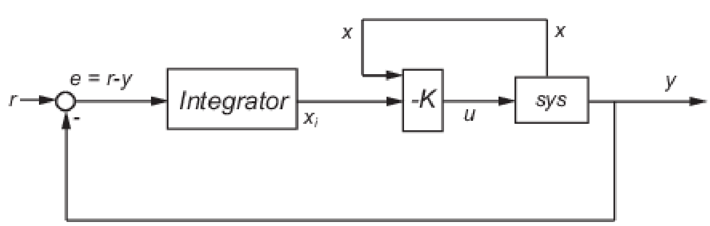
The solution will have format $u[k] = -Kz[k]$

where

$$K = \left(\Gamma_{aug}^T P \Gamma_{aug} + R\right)^{-1} \left(\Gamma_{aug}^T P \phi_{aug} + N^T\right)$$
$$P = \phi_{aug}^T P \phi_{aug} - \left(\Gamma_{aug}^T P \phi_{aug} + N\right) \left(\Gamma_{aug}^T P \Gamma_{aug} + R^{-1} \left(\Gamma_{aug}^T P \phi_{aug} + N^T\right) + Q\right.$$

4.2. Linear Quadratic Integral (LQI) Design

LQI design is capable of **disturbance rejection**



For system

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

The state-feedback control is of the form

$$u = -K\left[x, x_i\right]$$

Minimizes the following cost functions:

- for continuous time: $J(u) = \int_0^\infty \left\{ z^T Q z + u^T R u + 2 z^T N u \right\} dt$
- for discrete time: $J(u) = \sum_{n=0}^\infty \left\{ z^T Q z + u^T R u + 2 z^T N u \right\}$

In discrete time, the LQI controller computes the **integrator output(integral tracking error)** x_i using the forward Euler formula

$$x_i[n+1] = x_i[n] + Ts(r[n] - y[n])$$

The augmented state-space is:

$$\begin{aligned} \begin{bmatrix} z[k+1] \\ x_i[k+1] \end{bmatrix} &= \begin{bmatrix} \phi_{aug} & 0 \\ C_{aug} & 1 \end{bmatrix} \begin{bmatrix} z[k] \\ x_i[k] \end{bmatrix} + \begin{bmatrix} \Gamma_{aug} \\ 0 \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} C & 0 \end{bmatrix} x[k] \end{aligned}$$

Summary

- Consideration when design controller for system with τ and h : mainly for $\tau < h$
  - augmented state-space
  - approximation when $\tau << h$
- Three ways of designing controller:
  - pole placement
  - LQR controller
  - LQI controller