Introduction to Group Cohomology

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Contents

Introduction

This work is the author's Honours thesis as an undergraduate student at Monash University in Australia. Its purpose is to introduce cohomology groups assuming as background little more than group, ring and field theory. The first three chapters are devoted to building up the necessary machinery from homological algebra. This includes module theory, basic category theory and the theory of derived functors. In chapter four these concepts are then applied to group cohomology. Whilst group cohomology did not historically arise out of the theory of derived functors, the author favored this approach nonetheless as the extra work required to understand homological algebra seemed a fair price to pay for a deeper understanding.

The cohomology theory of groups arose from both topological and algebraic sources. As an example from algebra, suppose that the product of two matrices f and g is zero, i.e. $g \circ f = 0$. Suppose also that $g \cdot v = 0$ for some n-dimensional column vector v. It is then not always true that $v = f \cdot u$ for some vector u. However, the failure to be able to conclude so can be measured by the *defect*

$$d = \operatorname{nullity} g - \operatorname{rank} f = n - \operatorname{rank} f - \operatorname{rank} g.$$

In modern language, we would call d the *homology module* and write $H = \ker g / \operatorname{im} f$.

On the topological side of things, it was Hurewicz who first introduced higher homotopy groups $\pi_n(X)$ for $n \geq 2$ of a topological space X. Homology groups arise quite naturally in algebraic topology. We recall that a triangulation of a space X is a simplicial complex K together with a homeomorphism $\phi : K \to X$. Within such a complex, we can pass from higher dimensional simplices to lower dimensional ones through a so-called boundary operator ∂_n. Quite intuitively, the boundary operator ∂_1 would send a 1-simplex or line segment AB of a simplicial complex to the point $B - A$. Similarly, we send a 2-simplex or solid triangle ABC to its boundary, a line segment, $BC - AC + AB$ via ∂_2. If we denote the subspace of all n-simplices of $\mathbb{R}^n$ by S_n, we thus have a natural way of constructing a chain from any n-dimensional simplex to the empty set:

$$\mathbf{C} : \cdots \to S_n \xrightarrow{\partial_n} S_{n-1} \to \cdots \to S_1 \xrightarrow{\partial_1} S_0 \xrightarrow{\partial_0} 0.$$

In fact, with the boundary operator defined as we hint to above, we could show even more: $\partial_n \partial_{n+1} = 0$. This is important because it tells us that the image of ∂_{n+1} lies in the kernel of ∂_n which enables us to form the n^{th} homology group (in the sense of a topological group) $H_n(\mathbf{C}) = \ker \partial_n / \operatorname{im} \partial_{n+1}$. Note that this quotient again really measures a defect, namely by how much the boundary operator fails to be "exact", that is, by how much the image and the kernel of two successive boundary operators differ. The study of these defects is at the heart of homological algebra which is said to have begun with the cohomology of groups.

Chapter 1

Modules

In this chapter we review some background material from module theory needed in all subsequent chapters. For the entire article, we will let Λ denote a ring with unity element $1_\lambda \neq 0$. We begin with the definition of a Λ-module.

1.1 Λ-modules

A *left* Λ*-module* is an abelian group A together with a ring homomorphism $\omega : \Lambda \to \operatorname{End} A$ where $\operatorname{End} A$ is the ring of endomorphisms of A. We will talk of Λ operating on A from the left and denote this action by simply writing λa. The following rules are then satisfied for $a, a_1, a_2 \in A$ and $\lambda, \lambda_1, \lambda_2 \in \Lambda$:

$$\begin{array}{ll} \textbf{M1:} & (\lambda_1 + \lambda_2)a = \lambda_1 a + \lambda_2 a \\ \textbf{M2:} & (\lambda_1\lambda_2)a = \lambda_1(\lambda_2 a) \\ \textbf{M3:} & 1_\lambda a = a \\ \textbf{M4:} & \lambda(a_1 + a_2) = \lambda a_1 + \lambda a_2. \end{array}$$

A Λ*-module homomorphism* $\phi : M \to N$ is defined as a homomorphism of abelian groups such that $\phi(\lambda m) = \lambda\phi(m)$ for $m \in M$ and $\lambda \in \Lambda$. We also define a sequence of Λ-modules and Λ-module homomorphisms

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

to be *exact at* M_i if $\operatorname{im} f_i = \ker f_{i+1}$. The sequence is said to be *exact* if it is exact at each M_i. We have in particular:

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is exact if and only if f is injective, g is surjective and g induces an isomorphism of $\operatorname{coker} f = M/f(M')$ onto M''. A sequence of the preceding type is called *short exact* and we will also often write $M' \overset{f}{\hookrightarrow} M \overset{g}{\twoheadrightarrow} M''$.

Let A, B, C, D be Λ-modules and let $\alpha, \beta, \gamma, \delta$ be Λ-module homomorphisms. We say that the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

is *commutative* if $\beta\alpha = \delta\gamma : A \to D$. We shall need the following

Lemma 1.1.1. *Let $A' \overset{\mu}{\hookrightarrow} A \overset{\varepsilon}{\twoheadrightarrow} A''$ and $B' \overset{\mu'}{\hookrightarrow} B \overset{\varepsilon'}{\twoheadrightarrow} B''$ be two short exact sequences. Suppose that in the commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "\mu"] \arrow[d, "\alpha'"] & A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[r] \arrow[d, "\alpha''"] & 0 \\ 0 \arrow[r] & B' \arrow[r, "\mu'"] & B \arrow[r, "\varepsilon'"] & B'' \arrow[r] & 0 \end{tikzcd}$$

any two of the three homomorphisms $\alpha', \alpha, \alpha''$ are isomorphisms. Then the third is an isomorphism also.

Proof. The proof is an exercise in diagram chasing using exactness and commutativity. Details can be found in [HS, p. 15]. □

Let $\mathrm{Hom}_\Lambda(A, B)$ denote the set of all Λ-module homomorphisms from A to B. We give this set an abelian group structure by defining for any Λ-module homomorphisms $f, g : A \to B$

$$(f + g)(a) = f(a) + g(a) \tag{1.1}$$

for all $a \in A$. Given Λ-module homomorphisms $\beta : B_1 \to B_2$ and $\alpha : A_2 \to A_1$ we obtain induced mappings

$$\beta_* : \mathrm{Hom}_\Lambda(A, B_1) \to \mathrm{Hom}_\Lambda(A, B_2), \quad \beta_*(f) = \beta f : A \to B_2$$

and

$$\alpha^* : \mathrm{Hom}_\Lambda(A_1, B) \to \mathrm{Hom}_\Lambda(A_2, B), \quad \alpha^*(f) = f\alpha : A_2 \to B. \tag{1.2}$$

Proposition 1.1.2. *Let $A' \overset{\mu}{\to} A \overset{\varepsilon}{\to} A'' \to 0$ and $0 \to B' \overset{\mu}{\to} B \overset{\varepsilon}{\to} B''$ be exact sequences of Λ-modules. For any Λ-modules A and B the induced sequences*

$$0 \to \mathrm{Hom}_\Lambda(A'', B) \overset{\varepsilon^*}{\to} \mathrm{Hom}_\Lambda(A, B) \overset{\mu^*}{\to} \mathrm{Hom}_\Lambda(A', B)$$
$$0 \to \mathrm{Hom}_\Lambda(A, B') \overset{\mu_*}{\to} \mathrm{Hom}_\Lambda(A, B) \overset{\varepsilon_*}{\to} \mathrm{Hom}_\Lambda(A, B'')$$

are exact.

Proof. This is again straightforward and can be found in [HS, p. 17]. □

Example 1.1.3. Let us calculate some Hom-groups. Let ϕ be an element of the respective Hom-group then

- $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}_n) \simeq \mathbb{Z}_n$ since ϕ is determined by where it sends $1 \in \mathbb{Z}$ and there are n choices;

- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \simeq \mathbb{Z}_{\gcd(m,n)}$ since $m\phi(1) = 0 \mod n$ iff $\frac{m}{d}\phi(1) = 0 \mod \frac{n}{d}$ iff $\phi(1) \in \{0, \frac{n}{d}, \ldots, \frac{(d-1)n}{d}\}$;
- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}$;
- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) = 0$ since $n\phi(1) = 0$ iff $\phi(1) = 0$;
- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$ because letting p be a prime not dividing $f(x)$ for some nonzero homomorphism $f : \mathbb{Q} \to \mathbb{Z}$ and some $x \in \mathbb{Q}$, we get $f(x) = f(x/p + \ldots + x/p) = pf(x/p)$ which is a contradiction;
- $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \simeq \mathbb{Q}$.

We now show how to give Hom a Λ-module structure. Let A be a right Λ-module and let G be an abelian group. Regarding A as an abelian group also, we can form the abelian group $\mathrm{Hom}_{\mathbb{Z}}(A, G)$ as described above. Using the right Λ-module structure of A we define

$$(\lambda\phi)(a) = \phi(a\lambda), \quad a \in A, \lambda \in \Lambda, \phi \in \mathrm{Hom}_{\mathbb{Z}}(A, G)$$

giving $\mathrm{Hom}_{\mathbb{Z}}(G, A)$ a left Λ-module structure. We can of course interchange "left" and "right" in the above construction. We obtain the following important proposition:

Proposition 1.1.4. *Let A be a left Λ-module and let G be an abelian group. Regard $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, G)$ as a left Λ-module via the right Λ-module structure of Λ. Then there exists an isomorphism of abelian groups*

$$\eta = \eta_A : \mathrm{Hom}_\Lambda(A, \mathrm{Hom}_{\mathbb{Z}}(\Lambda, G)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(A, G).$$

Moreover, for every Λ-module homomorphism $\alpha : A \to B$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(B, \mathrm{Hom}_{\mathbb{Z}}(\Lambda, G)) \arrow[r, "\eta_B"] \arrow[d, "\alpha^*"'] & \mathrm{Hom}_{\mathbb{Z}}(B, G) \arrow[d, "\alpha^*"] \\
\mathrm{Hom}_\Lambda(A, \mathrm{Hom}_{\mathbb{Z}}(\Lambda, G)) \arrow[r, "\eta_A"] & \mathrm{Hom}_{\mathbb{Z}}(A, G)
\end{tikzcd}$$

is commutative.

Proof. Given a Λ-module homomorphism $\phi : A \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, G)$, define a homomorphism of abelian groups $\phi' : A \to G$ by $\phi'(a) = (\phi(a))(1), a \in A$. Conversely, given a homomorphism of abelian groups $\phi : A \to G$, define a Λ-module homomorphism $\psi' : A \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, G)$ by $(\psi'(a))(\lambda) = \psi(\lambda a), a \in A, \lambda \in \Lambda$. The rest is just definition checking. $\square$

1.2 Direct Sums and Products

Let $\{A_j\}, j \in J$ be a family of Λ-modules indexed by J. We define a Λ-module called the *direct sum* $\bigoplus_{j \in J} A_j$ of the modules A_j as follows: an element of $\bigoplus_{j \in J} A_j$ is a sequence $(a_j)_{j \in J}$ with $a_j \in A_j$ and $a_j = 0$ for all but a finite number of subscripts. Addition is defined by $(a_j) + (b_j) = (a_j + b_j)$ and the Λ-module operation by $\lambda(a_j) = (\lambda a_j)$. We also define *injections* $\iota_k : A_k \to \bigoplus_{j \in J} A_j$ by $\iota_k(a_k) = (b_j)$ with $b_j = 0$ when $j \neq k$ and $b_k = a_k$ for $a_k \in A_k$.

Proposition 1.2.1. *Let M be a Λ-module and let $\{\psi_j : A_j \to M\}, j \in J$ be a family of Λ-module homomorphisms. Then there exists a unique homomorphism $\psi = \langle \psi_j \rangle : \bigoplus_{j\in J} A_j \to M$, such that the diagram*

$$\begin{tikzcd} A_j \arrow[d, "\iota_j"'] \arrow[dr, "\psi_j"] & \\ \bigoplus_{j\in J} A_j \arrow[r, dashed, "\psi"] & M \end{tikzcd}$$

is commutative for all $j \in J$.

Proof. We define $\psi((a_j)) = \sum_{j\in J} \psi_j(a_j)$. This is well-defined since all but finitely many terms in this sum are zero. □

We remark that the direct sum together with its injections is unique up to isomorphism but omit the proof which just uses the above property also known as a *universal property*. Similarly, we define the *direct product* $\prod_{j\in J} A_j$ of a family of Λ-modules $\{A_j\}, j \in J$ indexed by J where the elements are again sequences $(a_j)_{j\in J}$ but with no restrictions imposed on the terms. We could thus have an infinite number of nonzero terms in such a sequence. Rather than injections, we define *projections* $\pi_k : \prod_{j\in J} A_j \to A_k$ by $\pi_k((a_j)) = a_k$ and we have the following proposition very similarly to (1.2.1):

Proposition 1.2.2. *Let M be a Λ-module and let $\{\phi_j : M \to A_j\}, j \in J$ be a family of Λ-module homomorphisms. Then there exists a unique homomorphism $\phi = \{\phi_j\} : M \to \prod_{j\in J} A_j$ such that for every $j \in J$ the diagram*

$$\begin{tikzcd} & A_j \\ M \arrow[ur, "\phi_j"] \arrow[r, dashed, "\phi"] & \prod_{j\in J} A_j \arrow[u, "\pi_j"'] \end{tikzcd}$$

is commutative.

Proof. We define $\phi(m) = (\phi_j(m))_{j\in J}$. □

Finally we prove that $\mathrm{Hom}_\Lambda(-,-)$ preserves sums and products.

Proposition 1.2.3. *Let A, B be Λ-modules and $\{A_j\}, \{B_j\}, j \in J$ be families of Λ-modules. Then there are isomorphisms*

$$\eta : \mathrm{Hom}_\Lambda\left(\bigoplus_{j\in J} A_j, B\right) \tilde{\to} \prod_{j\in J} \mathrm{Hom}_\Lambda(A_j, B)$$

$$\zeta : \mathrm{Hom}_\Lambda\left(A, \prod_{j\in J} B_j\right) \tilde{\to} \prod_{j\in J} \mathrm{Hom}_\Lambda(A, B_j).$$

Proof. Given $\psi : \bigoplus A_j \to B$, define $\eta(\psi) = (\psi\iota_j)_{j\in J}$. Conversely we use the universal property of the direct sum since a given family $\{\psi_j : A_j \to B\}$ then gives rise to a map $\psi : \bigoplus A_j \to B$. The proof for the ζ-isomorphism is similar but uses the universal property of the direct product. □

1.3 Free and Projective Modules

Let A be a Λ-module and let S be a subset of A. If the set

$$A_0 = \left\{ \sum_{s \in S} \lambda_s s : \lambda_s \in \Lambda, \lambda_s = 0 \text{ for all but finitely many } s \in S \right\} \subseteq A$$

is all of A, we say that S is a *set of generators* of A. If A admits a finite set of generators, we say it is *finitely generated.* A set S of generators of A is called a *basis* if every $a \in A$ is uniquely expressible as an element of A_0. This is equivalent to saying that S must be *linearly independent.* Finally, if S is a basis of the Λ-module P, then P is said to be *free on* S or just *free* if we do not specify S. Alternatively, we could say that P is free if and only if $P \simeq \bigoplus_{s \in S} \Lambda$ by the following proposition:

Proposition 1.3.1. *Suppose the Λ-module P is free on S. Then $P \simeq \bigoplus_{s \in S} \Lambda_s$ where $\Lambda_s = \Lambda$ as a left module for $s \in S$. Conversely, $\bigoplus_{s \in S} \Lambda_s$ is free on the set $\{1_{\Lambda_s}, s \in S\}$.*

Proof. Define $\phi : P \to \bigoplus \Lambda_s$ by $\phi(a) = (\lambda_s)_{s \in S}$ where $a = \sum_{s \in S} \lambda_s s$ is the unique expression for a in terms of the given basis. Given $s \in S$ we also define $\psi_s : \Lambda_s \to P$ by $\psi_s(\lambda_s) = \lambda_s s$. By the universal property of the direct sum the family $\{\psi_s\}, s \in S$ gives rise to a map $\psi : \bigoplus_{s \in S} \Lambda_s \to P$. It is clear that ϕ and ψ are mutual inverses. $\square$

We will also need the following property:

Proposition 1.3.2. *Let P be free on S. To every Λ-module M and to every function f from S into the set underlying M, there is a unique Λ-module homomorphism $\phi : P \to M$ extending f.*

Proof. Let $f(s) = m_s$. Define $\phi(a) = \phi\left(\sum \lambda_s s\right) = \sum \lambda_s m_s$. $\square$

This yields two very important results:

Proposition 1.3.3. *Every Λ-module A is a quotient of a free module P.*

Proof. Let S be a set of generators of A and let $P = \bigoplus_{s \in S} \Lambda_s$ with $\Lambda_s = \Lambda$. As P is free on $W = \{1_{\Lambda_s}, s \in S\}$, we define $f : W \to A$ by $f(1_{\Lambda_s}) = s$ and construct an extension $\phi : P \to A$ of f by $\phi(\sum_{s \in S} \lambda_s 1_s) = \sum_{s \in S} \lambda_s s$ which shows that ϕ is indeed surjective and thus completes the proof. $\square$

Proposition 1.3.4. *Let P be a free Λ-module. To every surjective homomorphism $\varepsilon : B \twoheadrightarrow C$ of Λ-modules and to every homomorphism $\gamma : P \to C$ there exists a homomorphism $\beta : P \to B$ such that $\varepsilon\beta = \gamma$.*

Proof. Let P be free on S. Since ε is surjective, we can find $b_s \in B$ such that for every $s \in S, \varepsilon(b_s) = \gamma(s)$. Then simply define β as the extension of the function $f(s) = b_s$. By the uniqueness of (1.3.2), we conclude $\varepsilon\beta = \gamma$. $\square$

Note that the last proposition states that $\mathrm{Hom}_\Lambda(P, -)$ gives rise to a short exact sequence when P is free. To see this, let $A \overset{\mu}{\hookrightarrow} B \overset{\varepsilon}{\twoheadrightarrow} C$ be a short exact sequence of Λ-modules. Then given a homomorphism $\gamma : P \to C$, we can find a homomorphism $\beta : P \to B$ such that $\varepsilon\beta = \gamma$. But this is exactly what we need to conclude that

$$0 \to \mathrm{Hom}_\Lambda(P, A) \overset{\mu_*}{\to} \mathrm{Hom}_\Lambda(P, B) \overset{\varepsilon_*}{\to} \mathrm{Hom}_\Lambda(P, C) \to 0 \tag{1.3}$$

is exact. Conversely, the exactness of (1.3) immediately implies (1.3.4). Modules possessing this property have a special name: a Λ-module P is *projective* if to any homomorphisms ε, γ with ε surjective there exists β making the below diagram commute.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\beta"'] \arrow[d, "\gamma"] \\ B \arrow[r, "\varepsilon"] & C \end{tikzcd}$$

For example, we see by (1.3.4) that every free module is projective.

Proposition 1.3.5. *For a Λ-module P the following are equivalent:*

i) *P is projective;*

ii) *for every short exact sequence, the corresponding* $\mathrm{Hom}_\Lambda(P, -)$*-sequence is short exact also;*

iii) *P is a direct summand in a free module.*

Proof. We already proved $i) \Leftrightarrow ii)$. So suppose P is projective. (1.3.3) constructs a surjective map $\phi : F \to P$ from a free module F to P. Since P is projective, there thus exists a map $\iota : P \to F$ such that $\phi\iota = 1_P$. But this shows that ι is injective whence P is a direct summand in $F \simeq \bigoplus \Lambda$. Conversely, suppose P is a direct summand in a free module. Then $P' \simeq P \oplus Q$ for some free Λ-module P' and some Λ-module Q. Hence P is projective by proposition (1.3.6) below. This completes $i) \Leftrightarrow iii)$. $\square$

Dualizing the concept of a projective module, we obtain the following definition: a Λ-module I is called *injective* if to any homomorphisms α, μ with μ injective there exists β making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[d, "\alpha"'] & B \arrow[dl, dashed, "\beta"] \\ I & \end{tikzcd}$$

We also state:

Proposition 1.3.6. *A direct sum (direct product) of modules is projective (injective) if and only if each summand (factor) is projective (injective).*

Proof. See [HS, pp. 24, 30]. $\square$

Chapter 2

Categories and Functors

This chapter is very heavy in definitions and many proofs of results will be omitted. We need, however, to go through these concepts to introduce the theory of derived functors upon which much of homological algebra is built.

2.1 Basic Category Theory

To define a *category* $\mathfrak{C}$ we must first be given three definitions of their own:

(1) a class of *objects* $A, B, C, \ldots$;

(2) to each pair of objects A, B of $\mathfrak{C}$, a set $\mathfrak{C}(A, B)$ of *morphisms* from A to B;

(3) to each triple of objects A, B, C, a *law of composition* $\mathfrak{C}(A, B) \times \mathfrak{C}(B, C) \to \mathfrak{C}(A, C)$.

Suppose we are given a morphism $f \in \mathfrak{C}(A, B)$ and a morphism $g \in \mathfrak{C}(B, C)$, then we will denote the composition law by writing gf for the morphism in $\mathfrak{C}(A, C)$. The rationale for this is that very often function composition will be this composition law. For $f \in \mathfrak{C}(A, B)$ we call A the *domain* and B the *codomain* of f. To be a category, $\mathfrak{C}$ must then obey the following axioms:

A1: The sets $\mathfrak{C}(A_1, B_1), \mathfrak{C}(A_2, B_2)$ are disjoint unless $A_1 = A_2, B_1 = B_2$.

A2: Given $f : A \to B, g : B \to C, h : C \to D$, then $h(gf) = (hg)f$.

A3: To each object A there is a morphism $1_A : A \to A$ such that, for any $f : A \to B, g : C \to A, f1_A = f, 1_Ag = g$.

We call **A2** the *associativity of composition* and **A3** the *existence of identities*. 1_A is called the *identity morphism* of A which is unique by **A3**. We also define a *zero object* 0 in $\mathfrak{C}$ to be an object with the property that, for any object $X \in \mathfrak{C}$, the sets $\mathfrak{C}(X, 0)$ and $\mathfrak{C}(0, X)$ both consist of exactly one element. However, not every category has a zero object. Take for instance the category of rings with unity element and ring homomorphisms.

An example of a category with zero object is the category of abelian groups with group homomorphisms: any one-element group is a zero object.

The zero object, if it exists, is unique and in any $\mathfrak{C}(X,Y)$ we have a morphism $X \to 0 \to Y$, called the *zero morphism* also denoted by 0, such that $0f = 0$ and $g0 = 0$ for any $g \in \mathfrak{C}(W,X)$ and any $f \in \mathfrak{C}(Y,Z)$. Finally, we will say that a morphism $f : A \to B$ of $\mathbb{C}$ is *invertible* if there exists a morphism $g : B \to A$ in $\mathbb{C}$ such that $gf = 1_A$ and $fg = 1_B$.

Remark 2.1.1. A few things to notice:

1. By **A1** we distinguish for instance between the functions $\sin : \mathbb{R} \to \mathbb{R}$ and $\sin : \mathbb{R} \to [-1,1]$ in category theory although one may argue that they are "the same".

2. A composition gf of morphisms is only defined if the codomain of f coincides with the domain of g.

3. The relation "$A \equiv B$ if there exists an invertible $f : A \to B$" is an equivalence relation on the objects of a given category. It is a categorical concept but has different names in different categories (isomorphisms of groups, homeomorphisms of spaces, 1-1 correspondences of sets).

Example 2.1.2. Let us list some examples of categories:

(a) The category $\mathfrak{S}$ of sets and functions;

(b) the category $\mathfrak{G}$ of groups and homomorphisms;

(c) the category $\mathfrak{Ab}$ of abelian groups and homomorphisms;

(d) the category $\mathfrak{R}_1$ of rings with unity element and ring homomorphisms preserving the unity element;

(e) the category $\mathfrak{M}^l_\Lambda$ of left Λ-modules, where Λ is an object of $\mathfrak{R}_1$.

Next we define transformations between categories. A *functor* $F : \mathfrak{C} \to \mathfrak{D}$ is a rule which associates with every object $X \in \mathfrak{C}$ an object $FX \in \mathfrak{D}$ and with every morphism $f \in \mathfrak{C}(X,Y)$ a morphism $Ff \in \mathfrak{C}(FX,FY)$ subject to the conditions

$$F(fg) = (Ff)(Fg), \quad F(1_A) = 1_{FA}.$$

Having defined functors in general, the notions of identity functor, composition of functors, invertible functors and isomorphic categories are all readily established.

Example 2.1.3. Let us also list some examples of functors:

(a) The *embedding* of a subcategory into the category the subcategory belongs to;

(b) the *free functor* $F : \mathfrak{S} \to \mathfrak{Ab}$ sending a set S to the free abelian group on S;

(c) in (1.1) we saw how to give $\mathrm{Hom}_\Lambda(A,B)$ an abelian group structure. If we keep A fixed then we have a functor $\mathrm{Hom}_\Lambda(A,-) : \mathfrak{M}^l_\Lambda \to \mathfrak{Ab}$.

Note that we have an asymmetry in the last of the above examples: if we keep B fixed, then $\mathrm{Hom}_\Lambda(-,B) : \mathfrak{M}^l_\Lambda \to \mathfrak{Ab}$ would not be a functor according to our initial definition since $\mathrm{Hom}_\Lambda(-,B)$ sends $\alpha : A_2 \to A_1$ to $\mathrm{Hom}_\Lambda(\alpha,B) = \alpha^* :$

$\mathrm{Hom}_\Lambda(A_1, B) \to \mathrm{Hom}_\Lambda(A_2, B)$ as in (1.2). To reestablish symmetry, we refine our definition and call functors obeying

$$f \in \mathfrak{C}(X, Y) \mapsto Ff \in \mathfrak{C}(FX, FY)$$

covariant functors and functors obeying

$$f \in \mathfrak{C}(X, Y) \mapsto Ff \in \mathfrak{C}(FY, FX)$$

contravariant functors. Thus, $\mathrm{Hom}_\Lambda(A, -)$ is covariant while $\mathrm{Hom}_\Lambda(-, B)$ is contravariant.

Let F, G be two functors from $\mathfrak{C}$ to $\mathfrak{D}$. A *natural transformation* η from F to G is a rule assigning to each $X \in \mathfrak{C}$ a morphism $\eta_X : FX \to GX$ in $\mathfrak{D}$ such that, for any morphism $f : X \to Y$ in $\mathbb{C}$, the diagram

$$\begin{tikzcd} FX \arrow[r, "\eta_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\eta_Y"] & GY \end{tikzcd}$$

is commutative. This is best understood by looking back at (1.1.4). We then see that the proposed isomorphism in that proposition is natural. If η_X is invertible for each X then η is called a natural equivalence and we write $F \simeq G$. Moreover, if there is a natural equivalence

$$\eta = \eta_{XY} : \mathfrak{D}(FX, Y) \tilde{\to} \mathfrak{C}(X, GY)$$

we say that F is *left adjoint* to G and G is *right adjoint* to F. For example, in proposition (1.1.4) we considered the functor $\mathrm{Hom}_\mathbb{Z}(\Lambda, -) : \mathfrak{Ab} \to \mathfrak{M}_\Lambda$. If we denote by $F : \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ the *forgetful functor* reducing a Λ-module to its underlying abelian group, then (1.1.4) states that there is a natural equivalence

$$\eta : \mathrm{Hom}_\Lambda(A, \mathrm{Hom}_\mathbb{Z}(\Lambda, C)) \tilde{\to} \mathrm{Hom}_\mathbb{Z}(FA, C), \quad A \in \mathfrak{M}_\Lambda, C \in \mathfrak{Ab}.$$

Hence, F is left adjoint to $\mathrm{Hom}_\mathbb{Z}(\Lambda, -)$ and $\mathrm{Hom}_\mathbb{Z}(\Lambda, -)$ is right adjoint to F. These definitions lead to the following useful

Theorem 2.1.4. *If $G : \mathfrak{D} \to \mathfrak{C}$ has a left adjoint then G preserves products and kernels.*

Proof. See [HS, p. 68]. □

As a special case, this shows that $\mathrm{Hom}_\mathbb{Z}(\Lambda, -)$ preserves products as was shown in generality in proposition (1.2.3).

We also define an *additive category* $\mathfrak{A}$ to be a category with zero object in which any two objects have a product and in which the morphism sets $\mathfrak{A}(A, B)$ are abelian groups such that the composition $\mathfrak{A}(A, B) \times \mathfrak{A}(B, C) \to \mathfrak{A}(A, C)$ is bilinear. For example, we have shown in part before that $\mathfrak{M}_\Lambda$ is an additive category. We remark that in an additive category finite sums and products are the same [Wei, p. 5]. Finally, let $F : \mathfrak{A} \to \mathfrak{B}$ be a functor between two additive categories. Then we define an *additive functor* to be a functor that preserves finite sums, or equivalently, preserves finite products. For details see [HS, p. 77]. As an example, we have shown in (1.2.3) that $\mathrm{Hom}_\Lambda(A, -)$ is additive.

2.2 Ext and Tor

In this section we shall introduce two new bifunctors $\mathrm{Ext}_\Lambda(-,-)$ and $\mathrm{Tor}_\Lambda(-,-)$ from the category of Λ-modules to the category of abelian groups.

A short exact sequence $R \overset{\mu}{\hookrightarrow} P \overset{\varepsilon}{\twoheadrightarrow} A$ of Λ-modules with P projective is called a *projective presentation* of A. Notice that (1.3.3) tells us that in $\mathfrak{M}_\Lambda$ every object has a projective presentation (just construct a short exact sequence from the given surjective map together with its kernel). Moreover, (1.1.2) tells us that a projective presentation induces for any Λ-module B an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(A,B) \overset{\varepsilon^*}{\to} \mathrm{Hom}_\Lambda(P,B) \overset{\mu^*}{\to} \mathrm{Hom}_\Lambda(R,B). \tag{2.1}$$

To given modules A and B and a chosen projective presentation $R \overset{\mu}{\hookrightarrow} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A, we can then associate the abelian group

$$\mathrm{Ext}^\varepsilon_\Lambda(A,B) = \mathrm{coker}(\mu^* : \mathrm{Hom}_\Lambda(P,B) \to \mathrm{Hom}_\Lambda(R,B)).$$

Elements in $\mathrm{Ext}^\varepsilon_\Lambda(A,B)$ are thus equivalence classes of Λ-module homomorphisms from R to B with two homomorphisms belonging to the same equivalence class if their difference extends to P. We note that a given homomorphism $\beta : B \to B'$ will map (2.1) into the corresponding sequence for B' yielding the induced map $\beta_* : \mathrm{Ext}^\varepsilon_\Lambda(A,B) \to \mathrm{Ext}^\varepsilon_\Lambda(A,B')$. $\mathrm{Ext}^\varepsilon_\Lambda(A,-)$ is then easily checked to be a covariant functor from $\mathfrak{M}_\Lambda$ to $\mathfrak{Ab}$. It can be shown that there exists a natural equivalence between functors $\mathrm{Ext}^\varepsilon_\Lambda(A,-)$ and $\mathrm{Ext}^{\varepsilon'}_\Lambda(A,-)$ for two different given projective presentations $P \overset{\varepsilon}{\twoheadrightarrow} A$ and $P' \overset{\varepsilon'}{\twoheadrightarrow} A$ of A. See [HS, p. 90] for the proof. We are thus justified to drop the superscript ε and just write $\mathrm{Ext}_\Lambda(A,-)$.

Given $\alpha : A' \to A$, it can be shown that $\mathrm{Ext}_\Lambda(-,B)$ is a contravariant functor. To do this, take projective presentations $P \twoheadrightarrow A$ and $P' \twoheadrightarrow A'$ of each of A and A' and note that since P' is projective, it induces a map π such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & R' \arrow[r] \arrow[d, "\sigma"] & P' \arrow[r] \arrow[d, "\pi"] & A' \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & R \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

is commutative. The map $\sigma : R' \to R$ we need to define $\alpha^* : \mathrm{Ext}_\Lambda(A,B) \to \mathrm{Ext}_\Lambda(A',B)$ is then just the restriction of π to R'. In this situation we say that π *lifts* α.

We also remark that we could construct a functor $\overline{\mathrm{Ext}}^\nu_\Lambda(-,B)$ by choosing an *injective presentation* of B, that is, an exact sequence $B \overset{\nu}{\hookrightarrow} I \overset{\eta}{\twoheadrightarrow} S$ with I injective, and defining $\overline{\mathrm{Ext}}^\nu_\Lambda(A,B)$ as the cokernel of the map $\eta_* : \mathrm{Hom}_\Lambda(A,I) \to \mathrm{Hom}_\Lambda(A,S)$. Again one shows that $\overline{\mathrm{Ext}}^\nu_\Lambda(-,-)$ is a bifunctor independent of the chosen injective presentation. Moreover, one can show that there is a natural equivalence between $\overline{\mathrm{Ext}}_\Lambda(-,-)$ and $\mathrm{Ext}_\Lambda(-,-)$ (see [HS, p. 96]) and we shall thus only use the notation $\mathrm{Ext}_\Lambda(-,-)$ for the remainder of the article.

Proposition 2.2.1. $\operatorname{Ext}_\Lambda(-,-)$ *preserves products and sum.*

Proof. Let $\{A_i\}$ be a family of Λ-modules. Choose a projective presentation $R_i \hookrightarrow P_i \twoheadrightarrow A_i$ of each A_i, then $\bigoplus R_i \hookrightarrow \bigoplus P_i \twoheadrightarrow \bigoplus A_i$ is a projective presentation of $\bigoplus A_i$ by (1.3.6). Using (1.2.3), the following diagram says it all:

$$\begin{tikzcd}
\operatorname{Hom}_\Lambda(\bigoplus A_i, B) \arrow[r] \arrow[d] & \operatorname{Hom}_\Lambda(\bigoplus P_i, B) \arrow[r] \arrow[d] & \operatorname{Hom}_\Lambda(\bigoplus R_i, B) \arrow[r, two heads] \arrow[d] & \operatorname{Ext}_\Lambda(\bigoplus A_i, B) \\
\prod \operatorname{Hom}_\Lambda(A_i, B) \arrow[r] & \prod \operatorname{Hom}_\Lambda(P_i, B) \arrow[r] & \prod \operatorname{Hom}_\Lambda(R_i, B) \arrow[r, two heads] & \prod \operatorname{Ext}_\Lambda(A_i, B)
\end{tikzcd}$$

A similar proof works for products. □

Example 2.2.2. Let us calculate Ext-groups for finite abelian groups. By the above proposition we only need to consider cyclic groups. Since $\mathbb{Z}$ is free as as a $\mathbb{Z}$-module, we choose $0 \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$ as projective presentation of $\mathbb{Z}$. It then follows immediately from the definition that $\operatorname{Ext}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}) = 0$. To compute $\operatorname{Ext}_\mathbb{Z}(\mathbb{Z}_r, \mathbb{Z})$ and $\operatorname{Ext}_\mathbb{Z}(\mathbb{Z}_r, \mathbb{Z}_q)$, we choose the projective presentation

$$\mathbb{Z} \overset{\mu}{\hookrightarrow} \mathbb{Z} \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}_r$$

where μ is multiplication by r. Taking (covariant!) $\operatorname{Hom}(-, \mathbb{Z})$ and using (1.1.3) we get the diagram

$$\begin{tikzcd}
\operatorname{Hom}_\mathbb{Z}(\mathbb{Z}_r, \mathbb{Z}) \arrow[r] \arrow[d] & \operatorname{Hom}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}) \arrow[r, "\mu^*"] \arrow[d] & \operatorname{Hom}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}) \arrow[r, two heads] \arrow[d] & \operatorname{Ext}_\mathbb{Z}(\mathbb{Z}_r, \mathbb{Z}) \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mu^*"] & \mathbb{Z} &
\end{tikzcd}$$

from which we conclude $\operatorname{Ext}_\mathbb{Z}(\mathbb{Z}_r, \mathbb{Z}) \simeq \mathbb{Z}/r\mathbb{Z} = \mathbb{Z}_r$. Similarly, from

$$\begin{tikzcd}
\operatorname{Hom}_\mathbb{Z}(\mathbb{Z}_r, \mathbb{Z}_q) \arrow[r] \arrow[d] & \operatorname{Hom}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}_q) \arrow[r, "\mu^*"] \arrow[d] & \operatorname{Hom}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}_q) \arrow[r, two heads] \arrow[d] & \operatorname{Ext}_\mathbb{Z}(\mathbb{Z}_r, \mathbb{Z}_q) \\
\mathbb{Z}_{\gcd(r,q)} \arrow[r] & \mathbb{Z}_q \arrow[r, "\mu^*"] & \mathbb{Z}_q &
\end{tikzcd}$$

we get $\operatorname{Ext}_\mathbb{Z}(\mathbb{Z}_r, \mathbb{Z}_q) \simeq \mathbb{Z}_q/r\mathbb{Z}_q = \mathbb{Z}_{\gcd(r,q)}$.

Next we introduce a handy tool that will be needed often.

Lemma 2.2.3 (Snake Lemma)**.** *Let the following commutative diagram, called a snake diagram, have exact rows.*

$$\begin{tikzcd}
 & A \arrow[r, "\mu"] \arrow[d, "\alpha"'] & B \arrow[r, "\varepsilon"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu'"] & B' \arrow[r, "\varepsilon'"] & C' &
\end{tikzcd}$$

Then there is a connecting homomorphism $\delta : \ker \gamma \to \operatorname{coker} \alpha$ *such that the following sequence is exact:*

$$\ker \alpha \overset{\mu_*}{\to} \ker \beta \overset{\varepsilon_*}{\to} \ker \gamma \overset{\delta}{\to} \operatorname{coker} \alpha \overset{\mu'_*}{\to} \operatorname{coker} \beta \overset{\varepsilon'_*}{\to} \operatorname{coker} \gamma.$$

Proof. This lemma is a standard result in commutative and homological algebra. For the proof consult [La, p. 158]. $\square$

From this result, we now obtain the Hom-Ext-*sequences*:

Proposition 2.2.4 (Hom-Ext-sequence in the second variable)**.** *Let A be a Λ-module and let $B' \overset{\phi}{\hookrightarrow} B \overset{\psi}{\twoheadrightarrow} B''$ be an exact sequence of Λ-modules. There exists a connecting homomorphism $\delta : \mathrm{Hom}_\Lambda(A, B'') \to \mathrm{Ext}_\Lambda(A, B')$ such that the following sequence is exact:*

$$\begin{aligned} 0 \to \mathrm{Hom}_\Lambda(A, B') &\overset{\phi_*}{\to} \mathrm{Hom}_\Lambda(A, B) \overset{\psi_*}{\to} \mathrm{Hom}_\Lambda(A, B'') \\ &\overset{\delta}{\to} \mathrm{Ext}_\Lambda(A, B') \overset{\phi_*}{\to} \mathrm{Ext}_\Lambda(A, B) \overset{\psi_*}{\to} \mathrm{Ext}_\Lambda(A, B''). \end{aligned}$$

Proof. To prove this, choose any projective presentation $R \overset{\mu}{\hookrightarrow} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A and apply (2.2.3) to the below commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & \mathrm{Hom}_\Lambda(A, B') \arrow[r, dashed, "\phi_*"] \arrow[d] & \mathrm{Hom}_\Lambda(A, B) \arrow[r, dashed, "\psi_*"] \arrow[d] & \mathrm{Hom}_\Lambda(A, B'') \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(P, B') \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P, B) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P, B'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(R, B') \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(R, B) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P, B'') \arrow[d] & \\
 & \mathrm{Ext}_\Lambda(A, B') \arrow[r, dashed, "\phi_*"] & \mathrm{Ext}_\Lambda(A, B) \arrow[r, dashed, "\psi_*"] & \mathrm{Ext}_\Lambda(A, B'') &
\end{tikzcd}$$

Note that the second line is exact by (1.3.5). To show the resulting sequence is independent of the chosen projective presentation of A, consult [HS, p. 101]. $\square$

One can also prove

Proposition 2.2.5 (Hom-Ext-sequence in the first variable)**.** *Let B be a Λ-module and let $A' \overset{\phi}{\hookrightarrow} A \overset{\psi}{\twoheadrightarrow} A''$ be an exact sequence of Λ-modules. There exists a connecting homomorphism $\delta : \mathrm{Hom}_\Lambda(A', B) \to \mathrm{Ext}_\Lambda(A'', B)$ such that the following sequence is exact:*

$$\begin{aligned} 0 \to \mathrm{Hom}_\Lambda(A'', B) &\overset{\psi^*}{\to} \mathrm{Hom}_\Lambda(A, B) \overset{\phi^*}{\to} \mathrm{Hom}_\Lambda(A', B) \\ &\overset{\delta}{\to} \mathrm{Ext}_\Lambda(A'', B) \overset{\psi^*}{\to} \mathrm{Ext}_\Lambda(A, B) \overset{\phi^*}{\to} \mathrm{Ext}_\Lambda(A', B). \end{aligned}$$

Proof. [HS, p. 102] $\square$

Note that these two propositions give us a new interpretation for Ext-groups: they extend exact sequences into longer exact sequences on the right.

Corollary 2.2.6. *The Λ-module A is projective if and only if $\mathrm{Ext}_\Lambda(A, B) = 0$ for all Λ-modules B.*

Proof. Suppose A is projective. Then $1 : A \tilde{\to} A$ is a projective presentation, whence $\mathrm{Ext}_\Lambda(A, B) = 0$ for all Λ-modules B. Conversely, suppose $\mathrm{Ext}_\Lambda(A, B) = 0$ for all B, then

$$0 \to \mathrm{Hom}_\Lambda(A, B') \to \mathrm{Hom}_\Lambda(A, B) \to \mathrm{Hom}_\Lambda(A, B'') \to 0$$

is exact by (2.2.4). By (1.3.5) A is projective. $\square$

Similarly, one can prove

Corollary 2.2.7. *The Λ-module B is injective if and only if* $\mathrm{Ext}_\Lambda(A, B) = 0$ *for all Λ-modules A.*

We now introduce two more functors one of which was already announced earlier: the tensor product and the Tor-functor.

Proposition 2.2.8 (Universal Property of the Tensor Product)**.** *Let M, N be Λ-modules. Then there exists a pair (T, g) consisting of an Λ-module T and an Λ-bilinear mapping $g : M \times N \to T$, with the following property: given any Λ-module P and any Λ-bilinear mapping $f : M \times N \to P$, there exists a unique Λ-linear mapping $f' : T \to P$ such that $f = f' \circ g$ (in other words, every bilinear function on $M \times N$ factors through T).*
Moreover, if (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism $j : T \to T'$ such that $j \circ g = g'$.

Proof. Let us prove uniqueness of the pair (T, g) up to isomorphism first. We have the following situation:

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[d, "g'"'] & T \arrow[dl, dashed, shift left, "j"] \\ T' \arrow[ur, dashed, shift left, "j'"] & \end{tikzcd}$$

Here the maps j and j' are given to us by the proposition and it is clear that they are mutual inverses so indeed $T \simeq T'$.
Next we prove existence. Let $C = \Lambda^{M \times N}$ be a free Λ-module. So elements in C are of the form $\Sigma_{i=1}^{n} \lambda_i (x_i, y_i)$ where $\lambda_i \in \Lambda, x_i \in M, y_i \in N$. Let $D \subseteq C$ be generated by all elements of the types

$$\begin{gathered} (x + x', y) - (x, y) - (x', y) \\ (x, y + y') - (x, y) - (x, y') \\ (\lambda x, y) - \lambda(x, y) \\ (x, \lambda y) - \lambda(x, y) \end{gathered}$$

Let $T = C/D$ and let $x \otimes y$ be the image of (x, y) in T. Then $(x + x') \otimes y = x \otimes y + x' \otimes y$ and similarly for all the other relations we would require for bilinearity. So $g : M \times N \to T$ such that $g(x, y) = x \otimes y$ is Λ-bilinear. Now let $f : M \times N \to P$ be a function and extend f by linearity to a function $\overline{f}$ on C. Since $\overline{f}$ vanishes on D it induces a well-defined Λ-module homomorphism $f' : T \to P$ such that $f'(x \otimes y) = f(x, y)$ which defines f' uniquely. So the pair (T, g) satisfies the conditions of the proposition. $\square$

Remark 2.2.9. The module T constructed above is called the *tensor product* of M and N and is denoted by $M \otimes_\Lambda N$. The subscript on the tensor product is extremely important as it tells us with respect to which ring the tensor product is bilinear just as the subscript for example on $\mathrm{Hom}_\Lambda(A,B)$ and $\mathrm{Hom}_\mathbb{Z}(A,B)$ told us whether to regard A and B as Λ-modules or as abelian groups.

There are several natural isomorphisms of particular importance:

Proposition 2.2.10. *Let M, N, P be (Λ, K)-bimodules. Then there are unique natural isomorphisms*

(a) $M \otimes_\Lambda N \to N \otimes_\Lambda M$;

(b) $(M \otimes_\Lambda N) \otimes_\Lambda P \to M \otimes_\Lambda (N \otimes_\Lambda P) \to M \otimes_\Lambda N \otimes_\Lambda P$;

(c) $\Lambda \otimes_\Lambda M \to M$;

(d) $(M \otimes_\Lambda N) \otimes_K P \to M \otimes_\Lambda (N \otimes_K P)$.

Proof. The proofs use the universal property of the tensor product. Details can be found in [AM, p. 26]. □

For any $\alpha : A \to A'$ we define an induced map $\alpha_* : A \otimes_\Lambda B \to A' \otimes_\Lambda B$ by $\alpha_*(a \otimes b) = (\alpha \otimes 1_B)(a \otimes b) = \alpha(a) \otimes b$ for $a \in A$ and $b \in B$. A similar definition holds for any $\beta : B \to B'$ and we quickly check that $- \otimes_\Lambda -$ becomes a bifunctor from $\mathfrak{M}_\Lambda$ to $\mathfrak{Ab}$.

Proposition 2.2.11. *For any right Λ-module A, the functor $A \otimes_\Lambda -$ is left adjoint to the functor* $\mathrm{Hom}_\mathbb{Z}(A, -)$.

Proof. We have to show that there exists a natural transformation η such that for any abelian group G and any left Λ-module B

$$\eta : \mathrm{Hom}_\mathbb{Z}(A \otimes_\Lambda B, G) \tilde{\to} \mathrm{Hom}_\Lambda(B, \mathrm{Hom}_\mathbb{Z}(A, G)).$$

Given a homomorphism $\phi : A \otimes_\Lambda B \to G$ of abelian groups we define a Λ-module homomorphism $\phi' : B \to \mathrm{Hom}_\mathbb{Z}(A, G)$ by $(\phi'(b))(a) = \phi(a \otimes b)$. Conversely, given a Λ-module homomorphism $\psi : B \to \mathrm{Hom}_\mathbb{Z}(A, G)$ define a homomorphism of abelian groups $\psi' : A \otimes_\Lambda B \to G$ by $\psi'(a \otimes b) = (\psi(b))(a)$. The rest is again just definition checking. □

As an immediate consequence of the above proposition and the dual to theorem (2.1.4) we have

Proposition 2.2.12. *i) Let $\{B_j\}_{j \in J}$ be a family of left Λ-modules and let A be a right Λ-module. Then there is a natural isomorphism*

$$A \otimes_\Lambda \left(\bigoplus_{j \in J} B_j \right) \tilde{\to} \bigoplus_{j \in J} (A \otimes_\Lambda B_j).$$

ii) If $B' \xrightarrow{\beta'} B \xrightarrow{\beta''} B'' \to 0$ is an exact sequence of left Λ-modules, then for any right Λ-module A, the sequence

$$A \otimes_\Lambda B' \xrightarrow{\beta'_*} A \otimes_\Lambda B \xrightarrow{\beta''_*} A \otimes_\Lambda B'' \to 0$$

is exact.

Similarly, one could of course show that $- \otimes_\Lambda B$ is left adjoint to $\mathrm{Hom}_\mathbb{Z}(B, -)$ and a proposition analogous to the one above would result.

A covariant functor F is said to be *left exact* if it transforms exact sequences of the form $0 \to A' \to A \to A''$ into exact sequences $0 \to FA' \to FA \to FA''$. Similarly, a contravariant functor F is said to be *left exact* if it transforms exact sequences of the form $A' \to A \to A'' \to 0$ into an exact sequence $0 \to FA'' \to FA \to FA'$. It is obvious how to define *right exactness* of a functor. As an example, $\mathrm{Hom}_\Lambda(-, B)$ is a contravariant left exact functor while $A \otimes_\Lambda -$ is a covariant right exact functor. We also define a left Λ-module B to be *flat* if tensoring by B transforms exact sequences into exact sequences. We have

Proposition 2.2.13. *Every projective module is flat.*

Proof. A projective module is a direct summand in a free module by (1.3.5). Since the tensor product preserves sums, it is thus sufficient to prove that free modules are flat. But free modules are again direct sums of their base ring and it is thus sufficient to prove that Λ as a left module is flat. But this is trivial since $A \otimes_\Lambda \Lambda \simeq A$. $\square$

We now define the functor Tor. Let A be a right Λ-module and let B be a left Λ-module. We can tensor a given projective presentation $R \overset{\mu}{\hookrightarrow} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A by B and define

$$\mathrm{Tor}_\varepsilon^\Lambda(A, B) = \ker(\mu_* : R \otimes_\Lambda B \to P \otimes_\Lambda B).$$

Just like Ext_Λ, we can make $\mathrm{Tor}_\varepsilon^\Lambda(A, -)$ into a covariant functor by defining, for a given map $\beta : B \to B', \beta_* : \mathrm{Tor}_\varepsilon^\Lambda(A, B) \to \mathrm{Tor}_\varepsilon^\Lambda(A, B')$ in the obvious way. Again, one shows that $\mathrm{Tor}_\varepsilon^\Lambda(A, -)$ is independent of the chosen projective presentation and just writes $\mathrm{Tor}^\Lambda(A, -)$. Given $\alpha : A \to A'$, we can also make $\mathrm{Tor}^\Lambda(-, B)$ into a covariant functor by lifting α in the same way we lifted α in the construction of $\mathrm{Ext}_\Lambda(-, B)$ thus defining $\alpha_* : \mathrm{Tor}^\Lambda(A, B) \to \mathrm{Tor}^\Lambda(A', B)$. As with Ext, one could define another covariant functor $\overline{\mathrm{Tor}}_\eta^\Lambda(-, B)$ via a projective presentation in the first variable. Note that we do not have to use an injective presentation since the tensor product is covariant in both variables. One shows independence of the chosen projective presentation η and constructs a natural equivalence between $\mathrm{Tor}^\Lambda(-, -)$ and $\overline{\mathrm{Tor}}^\Lambda(-, -)$. From now on, we will thus simply write $\mathrm{Tor}^\Lambda(-, -)$.

Similarly to (2.2.4) and (2.2.5), one can show that for given exact sequences $B' \overset{\kappa}{\hookrightarrow} B \overset{\nu}{\twoheadrightarrow} B''$ and $A' \overset{\kappa}{\hookrightarrow} A \overset{\nu}{\twoheadrightarrow} A''$ there are exact sequences

$$\begin{aligned} \mathrm{Tor}^\Lambda(A, B') \overset{\kappa_*}{\to} \mathrm{Tor}^\Lambda(A, B) \overset{\nu_*}{\to} \mathrm{Tor}^\Lambda(A, B'') \overset{\delta}{\to} A \otimes_\Lambda B' \\ \overset{\kappa_*}{\to} A \otimes_\Lambda B \overset{\nu_*}{\to} A \otimes_\Lambda B'' \to 0 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Tor}^\Lambda(A', B) \overset{\kappa_*}{\to} \mathrm{Tor}^\Lambda(A, B) \overset{\nu_*}{\to} \mathrm{Tor}^\Lambda(A'', B) \overset{\delta}{\to} A' \otimes_\Lambda B \\ \overset{\kappa_*}{\to} A \otimes_\Lambda B \overset{\nu_*}{\to} A'' \otimes_\Lambda B \to 0. \end{aligned}$$

We could thus think of Tor as extending exact sequences into longer exact sequences on the left. In analogy to (2.2.6), we finally prove

Proposition 2.2.14. *If A or B is projective, then* $\mathrm{Tor}^\Lambda(A, B) = 0$.

Proof. If A is projective, then A is flat by (2.2.13). Hence

$$0 \to A \otimes_\Lambda S \to A \otimes_\Lambda Q \to A \otimes_\Lambda B \to 0$$

is exact. Thus $\operatorname{Tor}^\Lambda(A, B) = 0$. Similarly for B. $\square$

Chapter 3

Derived Functors

In this chapter we generalize the functors Ext and Tor introduced in the last chapter in the theory of derived functors. This is the last and most fundamental step on our road to group cohomology as the (co)homology groups will be defined precisely through these derived functors.

3.1 Complexes

We define a chain complex $\mathbf{C} = \{C_n, \partial_n\}$ over Λ as a family $\{C_n\}_{n\in\mathbb{Z}}$ of Λ-modules together with a family of Λ-module homomorphisms $\{\partial_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ such that $\partial_n\partial_{n+1} = 0$:

$$\mathbf{C} : \cdots \to C_{n+1} \overset{\partial_{n+1}}{\to} C_n \overset{\partial_n}{\to} C_{n-1} \to \cdots .$$

A homomorphism ∂_n is called a *differential* or *boundary operator*. The kernel of ∂_n is called the *module of n-cycles* of $\mathbf{C}$ and denoted by $Z_n = Z_n(\mathbf{C})$. The image of $\partial_{n+1} : C_{n+1} \to C_n$ is the *module of n-boundaries* of $\mathbf{C}$ and written $B_n = B_n(\mathbf{C})$. Because $\partial_n\partial_{n+1} = 0$, we can form

$$H(\mathbf{C}) = \{H_n(\mathbf{C})\}, \quad \text{where } H_n(\mathbf{C}) = Z_n/B_n, \quad n \in \mathbb{Z}$$

called the n^{th} *homology module* of $\mathbf{C}$.

If we reindex the above definition with superscripts $C^n = C_{-n}$, we obtain the following dual notion. A *cochain complex* $\mathbf{C} = \{C^n, \partial^n\}$ over Λ is a family $\{C^n\}_{n\in\mathbb{Z}}$ of Λ-modules together with a family of Λ-module homomorphisms $\{\partial^n : C^n \to C^{n+1}\}_{n\in\mathbb{Z}}$ such that $\partial^n\partial^{n-1} = 0$:

$$\mathbf{C} : \cdots \to C^{n-1} \overset{\partial^{n-1}}{\to} C^n \overset{\partial^n}{\to} C^{n+1} \to \cdots .$$

Again, ∂^n is called a *differential* or *coboundary operator*. $Z^n = \ker \partial^n$ is the *module of n-cocycles* and $B^n = \operatorname{im} \partial^{n-1}$ is the *module of n-coboundaries* and we define

$$H(\mathbf{C}) = \{H^n(\mathbf{C})\}, \quad \text{where } H^n(\mathbf{C}) = Z^n/B^n, \quad n \in \mathbb{Z}$$

is the n^{th} *cohomology module* of $\mathbf{C}$.

We now show that $H(-)$ can be made into a functor. Define a *chain map* $\boldsymbol{\phi} : \mathbf{C} \to \mathbf{D}$ as a family $\{\phi_n : C_n \to D_n\}_{n\in\mathbb{Z}}$ of homomorphisms such that, for all n, the diagram

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "\phi_n"] & C_{n-1} \arrow[d, "\phi_{n-1}"] \\ D_n \arrow[r, "\tilde{\partial}_n"] & D_{n-1} \end{tikzcd}$$

is commutative.

A chain map $\boldsymbol{\phi} : \mathbf{C} \to \mathbf{D}$ induces a well-defined morphism $H(\boldsymbol{\phi}) = \boldsymbol{\phi}_* : H(\mathbf{C}) \to H(\mathbf{D})$. To see this, suppose that $x, y \in H_{n-1}(\mathbf{C})$. We then need to show that whenever $x - y \in \operatorname{im} \partial_n$, i.e. $\partial_n(z) = x - y$ for some $z \in C_n$ then $\phi_{n-1}(x-y) \in \operatorname{im} \tilde{\partial}_n$. But since the above diagram commutes, we get $\phi_{n-1}(x-y) = \phi_{n-1} \circ \partial_n(z) = \tilde{\partial}_n \circ \phi_n(z)$. Hence $\phi_{n-1}(x-y) \in \tilde{\partial}_n$.

With this definition, it can be checked that $H(-)$ becomes a functor called the *(co)homology functor* from the category of (co)chain complexes over Λ to the category of Λ-modules. Because $H(-)$ is actually a collection of functors, we give it a special name: we call $H(-)$ a *(co)homological δ-functor* (see section (4.9) for the precise definition).

Let us now restrict our attention to positive chain complexes, that is, chain complexes of the form

$$\mathbf{C} : \cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$$

with $C_n = 0$ for $n < 0$. Such a chain complex is called *projective* if C_n is projective for all $n \geq 0$. Moreover, we call it *acyclic* if $H_n(\mathbf{C}) = 0$ for $n \geq 1$. Note then that C is acyclic if and only if the sequence

$$\cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to H_0(\mathbf{C}) \to 0$$

is exact. A projective and acyclic complex

$$\mathbf{P} : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0$$

together with an isomorphism $H_0(\mathbf{P}) \tilde{\to} A$ is called a *projective resolution* of A. We say that an *abelian category* $\mathfrak{C}$, that is, a category in which every map has a kernel and a cokernel (for a precise definition see [Wei, p. 6]), has *enough projectives* if for every object A of $\mathfrak{C}$ there is a surjection $P \twoheadrightarrow A$ with P projective. We then claim that in an abelian category $\mathfrak{C}$ with enough projectives, every object $X \in \mathfrak{C}$ has a projective resolution. We illustrate the proof in the category $\mathfrak{M}_\Lambda$:

Proposition 3.1.1. *To every Λ-module A there exists a projective resolution.*

Proof. We have seen in (1.3.3) that we can find a projective presentation $R_1 \hookrightarrow P_0 \overset{\varepsilon_0}{\twoheadrightarrow} A$ of each Λ-module A. We then simply continue to find projective presentations $R_2 \hookrightarrow P_1 \twoheadrightarrow R_1$ of R_1 and of R_2 etc. We obtain a projective resolution by splicing these sequences together. That is

$$
\begin{tikzcd}[column sep=small]
 & & & & 0 & & 0 \arrow[dl] & & 0 & & 0 \arrow[dl] \\
 & & & & & R_2 \arrow[ul] \arrow[dl] & & & & R_4 \arrow[ul] \arrow[dl] & \\
0 & A \arrow[l] & P_0 \arrow[l, "\varepsilon_0"'] & & P_1 \arrow[ll, "\partial_1"'] \arrow[dl] & & P_2 \arrow[ll, "\partial_2"'] \arrow[ul] & & P_3 \arrow[ll, "\partial_3"'] \arrow[dl] & & \cdots \arrow[ll, "\partial_4"'] \arrow[ul, "N"'] \\
 & & & R_1 \arrow[ul] \arrow[dl] & & & & R_3 \arrow[ul] \arrow[dl] & & & \\
 & & 0 & & 0 \arrow[ul] & & 0 & & 0 \arrow[ul] & &
\end{tikzcd}
$$

□

Dually, one can also speak of *injective resolutions* of a given Λ-module A, that is, positive acyclic cochain complexes

$$\mathbf{I} : 0 \to I_0 \to I_1 \to I_2 \to \cdots \to I_n \to I_{n+1} \to \cdots,$$

with each I_n injective, $H^n(\mathbf{I}) = 0$ for $n \geq 1$ and an isomorphism $H^0(\mathbf{I}) \xrightarrow{\sim} A$. We will say a category $\mathfrak{C}$ has *enough injectives* if every object has an injective presentation. This would be enough to guarantee that every object also has an injective resolution. It can be shown that every Λ-module is a submodule of an injective Λ-module [HS, p. 35]. Hence $\mathfrak{M}_\Lambda$ has enough injectives.

3.2 Derived Functors

We are now ready to talk about derived functors. Let $T : \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be an additive covariant functor, then $L_nT : \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ for $n \geq 0$ is called the n^{th} *left derived functor* of T. The value of L_nT on a Λ-module A is computed by choosing a projective resolution $\mathbf{P}$ of A, forming the complex $T\mathbf{P}$ and taking homology. Then define $L_nTA = H_n(T\mathbf{P})$. That this definition makes sense and is independent of the chosen projective presentation can in fact be verified with the tools we have developed thus far but in order to move on quickly to group cohomology we refer the interested reader to [HS, p. 132]. Instead, we study some properties of these functors.

Proposition 3.2.1. *Let $T : \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be right exact. Then L_0T and T are naturally equivalent.*

Proof. Let $\mathbf{P}$ be a projective resolution of A. Then $\cdots \to P_1 \to P_0 \to A \to 0$ is exact and since T is right exact $\cdots \to TP_1 \to TP_0 \to TA \to 0$ is exact also. Hence $H_0(T\mathbf{P}) \simeq TA$. It is not hard to see that this isomorphism is natural. □

Proposition 3.2.2. *For P a projective Λ-module $L_nTP = 0$ for $n \geq 1$.*

Proof. Clearly $\mathbf{P} : \cdots \to 0 \to P_0 \to 0$ with $P_0 = P$ is a projective resolution of P. □

Similarly, we also define right derived functors: let $T : \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ again be an additive covariant functor, then $R^nT : \mathfrak{M}_\Lambda \to \mathfrak{Ab}, n \geq 0$ is called the n^{th} *right derived functor*. For any Λ-module A, the abelian group R^nTA is computed by taking an injective resolution $\mathbf{I}$ of A, forming the cochain complex $T\mathbf{I}$ and taking cohomology, i.e. $R^nTA = H^n(T\mathbf{I}), n \geq 0$.

Proposition 3.2.3. *For I an injective Λ-module, $R^nTI = 0$ for all $n \geq 1$. Moreover, if T is left exact, then R^0T is naturally equivalent to T.*

Proof. Same strategy as for the last two results. $\square$

We now return to the idea of interpreting Ext and Tor as functors extending short exact sequences into longer ones. As we are hoping to generalize these functors, one may expect the left and right derived functors to extend short exact sequences into even longer ones. First we show how to construct long exact (co)homology sequences.

Theorem 3.2.4. *Given a short exact sequence $\boldsymbol{A} \overset{\phi}{\hookrightarrow} \boldsymbol{B} \overset{\psi}{\twoheadrightarrow} \boldsymbol{C}$ of (co)chain complexes, there exists a morphism $\delta : H(\boldsymbol{C}) \to H(\boldsymbol{A})$ such that the triangle*

$$\begin{tikzcd}
H(\boldsymbol{A}) \arrow[rr, "\phi_*"] & & H(\boldsymbol{B}) \arrow[dl, "\psi_*"] \\
& H(\boldsymbol{C}) \arrow[ul, "\delta"] &
\end{tikzcd}$$

is exact. As in previous cases, we call δ the connecting homomorphism.

Proof. We prove this result for chain complexes only since the proof for cochains is analogous. To ease notation, let us write ∂_n for all differentials of the given chain complexes. Note that for every chain complex $\mathbf{C}$ a differential $\partial_n : C_n \to C_{n-1}$ induces a map $\tilde{\partial}_n : \operatorname{coker} \partial_{n+1} = C_n/\operatorname{im} \partial_{n+1} \twoheadrightarrow C_n/\ker \partial_n \simeq \operatorname{im} \partial_n \subseteq \ker \partial_{n-1}$. This also shows that $\ker \tilde{\partial}_n = \ker \partial_n/\operatorname{im} \partial_{n+1} = H_n(\mathbf{C})$ and $\operatorname{coker} \tilde{\partial}_n = \ker \partial_{n-1}/\operatorname{im} \partial_n = H_{n-1}(\mathbf{C})$. Applying the snake lemma (2.2.3) to

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "\phi_n"] \arrow[d, "\partial_n"] & B_n \arrow[r, "\psi_n"] \arrow[d, "\partial_n"] & C_n \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\phi_{n-1}"] & B_{n-1} \arrow[r, "\psi_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

and using the above induced map $\tilde{\partial}_n$ we get the following commutative diagram

$$\begin{tikzcd}
& H_n(\mathbf{A}) \arrow[r] \arrow[d] & H_n(\mathbf{B}) \arrow[r] \arrow[d] & H_n(\mathbf{C}) \arrow[d] & \\
& \operatorname{coker} \partial_{n+1} \arrow[r] \arrow[d, "\tilde{\partial}_n"] & \operatorname{coker} \partial_{n+1} \arrow[r] \arrow[d, "\tilde{\partial}_n"] & \operatorname{coker} \partial_{n+1} \arrow[r] \arrow[d, "\tilde{\partial}_n"] & 0 \\
0 \arrow[r] & \ker \partial_{n-1} \arrow[r] \arrow[d] & \ker \partial_{n-1} \arrow[r] \arrow[d] & \ker \partial_{n-1} \arrow[d] & \\
& H_{n-1}(\mathbf{A}) \arrow[r] & H_{n-1}(\mathbf{B}) \arrow[r] & H_{n-1}(\mathbf{C}) &
\end{tikzcd}$$

with exact rows. We apply the snake lemma again to deduce the existence of the connecting homomorphism $\delta_n : H_n(\mathbf{C}) \to H_{n-1}(\mathbf{A})$. $\square$

We next show yet another way of constructing projective resolutions.

Lemma 3.2.5 (Horseshoe Lemma)**.** *Suppose we are given a commutative diagram*

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ \cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\varepsilon'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\ & & & & A \arrow[d, "\pi_A"] & \\ \cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\varepsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\ & & & & 0 & \end{tikzcd}$$

where the column is exact and the rows are projective resolutions. Set $P_n = P_n' \oplus P_n''$. Then the P_n assemble to form a projective resolution $\boldsymbol{P}$ of A and the right-hand column lifts to an exact sequence of complexes

$$\boldsymbol{P'} \overset{\iota}{\hookrightarrow} \boldsymbol{P} \overset{\pi}{\twoheadrightarrow} \boldsymbol{P''},$$

where $\iota_n : P_n' \to P_n$ and $\pi_n : P_n \to P_n''$ are the natural inclusion and projection respectively.

Proof. The name of this lemma stems from the fact that we have to fill in the horseshoe-shaped diagram. To do this, lift ε'' to a map $\beta : P_0'' \to A$ such that $\pi_A \beta = \varepsilon''$. By the universal property of the direct sum, we form $\varepsilon = \langle \beta, \iota_A \varepsilon' \rangle : P_0 \to A$ such that the diagram below commutes.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \ker \varepsilon' \arrow[r] \arrow[d] & P_0' \arrow[r, "\varepsilon'"] \arrow[d] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\ 0 \arrow[r] & \ker \varepsilon \arrow[r] \arrow[d] & P_0 \arrow[r, "\varepsilon"] \arrow[d] & A \arrow[d, "\pi_A"] & \\ 0 \arrow[r] & \ker \varepsilon'' \arrow[r] \arrow[d] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Since the right two columns of this diagram are short exact, we can apply the snake lemma to get the exact sequence

$$\ker \varepsilon'' \to \ker \varepsilon \to \ker \varepsilon' \to 0 \to \operatorname{coker} \varepsilon \to 0$$

which shows that the sequence of kernels is short exact and that $\operatorname{coker} \varepsilon = 0$, so that ε is surjective. This finishes the initial step and we find ourselves in the situation

$$\begin{tikzcd} & & 0 \arrow[d] & \\ \cdots \arrow[r] & P_1' \arrow[r, "\partial'"] & \ker \varepsilon' \arrow[r] \arrow[d] & 0 \\ & & \ker \varepsilon \arrow[d] & \\ \cdots \arrow[r] & P_1'' \arrow[r, "\partial''"] & \ker \varepsilon'' \arrow[r] \arrow[d] & 0 \\ & & 0 & \end{tikzcd}$$

from which we proceed inductively. $\square$

Using the last two results, we can now produce long exact sequences of derived functors.

Theorem 3.2.6. *Let $T : \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be an additive functor and let $A' \overset{\alpha'}{\hookrightarrow} A \overset{\alpha''}{\twoheadrightarrow} A''$ be a short exact sequence. Then there exist connecting homomorphisms*

$$\delta_n : L_nTA'' \to L_{n-1}TA', \quad n \geq 1$$

such that the following sequence is exact:

$$\cdots \to L_nTA' \overset{\alpha'_*}{\to} L_nTA \overset{\alpha''_*}{\to} L_nTA'' \overset{\delta_n}{\to} L_{n-1}TA' \to \cdots$$
$$\cdots \to L_1TA'' \overset{\delta_1}{\to} L_0TA' \overset{\alpha'_*}{\to} L_0TA \overset{\alpha''_*}{\to} L_0TA'' \to 0.$$

Proof. From the horseshoe lemma we obtain a short exact sequence of complexes $\mathbf{P}' \overset{\alpha'}{\hookrightarrow} \mathbf{P} \overset{\alpha''}{\twoheadrightarrow} \mathbf{P}''$, where $\mathbf{P}', \mathbf{P}, \mathbf{P}''$ are projective resolutions of A', A, A'' respectively. Since T is additive and since $P_n = P'_n \oplus P''_n$ for all $n \geq 0$, the sequence $T\mathbf{P}' \hookrightarrow T\mathbf{P} \twoheadrightarrow T\mathbf{P}''$ is short exact also. Hence (3.2.4) yields the definition of

$$\delta_n : H_n(T\mathbf{P}'') \to H_{n-1}(T\mathbf{P}')$$

and the exactness of the sequence. We skip the proof that this definition is independent of the chosen projective resolutions. For details see [Wei, p. 10]. □

We will also need to form right derived functors from contravariant functors. To do this, let $S : \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be contravariant and additive. In order to compute R^nSA for a given Λ-module A, we need to obtain a cochain complex. We thus choose a projective resolution $\mathbf{P}$ of A, form the cochain complex $S\mathbf{P}$ and take cohomology $R^nSA = H^n(S\mathbf{P})$. Analogously, we obtain left derived functors of contravariant functors via injective resolutions (since these will turn into chain complexes allowing us to define homology). A good way to remember these procedures is to recall that left derived functors are always associated with chain complexes and homology, whereas right derived functors are associated with cochain complexes and cohomology. We thus choose, depending on whether the functor we are looking at is covariant or contravariant, a resolution that will yield a chain or cochain complex under the given functor depending on whether we want to compute homology or cohomology. For example, we have shown before that $\mathrm{Hom}_\Lambda(-, B)$ is an additive contravariant functor. We could thus study its right derived functors, i.e. the cohomology groups associated with a $\mathrm{Hom}_\Lambda(\mathbf{P}, B)$-complex where $\mathbf{P}$ is a projective resolution of a given Λ-module A. This is what we want because the contravariant nature of $\mathrm{Hom}_\Lambda(-, B)$ will turn a projective resolution into a cochain complex. In fact, we define

$$\mathrm{Ext}^n_\Lambda(-, B) = R^n(\mathrm{Hom}_\Lambda(-, B)) \text{ for } n \geq 0. \tag{3.1}$$

Since $\mathrm{Hom}_\Lambda(-, B)$ is left exact, it follows from (3.2.3) that $\mathrm{Ext}^0_\Lambda(A, B) = \mathrm{Hom}_\Lambda(A, B)$. It can further be shown that $\mathrm{Ext}^1_\Lambda(A, B) = \mathrm{Ext}_\Lambda(A, B)$ [HS, p. 139] thus justifying our notation. From the dual of (3.2.6) we obtain for every given short exact sequence $A' \hookrightarrow A \twoheadrightarrow A''$ a long exact sequence

$$0 \to \mathrm{Hom}_\Lambda(A'', B) \to \mathrm{Hom}_\Lambda(A, B) \to \mathrm{Hom}_\Lambda(A', B) \overset{\delta^0}{\to} \cdots \tag{3.2}$$
$$\cdots \to \mathrm{Ext}^n_\Lambda(A'', B) \to \mathrm{Ext}^n_\Lambda(A, B) \to \mathrm{Ext}^n_\Lambda(A', B) \overset{\delta^n}{\to} \mathrm{Ext}^{n+1}_\Lambda(A'', B) \to \cdots .$$

This sequence is called the *long exact* Ext-*sequence in the first variable.* One can also show that $\operatorname{Ext}^n_\Lambda(-,-)$ is a bifunctor covariant in the second variable and that for every short exact sequence $B' \hookrightarrow B \twoheadrightarrow B''$ there is a long exact sequence

$$0 \to \operatorname{Hom}_\Lambda(A,B') \to \operatorname{Hom}_\Lambda(A,B) \to \operatorname{Hom}_\Lambda(A,B'') \xrightarrow{\delta_0} \cdots \tag{3.3}$$
$$\cdots \to \operatorname{Ext}^n_\Lambda(A,B') \to \operatorname{Ext}^n_\Lambda(A,B) \to \operatorname{Ext}^n_\Lambda(A,B'') \xrightarrow{\delta_n} \operatorname{Ext}^{n+1}_\Lambda(A,B') \to \cdots .$$

This sequence is called the *long exact* Ext-*sequence in the second variable.*

One could now of course consider the covariant additive functor $\operatorname{Hom}_\Lambda(A,-)$ and form its right derived functors. It turns out that these functors are naturally equivalent to the ones just discussed. We could thus write

$$\operatorname{Ext}^n_\Lambda(A,B) = R^n(\operatorname{Hom}_\Lambda(-,B))(A) = R^n(\operatorname{Hom}_\Lambda(A,-))(B) \text{ for } n \geq 0. \tag{3.4}$$

We then have the following noteworthy result:

Proposition 3.2.7. *If P is projective and if I is injective, then*

$$\operatorname{Ext}^n_\Lambda(P,B) = 0 = \operatorname{Ext}^n_\Lambda(A,I) \textit{ for } n \geq 1.$$

Proof. The first assertion is immediate from (3.2.2) and the second from (3.2.3) using the just mentioned natural equivalence. □

Next, we look at the additive covariant functor $A \otimes_\Lambda -$. We wish to study its homology groups and thus define

$$\operatorname{Tor}^\Lambda_n(A,-) = L_n(A \otimes_\Lambda -) \text{ for } n \geq 1.$$

Since $A \otimes_\Lambda -$ is right exact, it follows from (3.2.1) that $\operatorname{Tor}^\Lambda_0(A,B) = A \otimes_\Lambda B$. It is again provable that $\operatorname{Tor}^\Lambda_1(A,B) = \operatorname{Tor}^\Lambda(A,B)$ hence justifying our notation. Given short exact sequences $B' \hookrightarrow B \twoheadrightarrow B''$ and $A' \hookrightarrow A \twoheadrightarrow A''$ and using (3.2.6) we also obtain two long exact sequences

$$\cdots \to \operatorname{Tor}^\Lambda_n(A,B') \to \operatorname{Tor}^\Lambda_n(A,B) \to \operatorname{Tor}^\Lambda_n(A,B'') \xrightarrow{\delta_n} \operatorname{Tor}^\Lambda_{n-1}(A,B) \to \cdots \tag{3.5}$$
$$\cdots \to \operatorname{Tor}^\Lambda_1(A,B'') \xrightarrow{\delta_1} A \otimes_\Lambda B' \to A \otimes_\Lambda B \to A \otimes_\Lambda B'' \to 0$$

and

$$\cdots \to \operatorname{Tor}^\Lambda_n(A',B) \to \operatorname{Tor}^\Lambda_n(A,B) \to \operatorname{Tor}^\Lambda_n(A'',B) \xrightarrow{\delta_n} \operatorname{Tor}^\Lambda_{n-1}(A',B) \to \cdots \tag{3.6}$$
$$\cdots \to \operatorname{Tor}^\Lambda_1(A'',B) \xrightarrow{\delta_1} A' \otimes_\Lambda B \to A \otimes_\Lambda B \to A'' \otimes_\Lambda B \to 0.$$

Again, we could define similar left derived functors using $- \otimes_\Lambda B$ and show that they are naturally equivalent to the ones just defined above. We then write

$$\operatorname{Tor}^\Lambda_n(A,B) = L_n(A \otimes_\Lambda -)(B) = L_n(- \otimes_\Lambda B)(A).$$

Our final result is

Proposition 3.2.8. *If P is projective, then*

$$\operatorname{Tor}^\Lambda_n(P,B) = 0 = \operatorname{Tor}^\Lambda_n(A,P) \textit{ for } n \geq 1.$$

Proof. (3.2.2). □

Chapter 4

The Cohomology of Groups

Intuitively speaking, cohomology groups measure by how much the Hom-functor changes a long exact sequence into a non-exact one. Note that this is precisely what the Ext-functor from the last chapter is doing. The theory of derived functors is thus exactly the tool we need.

4.1 G-modules

Let G be a group written multiplicatively. A G-module is an abelian group A together with a group homomorphism $\sigma : G \to \operatorname{Aut} A$ where $\operatorname{Aut} A$ is the group of automorphisms of A. This means that group elements act as automorphisms on A. We shall denote this action by writing $g \cdot a$ or just ga if there is no ambiguity where $g \in G$ and $a \in A$. Let us say that the action of G on A is trivial if $ga = a$ for all $g \in G$ and $a \in A$. For example, the integers (in fact all abelian groups) can be regarded as a trivial G-module.

Also define the *group ring* $\mathbb{Z}G$ of G as the free abelian group with basis the elements of G and multiplication of basis elements provided by the group law on G. The elements of $\mathbb{Z}G$ are finite sums $\sum_{g\in G} n_g g$, where $n_g \in \mathbb{Z}$, i.e. all but finitely many $n_g \neq 0$. Multiplication is defined by

$$\left(\sum_{g\in G} n_g g\right)\left(\sum_{g'\in G} m'_g g'\right) = \sum_{g,g'\in G} n_g m'_{g'} g g'. \tag{4.1}$$

We will now show that G-modules can be identified with $\mathbb{Z}G$-modules.

Proposition 4.1.1. *Let R be a ring. Any function f from a group G into R s.t. $f(xy) = f(x)f(y)$ and $f(1) = 1_R$ can be extended uniquely to a ring homomorphism $f' : \mathbb{Z}G \to R$ such that $f'\iota = f$ where ι is the obvious embedding of G into $\mathbb{Z}G$.*

Proof. Define $f'\left(\sum_{g\in G} n_g g\right) = \sum_{g\in G} n_g f(g)$ from which the result follows. □

Given a G-module with group homomorphism $\sigma : G \to \operatorname{Aut} A$ and using the fact that the group $\operatorname{Aut} A$ can be seen as a subset of the ring $\operatorname{End} A$ of all endomorphisms of A, we extend σ to $\sigma' : \mathbb{Z}G \to \operatorname{End} A$ using the above universal property, making A into a module over $\mathbb{Z}G$. Conversely, if A is a module over $\mathbb{Z}G$, we note that any ring homomorphism takes invertible elements into invertible elements and all elements of G are invertible. So restricting the associated $\mathbb{Z}G$-module ring homomorphism to G yields the required group homomorphism of a G-module. Hence the concepts of G-module and $\mathbb{Z}G$-module are equivalent.

One map is of particular interest when studying group cohomology. Define $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$, called the *augmentation map*, by

$$\varepsilon\left(\sum_{g\in G} n_g g\right) = \sum_{g\in G} n_g.$$

The kernel of ε, denoted by IG, is called the *augmentation ideal* of G.

Lemma 4.1.2. *i) As an abelian group, IG is free on the set $W = \{g-1 : 1 \neq g \in G\}$.*

ii) Let S be a generating set for G. Then, as G-module, IG is generated by $S - 1 = \{s-1 : s \in S\}$.

Proof. i) Since $\{1\} \cup \{g-1 : 1 \neq g \in G\}$ is a basis for $\mathbb{Z}G$ as a free $\mathbb{Z}$-module, it follows that W is linearly independent. To see it spans IG, let $\sum_{g\in G} n_g g \in IG$, then $\sum_{g\in G} n_g g = \sum_{g\in G} n_g(g-1)$ proving *i)*.
ii) Since $xy - 1 = x(y-1) + (x-1)$ and $x^{-1} - 1 = -x^{-1}(x-1)$ and since any $g \in G$ can be written as $g = s_1^{\pm 1} \cdots s_k^{\pm 1}, s_i \in S$, it follows that any $g-1$ belongs to the module generated by $S - 1$. $\square$

We end this section with a lemma we shall need later on.

Lemma 4.1.3. *Let H be a subgroup of G. Then $\mathbb{Z}G$ is free as a H-module.*

Proof. Choose $\{g_i\}$, $g_i \in G$, a system of representatives of the left cosets of H in G. Since cosets induce a partition of G, we can write G as the disjoint union of g_iH. But it is clear that each part of $\mathbb{Z}G$ linearly spanned by g_iH for a fixed i is an H-module isomorphic to $\mathbb{Z}H$. Hence $\mathbb{Z}G \simeq \bigoplus \mathbb{Z}H$. $\square$

4.2 Definition of (Co)Homology

For convenience, we use A', A, A'' to denote left G-modules only and B', B, B'' to denote right ones. Regard $\mathbb{Z}$ as a trivial G-module and define the n^{th} *cohomology group* of G with coefficients in a G-module A by

$$H^n(G, A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A).$$

Also define the n^{th} *homology group* of G with coefficients in a G-module B by

$$H_n(G, B) = \mathrm{Tor}_n^{\mathbb{Z}G}(B, \mathbb{Z}).$$

From the discussion in past chapters it is clear that both $H^n(G, -)$ and $H_n(G, -)$ are covariant functors and that these groups are computed by taking a projective G-module resolution $\mathbf{P}$ of $\mathbb{Z}$, forming the complexes $\mathrm{Hom}_{\mathbb{Z}G}(\mathbf{P}, A)$ and $B \otimes_{\mathbb{Z}G} \mathbf{P}$ and calculating their co(homology).

Also from the discussion in past chapters, we have the following results:

Result 4.2.1. By (3.3), to a short exact sequence $A' \hookrightarrow A \twoheadrightarrow A''$ of G-modules there is a long exact cohomology sequence

$$0 \to H^0(G, A') \to H^0(G, A) \to H^0(G, A'') \to H^1(G, A') \to \cdots$$
$$\cdots \to H^n(G, A') \to H^n(G, A) \to H^n(G, A'') \to H^{n+1}(G, A') \to \cdots.$$

Result 4.2.2. By (3.5), to a short exact sequence $B' \hookrightarrow B \twoheadrightarrow B''$ there is a long exact homology sequence

$$\cdots \to H_n(G, B') \to H_n(G, B) \to H_n(G, B'') \to H_{n-1}(G, B') \to \cdots$$
$$\cdots \to H_1(G, B'') \to H_0(G, B') \to H_0(G, B) \to H_0(G, B'') \to 0.$$

Result 4.2.3. It follows from (3.2.7) and (3.2.8) that if A is injective and B is projective (or flat) then $H^n(G, A) = 0 = H_n(G, B)$ for all $n \geq 1$.

Example 4.2.4. As an example, we calculate the (co)homology groups of $H^n(G, A)$ and $H_n(G, B)$ when G is trivial. Note that if $G = \{1\}$, then $H^n(G, A) = \mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}, A)$. But since $\mathbb{Z}$ is free as a $\mathbb{Z}$-module, we know from (3.2.7) that $H^n(G, A) = \mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}, A) = 0$ for all $n \geq 1$. We also see that $H^0(G, A) = \mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}, A) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \simeq A$ by (3.2.3). Similarly, $H_n(G, B) = \mathrm{Tor}^{\mathbb{Z}}_n(B, \mathbb{Z}) = 0$ for all $n \geq 1$ by (3.2.8) and $H_0(G, B) = \mathrm{Tor}^{\mathbb{Z}}_0(B, \mathbb{Z}) \simeq B \otimes_{\mathbb{Z}} \mathbb{Z} \simeq B$ by (3.2.1). Hence we have determined all (co)homology groups.

4.3 The Zeroth (Co)Homology Groups

As we already saw in the last calculation, $H^0(G, A) \simeq \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$. Such a G-module homomorphism ϕ is determined by the image of $1_{\mathbb{Z}}$, $\phi(1) = a$. But since ϕ is a G-module homomorphism, $g \cdot a = g \cdot \phi(1) = \phi(g \cdot 1) = \phi(1) = a$ since g is a group automorphism of A. In fact, it is not hard to see that ϕ as a homomorphism of abelian groups is a G-module homomorphism if and only if $\phi(1) = a$ remains fixed under the action of G. Let us then write $A^G = \{a \in A : g \cdot a = a \text{ for all } g \in G\}$ and we have

$$H^0(G, A) \simeq \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A) \simeq A^G. \tag{4.2}$$

A^G is called the *invariant subgroup* of A.

Next let us look at H_0. We have $H_0(G, B) \simeq B \otimes_{\mathbb{Z}G} \mathbb{Z} \simeq B \otimes_{\mathbb{Z}G} (\mathbb{Z}G/IG) \simeq B/B \cdot IG$ where $B \cdot IG = \{b(g-1) : b \in B, g \in G\}$. Let us denote $B/B \cdot IG$ by B_G then

$$H_0(G, B) \simeq B \otimes_{\mathbb{Z}G} \mathbb{Z} \simeq B_G. \tag{4.3}$$

B_G is called the *coinvariant subgroup* of A.

Example 4.3.1. We have $H_0(G, \mathbb{Z}) = \mathbb{Z}/\mathbb{Z} \cdot IG = \mathbb{Z}$ since every element of G acts trivially on $\mathbb{Z}$ and so $Z \cdot IG = 0$. As another example, $H_0(G, \mathbb{Z}G) = \mathbb{Z}G/\mathbb{Z}G \cdot IG = \mathbb{Z}G/IG \simeq \mathbb{Z}$ since IG is an ideal of $\mathbb{Z}G$.

4.4 The First (Co)Homology Groups with Trivial Coefficient Modules

The first (co)homology groups can be calculated by choosing G-free presentations rather than G-free resolutions of $\mathbb{Z}$ since, as we showed, $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, A) \simeq \mathrm{Ext}_{\mathbb{Z}G}(\mathbb{Z}, A)$. Let us take the obvious projective resolution to do so:

$$IG \overset{\iota}{\hookrightarrow} \mathbb{Z}G \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}. \tag{4.4}$$

We get exact sequences

$$0 \to H_1(G, B) \to B \otimes_{\mathbb{Z}G} IG \overset{\iota_*}{\to} B \otimes_{\mathbb{Z}G} \mathbb{Z}G \to H_0(G, B) \to 0,$$
$$0 \to H^0(G, A) \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \overset{\iota^*}{\to} \mathrm{Hom}_{\mathbb{Z}G}(IG, A) \to H^1(G, A) \to 0.$$

This implies

$$H_1(G, B) \simeq \ker(\iota_* : B \otimes_{\mathbb{Z}G} IG \to B), \tag{4.5}$$
$$H^1(G, A) \simeq \mathrm{coker}(\iota^* : A \to \mathrm{Hom}_{\mathbb{Z}G}(IG, A)). \tag{4.6}$$

The maps ι_* and ι^* are the composition of the usual induced maps with the respective isomorphism, i.e.

$$\iota_*(b \otimes (g-1)) = bg - b = b(g-1), \tag{4.7}$$
$$\iota^*(a)(g-1) = ga - a = (g-1)a \tag{4.8}$$

where $a \in A, b \in B$ and $g \in G$.

Let us now compute these groups when A and B are trivial G-modules. Let us start with H_1. If B is a trivial G-module then ι_* is the zero homomorphism and hence $H_1(G, B) \simeq B \otimes_{\mathbb{Z}G} IG$. To compute this group, we look at the kernel of the map $B \otimes_{\mathbb{Z}} IG \to B \otimes_{\mathbb{Z}G} IG$. This kernel is generated by elements of the form $b \otimes g'(g-1) - bg' \otimes (g-1)$ for $b \in B$ and $g, g' \in G$. But $bg' \otimes (g-1) = b \otimes (g-1)$ since B is trivial. Hence the kernel of this map is generated by all elements of the form $b \otimes (g'-1)(g-1)$ which is just $B \otimes_{\mathbb{Z}} (IG)^2$. Hence $B \otimes_{\mathbb{Z}G} IG \simeq B \otimes_{\mathbb{Z}} IG/(IG)^2$. Finally, let $G_{ab} = G/G'$ denote the quotient of G by its *commutator subgroup* $G' = [G, G]$, i.e. the subgroup generated by all elements of the form $g^{-1}g'^{-1}gg', g, g' \in G$. By the lemma following this paragraph we then have

$$H_1(G, B) \simeq B \otimes_{\mathbb{Z}} \left(IG/(IG)^2\right) \simeq B \otimes_{\mathbb{Z}} (G/G') = B \otimes_{\mathbb{Z}} G_{ab} \tag{4.9}$$

and in particular when $B = \mathbb{Z}$,

$$H_1(G, \mathbb{Z}) \simeq G/G' = G_{ab}$$

which is well known to topologists. This is in fact the proof that the first homology group is the quotient of the fundamental group by its commutator subgroup. Here is the promised lemma:

Lemma 4.4.1. *$IG/(IG)^2 \simeq G_{ab}$*

Proof. For details see [HS, p. 192]. The idea is to define a map on the set W given by (4.1.2), extend this map to IG and then use the fact that

$$(g-1)(g'-1) = (gg'-1) - (g-1) - (g'-1)$$

for all $g \in G$ to show that this map and an inverse to it defined similarly induce the required homomorphisms needed to prove the lemma via the first homomorphism theorem. $\square$

Let us now turn to H^1. Again, ι^* will be the zero homomorphism if A is a trivial G-module. Hence $H^1(G,A) \simeq \mathrm{Hom}_{\mathbb{Z}G}(IG,A)$. Moreover, $\phi : IG \to A$ is a G-module homomorphism if and only if $\phi(g(g'-1)) = g\phi(g'-1) = \phi(g'-1)$ for $g, g' \in G$. Hence if and only if $\phi((g-1)(g'-1)) = 0$. Thus by the previous lemma,

$$H^1(G,A) \simeq \mathrm{Hom}_{\mathbb{Z}}(IG/(IG)^2, A) \simeq \mathrm{Hom}_{\mathbb{Z}}(G_{ab}, A). \tag{4.10}$$

Note that this implies $H^1(G,A) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_1(G,\mathbb{Z}),A)$ whenever A is a trivial G-module.

4.5 Shapiro's Lemma, Cyclic and Free Groups and the Tate Cohomology

Let H be a subgroup of G and let A be a left H-module. As we discussed before, $\mathbb{Z}G \otimes_{\mathbb{Z}H} A$ and $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$ have a left $\mathbb{Z}G$-module structure. We define $\mathbb{Z}G \otimes_{\mathbb{Z}H} A = \mathrm{Ind}_H^G(A)$ and call this $\mathbb{Z}G$-module the *induced G-module*. We also define $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A) = \mathrm{Coind}_H^G(A)$ and call it the *coinduced G-module*.

Lemma 4.5.1 (Shapiro's Lemma)**.** *Let H be a subgroup of G and let A be an H-module, then $H_n(G, \mathrm{Ind}_H^G(A)) \simeq H_n(H,A)$ and $H^n(G, \mathrm{Coind}_H^G(A)) \simeq H^n(H,A)$.*

Proof. By (4.1.3) we know that $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module. Hence, any projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$, $P \to \mathbb{Z}$, will also be a projective $\mathbb{Z}H$-resolution of $\mathbb{Z}$. But we have $P \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}H} A) \simeq P \otimes_{\mathbb{Z}H} A$. Hence

$$\begin{aligned} H_n(G, \mathrm{Ind}_H^G(A)) &= \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}G \otimes_{\mathbb{Z}H} A, \mathbb{Z}) \\ &\simeq \mathrm{Tor}_n^{\mathbb{Z}H}(A, \mathbb{Z}) \\ &= H_n(H,A). \end{aligned}$$

Similarly, we have $\mathrm{Hom}_{\mathbb{Z}G}(P, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}H, A)) \simeq \mathrm{Hom}_{\mathbb{Z}H}(P,A)$ by an obvious generalization of (1.1.4). Hence,

$$\begin{aligned} H^n(G, \mathrm{Coind}_H^G(A) &= \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)) \\ &\simeq \mathrm{Ext}^n_{\mathbb{Z}H}(\mathbb{Z}, A) \\ &= H^n(H,A). \end{aligned}$$

$\square$

Applying this result to $H = 1$, i.e. A is an abelian group in this case, we get

Corollary 4.5.2.

$$H_n(G, \mathbb{Z}G \otimes_{\mathbb{Z}} A) = H^n(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A)) = \begin{cases} A & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Proof. We use Shapiro and then the problem is just (4.2.4). □

We next prove another important result with the aid of Shapiro's lemma that we shall need in due course.

Proposition 4.5.3. *If G is a finite group, then $H^n(G, \mathbb{Z}G \otimes_{\mathbb{Z}} A) = 0$ for $n \neq 0$ and for all abelian groups A.*

To prove this we go on a little tangent. Let $\alpha : R \to S$ be a ring homomorphism. S as an S-module can also be regarded as a right R-module via α. Define $s \cdot r = s\alpha(r)$. This R-module is said to be obtained by *restriction of scalars*.

We can also *"extend scalars"* in the following way. Let M be any left R-module and consider $S \otimes_R M$ where S has the above right R-module structure. Since the natural action of S on itself commutes with this right R-module action, $S \otimes_R M$ also has a left S-module structure given by $s \cdot (s' \otimes_R m) = ss' \otimes_R m$. We have a natural inclusion map $\iota : M \to S \otimes_R M$ given by $m \mapsto (1, m)$. ι is an R-module map since $\iota(rm) = 1 \otimes_R rm = \alpha(r) \otimes_R m = \alpha(r) \cdot (1 \otimes_R m) = \alpha(r) \cdot \iota(m)$ and we have the following universal property:

Proposition 4.5.4. *Given an S-module N and an R-module homomorphism $f : M \to N$, there exists a unique S-module map $g : S \otimes_R M \to N$ such that the diagram*

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "f"'] & S \otimes_R M \arrow[dl, dashed, "g"] \\ N & \end{tikzcd}$$

commutes.

Proof. Note that if such a g exists, $g(s \otimes_R m) = sg(1 \otimes_R m) = sg(\iota(m)) = sf(m)$ which immediately shows uniqueness and also tells us to define g as $g(s \otimes_R m) = sf(m)$ which is clearly an S-module homomorphism.
Let us check this definition is well-defined. Suppose $s \otimes_R m = s' \otimes_R m'$, i.e. either $s' \otimes_R m' = (s \cdot r) \otimes_R m'$ such that $rm' = m$ or $s' \otimes_R m' = s' \otimes_R rm$ such that $s' \cdot r = s$. Then either

$$\begin{aligned} g(s' \otimes_R m') &= g((s \cdot r) \otimes_R m') \\ &= (s \cdot r) f(m') \\ &= s(r \cdot f(m')) \\ &= sf(rm') \quad \text{since } f \text{ is an } R\text{-module homomorphism} \\ &= sf(m) \end{aligned}$$

or

$$\begin{aligned} g(s' \otimes_R m') &= g(s' \otimes_R rm) \\ &= s' f(rm) \\ &= s'(r \cdot f(m)) \quad \text{since } f \text{ is an } R\text{-module homomorphism} \\ &= (s' \cdot r) f(m) \\ &= sf(m) \end{aligned}$$

which shows that g is indeed well-defined. We also have $g\iota(m) = g(1 \otimes_R m) = f(m)$ which was the last missing link in proving the proposition. $\square$

Having such an S-module extension homomorphism g to a given R-module homomorphism f allows us to prove the following key lemma.

Lemma 4.5.5. *If* $[G : H] < \infty$ *then* $\mathrm{Ind}_H^G(A) \simeq \mathrm{Coind}_H^G(A)$.

Proof. We define an H-module homomorphism $\phi_0 : A \to \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$ by

$$\phi_0(a)(g) = \begin{cases} ga & g \in H \\ 0 & g \notin H. \end{cases}$$

That this definition truly yields an H-module homomorphism is quickly checked by recalling that the left H-module structure on $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$ is given by $h \cdot \psi(g) = \psi(gh)$ for $\psi \in \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$ and $h \in H$. We now regard $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$ as a left $\mathbb{Z}G$-module and use the above universal property to extend ϕ_0 to a unique $\mathbb{Z}G$-module homomorphism $\phi : \mathbb{Z}G \otimes_{\mathbb{Z}H} A \to \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$ that makes the diagram

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "\phi_0"'] & \mathbb{Z}G \otimes_{\mathbb{Z}H} A \arrow[dl, dashed, "\phi"] \\ \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A) & \end{tikzcd}$$

commute. But we saw that $\mathbb{Z}G \simeq \bigoplus \mathbb{Z}H$. Hence

$$\begin{aligned} \mathbb{Z}G \otimes_{\mathbb{Z}H} A &\simeq (\bigoplus \mathbb{Z}H) \otimes_{\mathbb{Z}H} A \\ &\simeq \bigoplus (\mathbb{Z}H \otimes_{\mathbb{Z}H} A) \\ &\simeq \bigoplus A \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A) &\simeq \mathrm{Hom}_{\mathbb{Z}H}(\bigoplus \mathbb{Z}G, A) \\ &\simeq \prod \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}H, A) \\ &\simeq \prod A. \end{aligned}$$

So ϕ is just the canonical inclusion of a sum into a product which is an epimorphism if $[G : H]$ is finite since then there will be exactly $[G : H]$ summands/factors. $\square$

Equipped with this lemma, proposition (4.5.3) now follows easily:

Proof of (4.5.3).

$$H^n(G, \mathbb{Z}G \otimes_{\mathbb{Z}} A) = H^n(G, \mathrm{Ind}_H^G(A)) \simeq H^n(G, \mathrm{Coind}_H^G(A)) = 0 \text{ if } n \neq 0$$

by (4.5.2). $\square$

Before we make further use of these results let us discuss the (co)homology of cyclic groups.

Let G be a finite group. Define the *norm element* N of the group ring $\mathbb{Z}G$ as the sum $N = \sum_{g\in G} g$. We claim that $N \in (\mathbb{Z}G)^G$. Indeed, let $h \in G$ then $hN = \sum_{g\in G} hg = \sum_{g'\in G} g' = N$. Similarly, $Nh = N$. For convenience, we will allow ourselves to omit the set over which the sum is to be taken when this is clear.

Lemma 4.5.6. *$H^0(G, \mathbb{Z}G) = (\mathbb{Z}G)^G = \mathbb{Z} \cdot N$ is a two sided ideal of $\mathbb{Z}G$ generated by N.*

Proof. The first equality has been shown before. To prove the second, let $a = \sum n_g g \in (\mathbb{Z}G)^G$, then $g'a = a$ for all $g' \in G$. Hence $\sum n_g g' g = \sum n_g g$ which can only be true for all $g' \in G$ if all coefficients n_g are equal. Hence $a = \sum n_g g = n \sum g = nN$ for some $n \in \mathbb{Z}$. So $(\mathbb{Z}G)^G \subseteq \mathbb{Z} \cdot N$. The reverse inclusion was discussed above. Hence the equality.
To see $\mathbb{Z} \cdot N$ is an ideal, let $\sum n_g g \in G$ and let $nN \in \mathbb{Z} \cdot N$ then $(nN)(\sum n_g g) = \sum n n_g N = mN$ for some $m \in \mathbb{Z}$. Similarly for $(\sum n_g g)(nN)$. Hence the claim. □

Remark 4.5.7. If G is infinite, then $H^0(G, \mathbb{Z}G) = (\mathbb{Z}G)^G = 0$ since all coefficients of an element in $(\mathbb{Z}G)^G$ are the same but only finitely many of them are nonzero.

Remark 4.5.8. If G is finite of order m and A is a trivial G-module then $(\sum g)a = ma$ for all $a \in A$.

For any G-module M, we define the *norm map* $N : M \to M$ by $m \mapsto \sum_{g\in G} gm$. We immediately see from the discussion before that $g'N(m) = N(m) = N(g'm)$ for all $g' \in G$. Hence $\operatorname{im} N \subseteq M^G$. Moreover, let $(g-1)m \in IG \cdot M$, then $N((g-1)m) = N(gm) - N(m) = N(m) - N(m) = 0$. Hence $IG \cdot M \subseteq \ker N$.

In particular, if we let $M = \mathbb{Z}G$ we get $IG \subseteq \ker N$. Conversely, let $\sum n_g g \in \mathbb{Z}G$ and suppose that $0 = N(\sum n_g g) = \sum n_g N$. Then we must have $\sum_{g\in G} n_g = 0$ i.e. $\sum n_g g \in \ker \varepsilon = IG$. Hence $IG = \ker N$. We also have $\operatorname{im} N \subseteq (\mathbb{Z}G)^G$ and if $a \in (\mathbb{Z}G)^G = \mathbb{Z} \cdot N$, then $a = nN = \sum_{g\in G} ng = (\sum g)(n \cdot 1_G) = N(n \cdot 1_G)$. Hence $a \in \operatorname{im} N$ and $\operatorname{im} N = (\mathbb{Z}G)^G$.

We are now ready to study cyclic groups. Let $G = C_m$ be the cyclic group of order m with generator σ. The norm in C_m is $N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{m-1}$. So $0 = \sigma^m - 1 = (\sigma - 1)N$ in $\mathbb{Z}C_m$. Hence the map $\sigma - 1 : \mathbb{Z}C_m \to \mathbb{Z}C_m$ satisfies $\operatorname{im} N \subseteq \ker(\sigma - 1)$.

By the previous discussion we have the following exact sequence:

$$0 \to IC_m \xrightarrow{\iota} \mathbb{Z}C_m \xrightarrow{N} \mathbb{Z} \cdot N \to 0.$$

We claim the following sequence is also exact:

$$0 \to \mathbb{Z} \cdot N \xrightarrow{\iota} \mathbb{Z}C_m \xrightarrow{\sigma-1} IC_m \to 0.$$

To see this, note that IC_m is generated by $\{(\sigma^i - 1) : 1 \leq i \leq m-1\}$. But we have $\sigma^i - 1 = (\sigma - 1)(\sum_{k=0}^{i-1} \sigma^k)$ which shows that $\sigma - 1$ is surjective. We have already shown that $\mathbb{Z} \cdot N = \operatorname{im} N \subseteq \ker(\sigma - 1)$. So we just need $\ker(\sigma - 1) \subseteq \mathbb{Z} \cdot N$. To see this, recall that $\mathbb{Z} \cdot N = (\mathbb{Z}C_m)^{C_m}$ so suppose $(\sigma - 1)x = 0$ for some $x \in \mathbb{Z}C_m$. Then $\sigma x = x$ hence $\sigma^j x = x$ for all j and so $x \in (\mathbb{Z}C_m)^{C_m}$.

We now obtain a periodic C_m-free resolution of $\mathbb{Z}$ by splicing these two sequences together:

$$\begin{tikzcd}[column sep=small]
 & & & & 0 & & 0 \arrow[dl] & & 0 & & 0 \arrow[dl] \\
 & & & & & \mathbb{Z}\cdot N \arrow[ul] \arrow[dl, "\iota"'] & & & & \mathbb{Z}\cdot N \arrow[ul] \arrow[dl, "\iota"'] & \\
0 & \mathbb{Z} \arrow[l] & \mathbb{Z}C_m \arrow[l, "\varepsilon"'] & & \mathbb{Z}C_m \arrow[ll, "\sigma-1"'] \arrow[dl, "\sigma-1"] & & \mathbb{Z}C_m \arrow[ll, "N"'] \arrow[ul, "N"'] & & \mathbb{Z}C_m \arrow[ll, "\sigma-1"'] \arrow[dl, "\sigma-1"] & & \cdots . \arrow[ll, "N"'] \arrow[ul, "N"'] \\
 & & & IC_m \arrow[ul, "\iota"] \arrow[dl] & & & & IC_m \arrow[ul, "\iota"] \arrow[dl] & & & \\
 & & 0 & & 0 \arrow[ul] & & 0 & & 0 \arrow[ul] & &
\end{tikzcd} \tag{4.11}$$

For any C_m-module A we then apply $A \otimes_{\mathbb{Z}C_m} -$ to find

$$0 \longleftarrow \mathbb{Z} \otimes_{\mathbb{Z}C_m} A \xleftarrow{\varepsilon_*} \mathbb{Z}C_m \otimes_{\mathbb{Z}C_m} A \xleftarrow{(\sigma-1)_*} \mathbb{Z}C_m \otimes_{\mathbb{Z}C_m} A \xleftarrow{N_*} \mathbb{Z}C_m \otimes_{\mathbb{Z}C_m} A \xleftarrow{(\sigma-1)_*} \cdots$$

and hence

$$0 \longleftarrow \mathbb{Z} \otimes_{\mathbb{Z}C_m} A \xleftarrow{\tilde{\varepsilon}_*} A \xleftarrow{\sigma-1} A \xleftarrow{N} A \xleftarrow{\sigma-1} \cdots$$

as can be readily checked using the usual definition of the isomorphism between $\mathbb{Z}C_m \otimes_{\mathbb{Z}C_m} A$ and A. Noting that $\ker(\sigma-1) = A^{C_m}$, one obtains the following homology groups:

$$H_n(C_m, A) = \begin{cases} A_{C_m} = A/A \cdot IC_m & \text{if } n = 0 \\ A^{C_m}/N(A) & \text{if } n \text{ is odd} \\ \ker N/(\sigma-1)A & \text{if } n \text{ is even .} \end{cases} \tag{4.12}$$

Next we apply $\mathrm{Hom}_{\mathbb{Z}C_m}(-, A)$ to the constructed resolution and find the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}C_m}(\mathbb{Z}, A) \xrightarrow{\varepsilon^*} \mathrm{Hom}_{\mathbb{Z}C_m}(\mathbb{Z}C_m, A) \xrightarrow{(\sigma-1)^*} \mathrm{Hom}_{\mathbb{Z}C_m}(\mathbb{Z}C_m, A) \xrightarrow{N^*} \cdots .$$

After using the standard isomorphism $\mathrm{Hom}_{\mathbb{Z}C_m}(\mathbb{Z}C_m, A) \simeq A$, we obtain

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}C_m}(\mathbb{Z}, A) \xrightarrow{\tilde{\varepsilon}^*} A \xrightarrow{\sigma-1} A \xrightarrow{N} A \xrightarrow{\sigma-1} \cdots$$

which yields the cohomology groups

$$H^n(C_m, A) = \begin{cases} A^{C_m} & \text{if } n = 0 \\ \ker N/(\sigma-1)A & \text{if } n \text{ is odd} \\ A^{C_m}/N(A) & \text{if } n \text{ is even .} \end{cases} \tag{4.13}$$

We have thus completely classified the (co)homology groups for cyclic groups.

Example 4.5.9. Taking $A = \mathbb{Z}$, i.e. a trivial C_m-module, we find recalling remark (4.5.8)

$$H_n(C_m, \mathbb{Z}) = \begin{cases} \mathbb{Z}/\mathbb{Z} \cdot IC_m = \mathbb{Z}/0 = \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{C_m}/N(\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} = C_m & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$H^n(C_m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n \text{ is odd} \\ C_m & \text{if } n \text{ is even .} \end{cases}$$

Let us now generalize the construction that led to (4.11). Given an arbitrary G-module M and the norm map N acting thereon, we noticed before that $\operatorname{im} N \subseteq M^G$ and $IG \cdot M \subseteq \ker N$. We thus obtain the sequence

$$\begin{tikzcd} & & M \arrow[d, "\pi"'] \arrow[dr, "N"] & & & \\ 0 \arrow[r] & \ker N/IG \cdot M \arrow[r, "\iota"'] & M/IG \cdot M \arrow[r, dashed, "\Psi=N"'] & M^G \arrow[r, "\pi"'] & M^G/N(M) \arrow[r] & 0 \end{tikzcd}$$

where π denotes the usual projection map. The map $\Psi = N$ is given to us by the first homomorphism theorem since $IG \cdot M \subseteq \ker N$. The theorem further tells us $\ker \Psi = \ker N/IG \cdot M$ and $\operatorname{im} \Psi = \operatorname{im} N$ which is exactly what we need for the above sequence to be exact. By (4.2) and (4.3) we can then write

$$0 \to \ker N/IG \cdot M \to H_0(G, M) \xrightarrow{N} H^0(G, M) \to M^G/N(M) \to 0. \tag{4.14}$$

Now let $M' \hookrightarrow M \twoheadrightarrow M''$ be an exact sequence of G-modules. Using (4.14) together with (4.2.1) and (4.2.2) we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd} \cdots \arrow[r] & H_1(G, M'') \arrow[r] & H_0(G, M') \arrow[r] \arrow[d, "N"'] & H_0(G, M) \arrow[r] \arrow[d, "N"'] & H_0(G, M'') \arrow[r] \arrow[d, "N"'] & 0 \\ & 0 \arrow[r] & H^0(G, M') \arrow[r] & H^0(G, M) \arrow[r] & H^0(G, M'') \arrow[r] & H^1(G, M') \arrow[r] & \cdots . \end{tikzcd} \tag{4.15}$$

We are almost in a position to apply the snake lemma (2.2.3) in order to obtain an interesting exact sequence called the *Tate complex*. However, before we can do so we need a slightly extended version:

Lemma 4.5.10 (Extended Snake Lemma)**.** *Every commutative diagram with exact rows*

$$\begin{tikzcd} & M' \arrow[r, "f"] \arrow[d, "d'"'] & M \arrow[r, "g"] \arrow[d, "d"'] & M'' \arrow[r] \arrow[d, "d''"'] & 0 \\ 0 \arrow[r] & N' \arrow[r, "f"'] & N \arrow[r, "g"'] & N'' & \end{tikzcd}$$

gives rise to an exact sequence

$$0 \to \ker f \to \ker d' \to \ker d \to \ker d'' \xrightarrow{\delta} \operatorname{coker} d' \to \operatorname{coker} d \to \operatorname{coker} d'' \to \operatorname{coker} g \to 0.$$

Proof. We have already proved exactness at all but two positions. Note that showing $\ker f \subseteq \ker d'$ will at once show exactness at $\ker d'$. So suppose that $x \in \ker f$, i.e. $f(x) = 0$, then $d \circ f(x) = 0 = f \circ d'(x)$. Hence $d'(x) = 0$ since f is injective and so $x \in \ker d'$. Similarly, note that it suffices to show that $d''(M'') \subseteq g(N)$ to prove exactness at $\operatorname{coker} d''$ by the way the transition maps between cokernels are naturally induced. The actual calculation is just as routine as the one above and will be omitted. □

Applying the extended snake lemma to (4.15) we obtain the following long exact sequence for $-\infty < n < \infty$

$$\cdots \to \hat{H}^n(G, M') \to \hat{H}^n(G, M) \to \hat{H}^n(G, M'') \xrightarrow{\delta} \hat{H}^{n+1} \to \cdots$$

where

$$\hat{H}^n(G, M) = \begin{cases} H^n(G, M) & \text{if } n > 0 \\ M^G/N(M) & \text{if } n = 0 \\ \ker N/IG \cdot M & \text{if } n = -1 \\ H_{-n-1}(G, M) & \text{if } r < -1. \end{cases}$$

The groups $\hat{H}^n(G, M)$ are called the *Tate cohomology groups*. Let us take a closer look at these groups. We claim:

Proposition 4.5.11. *If G is finite and P is a projective G-module, then $\hat{H}^n(G, P) = 0$ for all n.*

Proof. It suffices to prove the result for free G-modules since every projective module is a direct summand in a free module and all operations involved in forming the Tate cohomology groups are additive. Such modules are of the form $P = \mathbb{Z}G \otimes_{\mathbb{Z}} F$ with F free abelian since P being G-free implies $P \simeq \bigoplus \mathbb{Z}G \simeq (\bigoplus \mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Z} \simeq \mathbb{Z}G \otimes_{\mathbb{Z}} (\bigoplus \mathbb{Z})$. We proved before in (4.5.2) and (4.5.3) using Shapiro that $H_n(G, \mathbb{Z}G \otimes_{\mathbb{Z}} A) = 0$ for $n \geq 1$ when A is abelian and $H^n(G, \mathbb{Z}G \otimes_{\mathbb{Z}} A) = 0$ for $n \geq 1$ when G is finite. It remains to show that $\hat{H}^0(G, P)$ and $\hat{H}^{-1}(G, P)$ also vanish.
Since $P^G = (\mathbb{Z}G)^G \otimes_{\mathbb{Z}} F = N(\mathbb{Z}G) \otimes_{\mathbb{Z}} F = N(P)$, we have $P^G/N(P) = \hat{H}^0(G, P) = 0$. On the other hand we see that $P_G = P/IG \cdot P = (\mathbb{Z}G \otimes_{\mathbb{Z}} F)/(IG \otimes_{\mathbb{Z}} F) = (\mathbb{Z}G/IG) \otimes_{\mathbb{Z}} F = \mathbb{Z} \otimes_{\mathbb{Z}} F = F$ and since we know that N acts on abelian groups such as F by multiplication by the order of the group G we have, $N(x) = 0 \Leftrightarrow x = 0$ for $x \in P_G$ whence $\hat{H}^{-1}(G, P) = \ker N/IG \cdot P = 0$ which completes the proof. □

We apply this result to the so-called process of *dimension-shifting*. Given a G-module A, choose a projective presentation $K \hookrightarrow P \twoheadrightarrow A$ of A. As in the construction of the Tate groups, we obtain the long exact sequence

$$\cdots \to \hat{H}^n(G, K) \to \hat{H}^n(G, P) \to \hat{H}^n(G, A) \to \hat{H}^{n+1}(G, K) \to \cdots .$$

But all $\hat{H}^n(G, P) = 0$. So we conclude $\hat{H}^n(G, A) = \hat{H}^{n+1}(G, K)$ for all $n \in \mathbb{Z}$. This result shows that the Tate cohomology theory is completely determined by any one of the functors $\hat{H}^n$.

The last proposition in this section will enable us to compute the (co)homology groups for free groups by showing that (4.4) is not just a G-free presentation but also a G-free resolution of $\mathbb{Z}$.

Proposition 4.5.12. *Let G be a free group on X, then IG is a free $\mathbb{Z}G$-module with basis set $X - 1 = \{x - 1 : x \in X\}$.*

Proof. For details see [Wei, p. 169]. The idea is to use the fact that we already proved in (4.4.1) that $W = \{g - 1 : 1 \neq g \in G\}$ is a basis for IG as a $\mathbb{Z}$-module. Weibel shows that $\{g(x - 1) : g \in G, x \in X\}$ is another $\mathbb{Z}$-basis from which clearly follows that $X - 1 = \{x - 1 : x \in X\}$ is a $\mathbb{Z}G$-basis. □

Corollary 4.5.13. *If G is a free group on X, then $H_n(G, A) = H^n(G, A) = 0$ for $n \geq 2$.*

Proof. $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$ is a G-free resolution of $\mathbb{Z}$. $\square$

4.6 Derivations, Semidirect Products and Hilbert's Theorem 90

In this section, we shall give an interpretation of $H^1(G, A)$.

A function $d : G \to A$ from a group G into a left G-module A with the property $d(xy) = d(x) + x \cdot d(y)$ for all $x, y \in G$ is said to be a *derivation* or *crossed homomorphism.*

The family $\mathrm{Der}(G, A)$ of all such derivations is an abelian group if we define $(d + d')(x) = d(x) + d'(x)$ and the zero derivation being the identity. We note that if $\alpha : A \to A'$ is a G-module homomorphism then $\alpha \circ d : G \to A'$ is again a derivation. This is just what we need to see that $\mathrm{Der}(G, -)$ is a functor from the category of G-modules to the category of abelian groups.

Remark 4.6.1. (1) Note that $d(1) = 0$ for any derivation d.

(2) The usual Leibniz rule would suggest $d(xy) = x \cdot d(y) + d(x) \cdot y$. The term "derivation" thus seems more reasonable if we think of A as having the trivial right G-module structure in addition to its left G-module structure.

(3) If A is a trivial G-module, then any derivation from G to A is just a group homomorphism.

Theorem 4.6.2. *There is a natural isomorphism between* $\mathrm{Der}(G, A)$ *and* $\mathrm{Hom}_G(IG, A)$.

Proof. Let $d : G \to A$ be a derivation. Define $\phi_d : IG \to A$ by $\phi_d(y - 1) = dy$ for $y \in G$. We need to show that ϕ_d is a G-module homomorphism. Let $x, y \in G$ then

$$\begin{aligned} \phi_d(x(y-1)) &= \phi_d((xy-1) - (x-1)) \\ &= d(xy) - d(x) \\ &= d(x) + x \cdot d(y) - d(x) \\ &= x \cdot d(y) \\ &= x \cdot \phi_d(y-1). \end{aligned}$$

Conversely, given a G-module homomorphism $\phi : IG \to A$, define $d_\phi : G \to A$ by $d_\phi(y) = \phi(y - 1)$. We must show that d_ϕ defined as such is a derivation:

$$\begin{aligned} d_\phi(xy) &= \phi(xy - 1) \\ &= \phi(x(y-1) + (x-1)) \\ &= x \cdot \phi(y-1) + \phi(x-1) \\ &= x \cdot d_\phi(y) + d_\phi(x). \end{aligned}$$

It is clear that $d \mapsto \phi_d$ and $\phi \mapsto d_\phi$ are mutual inverses. Hence the isomorphism holds. Denote this isomorphism by η then η is easily seen to be natural: let $\alpha : A \to A'$ be a G-module homomorphism. Then $\eta(\alpha \circ d)(y-1) = \alpha \circ d(y) = \alpha \circ \eta(d)(y-1)$ and the diagram

$$\begin{tikzcd} \mathrm{Der}(G,A) \arrow[r, "\eta"] \arrow[d, "\alpha_*"'] & \mathrm{Hom}_G(IG,A) \arrow[d, "\alpha_*"] \\ \mathrm{Der}(G,A') \arrow[r, "\eta"] & \mathrm{Hom}_G(IG,A') \end{tikzcd}$$

commutes. □

From (4.6) and (4.8), we know that $H^1(G,A)$ is the quotient of $\mathrm{Hom}_G(IG,A)$ by the subgroup of homomorphisms $\phi : IG \to A$ of the form $\phi(x-1) = x \cdot a - a$ for some $a \in A$. The derivation the above isomorphism sends this homomorphism to is $d_\phi(x) = \phi(x-1) = x \cdot a - a = (x-1) \cdot a$. Derivations of this kind shall be called *inner* or *principal derivations* and we denote the set of all such derivations by $\mathrm{IDer}(G,A)$. We thus have

$$H^1(G,A) \simeq \mathrm{Der}(G,A)/\,\mathrm{IDer}(G,A). \tag{4.16}$$

Example 4.6.3. If A is a trivial G-module, then $H^1(G,A) \simeq \mathrm{Der}(G,A) \simeq \mathrm{Hom}_{\mathbb{Z}}(G,A)$.

Example 4.6.4. Let σ, the generator of C_2, act on $\mathbb{Z}$ by $\sigma \cdot n = -n$. Any derivation $d : C_2 \to \mathbb{Z}$ is then determined by $d(\sigma)$. So $\mathrm{Der}(C_2,\mathbb{Z}) \simeq \mathbb{Z}$. For any inner derivation we have $d(\sigma) = (\sigma - 1)a = -2a$ for some $a \in \mathbb{Z}$. Hence $\mathrm{IDer}(C_2,\mathbb{Z}) \simeq 2\mathbb{Z}$. Thus $H^1(C_2,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = C_2$ which agrees with (4.5.9).

We now define a product closely related to derivations as we shall see later on. Given a group G and a G-module A, define their *semidirect product* $A \rtimes G$ by taking $A \times G$ as the underlying set and defining a product by $(a,x).(a',x') = (a + xa', xx')$. It is easy to show that this definition is associative, that $(0,1)$ is an identity element and that $(a,x)^{-1} = (-x^{-1}a, x^{-1})$. The semidirect product is thus a group.

Define an injection $\iota : A \to A \rtimes G$ by $\iota(a) = (a,1)$ and a projection $p : A \rtimes G \to G$ by $p(a,x) = x$. Note that both of these are group homomorphisms since $\iota(a+b) = (a+b,1) = (a,1).(b,1) = \iota(a)\iota(b)$ and $p((a,g).(b,g')) = p(a+gb, gg') = gg' = p(a,g)p(b,g')$. However, this is not the case for the other possible projection $q : A \rtimes G \to A$ defined by $q(a,g) = a$ since $q((a,g).(b,g')) = a + gb$ but $q(a,g) + q(b,g') = a + b$. Further, $\iota(A)$ is a normal subgroup of $A \rtimes G$ since $(a,x).\iota(a).(a,x)^{-1} = (xa,1) = \iota(xa)$ for all $x \in G$ and $a \in A$. Hence $A \overset{\iota}{\hookrightarrow} A \rtimes G \overset{p}{\twoheadrightarrow} G$ is exact.

Example 4.6.5. Let x, the generator of C_2, act on C_m by $x \cdot a = -a$. We then easily check that

$$C_m \rtimes C_2 \simeq \{\sigma, \tau : \sigma^m = 1 = \tau^2, \sigma\tau = \tau\sigma^{m-1}\} = D_m$$

where D_m is the *dihedral group* of a regular m-gon with $2m$ elements. From our calculations for cyclic groups we get

$$H^1(C_2, C_m) = C_m/2C_m = C_{\gcd(2,m)} = \begin{cases} 0 & \text{if m is odd} \\ C_2 & \text{if m is even .} \end{cases}$$

Thus, if m is odd, all derivations are principal and if m is even, there is one non-trivial "outer" derivation.

Let us now apply our knowledge about derivations to an important theorem from Galois cohomology.

Let L be a finite Galois extension of the field K and let G be its Galois group. Then both L regarded as a group under addition and $L^\times$ regarded as a group under multiplication are G-modules, the action being that of the K-automorphisms. Notice that the norm map N in $L^\times$ then has the form $N = \prod_{\sigma \in G} \sigma$. We will need the following

Proposition 4.6.6 (Dedekind's Lemma)**.** *If K and L are fields, then every set of distinct monomorphisms from K into L is linearly independent over L.*

Proof. By contradiction, suppose that

$$\sum a_i \lambda_i(x) = 0 \text{ for all } x \in K \tag{4.17}$$

where λ_i are distinct monomorphisms and $a_i \in L$. Without loss of generality, $a_i \neq 0$ for all i and there must be an equation like (4.17) with a least nonzero number of terms. Let this number be n. Since $\lambda_1 \neq \lambda_n$, there exists $y \in K$ such that $\lambda_1(y) \neq \lambda_n(y)$. Hence $y \neq 0$ and we have $\sum a_i \lambda_i(yx) = 0$ which implies $\sum a_i \lambda_i(y) \lambda_i(x) = 0$. Hence

$$\lambda_1(y) \sum a_i \lambda_i(x) - \sum a_i \lambda_i(y) \lambda_i(x) = \sum_{i=2}^{n} a_i(\lambda_1(y) - \lambda_i(y))\lambda_i(x) = 0.$$

But since $a_n(\lambda_1(y) - \lambda_n(y)) \neq 0$ this is an equation of the form (4.17) with fewer than n terms which is a contradiction. $\square$

We are now ready to prove a theorem due to Emmy Noether.

Theorem 4.6.7. *Let $L : K$ be a finite Galois extension with Galois group G. Then $H^1(G, L^\times) = 0$.*

Proof. Since by (4.16) $H^1(G, L^\times) \simeq \operatorname{Der}(G, L^\times)/\operatorname{IDer}(G, L^\times)$, it suffices to show that every derivation from G to $L^\times$ is principal. In multiplicative notation (recall that we regard $L^\times$ as a multiplicative G-module) this means that for every function $\phi : G \to L^\times$ obeying $\phi(\tau\sigma) = \phi(\tau)(\tau \cdot \phi(\sigma))$ we must find a $c \in L^\times$ such that $\phi(\sigma) = \sigma \cdot c/c$ for $\sigma, \tau \in G$. Note first that by Dedekind's Lemma the K-automorphisms $\sigma : L^\times \to L^\times$ of G are linearly independent over $L^\times$. Hence $\sum_{\sigma \in G} \phi(\sigma) \cdot \sigma$ is not the zero map for all $\phi \in \operatorname{Der}(G, L^\times)$. Hence we can find an $a \in L^\times$ such that $0 \neq \sum_{\sigma \in G} \phi(\sigma)(\sigma \cdot a) =: b \in L^\times$. But then letting $\tau \in G$,

$$\begin{aligned} \tau \cdot b = \sum_{\sigma \in G} (\tau \cdot \phi(\sigma))(\tau\sigma \cdot a) &= \sum \phi(\tau)^{-1} \phi(\tau)(\tau \cdot \phi(\sigma))(\tau\sigma \cdot a) \\ &= \sum \phi(\tau)^{-1} \phi(\tau\sigma)(\tau\sigma \cdot a) \\ &= \phi(\tau)^{-1} b. \end{aligned}$$

Hence $\phi(\tau) = b/\tau \cdot b = \tau \cdot b^{-1}/b^{-1}$ which shows ϕ is principal. $\square$

The above theorem is a generalization of a famous theorem due to Hilbert.

Corollary 4.6.8 (Hilbert's Theorem 90)**.** *Let $L : K$ be a Galois extension with cyclic Galois group generated by σ and let $x \in L^\times$. Then $N(x) = 1$ implies there exists $y \in L^\times$ such that $x = \sigma \cdot y/y$.*

Proof. We saw in (4.13) that $H^1(C_m, L^\times) = \ker(N : L^\times \to L^\times)/(\sigma - 1)L^\times$. So (4.6.7) immediately yields the result. $\square$

We could thus regard (4.6.7) as Hilbert's Theorem 90 in the language of homological algebra. One may wonder about analogues for higher values of n. The following example shows that the conclusion does not generally hold.

Example 4.6.9. Consider the simple field extension $\mathbb{C} : \mathbb{R}$ and let G denote its Galois group. Then $G = \{id, conj\} \simeq C_2$ where id is the identity map on $\mathbb{C}$ and $conj$ is the complex conjugation map. Let $x + iy \in \mathbb{C}^\times$ then we have $N(x + iy) = (x + iy)(x - iy) = x^2 + y^2$. Therefore $N(\mathbb{C}^\times) = \mathbb{R}^+$ where $\mathbb{R}^+$ are the positive real numbers. We also see that $(\mathbb{C}^\times)^G = \mathbb{R}^\times$. So again by (4.13) we have $H^2(G, \mathbb{C}^\times) = (\mathbb{C}^\times)^G/N(\mathbb{C}^\times) = \mathbb{R}^\times/\mathbb{R}^+ \simeq C_2$.

However, we do have the following result.

Proposition 4.6.10. *Let $L : K$ be a finite Galois extension with Galois group G. Then $H^n(G, L) = 0$ for all $n > 0$.*

Proof. The Normal Basis Theorem found for instance in [La, p. 312] states that there exists an $x \in L$ such that $\{\sigma \cdot x : \sigma \in G\}$ is a basis for L as a K-vector space. Such a basis, called *normal*, clearly defines an isomorphism of G-modules $KG \to L$ by $\sum_{\sigma \in G} a_\sigma \sigma \mapsto \sum_{\sigma \in G} a_\sigma (\sigma \cdot x)$. But $KG \simeq KG \otimes_K K \simeq \mathbb{Z}G \otimes_{\mathbb{Z}} K$. Hence by Shapiro's lemma (4.5.2) we obtain $H^n(G, L) = H^n(G, \mathbb{Z}G \otimes_{\mathbb{Z}} K) = 0$ for all $n > 0$. $\square$

4.7 The Standard Resolution

Let $\bar{B}_n, n \geq 0$ be the free abelian group on the set of all $(n+1)$-tuples $(y_0, \ldots, y_n)$ of elements of G where we will write G multiplicatively as usual. Define a left G-module structure in $\bar{B}_n$ by

$$y(y_0, \ldots, y_n) = (yy_0, \ldots, yy_n), \quad y \in G.$$

Define a differential $\partial_n : \bar{B}_n \to \bar{B}_{n-1}$ by

$$\partial_n(y_0, \ldots, y_n) = \sum_{i=0}^{n} (-1)^i (y_0, \ldots, \hat{y_i}, \ldots, y_n) \tag{4.18}$$

where $\hat{y_i}$ indicates that y_i is to be omitted. Noting that $\bar{B}_0 = \mathbb{Z}G$, we also have the usual augmentation map $\varepsilon : \bar{B}_0 \to \mathbb{Z}$ defined by $\varepsilon(y) = 1$ for all $y \in G$. We now want to show that

$$\bar{\mathbf{B}} : \cdots \to \bar{B}_n \xrightarrow{\partial_n} \bar{B}_{n-1} \to \cdots \to \bar{B}_1 \xrightarrow{\partial_1} \bar{B}_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is indeed a G-free resolution of $\mathbb{Z}$.

To do this we introduce the *bar notation*. Note that we can rewrite

$$(y_0, \ldots, y_n) = y_0[y_0^{-1}y_1 | y_1^{-1}y_2 | \ldots | y_{n-1}^{-1}y_n] \tag{4.19}$$

where

$$[y_1 | \cdots | y_n] = (1, y_1, y_1y_2, \ldots, y_1 \cdots y_n).$$

In bar notation it thus becomes clear that each $\bar{B}_n$ is a free G-module on basis elements given by the symbols $[y_1|\cdots|y_n]$, i.e. basis elements of the form $(1, x_1, \ldots, x_n)$ where $x_i \in G$. For example, $\bar{B}_0$ is the free G-module on the symbol $[\,] = 1$ and $\varepsilon[\,] = 1$. We also rewrite the differential in bar notation as

$$\begin{aligned}
\partial_n[y_1, \ldots, y_n] &= \partial_n(1, y_1, \ldots, y_1 \cdots y_n) \\
&= (y_1, \ldots, y_1 \cdots y_n) + \sum_{i=1}^{n-1}(-1)^i(1, y_1, \ldots, \widehat{y_1 \cdots y_i}, \ldots, y_1 \cdots y_n) \\
&\quad +(-1)^n(1, y_1, \ldots, y_1 \cdots y_{n-1}) \\
&= y_1[y_2|\ldots|y_n] + \sum_{i=1}^{n-1}(-1)^i[y_1|\ldots|y_iy_{i+1}|\ldots|y_n] \\
&\quad +(-1)^n[y_1|\ldots|y_{n-1}].
\end{aligned}$$

Next we return to the original notation to show that $\bar{\mathbf{B}}$ is acyclic. It is easy to see that ε and ∂_n are indeed G-module homomorphisms. Moreover,

$$\begin{aligned}
\partial_{n-1}\partial_n(y_0, \ldots, y_n) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i(y_0, \ldots, \hat{y}_i, \ldots, y_n)\right) \\
&= \sum_{j=0}^{n-1}(-1)^j\sum_{i=0}^{n}(-1)^i(y_0, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_n) \\
&= 0
\end{aligned}$$

since the terms $(y_0, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_n)$ and $(y_0, \ldots, \hat{y}_j, \ldots, \hat{y}_i, \ldots, y_n)$ always have opposite signs and occur in pairs. Hence $\operatorname{im}\partial_n \subseteq \ker\partial_{n-1}$. We also have $\varepsilon\partial_1(y_0, y_1) = \varepsilon(y_1 - y_0) = 1 - 1 = 0$ which shows $\operatorname{im}\partial_1 \subseteq \ker\varepsilon$. To show the reverse inclusions, we define a G-module homomorphism $h_n : \bar{B}_n \to \bar{B}_{n+1}$ for all $n \geq 0$ by $(y_0, \ldots, y_n) \mapsto (1, y_0, \ldots, y_n)$ and show

$$\partial_1 h_0 + h_{-1}\varepsilon = 1_0 \quad \text{ and } \quad \partial_{n+1}h_n + h_{n+1}\partial_n = 1_n$$

for all $n \geq 1$ where 1_n here denotes the identity function on $\bar{B}_n$ and where we define $h_{-1}(1) = 1$.

To show the first of these equalities we note that $\partial_1 h_0(y) = \partial_1(1, y) = y - 1$ and $h_{-1}\varepsilon(y) = h_{-1}(1) = 1$. Hence the sum of the two is just y. It now follows that if $y \in \ker\varepsilon$, then $\partial_1(h_0(y)) = y$, i.e. $y \in \operatorname{im}\partial_1$. Hence $\operatorname{im}\partial_1 = \ker\varepsilon$. The second equality follows by a similar calculation:

$$\begin{aligned}
\partial_{n+1}h_s(y_0, \ldots, y_n) &= \partial_{n+1}(1, y_0, \ldots, y_n) \\
&= \sum_{i=0}^{n+1}(-1)^i(1, y_0, \ldots, \hat{y}_i, \ldots, y_n) \\
&= (y_0, \ldots, y_n) + \sum_{i=1}^{n+1}(-1)^i(1, y_0, \ldots, \hat{y}_i, \ldots, y_n)
\end{aligned}$$

and

$$\begin{aligned} h_{s-1}\partial_n(y_0,\dots,y_n) &= h_{s-1}\left(\sum_{i=0}^{n}(-1)^i(y_0,\dots,\hat{y}_i,\dots,y_n)\right) \\ &= -\sum_{i=1}^{n+1}(-1)^i(1,y_0,\dots,\hat{y}_i,\dots,y_n). \end{aligned}$$

Hence $\partial_{n+1}h_n + h_{n+1}\partial_n = 1_n$ as claimed. Similar to the previous case, it now follows that if $y \in \ker\partial_n$ then $\partial_{n+1}(h_n(y)) = y$ whence $y \in \operatorname{im}\partial_{n+1}$ and $\operatorname{im}\partial_n = \ker\partial_{n-1}$. This finishes the proof that $\bar{\mathbf{B}}$ is a free resolution of $\mathbb{Z}$. The complex $\bar{\mathbf{B}}$ is known as the *non-normalized standard resolution*. When the bar notation is used throughout, this complex is also often referred to as the *non-normalized bar resolution* but the two are of course equivalent.

Remark 4.7.1. We remark that the function h_n defined above is also known as a *contracting homotopy* but we will not define this term explicitly here.

Before we apply this resolution to group cohomology we need to normalize it. We will now make precise what normalization in this context means. Let $D_n \subseteq \bar{B}_n$ be the subgroup generated as a $\mathbb{Z}$-module by the $(n+1)$-tuples $(y_0,\dots,y_n)$ such that $y_i = y_{i+1}$ for at least one value of $i = 0,\dots,n-1$. Such an element will be called *degenerate*. Note that $D_0 = 0$. From the bar notation (4.19) we see that D_n is generated as a G-submodule of $\bar{B}_n$ by all degenerate $(n+1)$-tuples with $y_0 = 1$. Using (4.19) again, let us write

$$(y_0,\dots,y_n) = y_0[y_0^{-1}y_1|y_1^{-1}y_2|\dots|y_{n-1}^{-1}y_n] = y_0[x_1|\dots|x_n].$$

It then becomes clear that saying $y_i = y_{i+1}$ in $(y_0,\dots,y_n)$ for at least one value of $i = 0,\dots,n-1$ is the same as saying $x_i = 1$ in $[x_1|\dots|x_n]$ for at least one value of $i = 1,\dots,n$. We now claim that

$$\mathbf{D} : \cdots \to D_n \to \cdots \to D_0 \to 0$$

is a subcomplex of $\bar{\mathbf{B}}$. To see this, let $(y_0,\dots,y_n)$ be degenerate, i.e. $y_j = y_{j+1}$ for some j. Then $\partial_n(y_0,\dots,y_n)$ is a linear combination of degenerate n-tuples plus the sum

$$(-1)^j(y_0,\dots,y_{j-1},y,y_{j+2},\dots,y_n) + (-1)^{j+1}(y_0,\dots,y_{j-1},y,y_{j+2},\dots,y_n)$$

where $y = y_j = y_{j+1}$. But this sum is of course equal to zero and so $\partial_n D_n \subseteq D_{n-1}$. It is also clear that the contracting homotopy h_n sends D_n to D_{n+1}. So we have shown that $\mathbf{D}$ is indeed a subcomplex of $\bar{\mathbf{B}}$. We now pass to the quotient complex $\bar{\mathbf{B}}/\mathbf{D}$ generated as a $\mathbb{Z}$-module by all $(n+1)$-tuples $(y_0,\dots,y_n)$ in which no $y_i = y_{i+1}$ or generated as a G-module by the symbols $[x_1|\dots|x_n]$ with no $x_i = 1$. Such a complex is said to be *normalized*. We will write $\bar{\mathbf{B}}/\mathbf{D} = \mathbf{B}$ and call $\mathbf{B}$ the *normalized standard* (or *bar*) *resolution*.

Let us now look at the cohomology groups. Following the usual procedure for calculating $H^n(G,A)$, we apply the $\operatorname{Hom}_G(-,A)$ functor to the above normalized bar resolution to obtain a cochain complex $\operatorname{Hom}_G(\mathbf{B},A)$. By definition, we are thus dealing with functions $\phi : G^n \to A$ which we will call *n-cochains* that vanish whenever $x_i = 1$ for some i in $(x_1, \ldots, x_n) \in G^n$. Using the differential of the standard resolution in bar notation we obtain an induced boundary operator $\partial^n = \partial_n^* : \mathrm{Hom}_G(B_n, A) \to \mathrm{Hom}_G(B_{n+1}, A)$ sending an n-cochain $\phi : B_n \to A$ to an $(n+1)$-cochain:

$$\begin{aligned}\partial^n(\phi)(x_1, \ldots, x_{n+1}) &= \phi(\partial_n(x_1, \ldots, x_{n+1})) \\ &= x_1\phi(x_2, \ldots, x_n) + \sum_{i=1}^{n-1}(-1)^i\phi(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) \\ &\quad +(-1)^n\phi(x_1, \ldots, x_n).\end{aligned}$$

We call the n-cochains such that $\partial^n\phi = 0$ *n-cocycles* and the $(n+1)$-cochains $\partial^{n-1}\phi$ *n-coboundaries*. We define $Z^n(G, A) = \ker \partial^n$ as the group of all n-cocycles of G with coefficients in A and $B^n(G, A) = \operatorname{im} \partial^{n-1}$ as the group of all n-coboundaries of G with coefficients in A. We then have by definition,

$$H^n(G, A) = Z^n(G, A)/B^n(G, A). \tag{4.20}$$

Example 4.7.2. A 0-cochain ϕ is a map $1 \to A$, i.e. an element of A. Let $\phi(1) = a$, then $(\partial^0\phi)(g) = ga - a$ where $g \in G$. Thus ϕ is a 0-cocycle iff $a \in A^G$. Hence $Z^0(G, A) = A^G$. We also see from this calculation that $B^1(G, A) = \operatorname{im} \partial^0 = \mathrm{IDer}(G, A)$. Finally we notice that a 1-cocycle is a function $\psi : G \to A$ such that $\psi(1) = 0$ and $(\partial^1\psi)(gh) = g\psi(h) - \psi(gh) + \psi(g) = 0$ for all $g, h \in G$, i.e. ψ is a derivation! Hence the bar resolution provides a direct proof that $H^1(G, A) \simeq \mathrm{Der}(G, A)/\,\mathrm{IDer}(G, A)$.

We will also need the following

Example 4.7.3.
- $B^2(G, A)$ is the set of all $\psi : G \times G \to A$ such that $\psi(1, g) = \psi(g, 1) = 0$ and $\psi(f, g) = (\partial^1\phi)(f, g) = f\phi(g) - \phi(fg) + \phi(f)$ for some $\phi : G \to A$ and $f, g \in G$.
- $Z^2(G, A)$ is the set of all $\psi : G \times G \to A$ such that $\psi(1, g) = \psi(g, 1) = 0$ and $(\partial^2\psi)(f, g, h) = f\psi(g, h) - \psi(fg, h) + \psi(f, gh) - \psi(f, g) = 0$ for every $f, g, h \in G$.

4.8 H^2 and Beyond

In this section we will give an interpretation of the second cohomology group. Let $A \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G$ be an exact sequence of groups with A abelian. As usual, we will write operations in A additively and in E and G multiplicatively. ι thus transfers sums into products. We claim that in this situation we have a natural action of G on A, making A a G-module. Note first that E naturally acts on A by conjugation since A is embedded as a normal subgroup of E (since A is the kernel of π). We then define the induced action of $G \simeq E/A$ on A by the following procedure: given $g \in G$, find $\tilde{g} \in E$ such that $\pi(\tilde{g}) = g$. Then define $\iota(ga) = \tilde{g}\iota(a)\tilde{g}^{-1}, a \in A$. We can rewrite this as

$$\tilde{g}\iota(a) = \iota(ga)\tilde{g}. \tag{4.21}$$

It is easy to check that the axioms for a G-module are satisfied (in particular, one would need to verify that $\iota((gh)a) = \iota(g(ha))$ for $g, h \in G$) and we define an *extension* of G by a G-module A to be the above exact sequence of groups together with the described G-module structure. Two extensions $A \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G$ and $A \overset{\iota}{\hookrightarrow} E' \overset{\pi}{\twoheadrightarrow} G$ are said to be *equivalent* if there exists a group homomorphism ψ making the following diagram commute.

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, dashed, "\psi"] & G \arrow[d, equal] \\ A \arrow[r] & E' \arrow[r] & G \end{tikzcd}$$

Note that ψ is necessarily an isomorphism by (1.1.1).

Suppose now we are given a set-theoretic *section* of π, i.e. a function $s : G \to E$ such that $\pi \circ s = 1_G$. We shall also assume the normalization condition that $s(1) = 1$. If s is a homomorphism then the extension is said to *split* and s is called a *splitting.* We know its structure by the following argument.

Proposition 4.8.1. *Under the assumptions in the last paragraph we have $E \simeq A \rtimes G$.*

Proof. To see this, use the obvious set-theoretic bijection $(a, g) \mapsto \iota(a)s(g)$ between $A \times G$ and E and compute the E-induced group law on $A \times G$ making this bijection an isomorphism of groups:

$$\begin{aligned} (a, g).(b, h) &= \iota(a)s(g)\iota(b)s(h) \\ &= \iota(a)\iota(gb)s(g)s(h) \quad \text{by (4.21)} \\ &= \iota(a + gb)s(gh) \\ &= (a + gb, gh) \end{aligned}$$

for $a, b \in A$ and $g, h \in G$. But this is by definition the semidirect product. Note that this is another proof that $A \rtimes G$ is indeed a group. □

We can deduce even more: in case G acts trivially on A, then the above calculation shows that E is isomorphic to the direct product $A \times G$ and the splittings are in 1-1 correspondence with homomorphisms $G \to A$. When we first introduced the semidirect product, we promised to later reveal a close connection to derivations. Here now we redeem this promise by proving

Corollary 4.8.2. *Let $A \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G$ be an extension of G by A giving rise to a given action of G on A and let $\mathcal{S}_E(G, A)$ be the set of splittings of this extension. Then there is a 1-1 correspondence*

$$\mathcal{S}_E(G, A) \leftrightarrow \mathrm{Der}(G, A).$$

Proof. By (4.8.1) we can analyze the generic extension $A \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G$ via the canonical split extension $0 \to A \to A \rtimes G \to G \to 1$. A function $s : G \to A \rtimes G$ with $\pi \circ s = 1_G$ then has the form $s(g) = (dg, g)$, where d is a function $G \to A$. We have $s(g)s(h) = (dg + g \cdot dh, gh)$, so s will be a homomorphism if and only if d is a derivation proving the claim. □

Let us now ask the question what happens if the section s is not a homomorphism. In fact, let us try to measure by how much s fails to be a homomorphism. For this purpose define a function $f : G \times G \to A$ by noting that both $s(gh)$ and $s(g)s(h)$

map to gh in G via π. Thus, they must differ by an element in $\ker \pi = \operatorname{im} \iota$. We then define f such that

$$s(g)s(h) = \iota(f(g,h))s(gh). \tag{4.22}$$

Note that normalization of s, i.e. $s(1) = 1$, implies

$$f(1,h) = 0 = f(g,1) \tag{4.23}$$

for all $g, h \in G$ so that f is normalized in the sense of the standard resolution. We call f the *factor set* associated to the extension $A \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G$ and the section s.

Next we will show that a given extension $A \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G$ can be completely recovered from the factor set and the G-module structure on A. As before, we use the usual set-theoretic bijection $A \times G \to E$ sending $(a,g) \mapsto \iota(a)s(g)$ to determine the E-induced group law on $A \times G$ making the set-theoretic bijection an isomorphism of groups. Let $(a,g),(b,h) \in A \times G$, then

$$\begin{aligned} \iota(a)s(g)\iota(b)s(h) &= \iota(a)\iota(gb)s(g)s(h) \quad \text{using (4.21)} \\ &= \iota(a+gb)\iota(f(g,h))s(gh) \\ &= \iota(a+gb+f(g,h))s(gh). \end{aligned}$$

Thus the group law on $A \times G$ is

$$(a,g).(b,h) = (a+gh+f(g,h), gh) \tag{4.24}$$

which looks like the product in $A \rtimes G$ "perturbed" by f. Let us denote $A \times G$ with the above group law by $A \rtimes_f G$. Then $E \simeq A \rtimes_f G$ and by composing ι and π with the constructed isomorphism we obtain a canonical inclusion $A \to A \rtimes_f G$ noting that $\iota(a) = \iota(a)s(1)$,

$$a \mapsto (a,1) \tag{4.25}$$

and a canonical projection $A \rtimes_f G \simeq E \to G$ by $\pi(a,g) = \pi(\iota(a)s(g)) = g$,

$$(a,g) \mapsto g \tag{4.26}$$

making the original extension $A \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G$ equivalent to the extension

$$0 \to A \to A \rtimes_f G \to G \to 0 \tag{4.27}$$

entirely defined in terms of G, A, f and the relations (4.24), (4.25) and (4.26).

Conversely, we now want to show that given a function $f : G \times G \to A$ satisfying (4.23), we can construct an extension with normalized section. To do this, we would like to define a group $A \rtimes_f G$ by (4.24). However, one checks quickly that associativity in $A \rtimes_f G$ is only warranted if f satisfies the following identity:

$$f(g,h) + f(gh,k) = gf(h,k) + f(g,hk) \quad \text{for} \quad g,h,k \in G. \tag{4.28}$$

We claim that imposing conditions (4.23) and (4.28) on f is enough to produce an extension. To prove this, note that (4.23) implies that $(0,1)$ is a two-sided identity under (4.24). We also compute inverses. Given $(a,g) \in A \rtimes_f G$, solve $(a,g)(b,g^{-1}) = (0,1)$ and $(b',g^{-1})(a,g) = (0,1)$ for b and b' to obtain a right and a left inverse and then use associativity given by (4.28) to get

$$-g^{-1}a - g^{-1}f(g,g^{-1}) = b = b' = -g^{-1} - f(g^{-1},g).$$

Then, define homomorphisms according to (4.25) and (4.26) and note that these make (4.27) exact. Finally, we just need to check that (4.27) gives rise to the given action (4.21) of G on A and that the factor set associated to (4.27) and the canonical section $g \mapsto (0,g)$ is the original function f. The calculations involved are routine and will be omitted. We have thus shown that there exists a 1-1 correspondence

$$\begin{pmatrix} \text{extensions } A \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G \\ \text{with a normalized section} \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{functions } G \times G \to A \\ \text{satisfying (4.23) and (4.28)} \end{pmatrix}.$$

Note now that we can rewrite (4.28) as

$$gf(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0 \tag{4.29}$$

which should look very familiar in view of example (4.7.3). Recalling that f is normalized, we then get

$$\begin{pmatrix} \text{extensions } A \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G \\ \text{with a normalized section} \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{normalized 2-cocycles of G} \\ \text{with coefficients in } A \end{pmatrix}.$$

Finally, we claim that choosing a different section for a given extension translates precisely into modifying the corresponding 2-cocycle f by a 2-coboundary. To see this, let $s : G \to A \rtimes_f G$ be a given section and note that any other section $s' : G \to A \rtimes_f G$ differs from s by an element in $\ker \pi = \operatorname{im} \iota$, i.e. $s'(g) = \iota(\phi(g))s(g)$ for some function $\phi : G \to A$ and for all $g \in G$. Hence,

$$\begin{aligned} s'(g)s'(h) &= \iota(\phi(g))s(g)\iota(\phi(h))s(h) \\ &= \iota(\phi(g) + g\phi(h))s(g)s(h) \\ &= \iota(\phi(g) + g\phi(h))\iota(f(g,h))s(gh) \\ &= \iota(\phi(g) + g\phi(h) + f(g,h))\iota(\phi(gh)^{-1})s'(gh) \\ &= \iota(\phi(g) + g\phi(h) + f(g,h) - \phi(gh))s'(gh). \end{aligned}$$

So if we denote the factor set corresponding to s' by f', we get

$$f'(g,h) = \phi(g) + g\phi(h) + f(g,h) - \phi(gh).$$

Let $\psi = f' - f : G \times G \to A$, then

$$\psi(g,h) = g\phi(h) - \phi(gh) + \phi(g)$$

which, together with the normalization satisfied by this ψ, is exactly the requirement for being an element in $B^2(G,A)$, the set of 2-coboundaries of G with coefficients in A (cf. example (4.7.3)) which proves the claim. We have therefore proved:

Theorem 4.8.3. *Let A be a G-module and let $\mathcal{E}(G,A)$ be the set of equivalence classes of extensions of G by A giving rise to a given action of G on A. Then there is a 1-1 correspondence*

$$\mathcal{E}(G,A) \leftrightarrow H^2(G,A).$$

This result gives a group-theoretic interpretation of $H^2(G, A)$. Notice how once again we connected the process of measuring a defect to computing cohomology groups. As a final remark, we state that these considerations can be taken further to give group-theoretic interpretations of H^n for $n \geq 3$. The essential tool used for these investigations are so-called *n-extensions* of G by A which are exact sequences of the form

$$0 \to A \to E_n \to \cdots \to E_1 \to G \to 0.$$

An extension is thus just a 1-extension.

4.9 Change of Groups and the Restriction-Inflation Sequence

We have already shown that, when G is fixed, $H_n(G, A)$ and $H^n(G, A)$ are covariant functors of the G-module A. In this last section, we show how to make group (co)homology functors into functors of their first variable. This process is also know as the *change of groups*.

We begin by making precise a definition we have encountered earlier: a covariant *homological δ-functor* between two categories $\mathfrak{A}$ and $\mathfrak{B}$ is a collection of additive functors $T_n : \mathfrak{A} \to \mathfrak{B}$ for $n \geq 0$, together with connecting morphisms $\delta_n : T_n(C) \to T_{n-1}(A)$ defined for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{A}$ such that the following two conditions hold:

1. For each short exact sequence as above, there is a long exact sequence

   $$\cdots \to T_{n+1}(C) \xrightarrow{\delta} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta} T_{n-1}(A) \to \cdots$$

   with the convention that $T_n = 0$ for $n < 0$. In particular, T_0 is right exact.

2. For each morphism of short exact sequences from $0 \to A' \to B' \to C' \to 0$ to $0 \to A \to B \to C \to 0$, the connecting morphisms give a commutative diagram

   $$\begin{tikzcd} T_n(C') \arrow[r, "\delta"] \arrow[d] & T_{n-1}(A') \arrow[d] \\ T_n(C) \arrow[r, "\delta"] & T_{n-1}(A), \end{tikzcd}$$

   i.e. δ is natural.

The definition for a covariant *cohomological δ-functor* is similar.

Example 4.9.1. We showed in (3.2.4) and (3.2.6) how to construct connecting homomorphisms δ for the (co)homology functors H_n, H^n and the left derived functors L_nT. In both cases, one can show that δ is natural thus proving that these functors indeed each form a δ-functor. For details see [Wei, p. 14, p. 45].

We need two more definitions. A *morphism* $S \to T$ of δ-functors is a system of natural transformations $S_n \to T_n$ (resp. $S^n \to T^n$) that commute with δ. Pictorially speaking, this definition says that there is a commutative ladder diagram connecting long exact sequences of S and T.

We shall say a homological δ-functor T is *universal* if, given any other δ-functor S and a natural transformation $f_0 : S_0 \to T_0$, there is a unique morphism $\{f_n : S_n \to T_n\}$ of δ-functors S and T that extends f_0. Similarly, a cohomological δ-functor T is *universal* if, given S and $f^0 : T^0 \to S^0$, there exists a unique morphism $T \to S$ of δ-functors extending f^0.

Theorem 4.9.2. *Assume $\mathfrak{A}$ has enough projectives. Then for any right exact functor $F : \mathfrak{A} \to \mathfrak{B}$, the derived functors L_nF form a universal δ-functor.*

Proof. We already know that L_nF is a δ-functor. Suppose that T_n is another homological δ-functor and that a natural transformation $\phi_0 : T_0 \to F$ (recall that $L_0F \simeq F$ by (3.2.1)) is given. We must show that ϕ_0 admits a unique extension to a morphism $\phi : T_n \to L_nF$ of δ-functors. We do this by induction. Suppose that $\phi_i : T_i \to L_iF$ are already defined for $0 \leq i < n$, and that they commute with all the respective δ_i's. Given A in $\mathfrak{A}$, we choose a projective presentation $K \hookrightarrow P \twoheadrightarrow A$ of A. Since $L_nF(P) = 0$ by (3.2.2), we get a commutative diagram

$$\begin{tikzcd}
 & T_n(A) \arrow[r, "\delta_n"] & T_{n-1}(K) \arrow[r] \arrow[d, "\phi_{n-1}"] & T_{n-1}(P) \arrow[d, "\phi_{n-1}"] \\
0 \arrow[r] & L_nF(A) \arrow[r, "\delta_n"] & L_{n-1}F(K) \arrow[r] & L_{n-1}F(P)
\end{tikzcd}$$

with exact rows. Using the commutativity of the right square, we quickly check that there is a unique map ϕ_n from $T_n(A)$ to $L_nF(A)$ commuting with the given δ_n's. For the proof that ϕ_n is indeed natural, we refer the reader to [Wei, p. 47]. $\square$

A similar theorem holds for right derived functors.

We are now ready to talk about the change of groups. Let $\rho : H \to G$ be a group homomorphism. Let $\rho^\#$ be the forgetful functor from $\mathfrak{M}_G$ to $\mathfrak{M}_H$ which is clearly an exact functor. Note that there is a natural surjection $(\rho^\# A)_H \twoheadrightarrow A_G$. Similarly, there is a natural injection $A^G \hookrightarrow (\rho^\# A)^H$. Let us write $T_n(A) = H_n(H, \rho^\# A)$ and $T^n(A) = H^n(H, \rho^\# A)$. We showed in (4.9.2) that left and right derived functors, in particular the group (co)homology functors, are universal δ-functors. Hence the just described natural surjection and injection uniquely extend to two morphisms $\rho_* = \mathrm{cor}_H^G$, called *corestriction*, and $\rho^* = \mathrm{res}_H^G$, called *restriction*, of δ-functors from $\mathfrak{M}_G$ to $\mathfrak{Ab}$:

$$\mathrm{cor}_H^G : H_n(H, \rho^\# A) \to H_n(G, A) \quad \text{and} \quad \mathrm{res}_H^G : H^n(G, A) \to H^n(H, \rho^\# A).$$

Now let H be a subgroup of G. We showed before that $\mathbb{Z}G$ is then $\mathbb{Z}H$-free (4.1.3). Hence as in the proof of Shapiro's lemma, we may use any projective G-module resolution $P \to \mathbb{Z}$ to compute the homology and cohomology groups of H. Let A be a G-module and recall the construction of the (co)homology functor. It then becomes clear that cor_H^G is just the homology $H_n(\alpha)$ of the chain map $\alpha : P \otimes_H A \to P \otimes_G A$ and that res_H^G is the cohomology $H^n(\beta)$ of the cochain map $\beta : \mathrm{Hom}_G(P, A) \to \mathrm{Hom}_H(P, A)$.

Example 4.9.3. Let H be the cyclic subgroup C_m of the cyclic group C_{mn}. It is then clear that $\mathrm{cor}_H^G : H_n(C_m, \mathbb{Z}) \to H_n(C_{mn}, \mathbb{Z})$ is the natural inclusion $C_m \hookrightarrow C_{mn}$ for n odd, while $\mathrm{res}_H^G : H^n(C_{mn}, \mathbb{Z}) \to H^n(C_m, \mathbb{Z})$ is the natural projection $C_{mn} \twoheadrightarrow C_m$ for n even. This follows from (4.12) and (4.13).

Let H be a normal subgroup of G and A a G-module. The composite maps

$$\inf : H^n(G/H, A^H) \overset{\text{res}}{\to} H^n(G, A^H) \to H^n(G, A) \quad \text{and}$$
$$\text{coinf} : H_n(G, A) \to H_n(G, A_H) \overset{\text{cor}}{\to} H_n(G/H, A_H)$$

are called the *inflation* and *coinflation* maps, respectively. We look back at the natural surjection and injection that gave rise to the restriction and corestriction map and see that when $n = 0$, $\inf : (A^H)^{G/H} \tilde{\to} A^G$ and $\text{coinf} : A_G \tilde{\to} (A_H)_{G/H}$.

Example 4.9.4. If A is trivial as an H-module, we see that inflation and restriction become the same map and coninflation and corestriction also become the same map. Thus by example (4.9.3) we see that for n odd, the map coinf is again just the natural inclusion of C_m into C_{mn} and inf is the natural projection C_{mn} onto C_m.

Let H be a normal subgroup of G and let A be a G-module. We will now show that the following composition is the zero map for $n \neq 0$:

$$H^n(G/H, A^H) \overset{\inf}{\to} H^n(G, A) \overset{\text{res}}{\to} H^n(H, A). \tag{4.30}$$

To see this we use the standard resolution. Let $f : \bigoplus_{i=1}^n (G/H) \to A^H$ be an n-cocycle. Then f induces the n-cocycle $f' : \bigoplus_{i=1}^n G \to \bigoplus_{i=1}^n G/H \to A^H \to A$ and the class of f' is the inflation of the class of f. On the other hand, if $\phi : \bigoplus_{i=1}^n G \to A$ is an n-cocycle, then the class of $\phi|_{\bigoplus_{i=1}^n H} : \bigoplus_{i=1}^n H \to A$ is the restriction of the class of ϕ. Hence if $\phi = f'$, it is clear that $f'|_{\bigoplus_{i=1}^n H} = 0$. We remark that (4.30) is called the *restriction-inflation sequence* and that, in general, it is not exact. We only remark that dually, there is also a *corestriction-coinflation sequence*

$$H_n(H, A) \overset{\text{cor}}{\to} H_n(G, A) \overset{\text{coinf}}{\to} H_n(G/H, A_H)$$

satisfying $\text{coinf} \circ \text{cor} = 0$.

Finally, let $\mathfrak{C}$ be the category of pairs (G, A), where G is a group and A is a G-module. Define a morphism in $\mathfrak{C}$ from (H, B) to (G, A) to be a pair $(\rho : H \to G, \phi : B \to \rho^\# A)$, where ρ is a group homomorphism and ϕ is an H-module homomorphism. ϕ induces a map from $H_n(H, B)$ to $H_n(H, \rho^\# A)$ without the need for change of groups. Composing the induced map with cor_H^G, we obtain a map from $H_n(H, B)$ to $H_n(G, A)$. It is then easy to check that the n^{th} group homology functor H_n is a covariant functor from $\mathfrak{C}$ to $\mathfrak{Ab}$. Similarly, we make the n^{th} group cohomology functor H^n into a contravariant functor by letting $\mathfrak{D}$ be the category with the same objects as $\mathfrak{C}$ but with morphisms from (H, B) to (G, A) of the form $(\rho : H \to G, \psi : \rho^\# A \to B)$. Applying res_H^G to $H^n(G, A)$ first gives a map $H^n(G, A) \to H^n(H, \rho^\# A)$. Composing with the obvious ψ-induced map, we obtain $H^n(G, A) \to H^n(H, B)$. Again, it is not hard to check that H^n becomes a functor from $\mathfrak{D}$ to $\mathfrak{Ab}$.

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