05_MPC

1. Conceptions

Definition: Feasible Set: Feasible Set

The feasible set $\mathcal{X}_N$ is defined as the set of **initial states** x for which the MPC problem with horizen N is feasible, i.e. $\mathcal{X}_N := \{x \mid \exists \left[u_0, \ldots, u_{N-1}\right]$ such that $Cu_i + Dx_i \leq b, i = 1, \ldots, N\})$

Definition: Recursive Feasibility

The MPC problem is called **recursively feasible**, if for all feasible initial states, feasibility is guaranteed at **every state along** the closed-loop trajectory

Definition: Lyapunov Stability

The equilibrium point at the origin of system

$$x_{k+1} = Ax_k + B\kappa\left(x_k\right) = f_\kappa\left(x_k\right)$$

is said to be **(Lyapunov) stable** in $\mathcal{X}$ if for every $\epsilon > 0,$ there exists a $\delta(\epsilon) >$ such that, for every $x(0) \in (X)$

$$\|x(0)\| \leq \delta(\epsilon) \Rightarrow \|x(k)\| < \epsilon \quad \forall k \in \mathbb{N}$$

2. Design a Stable MPC

Problem of Standard MPC Model

Model: MPC Model Without Terminal Set

$$\begin{aligned} \min_{x,u} \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \\ \text{s.t. } x_{i+1} = Ax_i + Bu_i \\ b \geq Cx_i + Du_i \end{aligned}$$

Because we only use Finite-Horizon MPC without terminal cost, there may be two big problems here:

- we cannot guarantee feasibility: the MPC problem may not have a solution now or later
- we cannot guarantee stability: the trajectory may not converge to the origin

Main Idea

Introduce **terminal cost** and **constraints** to explicitly ensure stability and feasibility.

Model: MPC with Terminal Cost

$$\begin{aligned} J^*(x) = \min_{x,u} & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T P x_N \text{ Terminal cost} \\ \text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\ & Cx_i + Du_i \leq b \\ & x_N \in \mathcal{X}_f \text{ Terminal constraint} \\ & x_0 = x \end{aligned}$$

Lyapunov Stability Analysis

Zero Terminal Cost Case

Assumption:

Terminal constraint $X_N = 0$

Property:

By doing:

- Assume feasibility of x and let $\left[u_0^*, u_1^*, \ldots, u_{N-1}^*\right]$ be the optimal control sequence computed at x
- At $x^+$the control sequence $\left[u_1^*, u_2^*, \ldots, u_{N-1}^*, 0\right]$ is feasible (apply 0 control input $\Rightarrow x_{N+} = 0$)

Then the controller meet **recursive feasibility** and we can prove that $J^*(x)$ is a Lyapunov function.

$$\begin{aligned} J^*\left(x_0\right) &= \sum_{i=0}^{N-1} I\left(x_i^*, u_i^*\right) \\ J^*\left(x_1\right) \leq J\left(x_1\right) &= \sum_{i=1}^{N} I\left(x_i^*, u_i^*\right) \\ &= \sum_{i=0}^{N-1} I\left(x_i^*, u_i^*\right) - I\left(x_0, u_0^*\right) + I\left(x_N, u_N\right) \\ &= J^*\left(x_0\right) - \underbrace{I\left(x, u_0^*\right)}_{\text{Subtract cost}} + \underbrace{I(0,0)}_{\text{at stage 0}} \text{ staying at 0} \end{aligned}$$

More General Terminal State Case

The terminal constraint $X_N = 0$ reduce the feasible set. We can somehow loose the constraints. We can prove that if the following assumptions hold:

- The stage cost is a **positive definite function**, i.e. it is strictly positive and only zero at the origin
- Th**e terminal set is invariant** under the local control law $\kappa_f(x)$

  $$x^+ = Ax + B\kappa_f(x) \in \mathcal{X}_f \quad \text{ for all } x \in \mathcal{X}_f$$

  All state and input constraints are satisfied in $\mathcal{X}_f$: terminal state

  $$\mathcal{X}_f \subseteq \mathbb{X}, \kappa_f(x) \in \mathbb{U} \text{ for all } x \in \mathcal{X}_f$$

  i.e. we can find a "virtual terminal controller”
- Terminal cost is a **continuous Lyapunov function** in the terminal set $\mathcal{X}_f$:

  $$V_f\left(x^+\right) - V_f(x) \leq -l\left(x, \kappa_f(x)\right) \text{ for all } x \in \mathcal{X}_f$$

Then we have the following theorem:

Proposition: MPC with Established Terminal Set

The closed-loop system under the MPC control law $u_0^*(x)$ is stable and the system $x^+ = Ax + Bu_0^*(x)$ is invariant in the feasible set $\mathbb{X}_N$.

Notes:

Although better than zero terminal constraints, this method still reduces the region of attraction.

Asymptotic Stability Analysis

Definition: Asymptotic Stability

Given a positively invariant set $\mathcal{X}$ including the origin as an interior-point, the equilibrium point at the origin of system $x_{k+1} = f_k\left(x_k\right)$ is said to be **asymptotically stable** in $\mathcal{X}$ if it is

- (Lyapunov) stable
- attractive in $\mathcal{X}$, i.e. $\lim_{k \to \infty} \|x_k\| = 0$ for all $x(0) \in \mathcal{X}$

Theorem:

If the continuous Lyapunov function additionally satisfies

$$V\left(x_{k+1}\right) - V\left(x_k\right) < 0 \quad \forall x \neq 0$$

then the closed loop system converges to the origin and is hence asymptotically stable.

Proposition: MPC with Established Terminal Set

Recall: Decrease of the optimal MPC cost was given by

$$J\left(x_{k+1}\right)-J\left(x_k\right) \leq -l\left(x_k, u_0^*\right)$$

where the stage cost was assumed to be positive and only 0 at 0 .

So, the closed-loop system under the MPC control law is asymptotically stable

Extension to Nonlinear MPC

Model: Nonlinear MPC Problem

$$\begin{aligned}
J^*(x)=\min_{x, u} & \sum_{i=0}^{N-1} I\left(x_i, u_i\right)+V_f\left(x_N\right) \\
\text{s.t. } & x_{i+1}=f\left(x_i, u_i\right) \\
& g\left(x_i, u_i\right) \leq 0 \\
& x_N \in \mathcal{X}_f \\
& x_0=x
\end{aligned}$$

Notes:

Previous assumptions on the terminal set and cost did not rely on linearity, so the result can be directly extended to nonlinear systems. However, computing the sets $\mathcal{X}_f$ and V_f may be very difficult in a nonlinear system.

Summary

Finite-horizon MPC may not be stable!
Finite-horizon MPC may not satisfy constraints for all time!

- An infinite-horizon provides stability and invariance.
- We 'fake' infinite-horizon by forcing the final state to be in an invariant set for which there exists an invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- These ideas extend to non-linear systems, but the sets are difficult to compute.

- **Finite-Horizon MPC** cannot guarantee feasibility and stability because the finite-horizon. We can solve this problem by introducing **terminal set and terminal cost**.
- However, then a big problem is how to **guarantee stability and feasibility** with terminal set and terminal cost.
- We have shown that for **zero terminal set**, it will be **stable and feasible**.
- And we can prove that more generally, if we can build a **final virtual controller and final cost meet 3 assumptions**, we will get a **feasible and stable** MPC controller.
- The next problem is, how can we **find the virtual controller and build a final cost** meet the assumptions.