CTMC

1. Course Notes

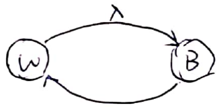
CTMC

- Model

Random Process $\{X(t), t\in[0,\infty)\}$ having countable states $S\subset\{0,1,2,\cdots\}$ and fulfilling Markovian property.

When the system reaches states i, it stays there for a random amount of time T_i, where T_i is an exponential random variable.

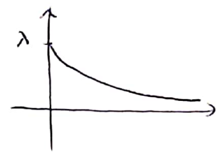
- Representation



- Mathematical Background

exponential distribution

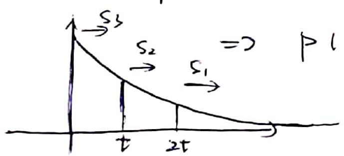
$$f(x;\lambda)=\begin{cases}\lambda e^{(\lambda x)} & x\geq 0\\ 0 & x<0\end{cases}$$



<1> Property:

① Memoryless

$$P(T>s+t\mid T>s)=P(T>t)$$



$$\Rightarrow P(T>(k+1)t\mid T>kt)=P(T>t)$$

$$\frac{S_1}{S_2}=\frac{S_2}{S_3}$$

∴ If we sample in discrete way: $t, 2t, \cdots, kt$.

It just looks the same as DTMC!

② Much devices in real life in fact exhibit constant failure rates

③ The exponential distribution can model time-to-failure for complex systems consisting of many components in series (but none of the individual components contribute significantly to the total failure density)

- Steady State of CTMC

• Method 1. Building upon DTMCs

① Steady state of DTMC: $\hat{\pi}=[\hat{\pi}_0,\cdots\hat{\pi}_m]$

② Assume $0<\sum_{k\in S}\frac{\hat{\pi}_k}{\lambda_k}<\infty$ → means $\lambda_k\neq 0$ → recurrent

③ $\pi_j=\lim_{t\to\infty}P(X(t)=j\mid X(0)=i)$

$$=\frac{\frac{\hat{\pi}_j}{\lambda_j}}{\sum_{k\in S}\frac{\hat{\pi}_k}{\lambda_k}}$$

• Method 2. State-Space Based:

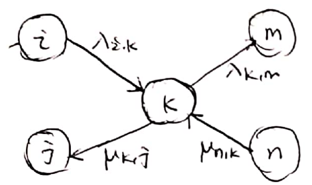
Assume: non-homogeneous CTMC

$$P_0 \xrightarrow{\lambda} P_1$$

$$\frac{dP_0}{dt} = -P_0 \cdot \underset{\sim}{\lambda}$$

$\hookrightarrow$ "rate parameter"

<1> Steady State flow



$$\frac{dP_k}{dt} = \sum_i \lambda_{i,k} \cdot P_i + \sum_n \mu_{n,k} P_n - \left(\sum_j \mu_{k,j} + \sum_m \lambda_{k,m}\right) P_k$$

<2> Steady State Condition

$$\frac{dP_k}{dt} = 0 \quad \text{for each } k$$

<3> Generation Matrix

$$g_{ij} = \begin{cases} \lambda_i P_{ij} & \text{if } i \neq j \\ -\lambda_i & \text{if } i = j \end{cases}$$

$$\therefore \frac{dP_k}{dt} = 0 \iff \Pi \cdot G = 0$$

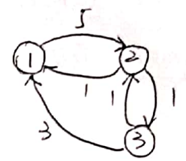
- Irreducibility and Aperiodicity

<1> A CTMC irreducible

$\Rightarrow$ steady state exists

Do not need aperiodical

- Example



$$G = \begin{bmatrix} -5 & 5 & 0 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix}$$

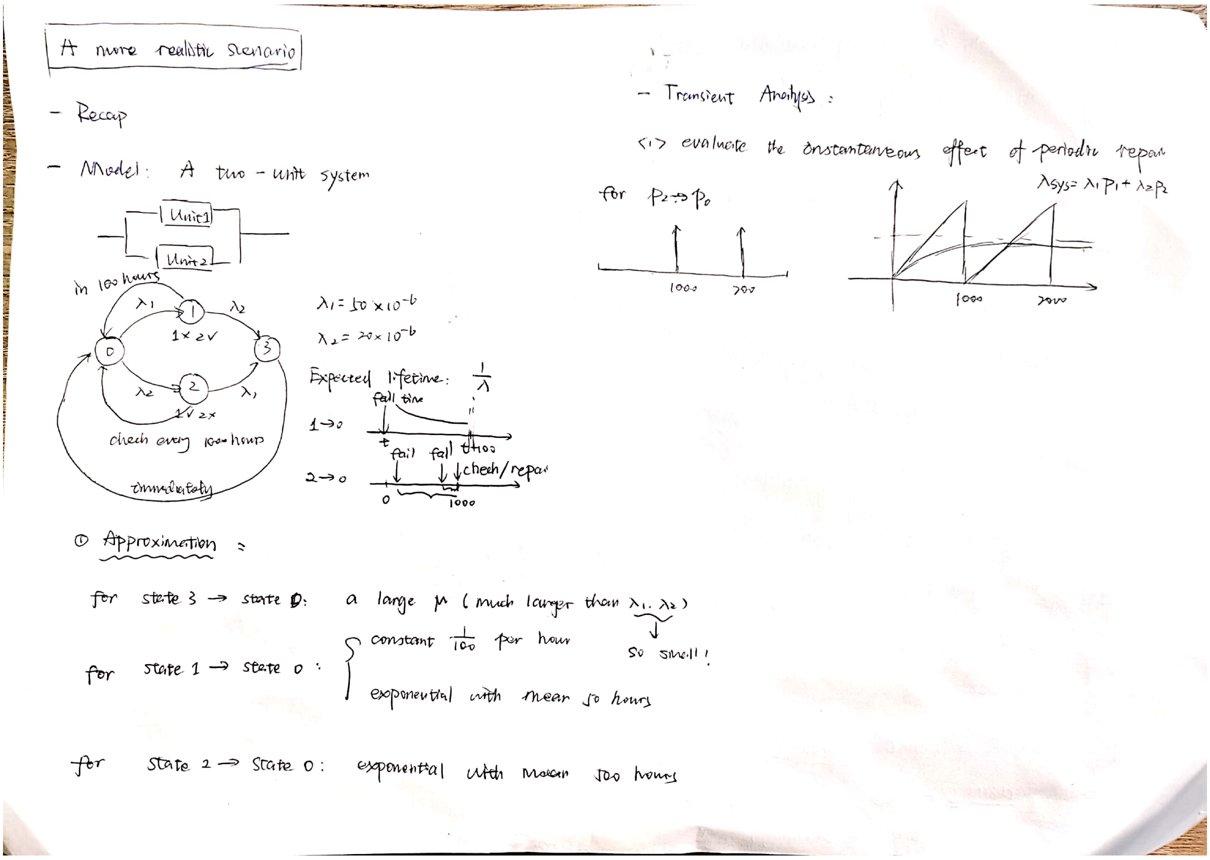
$$\therefore \Pi \cdot G = 0 \quad \& \quad \sum \Pi = 1$$

$$\therefore [\pi_1, \pi_2, \pi_3] \cdot \begin{bmatrix} -5 & 5 & 0 & 1 \\ 1 & -2 & 1 & 1 \\ 3 & 1 & -4 & 1 \end{bmatrix} = [0, 0, 0, 1]$$

$$\therefore \begin{bmatrix} -5 & 1 & 3 & 0 \\ 5 & -2 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 6 & 8 & 5 \\ 0 & -7 & -4 & -5 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 32 & 5 \\ 0 & 0 & -32 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 32 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\pi_3 = \frac{5}{32}, \quad \pi_2 = \frac{20}{32}, \quad \pi_1 = \frac{7}{32}$$



2. Introduction

2.1. Definition

A continuous-time Markov chain $X(t)$ is defined by two components: **a jump chain**, and a set of **holding time parameters** λi. The jump chain consists of a countable set of states $S \subset 0, 1, 2, \cdots$ along with transition probabilities p_{ij}. We assume p_{ii}=0, for all non-absorbing states $i \in S$. We assume

1. if $X(t) = i$, the time until the state changes has Exponential(λ_i) distribution
2. if $X(t) = i$, the next state will be j with probability p_{ij}

2.2. Inherent Property

Because of **Exponential Distribution,** The process inherently satisifies the Markov Property because of the **Memoryless** Property of Exponential Distribution.

for all $0 \leq t_1 < t_2 < \cdots < t_n < t_{n+1}$, we have

$$P\left(X\left(t_{n+1}\right)=j \mid X\left(t_n\right)=i, X\left(t_{n-1}\right)=i_{n-1}, \cdots, X\left(t_1\right)=i_1\right) \\ = P\left(X\left(t_{n+1}\right)=j \mid X\left(t_n\right)=i\right)$$

2.3. Transition Matrix

For a continuous-time Markov chain, we define the transition matrix $P(t)$. The (i, j)th entry of the transition matrix is given by

$$P_{ij}(t) = P(X(t) = j \mid X(0) = i)$$

So the transition matrix satisfies the following properties:

1. P(0) is equal to the identity matrix, $P(0) = I$
2. the rows of the transition matrix must sum to

   1

$$\sum_{j \in S} p_{ij}(t) = 1, \quad \text{for all } t \geq 0$$

3. for all $s, t \geq 0$:

$$P(s+t) = P(s)P(t)$$

3. Stationary and Limiting Distributions

3.1. Stationary Distribution

Let $X(t)$ be a continuous-time Markov chain with transition matrix $P(t)$ and state space $S = 0, 1, 2, \cdots$. A probability distribution π on S, i.e, a vector $\pi = [\pi_0, \pi_1, \pi_2, \cdots]$, where $\pi_i \in [0, 1]$ and

$$\sum_{i \in S} \pi_i = 1$$

is said to be a **stationary distribution** for $X(t)$ if

$$\pi = \pi P(t), \text{for all } t \geq 0$$

If a CTMC is irreducible $\Rightarrow$ Steady/Staionary Distributed exists

3.2. Limiting Distributions

The probability distribution $\pi = [\pi_0, \pi_1, \pi_2, \cdots]$ is called the limiting distribution of the continuous-time Markov chain $X(t)$ if for all $i, j \in S$

$$\pi_j = \lim_{t \to \infty} P(X(t) = j \mid X(0) = i)$$

and we have

$$\sum_{j \in S} \pi_i = 1$$

3.3. Method to obtain Limiting distributions

Let $X(t), t \geq 0$ be a continuous-time Markov chain with an irreducible positive recurrent jump chain. Suppose that the unique stationary distribution of the jump chain is given by

$$\tilde{\pi} = [\tilde{\pi}_0, \tilde{\pi}_1, \tilde{\pi}_2, \cdots].$$

Further assume that (means **no recurrent state**)

$$0 < \sum_{k \in S} \frac{\tilde{\pi}_k}{\lambda_k} < \infty.$$

Then

$$\pi_j = \lim_{t \to \infty} P(X(t) = j | X(0) = i) = \frac{\frac{\tilde{\pi}_j}{\lambda_j}}{\sum_{k \in S} \frac{\tilde{\pi}_k}{\lambda_k}}.$$

for all $i, j \in S$. That is, $\pi = [\pi 0, \pi 1, \pi 2, \cdots]$ is the **limiting distribution** of $X(t)$
.

4. Generation Matrix

4.1. Definition

For a continuous-time Markov chain, we define the **generator matrix** G. The (i, j) th entry of the transition matrix is given by

$$g_{ij} = \begin{cases} \lambda_i p_{ij} & \text{if } i \neq j \\ -\lambda_i & \text{if } i = j \end{cases}$$

4.2. Forward and Backward Equations

Forward and Backward Equations describes the relations between generation matrix and transition matrix. It is apparent through the derivative of exponenetial distribution.

The **forward equations** state that

$$P'(t) = P(t)G,$$

which is equivalent to

$$p'_{ij}(t) = \sum_{k \in S} g_{ik} p_{kj}(t), \text{ for all } i, j \in S.$$

The **backward equations** state that

$$P'(t) = GP(t),$$

which is equivalent to

$$p'_{ij}(t) = \sum_{k \in S} g_{ik} p_{kj}(t), \text{ for all } i, j \in S.$$

4.3. The second method to Obtain Statoinary Distribution

Consider a continuous Markov chain $X(t)$ with the state space S and the generator Matrix G. The probability distribution π on S is a stationary distribution for $X(t)$ if and only if it satisfies

$$\pi G = 0.$$

That means for each state: **flow in= flow out**

4.4. Transition Rate Diagram

A continuous-time Markov chain can be shown by its **transition rate diagram**. In this diagram:

1. the values g_{ij} are shown on the edges. The values of g_{ii} are not usually shown because they are implied by the other values, i.e.

$$g_{ii} = -\sum_{j \neq i} g_{ij}\,.$$