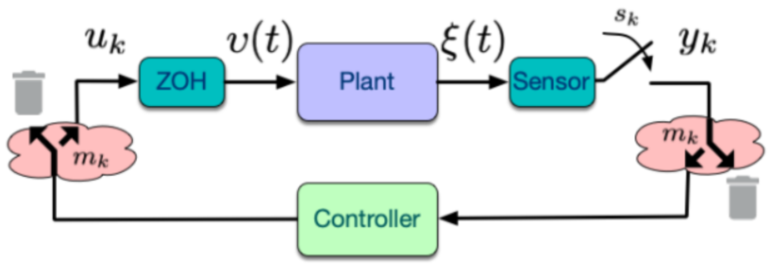
02_NCS with packet losses and protocols

1. Modeling of NCS with packet losses



1.1. Assumptions

- No Delays
- Packet Losses Model:
  - $$m_k = \begin{cases} 0, & \text{if no packet loss at time } k \\ 1, & \text{if packet is lost at time } k \end{cases}$$
  - Assume two sides' m_k are synchronized

1.2. Controller Schemes

- Zero-order-hold: $v(t) = u_k, t \in [s_k, s_{k+1})$

1.2.1. To-Hold Mechanism (event-driven)

$$u_k = \begin{cases} -\bar{K}x_k, & \text{if } m_k = 0 \\ u_{k-1}, & \text{if } m_k = 1 \end{cases}$$

1.2.2. To-Zero Mechanism (time-driven)

$$u_k = \begin{cases} -\bar{K}x_k, & \text{if } m_k = 0 \\ 0, & \text{if } m_k = 1 \end{cases}$$

1.3. Process Modeling

Sampled-Data System, To-Zero Mechanism

With the controller $u_k = -\bar{K}x_k$ and the "to zero" mechanism:

$$x_{k+1} = F^{cl}_{m_k}(h)x_k = \begin{cases} (F(h) - G(h)\bar{K})x_k & =: F^{cl}_0(h)x_k \quad \text{if } m_k = 0 \\ F(h)x_k & =: F^{cl}_1(h)x_k \quad \text{if } m_k = 1 \end{cases}$$

Sampled-Data System, To-Hold mechanism

1.4. Models of Dropout Sequences

We need to model the "**discrete dynamics**" of the NCS:

$$m_{k+1} = f\left(m_k, m_{k-1}, \ldots\right)$$

Deterministic m Discrete Dynamics

Here we will use ω**-automation**

An automaton is a 5-tuple $M = (Q, \Sigma, \delta, Q_0, F)$, where

- Q is a finite or countable set of discrete states,
- Σ is a finite or countable set of discrete inputs, the input alphabet, $
- $ is the transition function,
- $Q_0 \subseteq Q$ is the set of **start states**, and
- $F \subseteq Q$ is the set of accept states.

Stochastic m Discrete Dynamics

1.5. Model of Control Systems

Deterministic Packet Loss

We consider a **Deterministic Special Case**: max. δ consecutive losses

$$m \vDash (0^* \circ (10)^* \circ (110)^* \circ (1110)^* \circ \cdots \circ (\overbrace{1 \cdots 1}^{\delta}0))^\omega .$$

So we can model a **closed-loop system-network** as:

- Discrete-time **switched system**

$$x_{I+1} = \tilde{F}^{cl}(h, h_l) x_I$$

- Packet-drops as extensions of sampling interval:

$$h_l \in \{h, 2h, \ldots, (\delta+1)h\}$$

- State $x_l := x(\bar{s}_l)$, $\bar{s}_{l+1} = \bar{s}_l + h_l$
- > ***So we transfer deterministic packet loss to time-varying system***
    - With the "to zero" strategy:

    $$x_{l+1} = \left(e^{Ah_l} - e^{A(h_1-h)}\int_0^h e^{As} B\bar{K} ds\right) x_l$$

    Here the actual process is:

        - First calculate after interval h: $x_l^{'} = \left(e^{Ah} - \int_0^h e^{As} B\bar{K} ds\right) x_l$
        - For time interval: $[t+h, t+h_l)$, control signal become zeros, so we have $x_{l+1} = e^{A(h_l-h)} x_l^{'}$

Stochastic Packet Loss

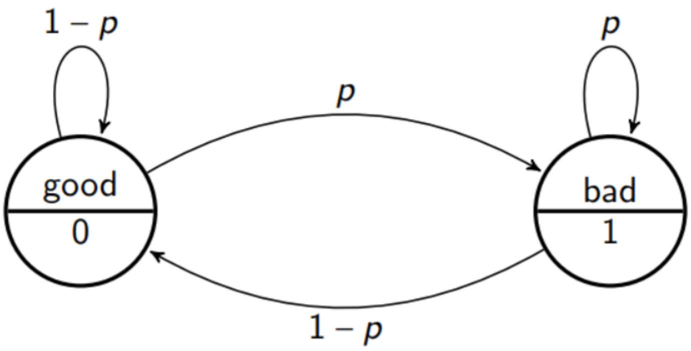
Here we use **Markov Decision Process** to model it

A Markov Decision Process (MDP) is a tuple $(\Sigma, \Gamma, Q, q_0, \delta, H)$, where:

- Σ is an input alphabet (finite non-empty set of symbols); Γ is an output alphabet (possibly, and often a subset of $\mathbb{R}$)
- Q is a finite, non-empty, set of states;
- $q_0 \in \mathbb{P}(Q)$ is the initial probability distribution of the states;
- $\delta : Q \times \Sigma \times Q \to [0, 1]$ is the state-transition relation;

- $H : Q \times \Sigma \times Q \to \Gamma$ is the output map or reward function.

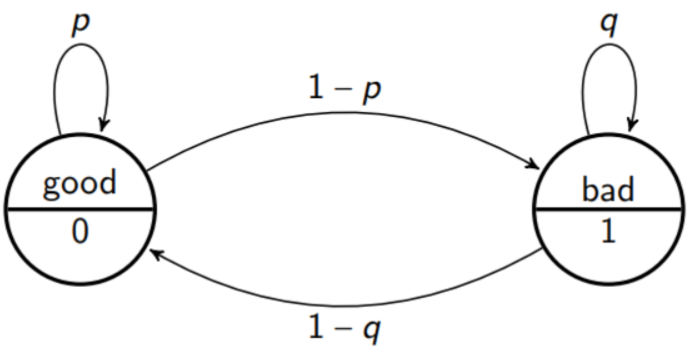
Bernoulli Model



Finite State $m_k \in \{0, 1\}$

$$\mathbb{P}\left(m_k = 1\right) = p$$

Gilbert-Elliot Model



Finite State $m_k \in \{0, 1\}$

$$\mathbb{P}\left(m_k = 0 \mid m_{k-1} = 0\right) = p_{00} = p, \mathbb{P}\left(m_k = 1 \mid m_{k-1} = 0\right) = p_{01} = 1 - p$$
$$\mathbb{P}\left(m_k = 1 \mid m_{k-1} = 1\right) = p_{11} = q, \mathbb{P}\left(m_k = 0 \mid m_{k-1} = 1\right) = p_{10} = 1 - q$$

2. Stability of NCS with Packet Loss

2.1. Deterministic Packet Loss

Total Model

$$h_l \in \{h, 2h, \ldots, (\delta+1)h\}$$

$$x_l := x\left(\bar{s}_l\right), \bar{s}_{l+1} = \bar{s}_l + h_l$$

- > ***we transfer deterministic packet loss to time-varying system***

    - With the "to zero" strategy:

$$x_{l+1} = \left(e^{Ah_l} - e^{A(h_1-h)} \int_0^h e^{As} B\bar{K} ds\right) x_l$$

Stability Analysis

> *Bounded Inter-sample Gain* $+$ ***Discrete-time system stability*** $\Rightarrow$ *Sampled-data stability*

Because:

$$h_l \in \mathcal{H} \Rightarrow h_l < \infty \Rightarrow \text{Bounded Inter-sample Gain}$$

2.2. Stochastic Stability Notions

Mean Square Stability(MSS), Exponentially MSS (EMSS), Uniformly EMSS(UEMSS)

A stochastic system is said to be **mean square stable (MSS)** if for all x_0 and all m_{-1} :

$$\lim_{k\to\infty} \mathbb{E}\left[\|x_k\|^2\right] = 0$$

If furthermore:

- $\mathbb{E}\left[\|x_k\|^2\right] \leq c\rho^k \|x_0\|^2$ for some $c \geq 0$ and $\rho \in [0, 1)$ it is said to be **Exponentially MSS (EMSS),**
- if c, ρ are independent of x_0 and m_{-1}, it is said to be **Uniformly EMSS (UEMSS)**

> *MSS: The final state's norm has a average 0*
>
> *EMSS: The state's norm 's average is bounded by exponential functions*
>
> *UEMSS: The state's norm's average is bounded by an unique exponential function*

Stochastic Stability (SS)

A stochastic system is said to be **stochastically stable (SS)** if for all x_0 and all m_{-1} :

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \|x_k\|^2\right] < \infty$$

> *The average energy of the systems is finite*

Almost Surely Stable (ASS)

A stochastic system is said to be **almost surely stable (ASS)** if for all x_0 and all m_{-1} :

$$\mathbb{P}\left[\lim_{k\to\infty} \|x_k\|\right] = 0.$$
$$= P\left[\lim_{k\to\infty} \|x_k\| = 0\right] = 1$$

Relation

For Markovian Jump Linear Systems (MJLS) the following holds:

$$\text{MSS} \Leftrightarrow SS \Leftrightarrow \text{EMSS} \Leftrightarrow \text{UEMSS} \Rightarrow \text{ASS}$$

> *ASS means the system has countable number of situation that may lead to a non-stable sequence (not converge to 0)*
>
> *MSS means the system has countable number of situation that may bead to a non-stable sequence and the non-stable sequence has a bounded deviation from fixed point.*

2.3. Stability of NCS with Stochastic Packet Loss

Overall Model

$$x_{k+1} = F^{cl}_{m_k}(h) x_k, \quad \mathbb{P}\left(m_k = j \mid m_{k-1} = i\right) = p_{ij}$$

Switched system, m controls **switching stochastically** (Markovian) : **Markovian Jump Linear System**

Stability Analysis (MJLS MSS)

A Markovian Jump Linear System:

$$x_{k+1} = A_{m_k} x_k, \quad \mathbb{P}\left(m_k = j \mid m_{k-1} = i\right) = p_{ij}$$

is **Mean Square Stable (MSS)** if there **exist** $P_i > 0, i = 0, \ldots, N$ that satisfy **any of** the following conditions:

- $P_i - A_i^T \left(\sum_{j=0}^N p_{ij} P_j\right) A_i > 0 \quad \text{for all } i = 0, \ldots, N$
- $P_j - A_j \left(\sum_{i=0}^N p_{ij} P_i\right) A_j^T > 0 \quad \text{for all } j = 0, \ldots, N$
- $P_i - \sum_{j=0}^N p_{ij} A_j^T P_j A_j > 0 \;\; \text{for all } i = 0, \ldots, N$
- $P_j - \sum_{i=0}^N p_{ij} A_i P_i A_i^T > 0 \quad \text{for all } j = 0, \ldots, N$

Note: In the case we consider $N = 1$, as $m_k \in \{0, 1\}$.

<u>**Understanding**</u>:

$$P_i - \sum_{j=0}^N p_{ij} A_j^T P_j A_j > 0 \quad \text{for all } i = 0, \ldots, N$$

$\Rightarrow$ The Lyapunov function $V\left(x_k, m_{k-1}\right) = x_k^T P_{m_{k-1}} x_k$ satisfies:

$$\mathbb{E}\left[V\left(x_{k+1}, x_k\right) \mid x_k\right] < V\left(x_k, m_{k-1}\right), \forall x_k \neq 0$$

For Bernoulli Probabilistic Transitions

$$x_{k+1} = A_{m_k}(h)x_k, \quad \mathbb{P}\left(m_k = j \mid m_{k-1} = i\right) = p_j$$

there exists $P > 0$ such that:

$$P_0 - (1-p)A_0^T P_0 A_0 - pA_1^T P_1 A_1 > 0$$
$$P_1 - pA_1^T P_1 A_1 - (1-p)A_0^T P_0 A_0 > 0$$

$\Leftrightarrow$ there exists $P > 0$ such that:

$$P - (1-p)A_0^T P A_0 - pA_1^T P A_1 > 0$$

For Gilbert-Elliot Model

$$P_i - \sum_{j=0}^{N} p_{ij} A_j^T P_j A_j > 0, \forall i, j \in \{0, 1\}$$

$\Leftrightarrow$

$$P_0 - pA_0^T P_0 A_0 - (1-p)A_1^T P_1 A_1 > 0$$
$$P_1 - qA_1^T P_1 A_1 - (1-q)A_0^T P_0 A_0 > 0$$

For B-Model and GE-Model with To-Zero Mechanism

MSS **if and only if**

- (Bernoulli) there exists $P > 0$ such that:

$$P - (1-p)F_0^{c/T} P F_0^{cl} - pF_1^{c/T} P F_1^{cl} > 0$$

- (Gilbert-Elliot) there exist $P_0, P_1 > 0$ such that:

$$P_0 - pF_0^{c/T} P_0 F_0^{cl} - (1-p)F_1^{c/T} P_1 F_1^{cl} > 0$$
$$P_1 - qF_1^{c/T} P_1 F_1^{cl} - (1-q)F_0^{c/T} P_0 F_0^{cl} > 0$$

where:

$$F(h) := e^{Ah}, \quad G(h) := \int_0^h e^{As} B ds$$
$$F_1^{cl} := F(h), \quad F_0^{cl} := (F(h) - G(h)\bar{K})$$

Almost Sure Stability Theorem for Bernoulli Model

The system:

$$x_{k+1} = e^{Ah} x_k + \int_0^h e^{As} B ds u_k$$

with the controller $u_k = -\bar{K} x_k$, "to zero" mechanism, and Bernoulli packet losses is ASS $\left(\mathbb{P}\left[\lim_{k\to\infty} \|x_k\| = 0\right] = 1\right)$ if there exists a $V : \mathbb{R}^n \to \mathbb{R}^+$ such that
$c_1 \|x\|^r \leq V(x) \leq c_2 \|x\|^r$, with $r \geq 1, 0 < c_1 \leq c_2$,
satisfying:

- $V\left(F_0^{cl} x\right) \leq \lambda V(x)$ for $\lambda \in [0, 1)$,
- $V\left(F_1^{cl} x\right) \leq L V(x)$ for some $L \geq 0$,
- $\lambda^{(1-p)} L^p < 1$.

> *Can be partly increase when loss happen, however, the average effect of decrease and increase should smaller than 1*

3. NCS with communication constraints: Protocol
