05_Switched Control

1. Introduction & Motivation for Hybrid Control

Motivation of Switched Controller

Theorem: Brockett's Necessary Condition

Consider system

$$\dot{x} = f(x,u) \quad \text{with } x \in \mathbb{R}^n, u \in \mathbb{R}^m, f(0,0) = 0$$

where f is smooth function
If system is **asymptotically stabilizable** (around $x = 0$) using **continuous feedback law** $u = \alpha(x)$,
then **image** of every open neighborhood of $(x,u) = (0,0)$ under f **contains open neighborhood** of $x = 0$

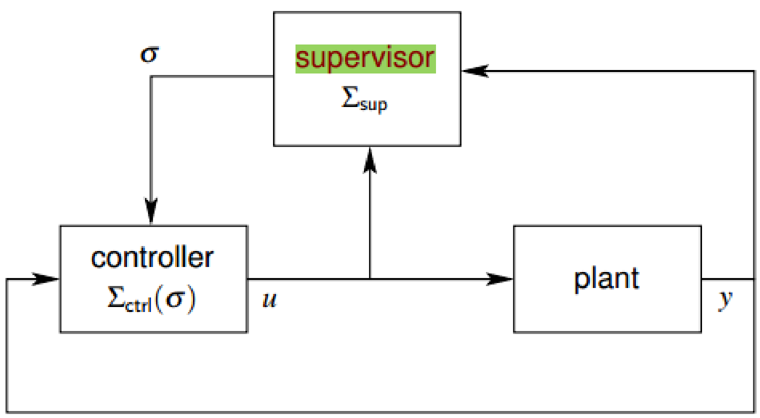
Example:

Here is an example (**non-holonomic integrator**) that the system cannot be stabilized by the continuous feedback law

- For non-holonomic integrator: $\dot{x} = u$
  $\dot{y} = v$
  $\dot{z} = xv - yu$
- Is asymptotically stabilizable (see later)
- Satisfies Brockett's necessary condition?
  - if $f_1 = f_2 = 0$ then $f_3 = 0$
  - hence, $(0,0,\varepsilon)$ cannot belong to image of f for any $\varepsilon \neq 0$
    $\rightarrow$ image of open neighborhood of $(x,y,z;u,v) = (0,0,0;0,0)$ under f does *not* contain open neighborhood of $(x,y,z) = (0,0,0)$
  - so non-holonomic integrator cannot be stabilized by *continuous* feedback

$\rightarrow$ hybrid control schemes necessary to stabilize it! hs_switched_ctrl.4

Structure of Switching Control



$\rightarrow$ shared controller state variables

Problem in Switching Control: Chattering

Main problem may appear in switching control controller is **chattering,** i.e. very fast switching. There are mainly two solutions:

- **Hysteresis Switching Logic**

- let $h > 0$, let π_σ be a performance criterion (to be minimized)
- if supervisor changes value of σ to q, then σ is held *fixed* at q until $\pi_p + h < \pi_q$ for some p

  $\to$ σ is set equal to p

  $\Rightarrow$ threshold parameter $h > 0$ prevents infinitely fast switching

- **Dwell-Time Switching Logic**

  once symbol σ is chosen by supervisor it remains constant for at least $\tau > 0$ time units (τ: "**dwell time**")

2. Stabilization of Switched Linear Systems

This part is mainly used to solve the **Problem C** in the last chapter:
Find switching rule σ as function of **time/state** such that closed-loop system is AS

Quadratic Stabilization via single Lyapunov Function

Select $\sigma(x) : \mathbb{R}^n \to I := \{1, 2, \ldots, N\}$ such that closed-loop system **has single quadratic Lyapunov function** $x^T P x$

Solution 1:

If some convex combination of A_i is stable:

$$A := \sum \alpha_i A_i \quad \left(\alpha_i \geqslant 0, \sum \alpha_i = 1\right) \text{ is stable}$$

Select $Q > 0$ and let $P > 0$ be solution of $A^T P + PA = -Q$

Then the switching rule should be:

$$i(x) := \arg\min x^T \left(A_i^T P + PA_i\right) x$$

Illustration:

- From $x^T(A^T P + PA)x = -x^T Qx < 0$ it follows that

$$\sum_i \alpha_i [x^T (A_i^T P + PA_i)x] \ < \ 0$$

- For each x there is at least one mode with $x^T(A_i^T P + PA_i)x < 0$ or stronger

$$\bigcup_{i \in I} \{x \mid x^T(A_i^T P + PA_i)x \leqslant -\frac{1}{N} x^T Qx\} \ = \ \mathbb{R}^n$$

Modified Switching Rule:(based on hysteresis switching logic)

- stay in mode i as long as $x^T\left(A_i^T P+P A_i\right) x \leqslant-\frac{1}{2 N} x^T Q x$
- when bound reached, switch to a new mode j that satisfies

$$x^T\left(A_j^T P+P A_j\right) x \leqslant-\frac{1}{N} x^T Q x$$

Property:

No conservatism for 2 modes

Stabilization via multiple Lyapunov functions

Main Idea

Main idea: Find function $V_i(x) = x^T P_i x$ that decreases for $\dot{x} = A_i x$ in some region

Define $\mathscr{X}_i := \{x \mid x^T[A_i^T P_i + P_i A_i]x < 0\}$

If $\bigcup_i \mathscr{X}_i = \mathbb{R}^n$, try to switch to satisfy multiple Lyapunov criterion to guarantee asymptotic stability.

Solution:

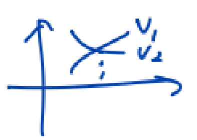
Find P_1 and P_2 such that they satisfy the coupled conditions:

$$x^T\left(P_1 A_1+A_1^T P_1\right) x<0 \text { when } x^T\left(P_1-P_2\right) x \geqslant 0, x \neq 0$$

and

$$x^T\left(P_2 A_2+A_2^T P_2\right) x<0 \text { when } x^T\left(P_2-P_1\right) x \geqslant 0, x \neq 0$$

Then $\sigma(t)=\arg \max \left\{V_i(x(t)) \mid i=1,2\right\}$ is stabilizing (V_σ will be **continuous decrease** (the reason to choose larger V))



S-procedure

The above question is hard to solve: it is **not an LMIs problem**. We can use S-procedure to solve it

If there exist $\beta_1, \beta_2 \geqslant 0$ such that

$$
\begin{aligned}
&-P_1 A_1 - A_1^T P_1 + \beta_1 (P_2 - P_1) > 0 \\
&-P_2 A_2 - A_2^T P_2 + \beta_2 (P_1 - P_2) > 0
\end{aligned}
$$

then $\sigma(t) = \arg\max i \{V_i(x(t)) \mid i = 1, 2\}$

Stabilization of switched linear systems with continuous inputs

Note:

Previous methods only finds the switching sequence, not the continuous input u

Switched linear system with inputs:

$$\dot{x} = A_i x + B_i u, i \in I = \{1, \ldots, N\}$$

Now both $\sigma : [0, \infty) \rightarrow I$ and feedback controllers $u = K_i x$ are to be determined

There are three levels of design:

- **Case 1**: Determine K_i such that closed loop is stable under **arbitrary switching** (assuming **we know mode**)!
- **Case 2**: Determine both $\sigma : [0, \infty) \rightarrow I$ and K_i
- **Case 3: (for PWL Systems):** Given σ as function of state, determine K_i

Case 1: Stabilization of switched linear system under arbitrary switching

$$\dot{x} = A_i x + B_i u, \ i \in I = \{1, \ldots, N\}$$

Sufficient condition: find common *quadratic* Lyapunov function $V(x) = x^T P x$ for some positive definite matrix P and $K_1, \ldots, K_N$

$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) < 0 \text{ for all } i = 1, \ldots, N \text{ and } P > 0$$

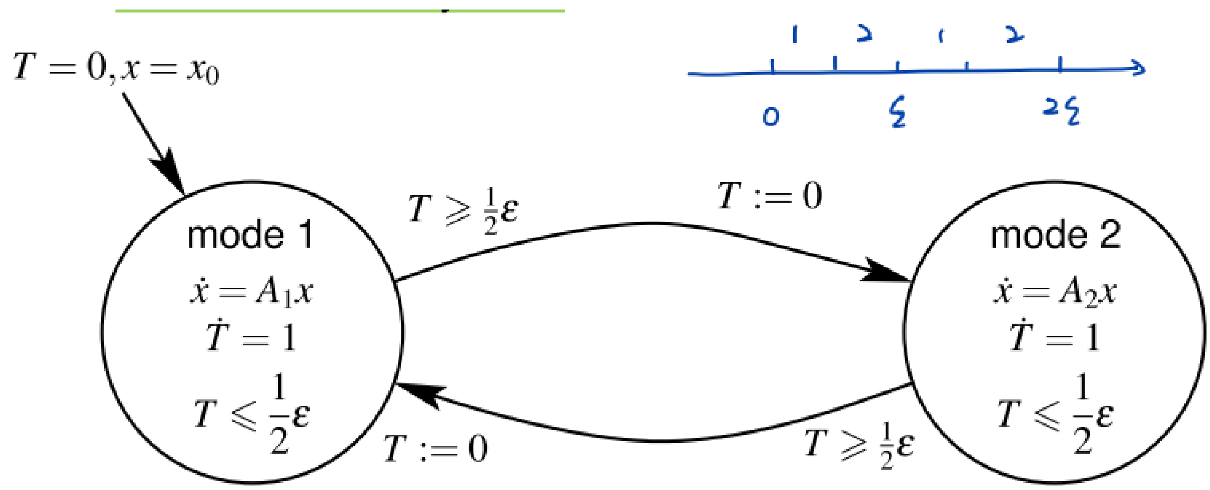
$\rightarrow$ LMIs (also for Cases 2 and 3)

$\longrightarrow$ state-based switching in this section, ... next ...

3. Time-Controlled Switching & pulse width modulation

If dynamical system **switches between several subsystems** → stability properties of total system may be **quite different from those of subsystems**

Time-Controlled Switching



For $\varepsilon \rightarrow 0$ solution of switched system tends to solution of $\dot{x} = \left(\frac{1}{2}A_1 + \frac{1}{2}A_2\right)x$ ("**averaged" system**)

- $$\begin{aligned} x(t_0+\tfrac{1}{2}\varepsilon) &= \exp(\tfrac{1}{2}\varepsilon A_1)x_0 = x_0 + \tfrac{\varepsilon}{2}A_1x_0 + \tfrac{\varepsilon^2}{8}A_1^2x_0 + \cdots \\ x(t_0+\varepsilon) &= \exp(\tfrac{1}{2}\varepsilon A_2)\exp(\tfrac{1}{2}\varepsilon A_1)x_0 \\ &= (I+\tfrac{\varepsilon}{2}A_2+\tfrac{\varepsilon^2}{8}A_2^2+\cdots)(I+\tfrac{\varepsilon}{2}A_1+\tfrac{\varepsilon^2}{8}A_1^2+\cdots)x_0 \\ &= (I+\varepsilon[\tfrac{1}{2}A_1+\tfrac{1}{2}A_2]+\tfrac{\varepsilon^2}{8}[A_1^2+A_2^2+2A_2A_1]+\cdots)x_0. \end{aligned}$$
- Compare with $\neq$
  $$\exp[\varepsilon(\tfrac{1}{2}A_1+\tfrac{1}{2}A_2)] = I+\varepsilon[\tfrac{1}{2}A_1+\tfrac{1}{2}A_2]+\tfrac{\varepsilon^2}{8}[A_1^2+A_2^2+A_1A_2+A_2A_1]+\cdots$$
  $\rightarrow$ same for $\varepsilon \approx 0$

- Possible that A_1, A_2 stable, whereas $1/2A_1 + 1/2A_2$ unstable, or vice versa
- And we can always compute a upper bound of ϵ
  - Minimal switching frequency found by computing eigenvalues of the mapping $\exp\left(\frac{1}{2}\varepsilon A_2\right)\exp\left(\frac{1}{2}\varepsilon A_1\right)$

Pulse-Width Modulation

Assume mode 1 followed during $h\epsilon$, and mode 2 during $(1-h)\epsilon$, then behavior of system is well approximated by system

$$\dot{x} = (hA_1 + (1-h)A_2)\,x$$

Parameter h might be considered **as control input**

- If h varies, should be on time scale that is **much slower** than the time scale of switching
- If mode 1 is “power on” and mode 2 is “power off”, then h is known as **duty ratio**

Formulation:

- System: $\dot{x} = f(x,u), \quad u \in \{0,1\}$
- Duty cycle: Δ (fixed)
- u is switched exactly one time from 1 to 0 in each cycle
- Duty ratio α: fraction of duty cycle for which $u = 1$

$$\begin{aligned} u(\tau) &= 1 \quad \text{for } t \leqslant \tau < t + \alpha\Delta \\ u(\tau) &= 0 \quad \text{for } t + \alpha\Delta \leqslant \tau < t + \Delta \end{aligned}$$

- Hence, $x(t+\Delta) = x(t) + \int_t^{t+\alpha\Delta} f(x(\tau),1)d\tau + \int_{t+\alpha\Delta}^{t+\Delta} f(x(\tau),0)d\tau$
- Ideal averaged model ($\Delta \to 0$):

$$\dot{x}(t) = \lim_{\Delta \to 0} \frac{x(t+\Delta) - x(t)}{\Delta} = \alpha f(x(t),1) + (1-\alpha) f(x(t),0)$$

4. Sliding Mode Control

Method

Model

- Consider $\dot{x}(t) = f(x(t), u(t))$ with u scalar
- Suppose **switching feedback control scheme**:

$$u(t) = \begin{cases} \phi_+(x(t)) & \text{if } h(x(t)) > 0 \\ \phi_-(x(t)) & \text{if } h(x(t)) < 0 \end{cases}$$

- Surface $\{x \mid h(x) = 0\}$ is called **switching surface**
- Let $f_+(x) = f\left(x, \phi_+(x)\right)$ and $f_-(x) = f\left(x, \phi_-(x)\right)$, then

$$\dot{x} = \frac{1}{2}(1+v) f_+(x) + \frac{1}{2}(1-v) f_-(x), \quad v = \operatorname{sgn}(h(x))$$

- Use solutions in Filippov's sense if "chattering"

Controller Design

Assume “desired behavior” whenever constraint $s(x) = 0$ is satisfied。

- Set $\{x \mid s(x) = 0\}$ is called **sliding surface**
- Find control law u such that

$$\frac{1}{2}\frac{d}{dt}s^2 \leqslant -\alpha|s| \quad S(x) \neq 0 \Rightarrow \text{ decreode}$$

  where $\alpha > 0$
  $\rightarrow$ squared "distance" to sliding surface decreases
  along all system trajectories
    - $S(x) = 0 \Rightarrow 0$
    - $S(x) \neq 0 \Rightarrow$ decrease quadratic

Properties

- **Quick succession** of switches may occur
    - embed sliding surface in thin **boundary layer**
    - smoothen discontinuity by **replacing** sgn **by steep sigmoid function**
    - Note: modifications may **deteriorate performance** of closed-loop system
- Main advantages:
    - conceptually simple
    - robustness w.r.t. uncertainty in system data
- excitation of unmodeled high-frequency modes

5. Stabilization By Switching Control: An Example

Here, we will use the **non-holonomic integrator** example to show how to design a stabilization controller by switching control

- System: $\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = xv - yu$
- Sliding mode control: $u = -x + y\,\mathrm{sgn}(z)$
  $v = -y - x\,\mathrm{sgn}(z)$

$\dot{V} = \frac{\partial V}{\partial x} \cdot \dot{x} + \frac{\partial V}{\partial y} \dot{y}$

- Switching surface: $z = 0$
- Lyapunov function for (x,y) subspace: $V(x,y) = \frac{1}{2}(x^2 + y^2)$

$$\Rightarrow \dot{V} = -x^2 + xy\,\mathrm{sgn}(z) - y^2 - xy\,\mathrm{sgn}(z) = -(x^2 + y^2) = -2V$$
$$\Rightarrow x, y \to 0$$

$(|z|)' = \mathrm{sgn}(z)$

- $\dot{z} = xv - yu = -(x^2 + y^2)\,\mathrm{sgn}(z) = -2V\,\mathrm{sgn}(z)$
  So $|z|$ will decrease and reach 0 provided that

$\frac{d(|z|)}{dt} = \mathrm{sgn}(z) \cdot \frac{dz}{dt}$
$= -2V.$

$$2\int_0^\infty V(\tau)d\tau > |z(0)|$$

?

$\to z$ will reach 0 in finite time

hs_switched_ctrl.26

$|z(\infty)| - |z(0)| = -2\int_0^\infty V(\tau)d\tau \Rightarrow |z(0)| = +2\int_0^\infty + |z(\infty)|$

- Since $V(t) = V(0)e^{-2t} = \frac{1}{2}(x^2(0) + y^2(0))e^{-2t}$
  condition for system to be asymptotically stable is

integral : $$\frac{1}{2}(x^2(0) + y^2(0)) \geqslant |z(0)|$$

$\to$ defines parabolic region $\mathscr{P} = \{(x,y,z) \mid 0.5(x^2 + y^2) \leqslant |z|\}$

- If initial conditions do *not* belong to $\mathscr{P}$ then sliding mode control asymptotically stabilizes system
- If initial state is inside $\mathscr{P}$:
  - first use control law (e.g., nonzero constant control) to steer system outside $\mathscr{P}$
  - then use sliding mode control

$\to$ *hybrid control scheme*

hs_switched_ctrl.27

Summary

- Introduction & Motivation:
  - Motivation: Continuous Control Law may not always be successful

    - Structure: **Supervisor part**
    - Problem: **Chattering:**
        - Setting a **threshold** in benefit or time
- Stabilization of Switched Linear Systems
    - Quadratic Stabilization via single Lyapunov Function
        - Convex-Hull → choose minimize
    - Stabilization via multiple Lyapunov Function
        - $\sigma(t) = \arg\max \{V_i(x(t)) \mid i = 1, 2\}$
    - S-procedure: **turn to LMIs**
- Stabilization of Switched Linear Systems with continuous input:
    - take K into consideration
- Time-Controlled Switching and pulse width modulation
    - Time-Controlled Switching: **average system** when ϵ is small
    - pulse width modulation: h and $1-h$
- Sliding Model Control:
    - Switching feedback control scheme + switching surface + **sliding surface (target)**