03_Reachablity, Observability and Similarity Transformation

1. Reachability and Observability

Definition: Reachability (Controllability)

System is **reachable** if for any $x_0 \in \mathbb{R}^n, x_f \in \mathbb{R}^n$, there exists input $u(\tau)$, with $\tau \in [0, t], 0 < t < \infty$, such that $x(t) = x_f$

- That is, **We can control system state using our input**
- $\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) = \mathbf{H}_1(s)\mathbf{U}(s)$

Definition: Observability

System is **observable** if any $x_0 \in \mathbb{R}^n$ can be derived from observation $y(\tau)$ within the interval $\tau \in [0, t], t > 0$

- $\mathbf{y} = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$
- **We can derive system state based on output**

2. Judgment of Reachability and Observability

Kalman Rank Condition

Reachability

Reachability Matrix

$$W_r = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

Kalman Rank Condition

(A, B) reachable **if and only if** rank

$$W_r = \text{rank}\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n(= \text{ number of rows })$$

- In MATLAB, use command `ctrb` over `ss` structure to obtain Reachability matrix

Observability

Observability Matrix

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Kalman Rank Condition

(A, C) observable **if and only if**

$$\text{rank} W_o = n(= \text{ number of columns})$$

- In MATLAB, use `obsv` over `ss` structure to obtain obs. matrix

Hautus Condition

Reachability

(A, B) is reachable **iff**

$$\text{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n, \text{ for all } \lambda \in \mathbb{C}$$

Observability

(A, C) is observable **iff**

$$\text{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n, \text{ for all } \lambda \in \mathbb{C}$$

Duality

(A, B) reachable $\Leftrightarrow$ $\left(A^T, B^T\right)$ observable

(A, C) observable $\Leftrightarrow$ $\left(A^T, C^T\right)$ reachable

- State feedback and observer design problems are closely related
- Both Reachability and observability are **invariant under state transformations**

Transfer Function Perspective

Theorem

- If there are **no zero-pole cancellations** in the transfer function of a **single-input single-output** system, then the system is both **controllable and observable**.
- If a **zero-pole cancellation** occurs, then the system is either **uncontrollable or unobservable** or both **uncontrollable and unobservable.**

Minimal realization

In control theory, given any transfer function, any state-space model that is **both controllable and observable** and has the **same input-output behaviour** as the transfer function is said to be a **minimal realization** of the transfer function.

Illustration

If there exists **zero-pole cancellations**, which means, there are **some information hidden** in the transfer function.

Note:

Use transfer function we can easily figure out the system **has some problem** on reachability and observability. Whether it is unreachable or unobservable still needs to be tested.

For example, based on:

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) = \mathbf{H}_1(s)\mathbf{U}(s)$$

If $[s\mathbf{I} - \mathbf{A}]^{-1}$ has **zero rows** or **linear independent rows**, which means uncontrollable.

Similar methods can be also used for observability.

3. Lack of Reachability/Observability

UC 1 - physical unreachability - uncoupled variables are not affected by input

UC 2 - parallel interconnection controlled by single input

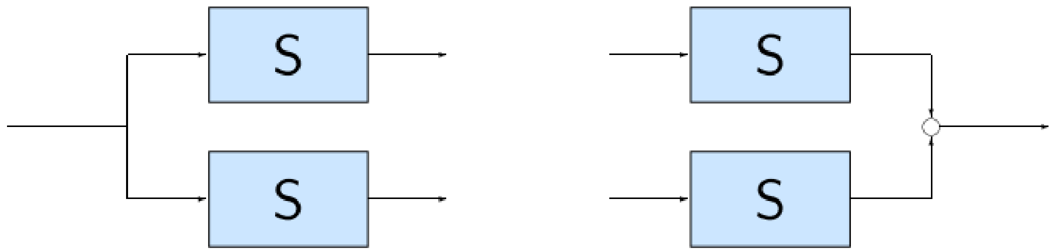
UO 1 - directly unmeasured variables are not fed back to measured ones

UO 2 - single variables cannot be extracted from global observation function

Example 1

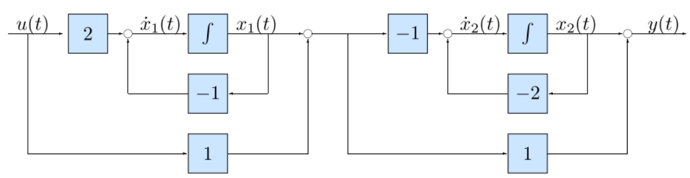
Consider identical systems in parallel:

two system with same State-Space Model



The first is not reachable, whereas the second is not observable

Example 2: Reachable but not observable



$G_1(s) = \frac{(s+3)}{(s+1)}$

$G_2(s) = \frac{(s+1)}{(s+2)}$

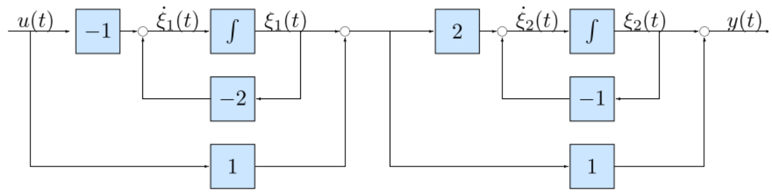
$$\begin{aligned}
\dot{x}_1(t) &= -x_1(t) + 2u(t) \\
\dot{x}_2(t) &= -x_1(t) - 2x_2(t) - u(t) \\
y(t) &= x_1(t) + x_2(t) + u(t)
\end{aligned}$$

Compute transfer function

$$\begin{aligned}
G(s) &= C(sI - A)^{-1}B + D \\
&= \frac{s^2 + 4s + 3}{s^2 + 3s + 2} \\
&= \frac{(s+3)(s+1)}{(s+2)(s+1)}
\end{aligned}$$

Pole-zero cancellation for $s = -1$.

Example 3: Observable but not reachable



$$G_1(s) = \frac{(s+1)}{(s+2)} \qquad G_2(s) = \frac{(s+3)}{(s+1)}$$

$$\begin{aligned}
\dot{\xi}_1(t) &= -2\xi_1(t) - u(t) \\
\dot{\xi}_2(t) &= 2\xi_1(t) - \xi_2(t) + 2u(t) \\
y(t) &= \xi_1(t) + \xi_2(t) + u(t)
\end{aligned}$$

$$\begin{aligned}
G(s) &= C(sI - A)^{-1}B + D \\
&= \frac{s^2 + 4s + 3}{s^2 + 3s + 2} \\
&= \frac{(s+3)(s+1)}{(s+2)(s+1)}
\end{aligned}$$

Pole-zero cancellation for $s = -1$.

4. Canonical Form

Unchangeable Property

Given state transformation $z(t) = Tx(t)$

$$\begin{aligned}
A' &= TAT^{-1} \qquad B' = TB \\
C' &= CT^{-1} \qquad D' = D
\end{aligned}$$

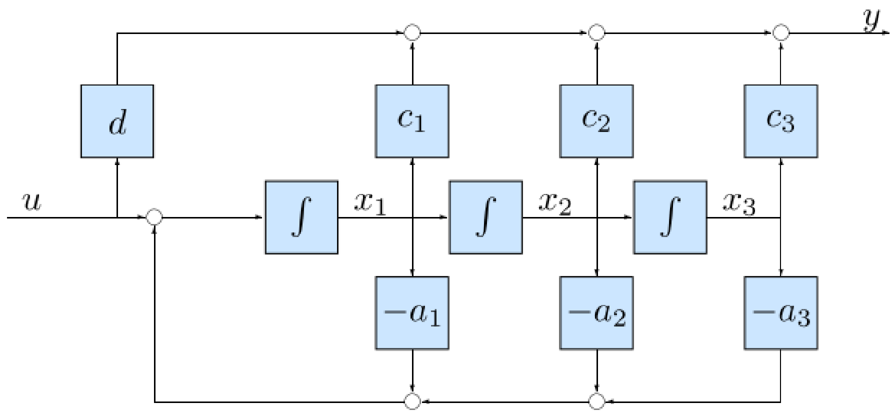
- If (A, B) is **reachable** $\Rightarrow (A', B')$ is **reachable**
- If (A, C) is **observable** $\Rightarrow (A', C')$ is **observable**

Reachable Canonical Form

Transform SISO model coordinates as $z = Tx$ to obtain the following:

$$A_r = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & 1 & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$C_r = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix}, \qquad D_r = [d],$$



- The characteristic polynomial is **the same as that of original model**:

$$\lambda(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

- (A_r, B_r) in reachable canonical form is reachable
- state transformations do not affect reachability
- If (A, B) is **reachable**, Then there **exists** a transformation T that **yields the reachable canonical form**
- $W_r =$ reachability matrix of original system.

  $\tilde{W}_r =$reachability matrix of system in reachable canonical form.

$$T = \tilde{W}_r W_r^{-1}$$

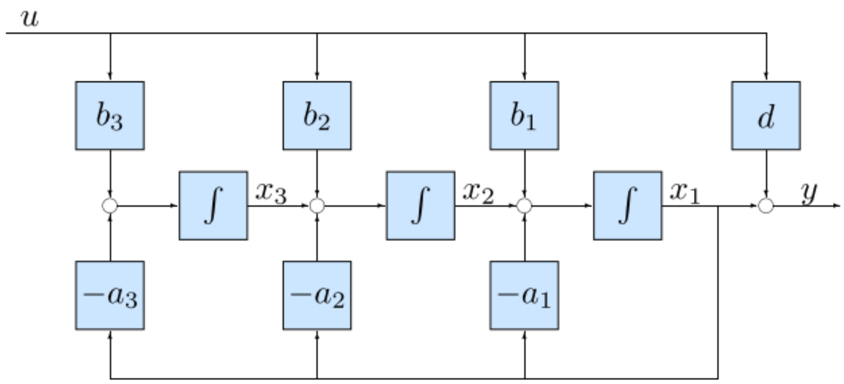
This conclusion can be easily proved by Kalman Rank Condition

Observable Canonical Form

Transform SISO model coordinates as $x = T\bar{x}$ to obtain

$$A_o = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B_o = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix},$$

$$C_o = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad D_o = [d]$$



- The characteristic polynomial is **the same as that of original model**:

$$\lambda(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

- (A_o, C_o) in observable canonical form is observable
- If (A, C) is **observabl**e, Then there **exists a transformation** T that yields the observable canonical form
- $W_o =$ observability matrix of original system.
  $\tilde{W} o =$ observability matrix of system in observable canonical form.

$$T = \tilde{W} o^{-1} W_o$$

Summary

- Reachability and Observability
  - Reachability: input → state
  - Observability: output → state
- Judgement
  - Kalman Rank Condition
  - Hautus Condition
  - Transfer Function Judgement
- Property
  - Duality
  - Unchangeable under similarity transformation
- Canonical Form
  - Reachable Canonical Form: can be used for controller design
  - Observable Canonical Form: can be used for observer design