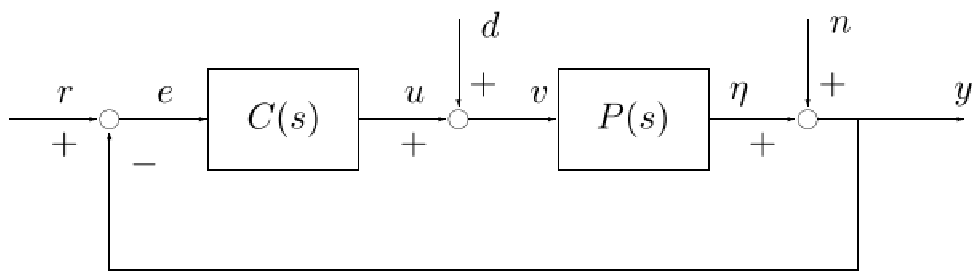
06_Robust Performance

1. Sensitivity Functions



$$\begin{bmatrix} y \\ \eta \\ v \\ u \\ e \end{bmatrix} = \begin{bmatrix} \frac{PC}{1+PC} & \frac{P}{1+PC} & \frac{1}{1+PC} \\ \frac{PC}{1+PC} & \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} & \frac{-C}{1+PC} \\ \frac{-PC}{1+CC} & \frac{-PC}{1+PC} & \frac{-C}{1+PC} \\ \frac{1}{1+PC} & \frac{-PC}{1+P} & \frac{-1}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

Sensitivity Function and Complementary Sensitivity Function

- **Sensitivity Function**: $S = \frac{1}{1+PC}$
- **Complementary Sensitivity Function**: $T = \frac{PC}{1+PC}$

- **Property:** $S + T = 1$

Gang of Four

- **Sensitivity Function**: $\frac{1}{1+PC}$
- **Load Sensitivity Function**: $\frac{P}{1+PC}$
- **Complementary Sensitivity Function**: $\frac{PC}{1+PC}$
- **Noise Sensitivity Function**: $\frac{C}{1+PC}$

Why Sensitivity Function

$$e = S(s)(r - P(s)d - n)$$

- minimizing sensitivity $\Rightarrow$ minimizing tracking error.
- **Sensitivity improvements in one frequency range must be paid for with sensitivity deteriorations in another frequency range**

2. Youla Parametrization

In control theory the **Youla–Kučera parametrization** (also simply known as **Youla parametrization**) is a formula that describes all **possible stabilizing feedback controllers** for a given plant P, as **function of a single parameter** Q.

Stable P Case

Stability Theorem with Gang of Four

Assume P is stable. The closed loop is stable if

$$\frac{1}{1+PC} \quad , \quad \frac{P}{1+PC} \quad , \quad \frac{C}{1+PC} \text{ and } \frac{PC}{1+PC}$$

are stable.

Controller Design

Now choose $C = \frac{Q}{1-PQ}$ for some function Q.

Then the closed loop system is **stable if and only if** Q is stable.

Illustration

$$\begin{aligned} \frac{C}{1+PC} &= Q & \frac{PC}{1+PC} &= PQ \\ \frac{1}{1+PC} &= 1 - PQ & \frac{P}{1+PC} &= P(1 - PQ) \end{aligned}$$

”Gang of four” is **stable if and only if** Q is stable.

Unstable P Case

Controller Design

Consider unstable system P and a stabilizing controller C_0.
Let A, B, F_0 and G_0 be stable transfer functions such that

$$P = \frac{B}{A} \quad , \quad C_0 = \frac{G_0}{F_0}$$

and

$$AF_0 + BG_0 = I$$

(This is called a **doubly coprime factorization**)

Now define **controller**

$$C = \frac{G_0 + QA}{F_0 - QB}$$

The system is then **stable** if Q is stable

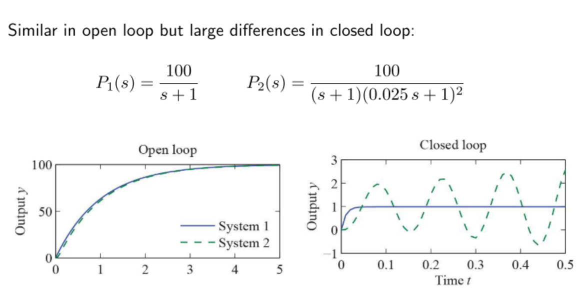
Illustration

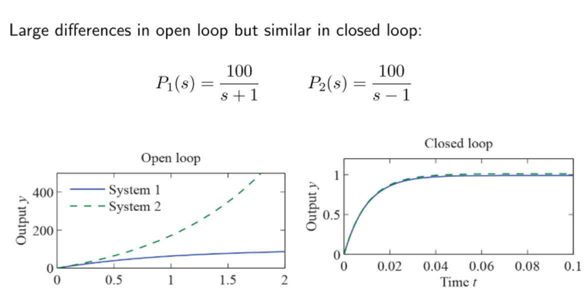
$$\frac{C}{1+PC} = A\left(G_0 + QA\right) \quad \frac{PC}{1+PC} = B\left(G_0 + QA\right)$$
$$\frac{1}{1+PC} = A\left(F_0 - QB\right) \quad \frac{P}{1+PC} = B\left(F_0 - QB\right)$$

3. Modeling Uncertainty

Comparison Between Open-Loop and Closed-Loop Behavior

- Small difference in open-loop $\not\Rightarrow$Small difference in closed-loop
- Large difference in open-loop $\not\Rightarrow$Large difference in closed-loop





Forms of Uncertainty

There are mainly two forms of model uncertainty

- **Parameter Uncertainty**

$$P(s) = \frac{1}{s+3} \quad , \quad P_{true}(s) = \frac{1}{s+3.1}$$

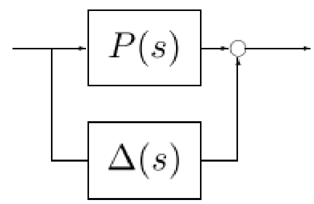
- **Unmodeled Dynamics**

$$P(s) = \frac{1}{s+3} \quad , \quad P_{true}(s) = \frac{1}{(s+3)(0.01s+1)}$$

Unmodeled Dynamics Uncertainty

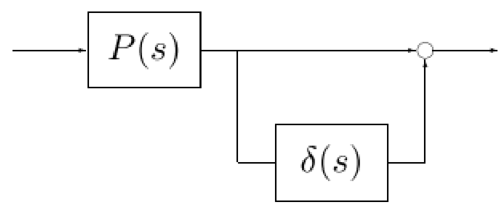
Additive Uncertainty

$$P_{\text{true}}(s) = P(s) + \Delta(s)$$
$$\Delta(s) = P_{\text{true}}(s) - P(s)$$



Multiplicative Uncertainty

$$P_{\text{true}}(s) = P(s)(1+\delta(s))$$
$$\delta(s) = \frac{P_{\text{true}}(s) - P(s)}{P(s)}$$



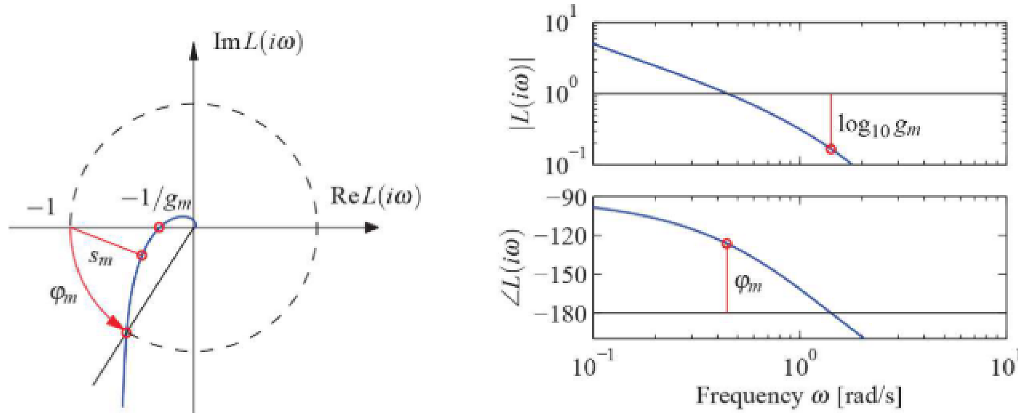
Properties

$$\Delta(s) = \delta(s)P(s)$$

4. Stability in the Presence of Uncertainty

Stability Margins

In classical theories, we have **Phase Margin** and **Gain Margin.** But for uncertainty modelling, it is not enough. We need to introduce **Stability Margin(Vector Stability Margin)**



Robust Stability Condition for Additive Uncertainty

Theorem

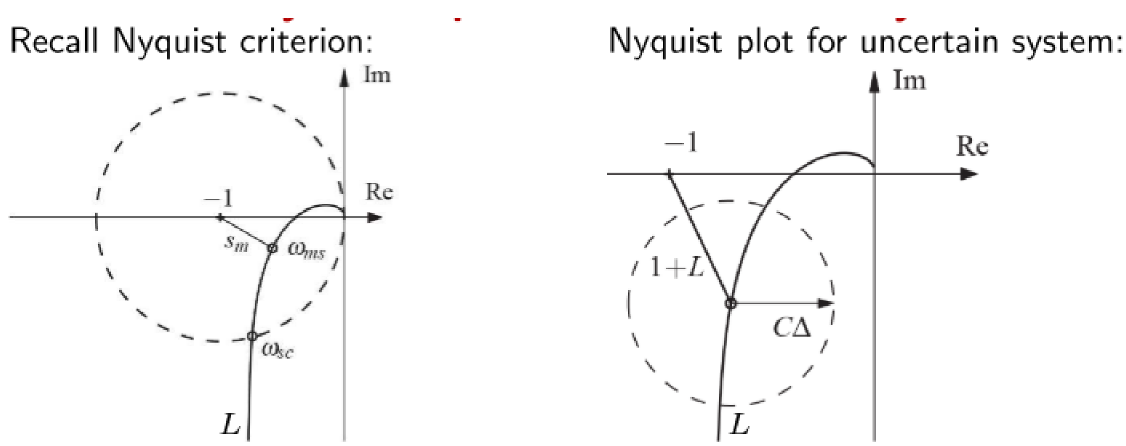
The system is **robust stable**, if

$$
\begin{aligned}
|C(i\omega)\Delta(i\omega)| < |1 + L(i\omega)| \quad , \quad \forall\omega \geq 0 \\
|\Delta(i\omega)| < \left|\tfrac{1+L(i\omega)}{C(i\omega)}\right| \quad , \quad \forall\omega \geq 0
\end{aligned}
$$

We can write it in the Infinity Norm: $\|H\|_\infty = \sup_\omega |H(i\omega)|$, then it will become:

$$
|\Delta(i\omega)| < \left|\frac{1 + L(i\omega)}{C(i\omega)}\right| \quad , \quad \forall\omega \geq 0 \Rightarrow \|\Delta CS\|_\infty < 1
$$

Illustration



$$
1 + L_{\text{true}} = 1 + P_{\text{true}}\, C = 1 + (P + \Delta)C = 1 + L + C\Delta
$$

- $1 + L$ is the distance between point on the phase trait and the $(-1, 0)$ point
- $C\Delta$ is the circle bound of uncertainty

Robust Stability Condition for Multiplicative Uncertainty

Theorem

The system is **robust stable**, if

$$|\delta(i\omega)| = \left|\frac{\Delta(i\omega)}{P(i\omega)}\right| < \left|\frac{1+L(i\omega)}{P(i\omega)C(i\omega)}\right| = \left|\frac{1}{T(i\omega)}\right| \quad , \quad \forall\omega \geq 0$$

We can present it in the Infinity Norm form:

$$|\delta(i\omega)| < \left|\frac{1}{T(i\omega)}\right| \quad , \quad \forall\omega \geq 0, \forall\omega \geq 0 \Longrightarrow \|\delta T\|_\infty < 1$$

Youla Parametrization and Robust Stability when P is stable

Now let $C = \frac{Q}{1-PQ}$ and Q is stable.

Then closed loop is stable for $\Delta = 0$.

- In the case of **additive uncertainty**, the condition for robust stability is

  $$\|\Delta(i\omega)C(i\omega)S(i\omega)\|_\infty < 1 \text{ so } \|\Delta Q\|_\infty < 1$$

  or Q is stable and $|Q(i\omega)| < \left|\frac{1}{\Delta(i\omega)}\right| \quad , \quad \forall\omega \geq 0$

- In the case of **multiplicative uncertainty**, the condition for robust stability is

  $$\|\delta(i\omega)T(i\omega)\|_\infty < 1 \text{ so } \|\delta PQ\|_\infty < 1$$

  or Q is stable and $|Q(i\omega)| < \left|\frac{1}{P(i\omega)\Delta(i\omega)}\right|, \forall\omega \geq 0$

Summary

- Sensitivity Function:
  - **Smaller Sensitivity → Smaller Track Error**
  - But we cannot make it small in a frequency, there will be **“sacrifice”**
- Youla Parametrization
  - Find Q and make it stable
- Modelling Uncertainty
  - Parameter Uncertainty

    - Unmodeled Uncertainty
        - Additive Uncertainty
        - Multiplicative Uncertainty
- Stability in the Presence of Uncertainty
    - Stability Margins: the smallest distance
    - Robust Stability Condition for Additive Uncertainty
    - Robust Stability Condition for Multiplicative Uncertainty
    - Youla Parametrization and Robust Stability when P is stable