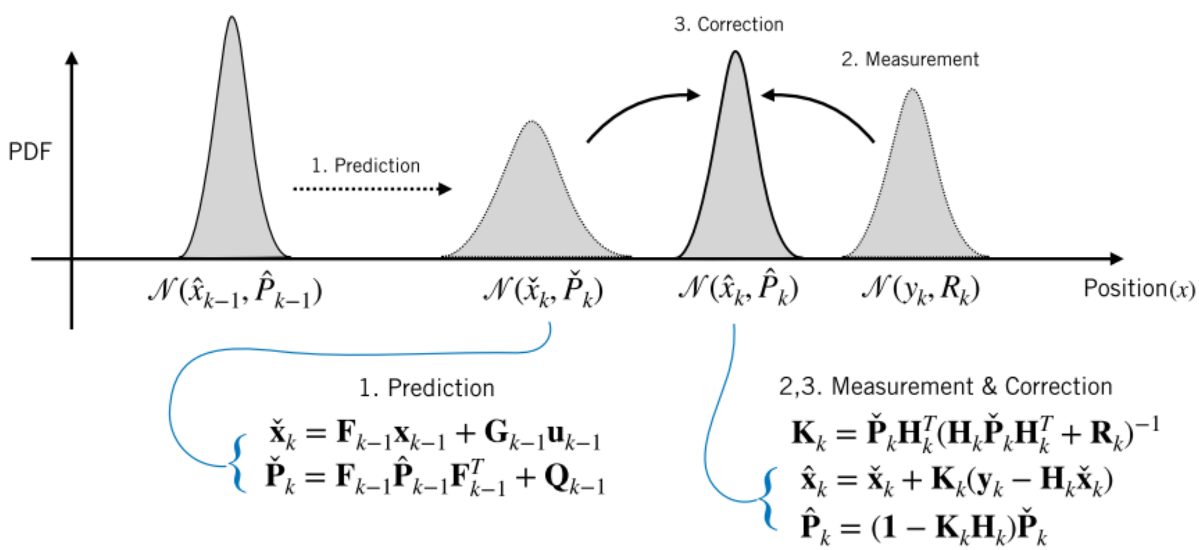
03_Linear_Kalman_Filter

1. Introduction



Kalman Filter include two steps:

- **Prediction** (using model)
- **Correction** (using Measurements)

2. Procedure

Model

System Model

$$
\begin{aligned}
\mathbf{x}_k &= \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1} + \mathbf{w}_{k-1} \\
\mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k
\end{aligned}
$$

Noise model

$$
\mathbf{v}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{R}_k\right) \quad \mathbf{w}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_k\right)
$$

Prediction Step

Update state estimation and covariance estimation based on system dynamical model

$$
\begin{aligned}
\check{\mathbf{x}}_k &= \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1} \\
\check{\mathbf{P}}_k &= \mathbf{F}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{F}_{k-1}^T + \mathbf{Q}_{k-1}
\end{aligned}
$$

Correction Step

- First Calculate the Optimal Gain (which can be derived from recursive least square, See 01_Least Squares)

$$
\mathbf{K}_k = \check{\mathbf{P}}_k\mathbf{H}_k^T\left(\mathbf{H}_k\check{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1}
$$

- Then correct the estimation based on: (See 01_Least Squares for covariance updating with two versions, here we put the simplified version)

$$
\begin{aligned}
\hat{\mathbf{x}}_k &= \check{\mathbf{x}}_k + \mathbf{K}_k\left(\mathbf{y}_k - \mathbf{H}_k\check{\mathbf{x}}_k\right) \\
\hat{\mathbf{P}}_k &= \left(\mathbf{1} - \mathbf{K}_k\mathbf{H}_k\right)\check{\mathbf{P}}_k
\end{aligned}
$$

4. Bias, Consistency and BLUE

Unbiased Filter

This filter is an unbiased if for all k,

$$
E\left[\hat{e}_k\right] = E\left[\hat{p}_k - p_k\right] = E\left[\hat{p}_k\right] - p_k = 0
$$

Consistent Filter

This filter is consistent if for all k,

$$
E\left[\hat{e}_k^2\right] = E\left[\left(\hat{p}_k - p_k\right)^2\right] = \hat{P}_k
$$

LKF-Unbiased

Property

So long as $E\left[\hat{\mathbf{e}}_0\right] = \mathbf{0}$, $E[\mathbf{v}] = \mathbf{0}$ $E[\mathbf{w}] = \mathbf{0}$, that is v and w are white, uncorrelated noise

The Kalman Filter is Unbiased

Proof:

$$\begin{aligned} E\left[\check{\mathbf{e}}_k\right] &= E\left[\mathbf{F}_{k-1}\check{\mathbf{e}}_{k-1} - \mathbf{w}_k\right] \\ &= \mathbf{F}_{k-1}E\left[\check{\mathbf{e}}_{k-1}\right] - E\left[\mathbf{w}_k\right] \\ &= \mathbf{0} \end{aligned}$$

$$\begin{aligned} E\left[\hat{\mathbf{e}}_k\right] &= E\left[\left(\mathbf{1} - \mathbf{K}_k\mathbf{H}_k\right)\check{\mathbf{e}}_k + \mathbf{K}_k\mathbf{v}_k\right] \\ &= \left(\mathbf{1} - \mathbf{K}_k\mathbf{H}_k\right)E\left[\check{\mathbf{e}}_k\right] + \mathbf{K}_k E\left[\mathbf{v}_k\right] \\ &= \mathbf{0} \end{aligned}$$

Note: this does not mean that the error on a given trial will be zero, but that, with enough trials, our expected error is zero!

LKF-Consistent

<u>Property</u>

If $E\left[\hat{\mathbf{e}}0\hat{\mathbf{e}}0^T\right] = \check{\mathbf{P}}_0 \; E[\mathbf{v}] = \mathbf{0} \; E[\mathbf{w}] = \mathbf{0}$, then the Kalman Filter is consistent

BLUE: Best Linear Unbiased Estimator

if we have white, uncorrelated zero-mean noise, the Kalman filter is the best (i.e., lowest variance) unbiased estimator that uses only a linear combination of measurements

Summary