04_Stability

1. Switched System

Model of Switched System

$$\dot{x} = f_\sigma(x)$$

$\{f_1(x), f_2(x), \ldots, f_N(x)\}$ family of smooth vector fields from $\mathbb{R}^n$ to $\mathbb{R}^n$
Switching signal $\sigma : [0, \infty) \to \{1, \ldots, N\}$ piecewise constant function

- of time $t : \sigma(t)$
- of state $x(t) : \sigma(x)$
- of time and state: $\sigma(t, x)$
- or extensions involving memory (like hysteresis)

Switched Linear Systems

One example of the Switched System is the Switched Linear System

Switched Linear System

$$\dot{x} = A_\sigma x$$

One special case in the Switched Linear System is the piece-wise or multi-modal linear systems

Piece-wise or multi-modal linear systems

Switching is only **state-dependent** $\dot{x} = A_i x$ when $x \in \mathscr{X}_i$

We assume the state-space is Well-posedness: cells form partitioning of the state space $\mathbb{R}^n$ (necessary condition only)

$$\bigcup i = 1^n \mathscr{X} i = \mathbb{R}^n \text{ and interior } (\mathscr{X} i) \cap \text{interior} (\mathscr{X}_j) = \varnothing (i \neq j)$$

One special case for the switched linear system is the PWA systems

Problems of Stability in Switched Systems

Global Uniform Asymptotic Stability (GUAS)

Global Asymptotic Stability + Uniform in σ

Generally, there are **three types** of questions need to be answered about the stability for the switched systems:

- A: Find conditions for which the switched system is GAS for any switching signal **(GUAS)**
- B: Show that the switched system is GAS for a **given switching strategy** or a class of switching strategies
- C: **Construct switching signal** that makes the switched system GAS (i.e. stabilization problem)

2. Lyapunov Theory for Smooth and Linear Systems

Basic knowledge for the Lyapunov Theory is already very familiar for us. Here we will just mention some converse part and complementary part.

Converse Theorem for Lyapunov GAS

If $x = 0$ is GAS equilibrium of $\dot{x} = f(x)$, then there **exists** radially unbounded Lyapunov Function $V(x)$

Lemmas for Stability of Linear Systems

The following statements are equivalent:

- $\dot{x} = Ax$ is **asymptotically stable**
- there is a **quadratic Lyapunov function** $V(x) = x^T P x$ for some positive definite matrix P such that $A^T P + PA < 0$

Moreover, for every asymptotically stable A and for any $Q > 0$ there is a $P > 0$ such that the following Lyapunov equality holds

$$A^T P + PA = -Q$$

Connection of stability of nonlinear system and its linearization

Let $x = a$ be equilibrium of $\dot{x} = f(x)$ (i.e., $f(a) = 0$) with $f : D \to \mathbb{R}^n$ continuously differentiable and D a neighborhood of a. Take

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=a}$$

- Equilibrium a is locally asymptotically stable, if A is asymptotically stable (i.e., all eigenvalues in open left half-plane)
- Equilibrium a is unstable (not stable), if there is an eigenvalue of A that lies in open right half-plane

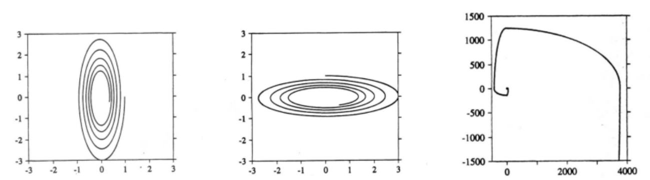
Note: no statements in case all eigenvalues in closed left half-plane

Combine Stable Dynamics

Combined System maybe unstable even if each subsystem is stable

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 < 0 \\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases}$$

$$A_1 = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix}; \; A_2 = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix} \text{ Eigenvalues } = -1 \pm 31.6j$$



$\rightarrow$ combined system unstable!

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3. Stability for Any Switching Signal: GUAS

Common Lyapunov Function Approach

Try to find one **shared Lyapunov function** that decreases along any of the sub-model

Definition: Common Lyapunov Function

A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called **common Lyapunov function** for $\dot{x} = f_\sigma(x)$ with $\sigma \in \{1, \ldots, N\}$ if

$$\dot{V}(x) = L_{f_i} V(x) = \frac{\partial V}{\partial x} f_i(x) < 0$$

when $x \neq 0$ and for all $i = 1, \ldots, N$

Theorem

If all smooth submodels **share** positive definite radially unbounded common Lyapunov function, then switched system is **globally uniformly asymptotically stable (GUAS)**

Converse Theorem

If switched system is GUAS, then all f_i share positive definite radially unbounded common Lyapunov function

Switched Linear Systems: Common Quadratic Lyapunov Function Approach

Definition: Quadratic Stability

There exists a quadratic Lyapunov function $V(x) = x^T Px$with $\dot{V}(x) \leqslant -\varepsilon\|x\|^2$ for some $\varepsilon > 0$

Theorem:

If we can solve the LMIs:

$$A_i^T P + PA_i < 0 \text{ for all } i = 1, \ldots, N \text{ and } P > 0$$

Then we can find an P which is the common quadratic Lyapunov function. Then the system is GUAS

Infeasibility test for common quadratic Lyapunov function

The set of LMls is **infeasible** (i.e., no quadratic stability) **if and only if** there exist positive definite matrices $R_i, i = 1, \ldots, N$ such that

$$\sum_{i=1}^{N} \left(A_i^T R_i + R_i A_i\right) > 0$$

Note:

For quadratic Lyapunov function theorem, there is not a converse version.

i.e. Asymptotic stability of switched linear system $\dot{x} = A_\sigma x \nRightarrow$existence of common quadratic Lyapunov Function.

Theorem: Computation of Common Quadratic Lyapunov Function

If matrices $\{A_1, \ldots, A_N\}$ **commute pairwise** (i.e., $A_i A_j = A_j A_i$) for all i, j and are all stable, then there **exists common quadratic Lyapunov function** $P = P_N$, that can be found from solving following set of Lyapunov equalities successively:

$$\begin{gathered} A_1^T P_1 + P_1 A_1 = -I \\ A_2^T P_2 + P_2 A_2 = -P_1 \\ A_3^T P_3 + P_3 A_3 = -P_2 \\ \vdots \\ A_N^T P_N + P_N A_N = -P_{N-1} \end{gathered}$$

4. Stability for Given Switching Signal

Multiple Lyapunov Approach

Here, we assume that there is no common Lyapunov Function (not GUAS).

Theorem 1:

Let switching times be given by $t_k, k = 0, 1, 2, \ldots$ and suppose that

$$V_{\sigma(t_{k-1})}\left(x\left(t_k\right)\right) = V_{\sigma(t_k)}\left(x\left(t_k\right)\right) \text{ for all } k = 1, 2, \ldots$$

V_σ is now continuous Lyapunov Function, then the switched system is GAS

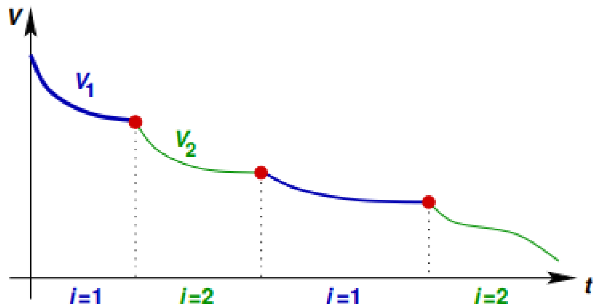

Illustration: energy keeps goes down after switching happens

More General Theorem:

Consider switched system with all sub-models $\dot{x} = f_i(x)$ GAS with corresponding Lyapunov function V_i

Suppose that for **every pair** of switching times $(t_k, t_l), k < l$ with $\sigma(t_k) = \sigma(t_l) = i$ and $\sigma(t_m) \neq i$ for $t_k < t_m < t_l$, we have

$$V_i\left(x\left(t_l\right)\right) - V_i\left(x\left(t_k\right)\right) \leqslant -\rho\left(\left\|x\left(t_k\right)\right\|\right) < 0,$$

then switched system is GAS

Illustration: energy keeps goes down in each mode

State-dependent Switching:

Assume no common quadratic Lyapunov Function

Single Lyapunov Function Approach

Instead of find V meet Lyapunov Conditions for all x, we can turn to find V such that $L_{f_i}V(x)$ is **only negative where $\dot{x} = f_i(x)$ can be active**

Multiple Lyapunov Function Approach

Also, we assume there is no common Lyapunov Function.

Find V_σ that is **continuous (at boundary) and strictly decreasing**

More General Set-up for Piece-wise Linear Systems

Several **relaxations** possible w.r.t. common quadratic Lyapunov Function.

- One can require that derivative $L_{f_i(x)}V(x)$ of $V(x) = x^T P x$ is **only negative in region where subsystem is active**
- One can use **multiple Lyapunov functions**, say $V_i(x) = x^T P_i x$, for each sub-model and "**connect them**" in a suitable way
- One can require that the Lyapunov function $V_i(x) = x^T P_i x$ is **only positive definite in its active region**

S-procedure

S-procedure is a method to deal with relaxation, we will introduce the S-procedure for the first relaxation.

- **Aim:** $V(x) = x^T P x, P > 0$ such that $x^T \left[A_i^T P + P A_i\right] x < 0$ for $0 \neq x \in \mathscr{X}_i$ (this is **not an LMI**, because not hold for all x)
- **Find:** $S_i(x)$ based on $\mathscr{X}_i$ with $S_i(x) \geqslant 0$ **when** $x \in \mathscr{X}_i$
- **Next:** search for $\beta \geqslant 0$ satisfying (an **uniform** β)

$$x^T A_i^T P x + x^T P A_i x + \beta S_i(x) < 0 \text{ for all } x$$

- **Result:** Since $S_i(x)$ might be negative outside $\mathscr{X}_i$, so **less conservative** than $A_i^T P + P A_i < 0$ (i.e., $x^T A_i^T P x + x^T P A_i x < 0$ for all x)
- **Computationally interesting**: $S_i(x) = x^T S_i x$, then LMI:
  Find $\beta_i \geqslant 0$ and $P > 0$ such that $A_i^T P + P A_i + \beta_i S_i < 0$

Illustration:

- For $x \in \mathscr{X}_i,\ x^T A_i^T P x + x^T P A_i x < 0, S_i(x) > 0$, the β sum can be smaller than 0.
- For $x \notin \mathscr{X}_i,\ x^T A_i^T P x + x^T P A_i x$ **may** ≥ 0, $S_i(x)$ may ≤ 0, the β sum can be smaller than 0.
- Actually, we are going to design P and S_i, by adding more variables S_i , we have a larger “freedom”

Summary

5. Summary

- Stability of submodels $\not\Rightarrow$ stability!
- Problem A: GAS for arbitrary switchings:
  - common Lyapunov function approach
  - piecewise linear: common quadratic Lyapunov function
- Problem B: GAS for specific switchings
  - multiple Lyapunov function: hard to verify in general case
  - state-dependent switching
    - decrease of Lyapunov function only in active region
    - multiple Lyapunov function (continuous over boundary)
  - Piecewise linear systems:
    - S-procedure: nice tool to get LMI

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- Switched Systems:
  - General Switched Systems
  - Switched Linear Systems, PWA
  - **Problem: stability of sub-models $\not\Rightarrow$ GS**
  - Problems: A, B, C
- Problem A: GUAS
  - Common Lyapunov Function Approach and Converse Theorem
  - piecewise linear: **common quadratic** Lyapunov Function
- Problem B: GAS for specific switching
  - multiple Lyapunov Function: **continuous or keep decreasing in each mode, hard to verify**
    - some relaxations
  - state-dependent switching:
    - decrease **only in active region**
    - multiple Lyapunov function (**continuous over boundary**)
    - other relaxations
  - Piecewise Linear Systems
    - S-procedure: nice tool to **get LMI**