Trace Operator

1. Definition

Definition:

$$\operatorname{tr}(\mathbf{A})=\sum_{i=1}^{n} a_{ii}=a_{11}+a_{22}+\cdots+a_{nn}$$

- the trace of a matrix is the **sum of its (complex) eigenvalues** (counted with multiplicities)

2. Properties

Basic Properties

$$\begin{aligned}\operatorname{tr}(\mathbf{A}+\mathbf{B}) &= \operatorname{tr}(\mathbf{A})+\operatorname{tr}(\mathbf{B}) \\ \operatorname{tr}(c\mathbf{A}) &= c\operatorname{tr}(\mathbf{A}) \\ \operatorname{tr}(\mathbf{A}) &= \operatorname{tr}\left(\mathbf{A}^{\top}\right) \\ \operatorname{tr}(\mathbf{A}\mathbf{B}) &= \operatorname{tr}(\mathbf{B}\mathbf{A}) \\ \operatorname{tr}\left(\mathbf{P}^{-1}(\mathbf{A}\mathbf{P})\right) &= \operatorname{tr}\left((\mathbf{A}\mathbf{P})\mathbf{P}^{-1}\right)=\operatorname{tr}(\mathbf{A})\end{aligned}$$

Trace of a Product

if A and B are two $m \times n$ matrix

$$\operatorname{tr}\left(\mathbf{A}^{\top}\mathbf{B}\right)=\operatorname{tr}\left(\mathbf{A}\mathbf{B}^{\top}\right)=\operatorname{tr}\left(\mathbf{B}^{\top}\mathbf{A}\right)=\operatorname{tr}\left(\mathbf{B}\mathbf{A}^{\top}\right)=\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}$$

Cyclic Property

$$\operatorname{tr}(\mathbf{ABCD})=\operatorname{tr}(\mathbf{BCDA})=\operatorname{tr}(\mathbf{CDAB})=\operatorname{tr}(\mathbf{DABC})$$

Trace of Kronecker Product

$$\operatorname{tr}(\mathbf{A}\otimes\mathbf{B})=\operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{B})$$

3. Derivatives

$$\frac{d}{d\boldsymbol{A}}(\mathrm{tr}(\boldsymbol{A}\boldsymbol{B})) = \boldsymbol{B}^T$$
$$\frac{d}{d\boldsymbol{A}}\left(\mathrm{tr}\left(\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{\boldsymbol{T}}\right)\right) = 2\boldsymbol{A}\boldsymbol{B}$$