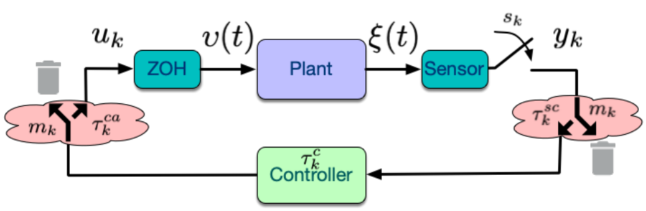
NCS with Multiple Imperfections and ETC

1. NCS with Generalized Imperfections



1.1. Assumptions

- Time-Varying Sampling:

$$h_k = (s_{k+1} - s_k) \in [h_{\min}, h_{\max}]$$

- Time-Varying Dealys

$$\tau_k = (\tau_k^{sc} + \tau_k^c + \tau_k^{ca}) \in [\tau_{\min}, \tau_{\max}]$$

- Packet Losses

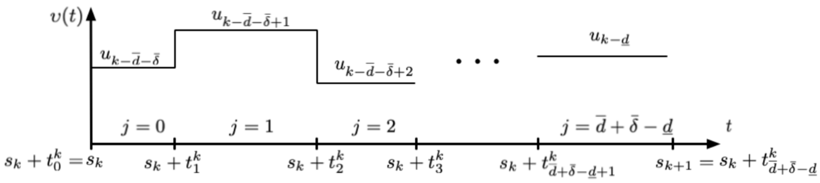
$$m_k = \begin{cases} 0, & \text{if no packet loss at time } k \\ 1, & \text{if packet is lost at time } k \end{cases}$$

  - with maximum $\bar{\delta}$ consecutive dropouts:

$$\sum_{v=k-\delta}^{k} m_c \leq \bar{\delta}$$

> *Continuous Uncertainty Sets*

1.2. Modeling



$$\underline{d}=\left\lfloor\frac{\tau_{\min}}{h_{\max}}\right\rfloor, \bar{d}=\left\lceil\frac{\tau_{\max}}{h_{\min}}\right\rceil$$

So, we have the state space model:

$$\begin{aligned}&\dot{\xi}(t)=A\xi(t)+Bv(t)\\&v(t)=u_{k+j-\bar{d}-\bar{\delta}}, \qquad \text{for } t\in\left[s_k+t_j^k, s_k+t_{j+1}^k\right)\end{aligned}$$

Deterministic

One can explicitly compute the actuation update times:

$$t_j^k=\phi\left(h_k,\tau_k,m_k,h_{k-1},\tau_{k-1},m_{k-1},\ldots\right).$$

and bounds $t_j^k \in [t_{j,min}, t_{j,\max}]$

$$t_{j,\min/\max}=\phi_{\min/\max}\left(\tau_{\min},\tau_{\max},h_{\min},h_{\max},\bar{\delta}\right).$$

General NCS model

$$\begin{aligned}x_{k+1}=&\Lambda\left(\theta_k\right)x_k+ && \text{Current state}\\&+M_{\bar{d}+\bar{\delta}-1}\left(\theta_k\right)u_{k-1}+\cdots+M_0\left(\theta_k\right)u_{k-\bar{d}-\bar{\delta}^+} && \text{Past Control Inputs}\\&+M_{\bar{d}+\bar{\delta}}\left(\theta_k\right)u_k && \text{Current Control Input}\end{aligned}$$

where

$$\Lambda\left(\theta_k\right)=e^{Ah_k}, M_j\left(\theta_k\right)=\begin{cases}\int_{h_k-t_{j+1}^k}^{h_k-t_j^k} e^{As}dsB & \text{if } \quad 0\leq j\leq\bar{d}+\bar{\delta}-\underline{d}\\ \qquad 0 & \text{if } \bar{d}+\bar{\delta}-\underline{d}<j\leq\bar{d}+\bar{\delta}\end{cases}$$

$$\theta_k:=\left(h_k,t_1^k,\ldots,t_{\frac{d}{d}+\bar{\delta}-\underline{d}}^k\right)$$

Extended State Model

$$x_k^e=\left[x_k^T u_{k-1}^T\ldots u_{k-\bar{d}-\bar{\delta}}^T\right]^T$$

If we consider a linear controller $u_k=-\bar{K}x_k$, then we will have $x_{k+1}^e=H(\theta_k)x_k^e$

$$
H(\theta_k) := \begin{bmatrix} \Lambda(\theta_k) - M_{\bar{d}+\bar{\delta}}(\theta_k)\bar{K} & M_{\bar{d}+\bar{\delta}-1}(\theta_k) & M_{\bar{d}+\bar{\delta}-2}(\theta_k) & \dots & M_1(\theta_k) & M_0(\theta_k) \\ -\bar{K} & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

1.3. Stability (Common Lyapunov Function)

The closed-loop NCS is Globally Asymptotically Stable if there exists $P > 0, \gamma \in (0,1)$ such that:

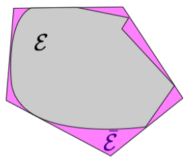
$$
H^T(\theta_k) P H(\theta_k) - P \leq -\gamma P, \quad \forall \theta_k \in \Theta
$$

However, there is a large problem: because θ_k belongs to a continuous set, so the LMIs set is a **infinite set**

2. Stability of General NCS model

2.1. From an infinite set to a finite set of LMIs

Using Approximation:



Embedding in **convex polytopic sets** $\overline{\mathcal{E}}$ the uncertain terms:

$$
\mathcal{E} := \left\{ \int_0^{h-\tau} e^{As} ds \mid \tau \in [\tau_{\min}, \tau_{\max}], h \in [h_{\min}, h_{\max}] \right\} \subseteq \overline{\mathcal{E}}
$$
$$
\mathcal{E} \subseteq \overline{\mathcal{E}} := \left\{ \sum_{i=1}^N \lambda_i E_i \mid \lambda_i \geq 0, \sum_{i=1}^N \lambda_i = 1 \right\}
$$

2.2. Polytopic Dynamical Model

Consider now the case we have at hand, an LPV system:

$$
x_{k+1}^e = H(\theta_k) x_k^e
$$

with matrix uncertainty set $\mathcal{H} = \{H(\theta) \mid \theta \in \Theta\}, |\Theta| = \infty$ (continuous).

By use the above method, we can use a polytopic overapproximation $\bar{\mathcal{H}}$

$$
x_{k+1}^e = \left(\sum_{i=1}^N \lambda_i H_i \right) x_k^e, \quad \lambda_i \geq 0, \sum_{i=1}^N \lambda_i = 1
$$

2.3. Stability of Polytopic Model

The polytopic model:

$$x_{k+1}^e = \left(\sum_{i=1}^N \lambda_i H_i\right) x_k^e, \quad \lambda_i \geq 0, \sum_{i=1}^N \lambda_i = 1$$

is globally asymptotically stable **if there exists** $P = P^T > 0, \gamma \in (0,1)$ such that the following finite set of LMIs are satisfied:

$$H_i^T P H_i - P \leq -\gamma P, \forall i = 1, \ldots, N$$

> *H_i is actually the vertex of the polytopic*
>
> - *This theorem allows us to reduce an infinite set of LMIs to a finite set of N LMIs*
> - *By making the overapproximation area tighter and tighter, we will gradually have a more "iff" condition*
> - *And then, there will be a **precision-complexity trade-off***

2.4. Over-Approximate use Real-Jordan Form

Jordan-Form

The Jordan normal form transformation of a matrix $A \in \mathbb{R}^{n \times n}$ is given by:

$$A = Q^{-1} J Q$$

with $Q \in \mathbb{R}^{n \times n}$ a matrix with the generalized eigenvectors of A as columns; and (assuming p distinct real eigenvalues):

$$\begin{aligned} & J = \operatorname{diag}\left(J_1, \ldots, J_p\right), \text{ with} \\ & J_i = \lambda_i, \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}, \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}, \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix} \end{aligned}$$

And the exponential of $A = Q^{-1} J Q$ is given by:

$$\begin{aligned} & e^{As} = Q^{-1} e^{Js} Q = Q^{-1} \operatorname{diag}\left(e^{J_1}, \ldots, e^{J_p}\right) Q, \text{ with} \\ & e^{J_i s} = e^{\lambda_i s}, e^{\lambda_i s} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, e^{\lambda_i s} \begin{bmatrix} 1 & s & \frac{s^2}{2!} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}, e^{\lambda_i s} \begin{bmatrix} 1 & s & \frac{s^2}{2!} & \ldots & \frac{s^{k-1}}{(k-1)!} \\ 0 & 1 & s & \ldots & \frac{s^{k-2}}{(k-2)!} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & s \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \end{aligned}$$

resulting (in the case of real eigenvalues) in:

$$e^{As} = Q^{-1} \left(\sum_{i=1}^{p} \sum_{j=0}^{q_i - 1} \frac{s^j}{j!} e^{\lambda_i s} S_{i,j} \right) Q$$

with q_i the number of time-varying parameters associated to λ_i.

Application of Jordan Form

For a given system (for example with only uncertain delays):

$$x_{k+1}^e = \begin{bmatrix} e^{Ah} & \int_0^{h-\tau_k} e^{As} ds B \\ 0 & 0 \end{bmatrix} x_k^e + \begin{bmatrix} \int_{h-\tau_k}^h e^{As} ds B \\ I \end{bmatrix} u_k =: F(\tau_k) x_k^e + G(\tau_k) u_k$$

Use Jordan Form we will get:

$$x_{k+1}^e = \left(F_0 + \sum_{i=1}^{r} \alpha_i\left(\tau_k\right) F_i\right) x_k^e + \left(G_0 + \sum_{i=1}^{r} \alpha_i\left(\tau_k\right) G_i\right) u_k$$

where $\alpha_i\left(\tau_k\right) = \frac{(h-\tau_k)^j}{j!} e^{\lambda_i(h-\tau_k)}$: by using Integration by parts, there are a lot of offset, then we will get this one

Polytopic Overapproximation

For above example, we now have:

$$\mathcal{F} = \left\{F_0 + \sum_{i=1}^{r} \alpha_i(\tau) F_i \mid \tau \in \left[\tau_{\min}, \tau_{\max}\right]\right\}$$
$$\mathcal{G} = \left\{G_0 + \sum_{i=1}^{r} \alpha_i(\tau) G_i \mid \tau \in \left[\tau_{\min}, \tau_{\max}\right]\right\}$$

Then we can obtain $\overline{\mathcal{F}} \supseteq \mathcal{F}$ and $\overline{\mathcal{G}} \supseteq \mathcal{G}$, by using:
$\bar{\alpha}_i = \max_{\tau \in [\tau_{\min}, \tau_{\max}]} \alpha_i(\tau)$ and $\underline{\alpha}_i = \min_{\tau \in [\tau_{\min}, \tau_{\max}]} \alpha_i(\tau)$

$$\overline{\mathcal{F}} = \left\{F_0 + \sum_{i=1}^{r} \delta_i F_i \mid \delta_i \in \left[\underline{\alpha}_i, \bar{\alpha}_i\right], i = 1, 2, \ldots, r\right\}$$
$$\overline{\mathcal{G}} = \left\{G_0 + \sum_{i=1}^{r} \delta_i G_i \mid \delta_i \in \left[\underline{\alpha}_i, \bar{\alpha}_i\right], i = 1, 2, \ldots, r\right\}$$

> *It is a **convex hull of a finite set of vertices (in the space of matrices)***
>
> *Then for LMIs, we can always **test the vertice**s instead of whole space*

2.5. Global Exponential Stability

Global exponential stability can be established if there exists

$$P = P^T > 0$$
$$\left(H_{F,j} - H_{G,j} K\right)^T P \left(H_{F,j} - H_{G,j} K\right) - P \leq -\gamma P, \quad \left(2^r + 1\right.$$

for all $\left(H_{F,j}, H_{G,j}\right) \in \mathcal{H}_{\mathcal{F}} \times \mathcal{H}_G$

2.6. An example

3. Other Reflection

One should always notes one point:

For a given NCS.

- If we proved the stability or design a stable controller, it means, under all circumstances considered by the model, the system always will accomplish the stability
- If we cannot proved the stability, that does not mean the system will definitely become unstable in one execution

4. Event-Triggered Control

4.1. Sampling Paradigms for Control

Time-triggered Control (TTC)

Sampling sequences **independent of plant's state**

Event-Triggered Control (ETC)

Sampling sequences **dependent of plant's state**:

- Sensor Measure Periodically, but may not send each time

$$
\begin{aligned}
& s_k=\Phi_{\mathrm{ETC}}\left(\xi, x_k, x_{k-1}, \ldots, x_0, s_{k-1}, \ldots, s_0\right) \\
& \text{e.g. } s_k=\inf \left\{t>s_{k-1} \left\|\xi(t)-x_{k-1}\right\|>\sigma\|\xi(t)\|\right\}
\end{aligned}
$$

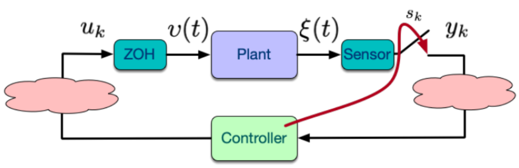
Periodic Event-Triggered Control (PETC)

Sampling sequences dependent of plant's state.

Sampling times subset of a periodic sampling sequence

$$
\begin{aligned}
& s_k=\Phi_{\mathrm{PETC}}\ \left(\xi, x_k, x_{k-1}, \ldots, x_0, s_{k-1}, \ldots, s_0\right) h, h \text{ constant}, \ \operatorname{Im}(f) \in \mathbb{N} \\
& \text{e.g. } s_k=\inf \left\{r h>s_{k-1} \mid\left\|\xi(r h)-x_{k-1}\right\|>\sigma\|\xi(r h)\|\right\}, r \in \mathbb{N}
\end{aligned}
$$

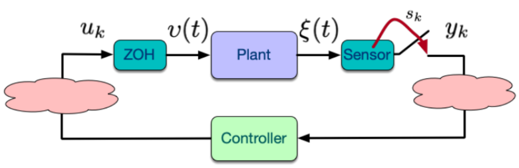
Self-Triggered Control (STC)



- Sampling sequences **dependent of previous sampled plant's state.**
- Sampling times often a **conservative prediction of those from ETC**.
- Sampling based on **information at the controller**.

$$
\begin{aligned}
& s_k=\Phi_{\mathrm{STC}}\left(x_k, x_{k-1}, \ldots, x_0, s_{k-1}, \ldots, s_0\right) \\
& \text{e.g. } s_k=h\left(x_k\right)
\end{aligned}
$$

4.2. Framework and Assumptions for Event-triggered Control



We will focus on **LTI plants:**

$$
\dot{\xi}(t)=A \xi(t)+B v(t)
$$

Controller is given: $v(t) = -K\xi(t)$, such that the continuous-time closed loop is GES.

- No delays and no quantization effects.
- Sensors measure continuously the state.

For the closed loop GES:

$\exists$ a Lyapunov function $V(x) = x^T Px$ with $P = P^T > 0$ such that

$$(A - BK)^T P(A - BK) = -Q$$

for some $Q > 0$.

$\frac{d}{dt}V(\xi(t)) = -x^T Qx$ can be regarded as a performance specification: a convergence rate

4.3. Objective

Determine a **triggering function** $s_k = \Phi_{\mathrm{ETC}}\ (\xi, x_k, x_{k-1}, \ldots, x_0, s_{k-1}, \ldots, s_0)$

such that:

- The resulting sampled-data closed-loop **remains GES**;
- A **minimum performance** can be specified;
- The time between events is always **lower bounded by a positive quantity**, i.e.:

$$s_{k+1} - s_k \geq \tau_{\min} > 0, \forall k$$

- If possible, implementation is **simple.**
- If possible, a **reduction on transmission**s can be guaranteed w.r.t. alternative implementations (e.g. TTC);

4.4. Controller Design

Minimum Performance

> *By finding above **triggered-condition**, we can guarantee that the performance criterion*
>
> - *Based on the **continuous model,** the system can **guarantee** a performance criterion*
> - *Based on triggered mode, although because of information latent, the origin performance may not be met, however, for a given expected performance, **we can meet it by adjust the triggered-condition (somewhat can be regarded as worst-case frequency or worst up-to-date extent of control signal)***

it is reasonable to **assume** that some **performance may need to be sacrificed**, then we can let the desired performance be:

$$\frac{d}{dt}V(\xi(t)) \leq -\sigma\xi(t)^T Q\xi(t)$$

with $\sigma \in (0, 1)$

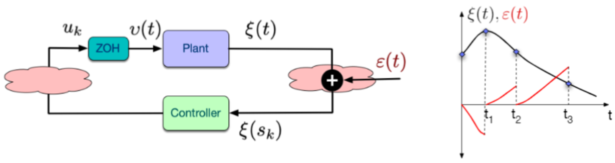
By using state-space function, we can transform it into:

$$\xi(t)^T \left(A^T P + PA\right)\xi(t) - \xi(t)^T PBK\xi\left(s_k\right) - \xi(s_k)^T (BK)^T P\xi(t) \leq -\sigma\xi(t)^T Q\xi(t)$$

Which means, the performance can be imposed by satisifies:

$$\phi\left(\xi(t), \xi\left(s_k\right)\right):=\begin{bmatrix}\xi(t)^T & \xi\left(s_k\right)^T\end{bmatrix}\begin{bmatrix}A^T P+P A+\sigma Q & -P B K \\ -(B K)^T P & 0\end{bmatrix}\begin{bmatrix}\xi(t) \\ \xi\left(s_k\right)\end{bmatrix} \leq 0 \Rightarrow s_{k+1}=\inf \left\{t>s_k \mid \phi\left(\xi(t), \xi\left(s_k\right.\right.\right.$$

Here, we use an extra variable $\epsilon(t)$ to represent **difference between last timestep and current time**



$$\begin{aligned}
\varepsilon(t) &= \xi\left(s_k\right)-\xi(t) \\
\dot{\xi}(t) &= (A-B K) \xi(t)-B K \varepsilon(t), \forall t \\
\dot{\varepsilon}(t) &= -(A-B K) \xi(t)+B K \varepsilon(t), \forall t \\
\varepsilon\left(s_k^{+}\right) &= 0
\end{aligned}$$

Then we can rewritten that:

$$-\xi(t)^T Q \xi(t)-\xi(t)^T P B K \varepsilon(t)-\varepsilon(t)^T(B K)^T P \xi(t) \leq-\sigma \xi(t)^T Q \xi(t)$$

So, the triggered threshold can be rewritten to:

$$\begin{gathered}
s_{k+1}=\inf \left\{t>s_k \mid \phi^e(\xi(t), \varepsilon(t)) \leq 0\right\}=: \Phi_{\mathrm{ETC}}^e(\xi, \varepsilon, s) \\
\phi^e(\xi(t), \varepsilon(t)):=\begin{bmatrix}\xi(t)^T & \varepsilon(t)^T\end{bmatrix}\begin{bmatrix}(1-\sigma) Q & P B K \\ (B K)^T P & 0\end{bmatrix}\begin{bmatrix}\xi(t) \\ \varepsilon(t)\end{bmatrix}
\end{gathered}$$

Minimum Sampling Time

> *With previous triggered-condition, we can also find a minimum interval time*

First, we transform previous performance condition

$$\begin{aligned}
& -\xi(t)^T Q \xi(t)-\xi(t)^T P B K \varepsilon(t)-\varepsilon(t)^T(B K)^T P \xi(t) \leq-\sigma \xi(t)^T Q \xi(t) \\
\Leftrightarrow & (1-\sigma) \xi(t)^T Q \xi(t)+2 \xi(t)^T P B K \varepsilon(t) \geq 0 \\
\Leftrightarrow & (1-\sigma) \xi(t)^T Q \xi(t) \geq-2 \xi(t)^T P B K \varepsilon(t)
\end{aligned}$$

And, it can be observed that:

$$\begin{aligned}
& (1-\sigma) \lambda_{\min }(Q)\|\xi(t)\|^2 \geq 2\|P B K\|\|\xi(t)\|\|\varepsilon(t)\| \\
\Leftrightarrow & (1-\sigma) \lambda_{\min }(Q)\|\xi(t)\| \geq 2\|P B K\|\|\varepsilon(t)\| \\
\Leftrightarrow & \tilde{\sigma}:=\frac{(1-\sigma) \lambda_{\min }(Q)}{2\|P B K\|} \geq \frac{\|\varepsilon(t)\|}{\|\xi(t)\|} \\
\Rightarrow & (1-\sigma) \xi(t)^T Q \xi(t) \geq-2 \xi(t)^T P B K \varepsilon(t)
\end{aligned}$$

Which means, the following triggering condition is **more conservative**

- **Conservative**: means in frequency, smaller time interval upper-bound

$$\begin{gathered}
\inf \left\{t>s_k \mid \tilde{\phi}^e(\xi(t), \varepsilon(t)) \leq 0\right\} \leq \inf \left\{t>s_k \mid \phi^e(\xi(t), \varepsilon(t)) \leq 0\right\} \\
\tilde{\phi}^e(\xi, \varepsilon):=\tilde{\sigma}\|\xi(t)\|-\|\varepsilon(t)\| \leq 0 \Leftrightarrow \frac{\|\varepsilon(t)\|}{\|\xi(t)\|} \geq \tilde{\sigma}
\end{gathered}$$

Here, we will show that there is a minimum inter-event time τ_{min},by showing there's a minimum time for $\varphi(t)=\frac{\|\varepsilon(t)\|}{\|\xi(t)\|}$to evolve from 0 to $\tilde{\sigma}$:

$$\varphi(t)=\frac{\|\varepsilon(t)\|}{\|\xi(t)\|}, \varphi(0)=0$$

Note that $\frac{d}{d t} \varepsilon(t)=-\frac{d}{d t} \xi(t)$

$$\frac{d}{dt}\varphi(t) = \frac{\sqrt{\varepsilon^T\varepsilon}}{\sqrt{\xi^T\xi}} = -\frac{\varepsilon^T\dot{\xi}}{\|\varepsilon\|\|\xi\|} - \frac{\xi^T\dot{\xi}}{\|\xi\|^2}\frac{\|\varepsilon\|}{\|\xi\|} \leq \frac{\|\dot{\xi}\|}{\|\xi\|} + \frac{\|\dot{\xi}\|}{\|\xi\|}\varphi \leq$$
$$\leq \|A - BK\| + (\|A - BK\| + \|BK\|)\varphi + \|BK\|\varphi^2$$

where we used $\|\dot{\xi}\| \leq \|A - BK\|\xi + \|BK\|\varepsilon$

because ϕ is bounded, so, the derivative is bounded, so there will always be a minimum interval

Comparing to TTC

- Accepts a trivial answer when **comparing with Periodic TTC** with constant h
- But in general may not hold for **more sophisticated TTC strategies**
- ETC is a **greedy** approach, that means it **only see 1-step forward**
    - Sometimes sampling a bit earlier gives long term reductions