:— title: “S3” author: “Manu” date: “20 February 2017” output: html_document —

S3 Group

The S3 group is the group of permutation on 3 elements. It has 6 elements. It is the group of all the automorphism on the set {1, 2, 3}.

Here is the full list of the elements of S3:

```
## e: ()
## a: (1,2)
## b: (1,3)
## c: (2,3)
## p: (1,2,3)
## p^2: (1,3,2)
```

The group <p> = {e, p, p2} is normal in S3:

```
all(p^a == p2, p^b == p2, p^c == p2)
```

```
## [1] TRUE
```

The classes of <p> in S3 are:

{e, p, p2}, {a, b, c}

The action of a on <p>: (p p2)

Subgroups of S3

is isomorphic to Z/3Z and it has no non-trivial subgroups.

If G is a subgroup of S3 containing

then G*

is a subgroup of S3/

. S3/

is a group of order 2 so it is isomorphic to Z/2Z and it has no non trivial subgroup. The only subgroups containing p are {e} and S3.

If G is a sub-group not including

. Then inter(G,

) is {e}. because inter(G,

) is a subgroup of

and it cannot be

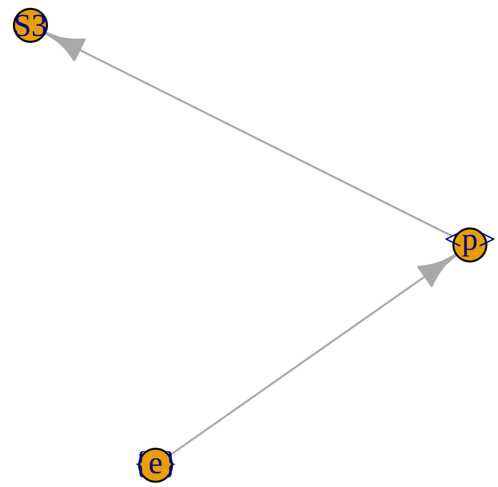
because G does no include

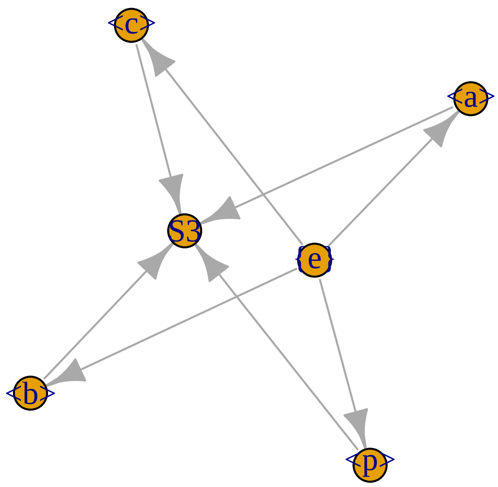
.

So G is included in {e, a, b, c}. G cannot contain more than 2 elements because any of products ab, bc, … will yield an element of

. So G is either {e, a}, {e, b}, {e, c}.

Conclusion: Here is the full list of the subgroups of S3: {e}, {e, a}, {e, b}, {e, c}, {e, p, p2}, {e, a, b, c, p, p2}


S3
p
e



Action of the group

- <p> Act on the 'edges' of a triangle {1, 2, 3}: {{1, 2}, {1, 3}, {2, 3}}

```
## [1] TRUE
```

The action of a can also be defined: a({1, 2}) == {1, 2} a{1, 3} == {2, 3} a{2, 3} == {1, 3}

Since a and p generate S3 we deduce that S3 acts on the edges. And the action is faithfull.

- The natural action of S3 on {1, 2, 3} is also faithfull.
- The action of S3 on S3/<p> is not faithfull. S3/<p> = {{e, p, p2}, {a, b, c}} Any element of <p> is sent to the identity. Any other element is sent to ({e, p, p2} {a, b, c})
- There is also an action of S3 on {a, b, c}.
- And an action of S3 on <p>

How to see if 2 actions are isomorphic?

An action is a group morphism phi1: $G \to Aut(S)$ where S is some set. Let phi2: $G \to Aut(S2)$ be another group action. 2 action are isomorphic if there exist a set isomorphism $f : S \to S2$ such that

$$\forall g \in G, f(phi1(g)(x)) = phi2(g)(f(x))$$

f(phi1(g1*g2)(x)) = f(phi1(g1)(phi1(g2)(x))) = phi2(g1)(f(phi1(g2)x)) = phi2(g1)(phi2(g2)(fx)) = phi2(g1g2)(fx)

Let's show that the action on {1, 2, 3} is isomorphic to the action on {{1, 2}, {1, 3}, {2, 3}} by taking f: 1 -> {2, 3} f: 2 -> {1, 3} f: 3 -> {1, 2}

f(p.1) = f(2) = {1, 3} p.f(1) = p.{2, 3} = {1, 3}

f(p.2) = f(3) = {1, 2} p.f(2) = p.{1, 3} = {1, 2}

f(a.1) = f(2) = {1, 3} a.f(1) = a.{2, 3} = {1, 3}

f(a.3) = f(3) = a.f(3) => f(3) = {1, 2}

f(b.2) = f(2) = b.f(2) => f(2) = {1, 3}

f(c.1) = f(1) = c.f(1) => f(1) = {2, 3}

So f is an isomorphism of actions.

Automorphism group of S3

Inner automorphisms

Conjugation by an element creates an automorphism. The conjugation by p has the following effect:

```
## a^p: (2,3)
## b^p: (1,2)
## c^p: (1,3)
## p^p: (1,2,3)
## p2^p: (1,3,2)
```

So conj(p) = (a c b)

Here is the conjugation by p2:

```
## a^p2: (1,3)
## b^p2: (2,3)
## c^p2: (1,2)
## p^p2: (1,2,3)
## p2^p2: (1,3,2)
```

So conj(p2) = (a b c)

Here is the conjugation by a:

```
## a^a: (1,2)
## b^a: (2,3)
## c^a: (1,3)
## p^a: (1,3,2)
## p2^a: (1,2,3)
```

So conj(a) = (b c) (p p2)

Here is the conjugation by b:

```
## a^b: (2,3)
## b^b: (1,3)
## c^b: (1,2)
## p^b: (1,3,2)
## p2^b: (1,2,3)
```

So conj(b) = (a c) (p p2)

Here is the conjugation by c:

```
## a^c: (1,3)
## b^c: (1,2)
## c^c: (2,3)
## p^c: (1,3,2)
## p2^c: (1,2,3)
```

So conj(c) = (a b)(p p2)

So together with the identity there are 6 inner automorphisms.

Structure of the automorphism group

We will show that Inner(S3) = S3 conj(a)^2 = e conj(p)^3 = e conj(p)^conj(a) = conj a-1 o conj p o conj a = conj a-1pa = conj p2 = conj(p)^2

Since the generators of Inner(S3) have the same relations as those of S3 we know the 2 groups are isomorphic.

Total group or isomorphism

We can show that there are no outer isomorphism. Indeed an automorphism maps elements to elements of the same order. There are thus 2 possibilities for mapping p and 3 possibilities for mapping a. Since a and p generate S3 this 2 images will define uniquely a mapping. There are at most 3*2 automorphisms and according to the previous paragraphs we already have 6 inner automorphisms. So all the automorphism are inner.

Matrix representation of an automorphism

Since an automorphism is uniquely defined by its image of the generators a and p we might be able to write them as a matrix. Indeed the reason we are able to write linear operators as matrix is because they are defined by their action on a basis which is a generator set of the whole vector space.

First lets try to write elements of S3 as vectors: we note a = (1, 0) and p = (0, 1) (0, x) * (1, m) = p^x * a * p^m = a * a p^x * a p^m = (1, 2*x+m)

(g1, n1) * (g2, n2) = g1n1g2n2 = g1g2inv(g2)n1g2n2 = g1g2n1^g2n2 = (g1g2, n1^g2*n2)

inv(g1) * (g2, n2) * g1 = inv(g1) * (g2*g1, n2^g1) = (g2^g1, n2^g1) conj(g1) = g1 e e g1