06_07_08_Stability_Theory

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Summary

1. Notions of Stability

Conceptions

Assume that the autonomous system $\dot{x} = f(x)$ with

$$f : D \rightarrow \mathbb{R}^n, \quad D = \text{ domain in } \mathbb{R}^n, \quad f \text{ locally Lipschitz,}$$

has an equilibrium at $x = \bar{x}$. Then this equilibrium is called

Definition: Stable

Stable if for any $\epsilon > 0$ there exists an $\delta > 0$ such that

$$\|x(t_0) - \bar{x}\| < \delta \Longrightarrow \|x(t) - \bar{x}\| < \epsilon, \forall t \geq t_0$$

Note:

- It is apparent that $\delta \leqslant \varepsilon$

- The $\|x(t_0) - \bar{x}\| < \delta$ means for all $x(t_0)$ in the circle

Definition: Asymptotically Stable

Asymptotically stable if it is stable and, for some $\delta > 0$,

$$\|x(t_0) - \bar{x}\| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} \|x(t) - \bar{x}\| = 0$$

- If $D = \mathbb{R}^n$, then it become **Globally Asymptotically Stable**
    - The globally **asymptotically stable equilibrium will be unique** (we can get contradiction if we have two asymptotically stable equilibriums)

Definition: Exponentially Stable

Exponentially stable if it is stable and, for some $\delta, k, \lambda > 0$,

$$\|x(t_0) - \bar{x}\| < \delta \Rightarrow \|x(t) - \bar{x}\| \leq k \|x(t_0)\| e^{-\lambda t}, \forall t \geq t_0.$$

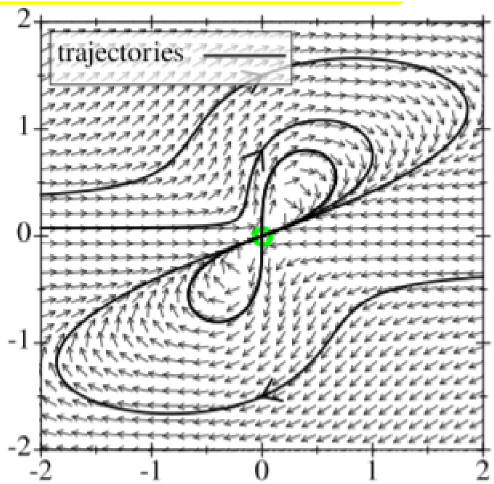
- If $D = \mathbb{R}^n$, then it become **Globally Exponentially Stable**
    - The **globally asymptotically stable equilibrium will be unique**

Relations

Exponentially Stable $\Rightarrow$ Asymptotically Stable $\Rightarrow$ Stable

- Orbits is an example that is stable but not asymptotically stable

Vignorad's Example: Attractive but not stable

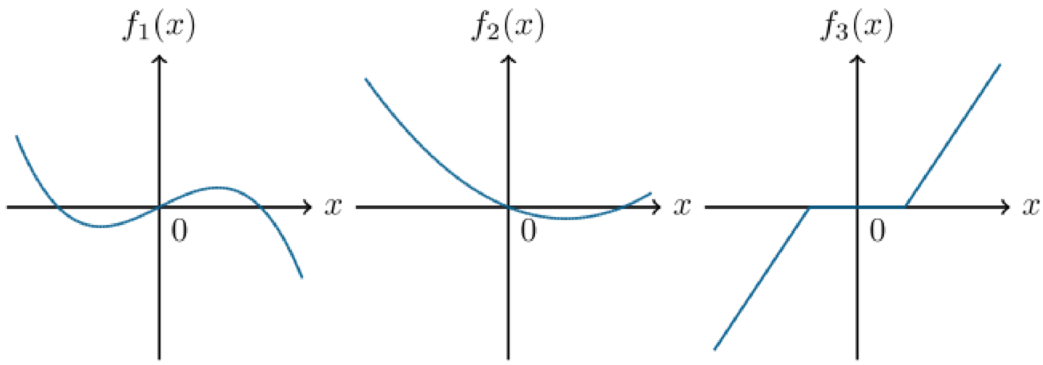


The origin is an unstable equilibrium of the system.

It attracts all trajectories, but not stable.

This example indicate that when considering asymptotically stable, we should first meet stable property.

Judgement of the One-Dimension System



The equilibrium point $\bar{x}$ is stable if and only if

$$\begin{cases} f(x) \geq 0, & x < \bar{x} \\ f(x) \leq 0, & x > \bar{x} \end{cases} \iff (x - \bar{x}) f(x) \leq 0$$

in a small environment of $\bar{x}$. Intuitively, this condition means that the system drives all states starting slightly

- For the first one, $x = 0$ is unstable
- For the second one $x = 0$ is stable
- For the third one, $x = 0$ is stable

2. Lyapunov's Direct Method

Derivative Along Trajectories

The derivative of a continuously differentiable function

$$V : D \to \mathbb{R}, \quad D \subseteq \mathbb{R}^n \text{ domain},$$

along the trajectories of a system $\dot{x} = f(x)$ is defined as

$$\dot{V} : D \to \mathbb{R}, \quad \dot{V}(z) = \sum_{i=1}^{n} \frac{\partial V(z)}{\partial z_i} f_i(z) = f(z) \Delta V(z)$$

Theorem for Stable Equilibrium

Theorem

We consider the system

$$\dot{x} = f(x), \quad f : D \to \mathbb{R}^n \text{ locally Lipschitz}, \quad D \subseteq \mathbb{R}^n \text{ domain.}$$

Let the system have an equilibrium point $\bar{x} \in D$. Furthermore, let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

- $V(\bar{x}) = 0$,
- $V(z) > 0$ for all $z \in D$ with $z \neq \bar{x}$,
- $\dot{V}(z) \leq 0$ for all $z \in D$.

Then, the equilibrium point $\bar{x}$ is stable.

Finding Lyapunov Functions

Definition: Lyapunov Candidate Function

A continuosly diff'able function $V : \mathbb{D} \to \mathbb{R}, D \subseteq \mathbb{R}^n$ domain,

$$V(\bar{x}) = 0 \quad \text{and} \quad V(z) > 0, \forall z \in D, z \neq \bar{x},$$

is called a **Lyapunov candidate function** (w.r.t. $\bar{x}$; usually, $\bar{x} = 0$).

Some Ways to Design Lyapunov Function

- Energy Arguments
- Linearization
- Choose a family of candidates, optimize parameters
- Polynomial systems: try SOSTOOLS

Theorem for Asymptotic Stability

Theorem: Asymptotically Stable

If, in addition to the assumptions in the basic Lyapunov theorem,

$$\dot{V}(z) < 0, \quad \forall z \in D, z \neq \bar{x},$$

then the equilibrium point $\bar{x}$ is **asymptotically stable**.

Theorem: Globally Asymptotically Stable

If, in addition to the assumptions in the result above,

$$D = \mathbb{R} \quad \text{and} \quad \|z\| \to \infty \Rightarrow |V(z)| \to \infty$$

then the equilibrium point $\bar{x}$ is **globally asymptotically stable.**

Theorem for Exponential Stability

Theorem: Exponential Stability

If, in addition to the assumptions in the basic Lyapunov theorem, there are constants $k_1, k_2, k_3, a > 0$ such that
$k_1 \|z - \bar{x}\|^a \leq V(z) \leq k_2 \|z - \bar{x}\|^a$ and $\dot{V}(z) \leq -k_3 \|y - \bar{x}\|^a, \quad \forall z \in D$
then the equilibrium point $\bar{x}$ is **exponentially stable.**

Theorem: Globally Exponential Stable

If, in addition to the assumptions in the result above, the domain is $D = \mathbb{R}^n$, then the equilibrium $\bar{x}$ is globally exponentially stable.

3. The Invariance Principle

Conceptions

Definition: Positively Invariant Set

A set M is an **positively invariant set** with respect to $\dot{x} = f(x)$ if

$$x(t_0) \in M \Rightarrow x(t) \in M, \forall t \geq t_0.$$

Definition: Invariant Set

A set M is an **invariant set** with respect to $\dot{x} = f(x)$ if it is positively invariant with respect to both $\dot{x} = f(x)$ and $\dot{x} = -f(x)$.

Notes:

- The system $\dot{x} = -f(x)$ is the **time-reversed versio**n of $\dot{x} = f(x)$
- If M_1 and M_2 are two invariant sets, then $M_1 \cup M_2$ also is an invariant set. There thus **exists a largest invariant set** that contains all other invariant sets.
- If $\bar{x}$ is an equilibrium point of $\dot{x} = f(x)$, then $\{\bar{x}\}$ is invariant.
- If $x(t)$ is a periodic orbit of $\dot{x} = f(x)$, then $\{x(t) : t \geq t_0\}$ is invariant.

LaSalle's Invariance Principle

Theorem: LaSalle's Invariance Principle

We consider the system $\dot{x} = f(x)$, where
$f : D \to \mathbb{R}^n, \quad D \subset \mathbb{R}^n$ domain, f locally Lipschitz.
Furthermore, let

- $\Omega \subset D$ be a compact set, positively invariant w.r.t. $\dot{x} = f(x)$;
- $V : D \to \mathbb{R}$ be a continuously differentiable function that satisfies $\dot{V}(z) \leq 0$ for all $z \in \Omega$;
- M be the largest invariant set in $E := \{z \in \Omega : \dot{V}(z) = 0\}$.

Then any solution starting in Ω approaches the invariant set M

$$x(t_0) \in \Omega \quad \Rightarrow \quad \lim_{t \to \infty} \inf_{p \in M} \|p - x(t)\| = 0$$

Note:

in $\mathbb{R}^n$, compact means bounded and closed

Barbashin-Krasovskii Theorem

Theorem: Barbashin-Krasovskii Theorem

Let $\bar{x}$ be an equilibrium of the system $\dot{x} = f(x)$,

where $f : D \to \mathbb{R}^n, \quad D \subset \mathbb{R}^n$ domain, f locally Lipschitz.
Furthermore, let V be a Lyapunov function that satisfies $\dot{V}(z) \leq 0$ for all $z \in D$. We denote the set of points where $\dot{V}$ vanishes by

$$S := \{x \in D : \dot{V}(x) = 0\}.$$

Then $\bar{x}$ is **asymptotically stable** if **only the trivial solution** stays in S

$$x(t) \in S, \forall t \geq t_0 \Rightarrow x(t) = \bar{x}.$$

If, in addition to the conditions above, furthermore $D = \mathbb{R}^n$ and $||z|| \to \infty \Rightarrow V(z) \to \infty$, then $\bar{x}$ is **globally asymptotically stable**.

4. Lyapunov’s Indirect Method

Theorem: Lyapunov’s Indirect Method

Assume that the autonomous system $\dot{x} = f(x)$ with

$$f : D \to \mathbb{R}^n, \quad D = \text{ domain in } \mathbb{R}^n,$$

has an equilibrium at $x = \bar{x}$. Also assume that f is continuously differentiable in a neighborhood of $\bar{x}$.
We denote the eigenvalues of the Jacobian at the equilibrium

$$A = \left[\frac{\partial f(\bar{x})}{\partial x}\right]$$

by $\lambda_1, \ldots, \lambda_n$. Then, $\bar{x}$ is

- **exponentially stable if and only if** $\Re\left[\lambda_i\right] < 0$ for all i.
- **unstable if** $\Re\left[\lambda_i\right] > 0$ for one or more i.

Note:

if $\Re\left[\lambda_i\right] = 0$, anything can happen except exponentially stable

5. Comparison Functions

$\mathcal{K}$ function and $\mathcal{K}_\infty$ function

A continuous function $\alpha : [0, a) \to \mathbb{R}$ belongs to

- class $\mathcal{K}$ if $\alpha(0) = 0$ and $\alpha(r_1) < \alpha(r_2)$ for all $r_1 < r_2$
- class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ with $a = \infty$ and $\alpha(r) \to \infty$ for $r \to \infty$

$\mathcal{KL}$ function

A continuous function

$$\beta : [0, a) \times [0, \infty) \to \mathbb{R}$$

belongs to class $\mathcal{KL}$ if

- for any fixed s, the function

  $$\alpha : [0, a) \to \mathbb{R}, \quad \alpha(r) := \beta(r, s)$$

  is in $\mathcal{K}$
- for any fixed r, $\beta(r, s_1) \geq \beta(r, s_2)$ for all $s_1 < s_2$
- for any fixed $r, \beta(r, s) \to 0$ for $s \to \infty$

Note:

- Monotonically Increase in s
- Monotonically Decrease in r

6. Lyapunov Theory for Time-varying System

In this part, we consider system:

$$\dot{x} = f(t, x), \quad f : [0, \infty) \times D \to \mathbb{R}^n, \quad D \subseteq \mathbb{R}^n \text{ domain},$$

Definition: Equilibrium Point

The point $\bar{x} \in D$ is an equilibrium point if $f(t, \bar{x}) = 0$ for all $t \geq 0$.

Definition: Stable

An equilibrium $\bar{x}$ is **stable** if for all $\epsilon > 0, t_0 \geq 0$ there is a $\delta > 0$ s.t.

$$\|x(t_0) - \bar{x}\| < \delta \Rightarrow \quad \|x(t) - \bar{x}\| < \epsilon, \quad \forall t \geq t_0$$

It is **asymptotically stable** if furthermore

$$\|x(t_0) - \bar{x}\| < \delta \quad \Rightarrow \quad \lim \|x(t) - \bar{x}\| = 0.$$

Note: δ may depend on both ϵ and t_0

<u>Definition: Uniform Stability</u>

Then $\bar{x}$ is called **<u>uniformly stable</u>** if there are $\delta > 0$ and $\alpha \in \mathcal{K}$ s.t.

$$\|x(t_0) - \bar{x}\| < \delta \Rightarrow \|x(t) - \bar{x}\| \leq \alpha(\|x(t_0) - \bar{x}\|), \forall t \geq t_0, t_0 \geq 0.$$

Note: δ and α have to be independent of t_0

<u>Theorem:</u>

Let $V : [0, \infty) \times D \rightarrow \mathbb{R}$ be a continuously differentiable function. Furthermore, let $W_1, W_2 : D \rightarrow \mathbb{R}$ be **continuous functions** with

$$W_{1,2}(\bar{x}) = 0, \quad W_{1,2}(z) > 0 \quad \forall z \neq \bar{x}$$

Then $\bar{x}$ **is uniformly stable** if, for all $t \geq 0$ and $z \in D$,

$$W_1(z) \leq V(t, z) \leq W_2(z), \quad \frac{\partial V(t, z)}{\partial t} + \frac{\partial V(t, z)}{\partial z} f(t, z) \leq 0$$

7. System with Input

Passive System

<u>Definition: Passive System</u>

The system is **<u>passive</u>** if there exists a continuously differentiable function (called the storage function)

$$V : D \rightarrow \mathbb{R}, \quad V(0) = 0, \quad V(z) \geq 0 \quad \forall z \neq 0$$

such that

$$u^T y \geq \dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x, u), \quad \forall x, u$$

Interpretation: the change in total energy cannot exceed the input energy.

<u>Theorem: Stable Passive System</u>

If the **storage function** of a passive system $\dot{x} = f(x, u)$ satisfies

$$V(z) > 0, \quad \forall z \neq 0,$$

then the **origin** of $\dot{x} = f(x, 0)$ is **stable**.

<u>Theorem: Asymptotically Stable of Passive System</u>

If a passive system $\dot{x} = f(x,u)$ satisfies

$$u^T y \geq \dot{V}(z) + \psi(z), \quad \forall z, u,$$

where ψ is a function that satisfies

$$\psi(0) = 0, \quad \psi(z) > 0, \quad \forall z \neq 0,$$

then the origin of $\dot{x} = f(x,0)$ is **asymptotically stable**.

Input-to-State Stability

Definition: Input-to-State Stability

The system is **input-to-state stable** if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any $t_0 \geq 0$, initial state $x\left(t_0\right)$ and bounded input $u(t)$,

$$\|x(t)\| \leq \max\left\{\beta\left(\left\|x\left(t_0\right)\right\|, t-t_0\right), \gamma\left(\sup_{t_0 \leq \tau \leq t}\|u(\tau)\|\right)\right\}$$

for all $t \geq t_0$. Here, solutions should exist for all times.

Definition: Locally ISS

Let $f(x,u)$ be **locally Lipschitz** in some neighborhood of $(x,u) = (0,0)$. Then there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\|x(t)\| \leq \max\left\{\beta\left(\left\|x\left(t_0\right)\right\|, t-t_0\right), \gamma\left(\sup_{t_0 \leq \tau \leq t}\|u(\tau)\|\right)\right\}$$

for all $x\left(t_0\right)$ and $u(t)$ in a neighborhood of $(x,u) = (0,0)$ if and only if the origin is an asymptotically stable equilibrium point of $\dot{x} = f(x,0)$.

Then such a system is called **locally ISS**

Property

- If $u(t)$ is bounded, then $x(t)$ is bounded.
- If $\lim_{t\to\infty} u(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$.
- The origin is globally asymptotically stable for $u = 0$.

Theorem

Let $f(x,u)$ be continuously differentiable and globally Lipschitz in (x,u). If the origin is a globally exponentially stable equilibrium of the system $\dot{x} = f(x,0)$, then $\dot{x} = f(x,u)$ is input-to-stable stable

Cascade Property

The cascade connection of two input-to-state stable systems is ISS.

Theorem: Lyapunov Theorem for ISS

The system from Slide 15 is input-to-state stable if there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x, u,$$

and

$$\|x\| \geq \rho(\|u\|) \quad \Rightarrow \quad \frac{\partial V(x)}{\partial x} f(x, u) \leq -W_3(x)$$

Here, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \rho \in \mathcal{K}$ *and* $W3 : \mathbb{R}^n \to \mathbb{R}$ is continuous with

$$W_3(0) = 0, \quad W_3(z) > 0, \quad \forall z \neq 0.$$

The function γ from the definition can chosen as $\gamma = \alpha^{-1} \circ \alpha_2 \circ \rho$.

Summary

- An equilibrium point can be
  - stable
  - asymptotically stable
  - exponentially stable
- Can be proven using Lypanunov's direct method
- Main challenge: Find a suitable Lyapunov function $V(x)$

- The invariance principle lets us show asymptotic stability even if $\dot{V}(z) = 0$
- Exploits the notion of invariant sets (in which solutions stay even if time is reversed)
- The Barbashin-Krasovskii theorem is a related result
- Lyapunov's indirect method $\Rightarrow$ stability analysis using linearization

- Introduced stability for equilibria of time-varying systems $\rightarrow \delta$ may depend on t_0
- Uniform stability is similar, but δ has to be independent of t_0!
- Several Lyapunov-type theorems for time-varying systems
  - Formulated using comparison functions (class $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$)
- Two classes of time-invariant systems with inputs: passive and input-to-state stable
  - Feedback connections of passive systems are passive
  - cascade connections of ISS systems are ISS