02_Fuzzy Systems and Fuzzy Clustering

1. Fuzzy Systems

1.1. Linguistic （Mamdani）fuzzy model

$$\text{if } x \text{ is } A \text{ then } y \text{ is } B$$
$$x \text{ is } A \rightarrow \text{antecedent(fuzzy proposition)}$$
$$y \text{ is } B \rightarrow \text{consequent(fuzzy proposition)}$$

1.2. Fuzzy Relational Model

$$\text{if } x \text{ is } A \text{ then } y \text{ is } B_1(0.1), B_2(0.8)$$

1.3. Takagi-sugeno Fuzz model

$$\text{if } x \text{ is } A \text{ then } y = f(x)$$

2. Mandani Fuzzy Systems

Given the if-then **rules** and an **input** fuzzy set, deduce the **corresponding output fuzzy set**

2.1. Fuzzy implication and Conjunctions

$$R : [0,1] \times [0,1] \rightarrow [0,1]$$
$$\mu_R(x,y) = \mathrm{I}(\mu_A(x), \mu_B(y))$$

"classical"	Kleene–Diene	$\mathrm{I}(a,b) = \max(1-a, b)$	implication
	Łukasiewicz	$\mathrm{I}(a,b) = \min(1, 1-a+b)$	
T-norms	Mamdani	$\mathrm{I}(a,b) = \min(a,b)$	conjunctions
	Larsen	$\mathrm{I}(a,b) = a \cdot b$	

same value as classical

2.2. Normal Inference

Inference with One Rule

1. construct implication relation

$$\mu_R(x,y) = \mathrm{I}(\mu_A(x), \mu_B(y))$$

2. use relational composition to derive B' from A'

$$B' = A' \circ R$$

Inference with Several Rules

1. Construct implication relation for each rule i

$$\mu_{R_i}(x,y) = \mathrm{I}(\mu_{A_i}(x), \mu_{B_i}(y))$$

2. Aggregate relations R_i into one

$$\mu_R(x,y) = \mathrm{aggr}(\mu_{R_i}(x,y))$$
$$\text{The aggr operator is the minimum for implications and the maximum for conjunctions}$$

3. Use relational composition to derive B' from A'

$$B' = A' \circ R$$

2.3. A simplier way: Mamdani Inference

1. **Compute the match** between the input and the antecedent membership functions (degree of fulfillment)
2. **Clip** the corresponding output fuzzy set for each rule by using the degree of fulfillment.
3. **Aggregate** output fuzzy sets of all the rules into one fuzzy set.

It can be seen as :

1. first finding the highest point (maximum fulfillment)
2. Then all inference result should less than the result of maximum fulfilment and at the same time meet the property of initial result.

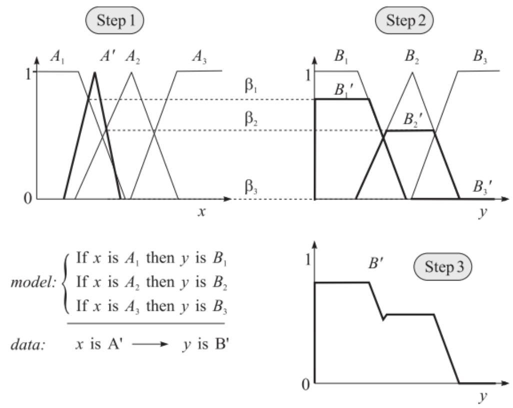


Figure 3.8.: A schematic representation of the Mamdani inference algorithm.

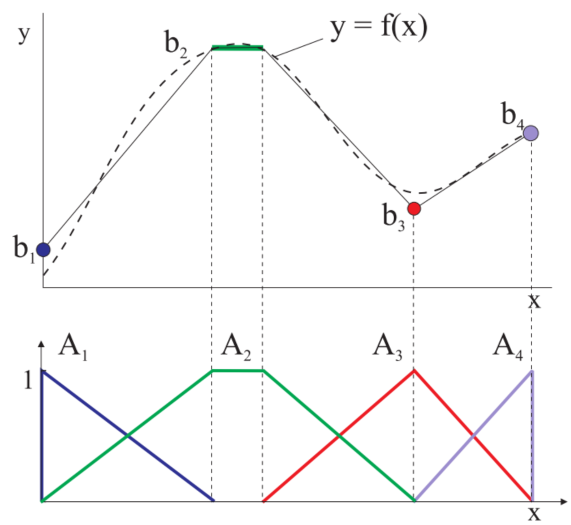
3. Singleton and Takagi-Sugeno Fuzzy System

3.1. Singletion Fuzzy model

$$\text{If } x \text{ is } A_i \text{ then } y = b_i$$

- Defuzzication/Infererence:

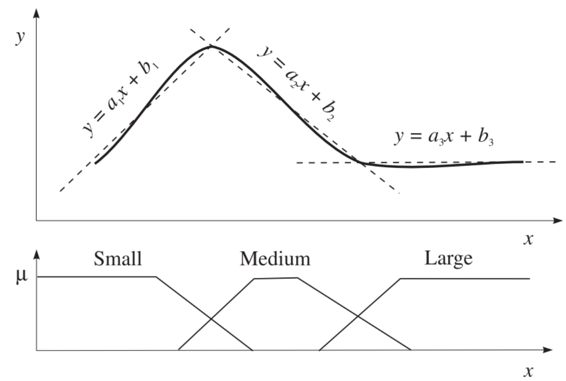
$$y = \frac{\sum_{i=1}^{K} \mu_{A_i}(x) b_i}{\sum_{i=1}^{K} \mu_{A_i}(x)}$$



3.2. Takagi-Sugeno (TS) Fuzzy model

$$\text{If } x \text{ is } A_i \text{ then } y_i = a_i x + b_i$$

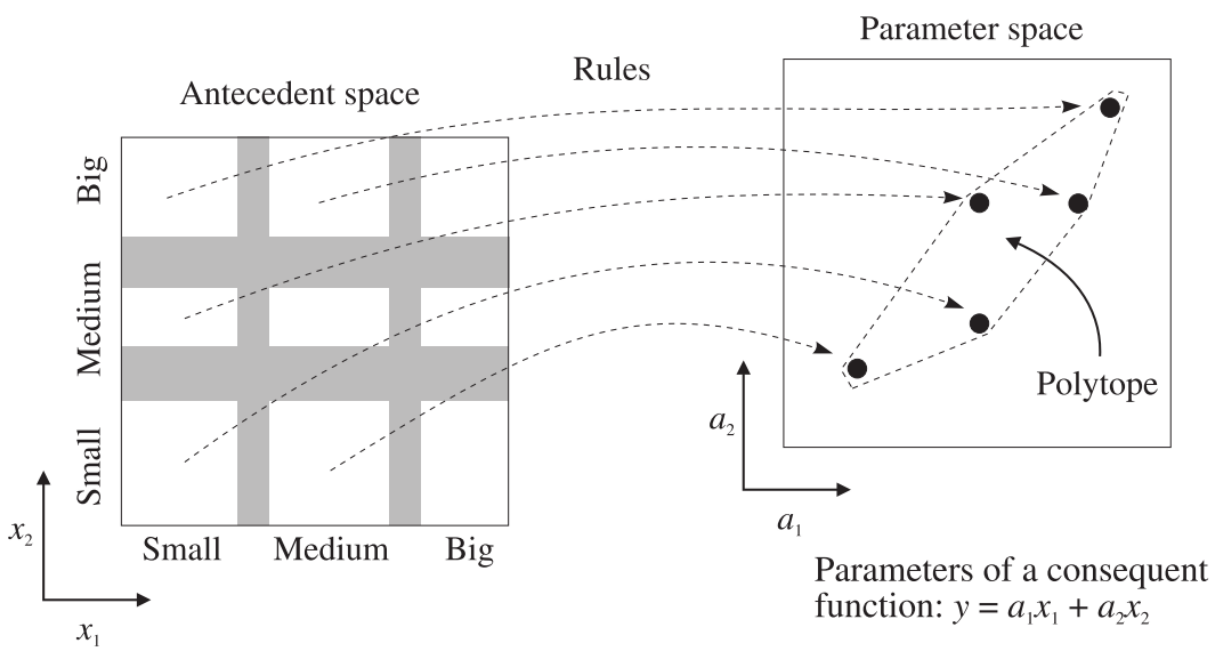
$$y = \frac{\sum_{i=1}^{K} \mu_{A_i}(x) y_i}{\sum_{i=1}^{K} \mu_{A_i}(x)} = \frac{\sum_{i=1}^{K} \mu_{A_i}(x) \left(a_i x + b_i\right)}{\sum_{i=1}^{K} \mu_{A_i}(x)}$$



- Quasi-Linear Property

$$y = \underbrace{\left(\sum_{i=1}^{K} \gamma_i(\boldsymbol{x})\boldsymbol{a}_i^T\right)}_{\boldsymbol{a}(\mathbf{x})^T}\boldsymbol{x} + \underbrace{\sum_{i=1}^{K} \gamma_i(\boldsymbol{x})b_i}_{b(\boldsymbol{x})}$$

linear in parameters a_i and b_i, pseudo-linear in x (LPV)

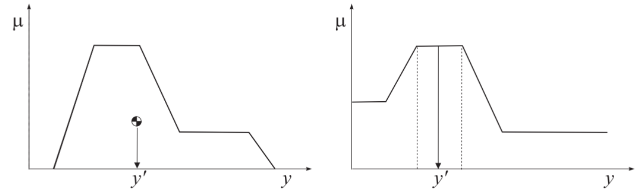


4. Defuzzification

Defuzzication convert a fuzzy set to a crisp value

- Center of Gravity Method:

$$y' = \frac{\sum_{j=1}^{F} \mu_{B'}(y_j) y_j}{\sum_{j=1}^{F} \mu_{B'}(y_j)}$$



(a) center of gravity (b) mean of maxima

5. Fuzzy Clustering

5.1. Problem Model

Given:

$$z_k = [z_{1k}, z_{2k}, \ldots, z_{nk}]^T \in \mathbb{R}^n, \quad k = 1, \ldots, N$$

Find:

- the fuzzy partition matrix

$$\boldsymbol{U} = \begin{bmatrix} \mu_{11} & \ldots & \mu_{1k} & \ldots & \mu_{1N} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ \mu_{c1} & \ldots & \mu_{ck} & \ldots & \mu_{cN} \end{bmatrix}$$

- the cluster centers

$$\boldsymbol{V} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_c\}, \quad \boldsymbol{v}_i \in \mathbb{R}^n$$

5.2. An optimization Approah

$$J(Z; V, U, A) = \sum_{i=1}^{c} \sum_{j=1}^{N} \mu_{i,j}^m d_{A_i}^2 (z_j, v_i)$$

$$\text{s.t.}$$

$$0 \le \mu_{i,j} \le 1, \quad i = 1, \ldots, c, j = 1, \ldots, N \quad \text{(membership degree)}$$

$$0 < \sum_{j=1}^{N} \mu_{i,j} < N, \quad i = 1, \ldots, c \quad \text{(no cluster empty)}$$

$$\sum_{i=1}^{c} \mu_{i,j} = 1, \quad j = 1, \ldots, N \text{ (total membership)}$$

Distance Matrics

- **Euclidean norm**

$$d^2(z_j, v_i) = (z_j - v_i)^T (z_j - v_i)$$

- **Inner-product norm**

$$d_{A_i}^2(z_j, v_i) = (z_j - v_i)^T A_i (z_j - v_i)$$

5.3. Fuzzy c-Means Algorithm

Repeat:

1. Compute cluster prototypes (means): $v_i = \frac{\sum_{k=1}^N \mu_{i,k}^m z_k}{\sum_{k=1}^N \mu_{i,k}^m}$
2. Calculate distances: $d_{ik} = (\mathbf{z}_k - \mathbf{v}_i)^T (\mathbf{z}_k - \mathbf{v}_i)$
3. Update partition matrix: $\mu_{ik} = \frac{1}{\sum_{j=1}^c (d_{ik}/d_{jk})^{1/(m-1)}}$

until $\|\Delta \boldsymbol{U}\| < \epsilon$