Multiple Linear Regression: practical aspects

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Recap

Recap: Ceteris Paribus

Fill in page 1 here: bit.ly/41R1YpL

Recap: Multiple regression

Fill in page 2 here: bit.ly/41R1YpL

Where does the regression equation come from?

- We do not make regression equations because we like how they look.
- We base them on theory:
  - on economic theory as well as underlining natural and biological processes.
- Here is one of many examples: Hedonic Model

Hedonic Model

To understand where the regression equation comes from, let us follow an example of:

Hedonic Model

- Any idea what this is?

Hedonic Model overview

Hedonic prices is an econometric approach of quantifying monetary values of **differentiated characteristics** (x_i) of goods and services, which are subjects of economic exchange (and stochastic variation u).

$$\text{Price} = f(x_1, x_2, \cdots, x_i, u)$$

For example, **agricultural land** has such characteristics as: …

Land quality (location, slope, soil salinity, nutrient content, irrigation availability, rainfall, climate) environmental limitation, farmers' accessibility, and other.

Hedonic equation is based on the theory that takes its roots to supply and demand.

Supply and demand theory: a structural approach

Demand function:

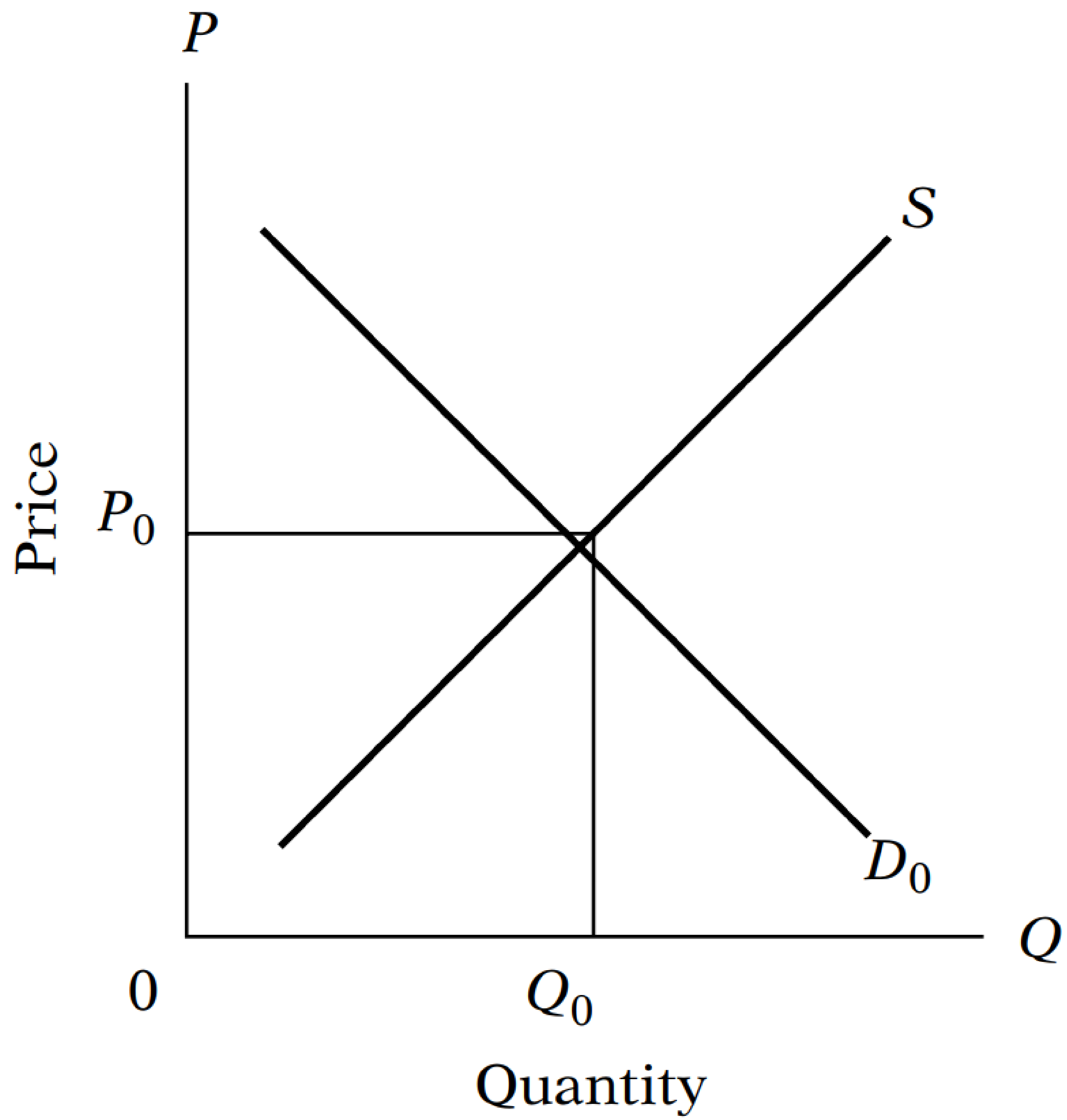
$$Q_t^D = \alpha_0 + \alpha_1 P_t + u_{1t}, \quad \text{with } \alpha_1 <$$

Supply function:

$$Q_t^S = \beta_0 + \beta_1 P_t + u_{2t}, \quad \text{with } \beta_1 >$$

Equilibrium condition:

$$Q_t^S = Q_t^D$$



Hedonic land prices model

Relies on the partial equilibrium framework (Palmquist, 1989), where

- $R(\cdot)$ - realized land price (rental of sales), are modeled from two sides:

Supply

- Land owner whats to **maximize own profit** from renting land out.
- Owner's **offer** function:

$$\phi(\hat{z}, \tilde{z}, \pi^{S'}, r, \beta)$$

$\hat{z}$ - land characteristics exogenous to land owner;

$\tilde{z}$ - land characteristics in control of land owner;

Demand

- Farmer whats to **maximize agricultural profit** from land
- Farmer's **bid** function"

$$\pi^{S'} + C(\hat{z}, \tilde{z}, r, \beta)$$

r - inputs prices;

β - technologies and opportunities such as credit availability;

$\pi^{S'}$ - expected profit of agricultural producers from land;

Structural model of the realized prices

Supply-demand equilibrium:

$$\phi(\hat{z}, \tilde{z}, \pi^{S^{'}}, r, \beta) = R = \pi^{S^{'}} + C(\hat{z}, \tilde{z}, r, \beta)$$

Observed prices R are the equilibrium between **bid** and **offer** (demand and supply).

This is called a Structural Model

Econometric modell

To explain causes behind price changes, we can deconstruct **structural model** into **reduced form** equations, which can be estimated:

Reduced form supply side

$$R = \phi(\hat{z}, \tilde{z}, \pi^{S'}, r, \beta) + e$$

Reduced form demand side

$$R = \pi^{S'} + C(\hat{z}, \tilde{z}, r, \beta) + e$$

When we run a hedonic prices model

$$R = R(\hat{z}, \tilde{z}, \pi^{S'}, r, \beta)$$

we estimate one of the reduced forms and select independent variables based on the hedonic prices theory (differentiated land qualities).

Relevance of the theory

1. Theory provides a rationale behind causal relationship

$$R = R(\hat{z}, \tilde{z}, \pi^{S'}, r, \beta)$$

2. Theory suggests a functional form.

3. Theory stipulates the dependent variable.

4. Theory specifies key determinants of the outcome:
    - AKA what our regressors/independnet variables.

What are the differentiated land characteristics?

In $R = R(\hat{z}, \tilde{z}, \pi^{S'}, r, \beta)$, what are these independent and dependent variables?

$\tilde{z}$ Affected by land owner:

- To enroll for subsidies or not.
- To install irrigation or not.
- Fertilize land
- Improve the landscape

$\hat{z}$ Not affected by land owner:

- Weather
- Location
- Restrictions

Example of a hedonic land prices model: `03c-hedonic-land-prices.Rmd`.

Problem

We would like to assess the effect of the “Conservation Reserve Program” (CPR) on the agricultural land prices in Minnesota in 2002-2011.

Conservation Reserve Program

- is a subsidy
- obligates farms NOT TO GROW ANY CROPS on the enrolled land
- pays monetary compensation in exchange;

Regression equation

$$\log(\text{acrePrice}) = \beta_0 + \beta_1 \text{crpPct} + \log(\beta_2 \text{acres}) + \beta_3 \text{region} \\ + \beta_4 \text{year} + \beta_4 \text{tillable} + \beta_5 \text{productivity} + \beta_6 \text{improvements} + e$$

- `acrePrice` - sale price in dollars per acre;
- `acres` - size of the farm in acres;
- `region` - region in the state Minnesota;
- `year` - year of the land sales translation;
- `crpPct` - the percentage of all farm acres enrolled in CRP;
- `tillable` - percentage of farm acreage that is rated arable by the assessor;
- `productivity` - average agronomic productivity scaled 1 to 100, with larger numbers for more productive land;
- `improvements` - percentage of property value due to improvements (infrastructure)

Regression results

	log(Price per acre)	log(Price per acre)
Intercept	6.099 *** (0.051)	6.507 *** (0.034)
Subsidy (0|1)	-0.00370 *** (0.00027)	-0.00488 *** (0.00017)
Area (log), acres	-0.0587 *** (0.0058)	-0.0913 *** (0.0051)
Tillable area, % (0-100)	0.00421 *** (0.00036)	0.00550 *** (0.00018)
Improvements, (0-100)	0.01567 *** (0.00062)	0.01414 *** (0.00033)
Productivity, (0-100)	0.0094 *** (0.0004)	
West Central (0|1)	0.633 *** (0.016)	0.746 *** (0.011)
Central (0|1)	0.885 *** (0.018)	1.041 *** (0.013)
South West (0|1)	0.734 *** (0.016)	1.019 *** (0.011)
South Central (0|1)	0.854 *** (0.017)	1.191 *** (0.011)
South East (0|1)	0.897 *** (0.018)	1.234 *** (0.012)
Num.Obs.	8770	17441
R2 Adj.	0.700	0.646

Note: ^^ Year-specific dummy variables are omitted. Heteroscedasticity consistent standard errors are reported in parentheses. P-values are coded as: * p=0.05, ** p=0.01, *** p<0.001

Goodness of fit: R^2 adjusted

Shows the share of variance explained by a model adjusted to the number of independent variables.

- If the number of independent variables increases, but variables do not explain y better, R^2 adjusted could shrink to 0 or a negative number.
- We generally want to have it as high as possible, however!
- R^2 adjusted has nothing to do with the coefficients' significance and their causal meaning.
- If the goal of regression is to explain causes, rather than predict outcomes, R^2 adjusted has not much relevance.

Practical example

Base on (Ashenfelter, Ashmore, & Lalonde, 1995)

> **What makes wine so expensive?**

youtu.be/8WMRj9mTQtI

Let us summarize the causes of wine prices

- Any guesses?
- Weather
- Any unobserved characteristics?
  - Art of the winemaker
  - Storage
  - Way of drinking

Let us see what regression tells us about wine

$$Price = \beta_0 + \beta_1 \text{Age} + \beta_2 \text{AGST} + \beta_3 \text{HarvestRain} + \beta_4 \text{WinterRain} + e$$

What are our expectations about signs of β s?

Variable

- `Price`: average market price for Bordeaux vintages according to a series of auctions (USD). The price is relative to the price of the 1961 vintage, regarded as the best one ever recorded.
- `WinterRain`: winter rainfall (in mm).
- `AGST`: Average Growing Season Temperature (in Celsius degrees).
- `HarvestRain`: harvest rainfall (in mm).
- `Age`: age of the wine, measured in 1983 as the number of years stored in a cask.

Let us regress wine in the class

> Exercise: `03-wine-regression.Rmd`

Multiple Linear Regression

> "Regression is the tool that masters pick up first, if only to provide a benchmark for more elaborate empirical strategies." (Angrist & Pischke, 2014)

Why regression?

> "… regression estimates are weighted averages of multiple matched comparisons of the sort constructed for the groups in our stylized matching matrix." (Angrist & Pischke, 2014)

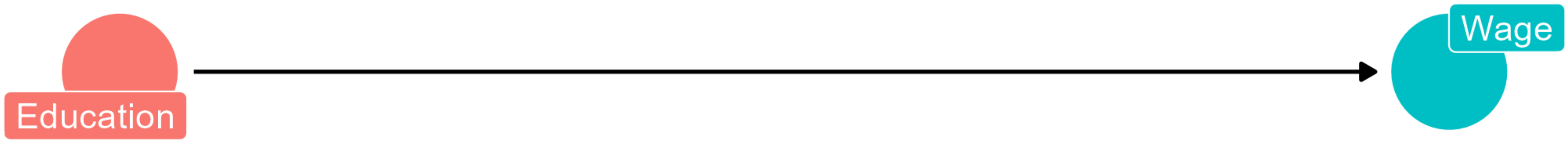
Wage ~ Education: equation

$$Y_i = \alpha + \beta \, P_i + e_i,$$

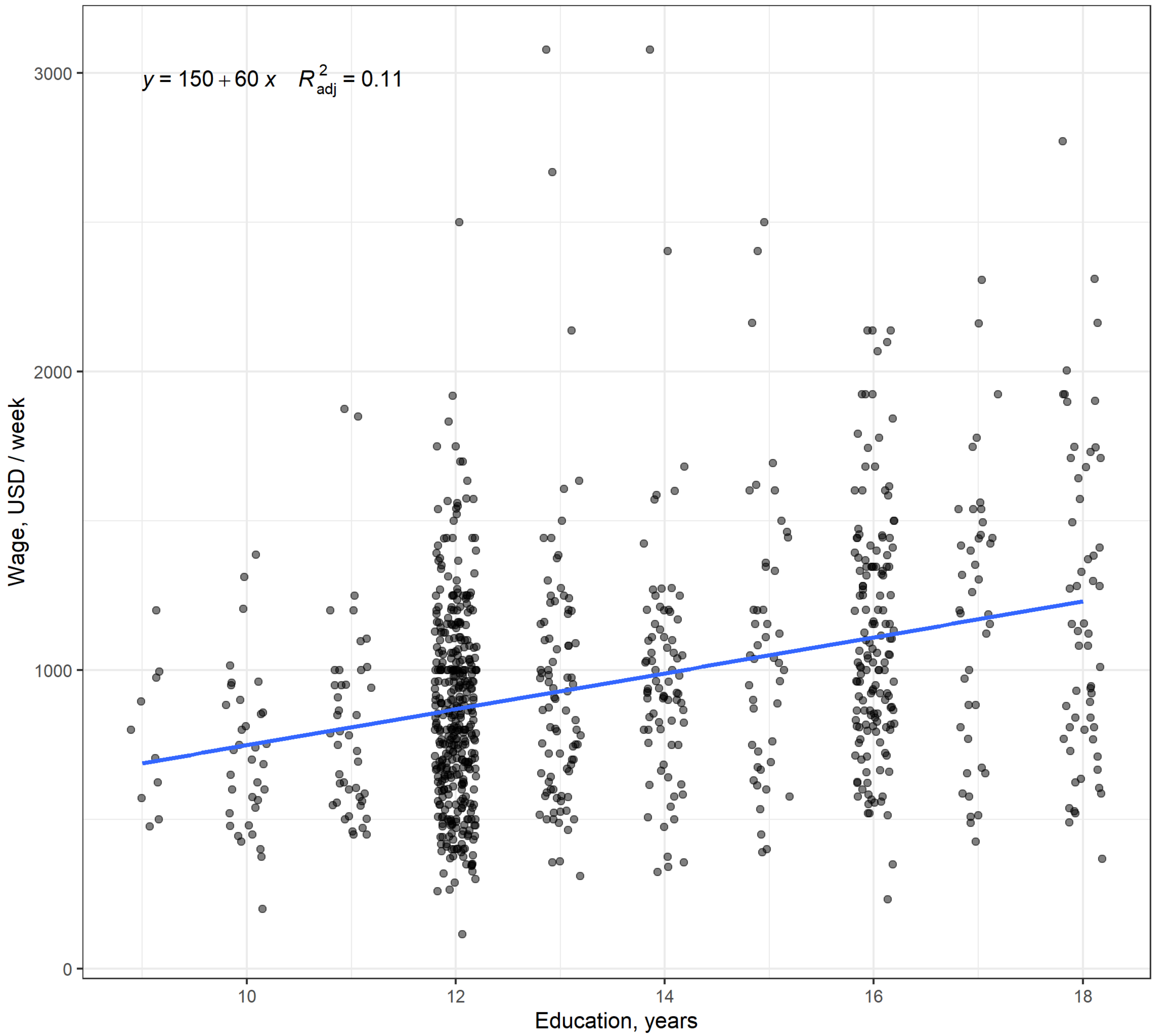
where:

- Y_i is wage in Euro per week
- P_i is education in years
- α is the intercept
- β is the slope or **causal effect of interest**

Wage ~ Education.

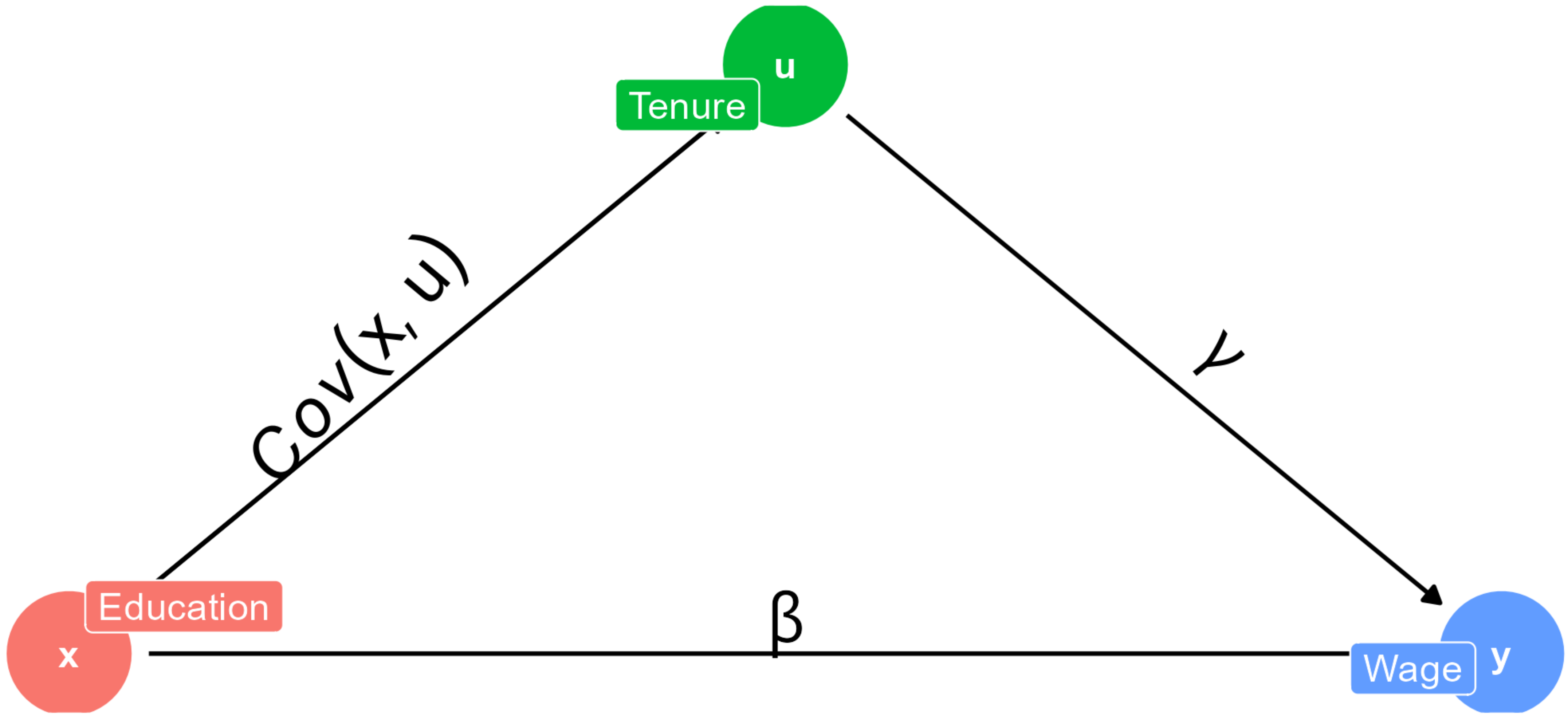


Relationship



- Interpret the effect of education on wage.
- Is this a causal effect on education on wage?
  - Explain why?

Wage ~ Education: Is there a ceteris paribus?



> Regression accounts for the observed (included) confounders by attributing variance in y to the variance in x (variable of interest) and u (control variables).

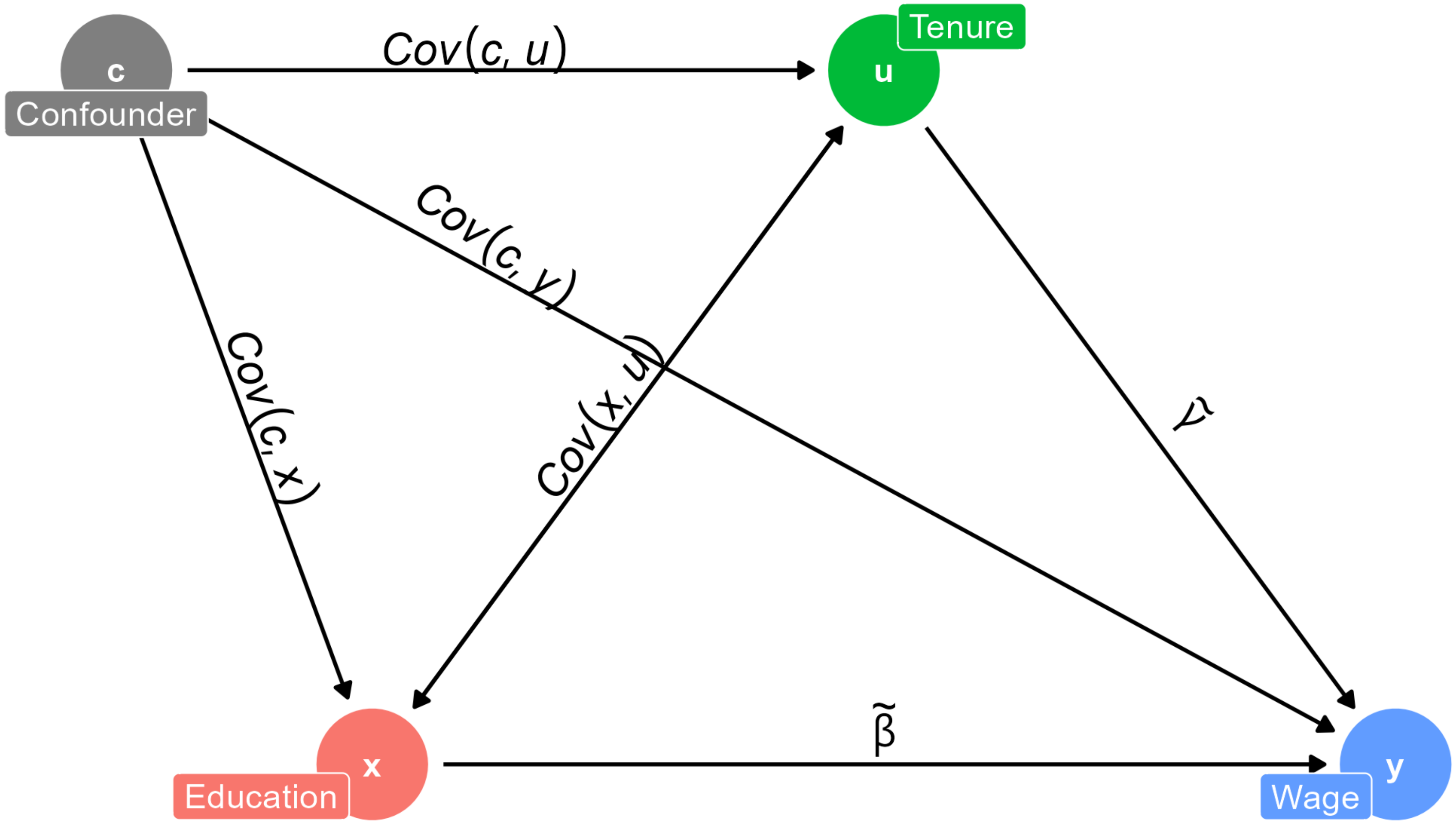
Why regression?

- In multiple regression, Ceteris Paribus is achieved by introducing **control variables** (A_i).

$$Y_i = \alpha + \beta\, P_i + \gamma\, A_i + e_i,$$

- Regression **controls** the variance in Y_i with observed P_i and A_i.
- In the context of the variable of interest (P_i):
    - Regression controls other variables (A_i) fixed,
    - ensuring that β reviels causal effect Ceteris Paribus.

Wage ~ Education: Really? Is there a ceteris paribus?



- Not every confounded could be observed or measures.
- There are unobserved ones:
  - …
- Ability, attitude, effort.

When a confounded correlated with outcome $Cov(c, y) \neq 0$ and other regressors $Cov(c, u) \neq 0$ and $Cov(c, x) \neq 0$.

- Estimates of β and γ are no longer ceteris paribus!
  - They are biased: $\tilde{\beta}$ and $\tilde{\gamma}$

Selection bias in regression analysis

> Absence of the Ceteris Paribus in a regression is called **omitted variable bias**

OVB: The long model

Supposed that our **ideal regression**

- the true model / population regression / **long model** is:

$$Y_i = \alpha^l + \beta^l P_i + \gamma A_i + e_i^l,$$

We cannot measure A_i, but:

- A_i has a causal effect on Y_i: $(E[Y_i|A_i] \neq 0)$, and
- A_i correlated with P_i: $(E[P_i|A_i] \neq 0)$:

OVB: The short model

Because of the **omitted variable**,

we cannot estimate the **long model**.

Instead, we estimate a **short model**:

$$Y_i = \alpha^s + \beta^s P_i + e_i^s$$

where omitted variable is implicit in the residuals:

$$e_i^s = e_i^l + A_i$$

Bias of variable omission

Omitted variable causes bias of all estimates!

This bias can be measured as OVB:

$$\text{OVB} = \beta^s - \beta^l$$

To be continued on the OVB in another week

How does regression fights selection bias?

Any ideas?

Any ideas?

We include control variables to reduce or defeat the omitted variable bias.

Wage ~ Education example

Introduction

We use data from (Blackburn and Neumark, 1992) on wage determinants. Variables present are:

- $wage$ - monthly earnings in USD;
- $educ$ - years of education;
- $exper$ - years of experiences;
- $black$ - dummy variable representing individuals which are not Caucasian;
- $female$ - dummy variable representing females;

Our goal is to identify the causal effect of education on wage estimating following equation:

$$\text{wage} = \beta_0 + \beta_1 \text{educ} + \beta_2 \text{exper} + \beta_3 \text{black} + \beta_4 \text{female} + e$$

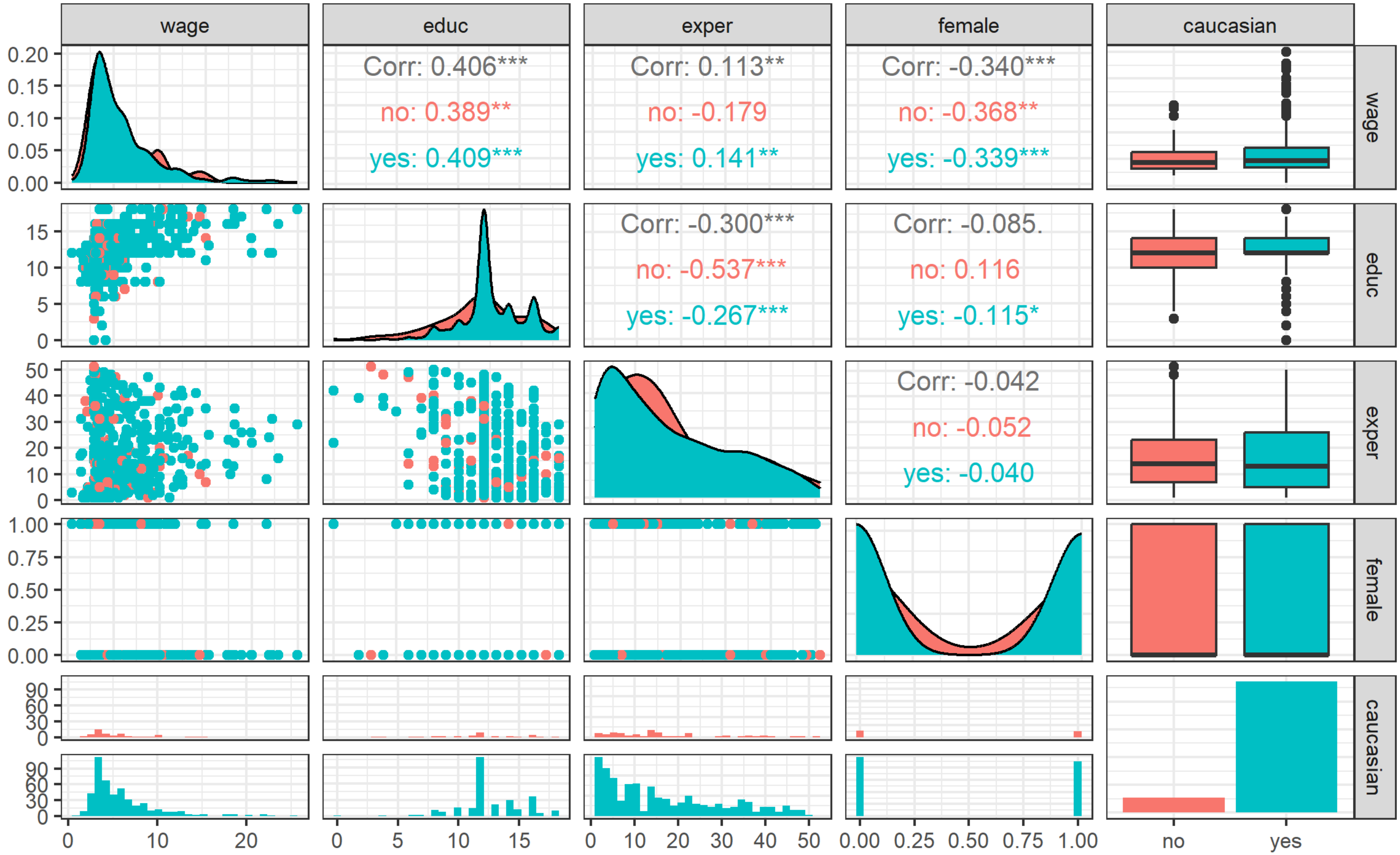
Loading data

```
Rows: 526
Columns: 7
$ wage      <dbl> 3.10, 3.24, 3.00, 6.00, 5.30, 8.75, 11.25, 5.00, 3.60, 18.18…
$ educ      <dbl> 11, 12, 11, 8, 12, 16, 18, 12, 12, 17, 16, 13, 12, 12, 12, 1…
$ exper     <dbl> 2, 22, 2, 44, 7, 9, 15, 5, 26, 22, 8, 3, 15, 18, 31, 14, 10,…
$ black     <dbl> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, …
$ white     <dbl> 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, …
$ female    <dbl> 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, …
$ caucasian <fct> yes, yes, yes, yes, yes, yes, yes, yes, yes, yes, yes, yes, …
```

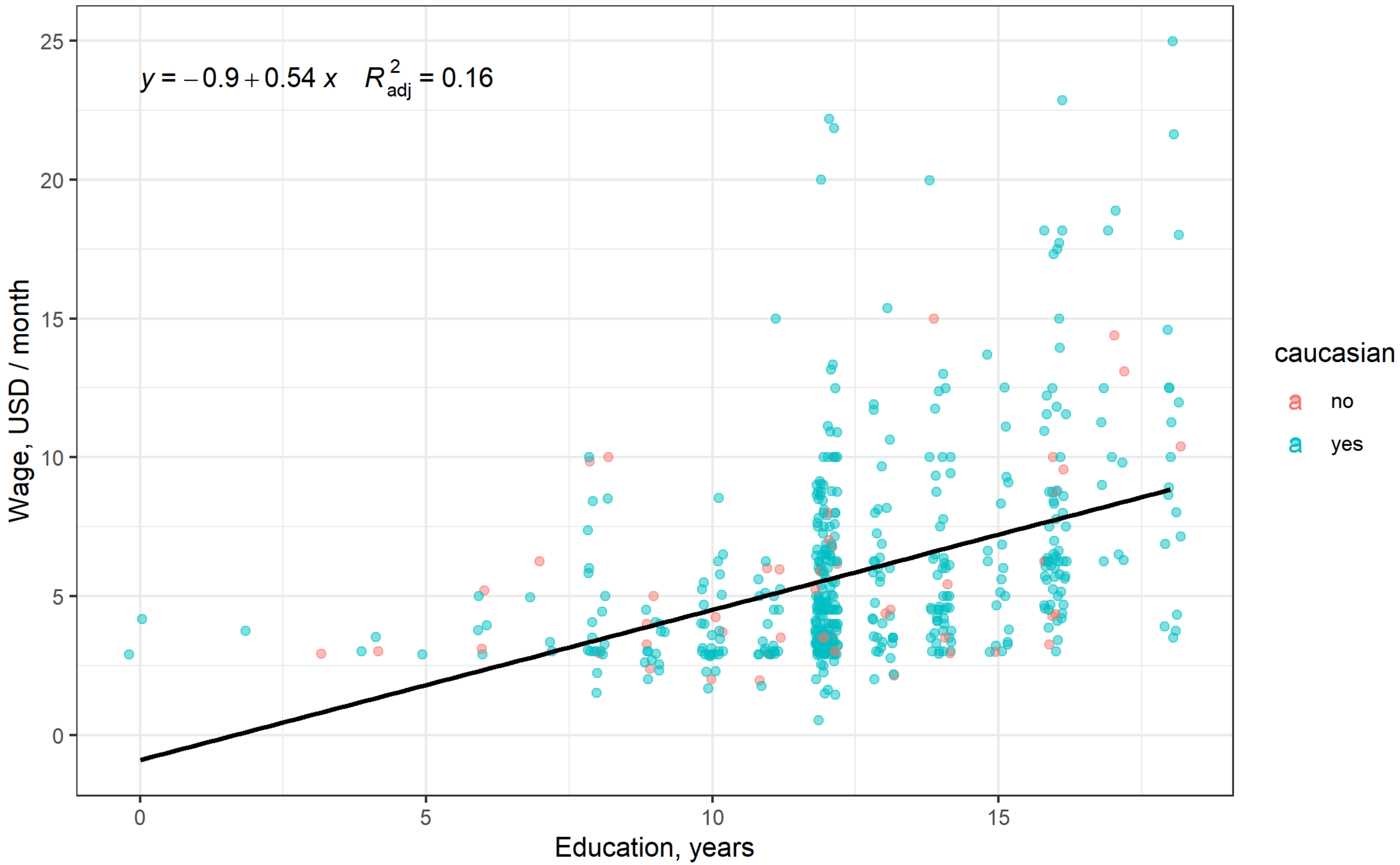
Exploratory data analysis (1/3)

	Unique (#)	Missing (%)	Mean	SD	Min	Median	Max
wage	241	0	5.9	3.7	0.5	4.7	25.0
educ	18	0	12.6	2.8	0.0	12.0	18.0
exper	51	0	17.0	13.6	1.0	13.5	51.0
black	2	0	0.1	0.3	0.0	0.0	1.0
white	2	0	0.9	0.3	0.0	1.0	1.0
female	2	0	0.5	0.5	0.0	0.0	1.0

Exploratory data analysis (2/3)



Exploratory data analysis (3/3)



Estimating regression

```r
mod1 <- lm(wage ~ educ + exper + black + female, data = wage_dta)
mod1
```

```
Call:
lm(formula = wage ~ educ + exper + black + female, data = wage_dta)

Coefficients:
(Intercept)         educ        exper        black       female  
   -1.71453      0.60175      0.06422     -0.08389     -2.15649  
```

Interpreting the results (1)

```r
summary(mod1)
```

```
Call:
lm(formula = wage ~ educ + exper + black + female, data = wage_dta)

Residuals:
    Min      1Q  Median      3Q     Max 
-6.3666 -1.9740 -0.4936  1.1248 14.8123 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -1.71453    0.76169  -2.251   0.0248 *  
educ         0.60175    0.05135  11.718  < 2e-16 ***
exper        0.06422    0.01041   6.168 1.39e-09 ***
black       -0.08389    0.44430  -0.189   0.8503    
female      -2.15649    0.27060  -7.969 1.01e-14 ***
```

Interpreting the results (2): fancy summary

	Model 1
(Intercept)	-1.715 (0.762)*
educ	0.602 (0.051)***
exper	0.064 (0.010)***
black	-0.084 (0.444)
female	-2.156 (0.271)***
Num.Obs.	526
R2 Adj.	0.304
F	58.341

Interpreting the results (3): Effect of a dummy variables

```r
ggpredict(mod1, term = c("educ"))
```

```
# Predicted values of wage

educ | Predicted |         95% CI
---------------------------------
   0 |     -1.66 | [-2.96, -0.37]
   3 |      0.14 | [-0.86,  1.14]
   5 |      1.35 | [ 0.54,  2.15]
   7 |      2.55 | [ 1.93,  3.17]
  10 |      4.35 | [ 3.99,  4.72]
  12 |      5.56 | [ 5.29,  5.83]
  14 |      6.76 | [ 6.46,  7.06]
  18 |      9.17 | [ 8.56,  9.78]

Adjusted for:
*  exper = 17.02
*  black =  0.10
* female =  0.48
```

```r
ggpredict(mod1, term = c("educ", "female"))
```

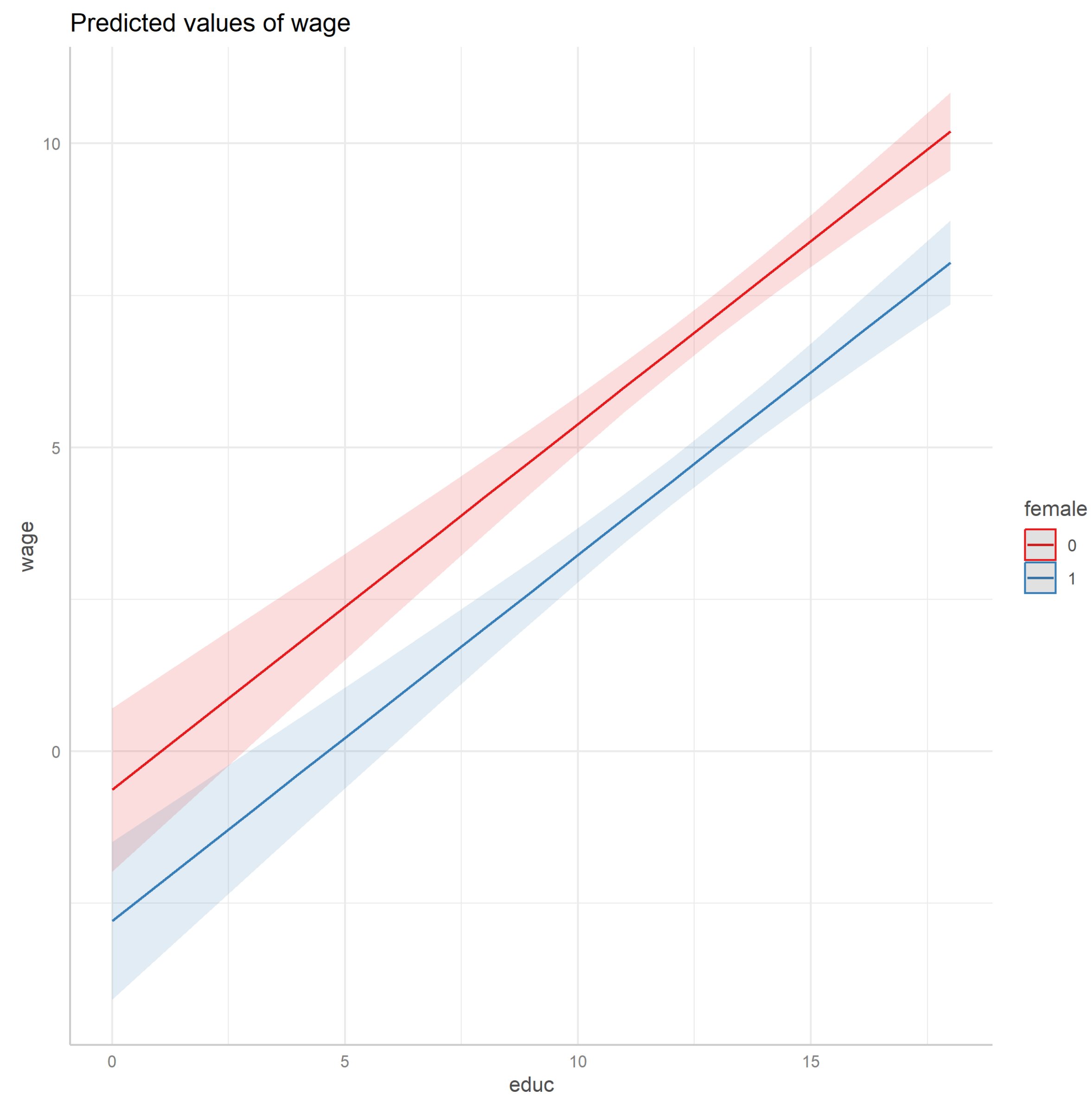
```
# Predicted values of wage

# female = 0

educ | Predicted |         95% CI
---------------------------------
   0 |     -0.63 | [-1.97,  0.71]
   4 |      1.78 | [ 0.82,  2.74]
   7 |      3.58 | [ 2.89,  4.27]
  10 |      5.39 | [ 4.92,  5.85]
  12 |      6.59 | [ 6.22,  6.96]
  18 |     10.20 | [ 9.57, 10.84]

# female = 1

educ | Predicted |         95% CI
---------------------------------
   0 |     -2.79 | [-4.08, -1.49]
   4 |     -0.38 | [-1.30,  0.54]
   7 |      1.43 | [ 0.77,  2.08]
  10 |      3.23 | [ 2.79,  3.67]
  12 |      4.43 | [ 4.05,  4.82]
```



Conclude

- Is model 1 a good predictor of wage based on education?
- Is the effect of education causal?

	Model 1
(Intercept)	-1.715 (0.762)*
educ	0.602 (0.051)***
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Elasticity

> Elasticity is a unit-less measure of change in one variable as a result of a change in the other.

Elasticity

Elasticity of y with response to x (x elasticity of y):

$$\epsilon = \frac{\partial y / y}{\partial x / x} = \frac{\partial y}{\partial x} \frac{x}{y}$$

$$\epsilon = \frac{\frac{y_2 - y_1}{y_1}}{\frac{x_2 - x_1}{x_1}}$$

Elasticity in a linear model

$$\text{wage} = \beta_0 + \beta_1 \text{educ} + \beta_2 \text{exper} + \beta_3 \text{black} + e$$

Let us compute elasticity of wage in response to educ:

$$\epsilon_{\text{wage,educ}} = \frac{\partial y}{\partial x} \frac{x}{y},$$

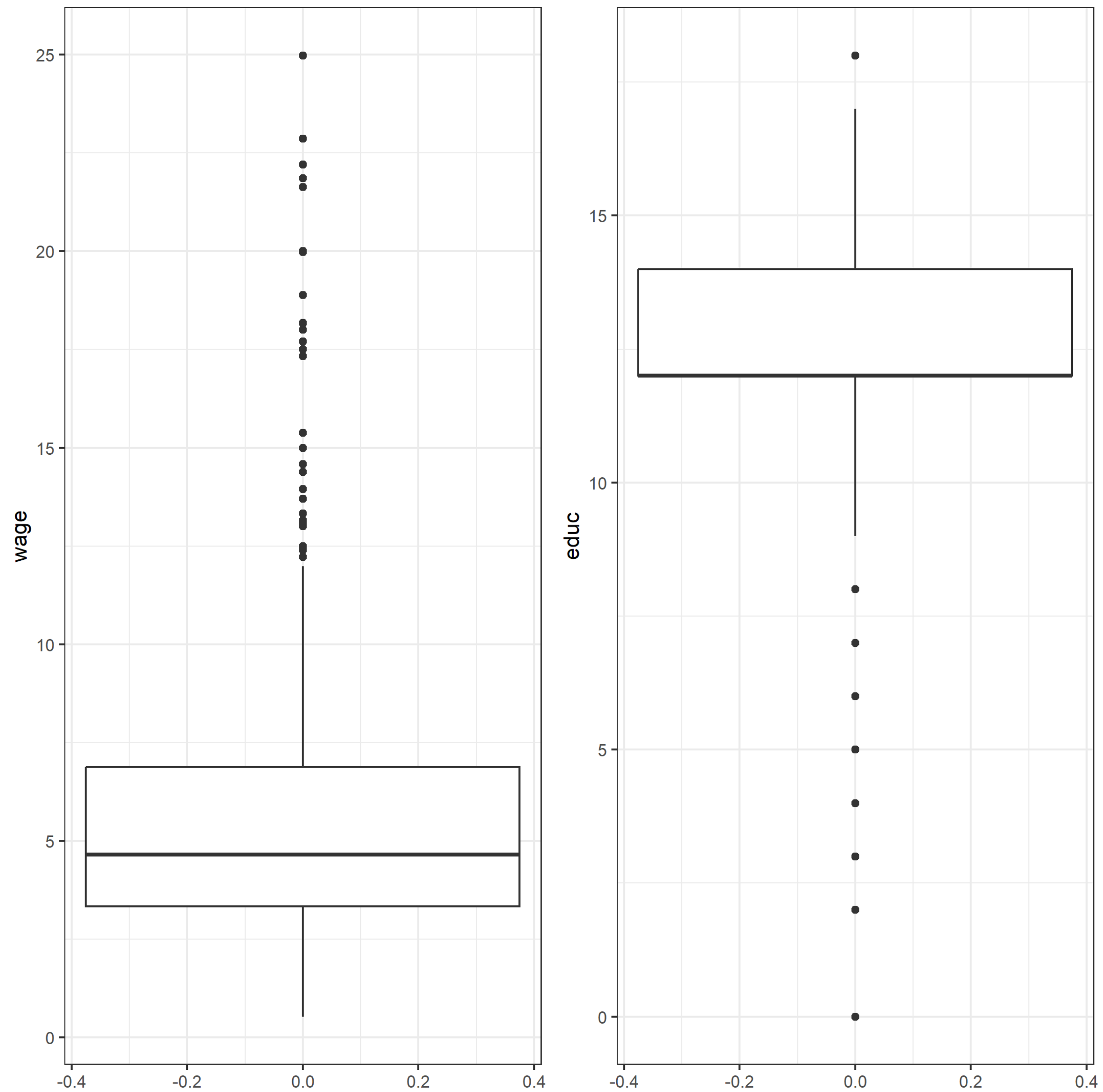
where: $\beta_1 = \frac{\partial y}{\partial x}$

Therefore, elasticity of wage depends on on the value of x and y.

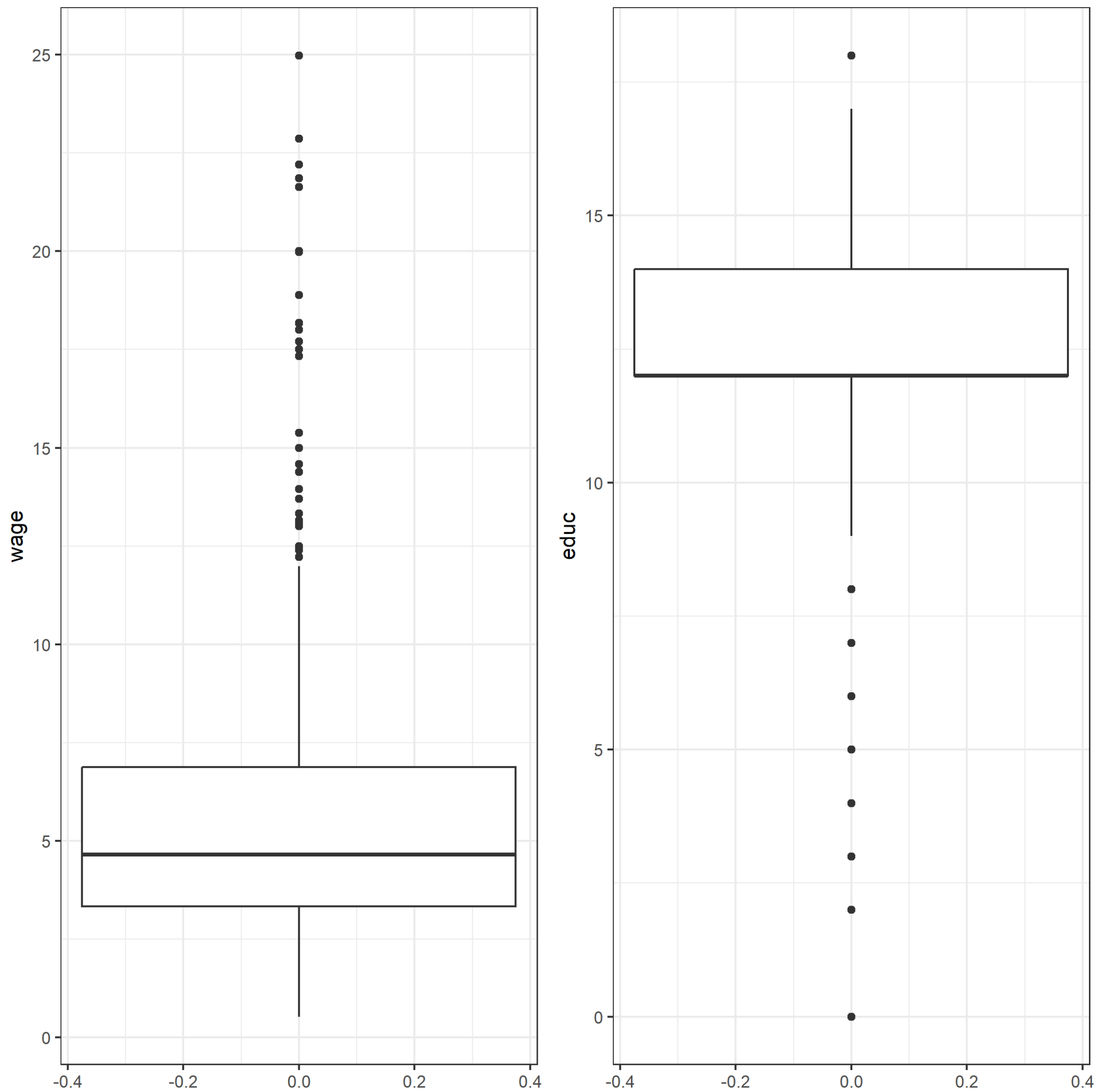
When elasticity depends on a valued of another variable, we evaluate it at mean (or other quantiles) values of these variables.

Eslimating elasticity in a linear model (1/2)

	wage	educ
mean	5.90	12.56
q2	3.33	12.00
median	4.65	12.00
q4	6.88	14.00



Eslimating elasticity in a linear model (1/2)



Elasticity at mean:

```
coef(mod1)[2] * (mean(wage_dta$educ) / mean
```

```
    educ
1.282137
```

Elasticity at 2nd, 3rd and 4th quartiles:

```
coef(mod1)[2] * (fivenum(wage_dta$educ)[[2]
```

```
    educ
2.168464
```

```
coef(mod1)[2] * (fivenum(wage_dta$educ)[[3]
```

```
  educ
1.5529
```

```
coef(mod1)[2] * (fivenum(wage_dta$educ)[[4]
```

```
    educ
1.224489
```

Regression assumptions

Go to the page 2 to refresh motivation for regression assumptions

- bit.ly/41R1YpL

Asymptotic properties of the OLS (simplified)

OLS results with consistent (unbiased) and efficient estimates of population parameters when sample size is finite ($n \rightarrow \infty$)

- $\hat{\beta} \rightarrow \beta$
- $Var(\hat{\beta}) \rightarrow 0$

When the sample size is finite and all Gauss-Markov assumptions are satisfied:

- $\hat{\beta}$ - estimates vary from sample to sample, but
- $Var(\hat{\beta})$ is distributed according to the `t` **- distribution**.
- Variances of two estimates $Var(\hat{\beta}_1)$ and $Var(\hat{\beta}_2)$ are distributed according to the `F` **- distribution**.

When GM assumptions are not satisfied:

- `t` and `F` distributions are no longer relevant and we cannot perform conduct inference.

Linearity

Linearity: meaning

- the expected value of a dependent variable is a straight-line function of the independent variable
- If linearity is violated:
  - **estimates are biased**
  - inappropriate representation of the dependent variable

Linearity: detection

- How to detect a non-linearity?
  - no accepted statistical tests, but
  - the visual inspection
- Typical plots:
  - Scatter plots of dependent and independent variables;
  - **observed** versus **predicted/fitted** values;
  - **residuals** versus **predicted/fitted** values;

Linearity: resolutions

1. (non) linear transformation to the dependent and/or independent variables;
   - **it does change the way how we must interpret coefficients**;
2. find a different independent variable;
3. propose a different functional form;

Common linear transformations

- Interaction term: $y = \beta_0 + \beta_1 x_1 \cdot x_2 + \beta_2 x_3 + e$
- Natural logarithm: $\log y = \beta_0 + \beta_2 \log x_1 + \beta_2 x_2 + \beta_3 \log x_3 + e$
- Power transformation and polynomial: $y = \beta_0 + \beta_2 x_1^2 + \beta_2 x_2^3 + \beta_3 \sqrt{x}_3 + e$
  - Box-Cox transformation.
  - Tailor expansion (Cobb-Douglas, Trans-log).
- Reciprocal: $\log y = \beta_0 + \beta_2 \frac{1}{x_1} + \beta_2 x_2 + \beta_3 \log x_3 + e$
- Standardized variables $\frac{y - \bar{y}}{S_y} = \beta_0 + \beta_1 \frac{x_1 - \bar{x}_1}{S_{x_1}} + \beta_2 \frac{x_2 - \bar{x}_2}{S_{x_2}} + e$

Log

Model	Dep. var.	Indep. var.	Equation	Slope	Interpretation	Elasticity
				$\frac{\partial y}{\partial x}$		$\frac{\partial y}{\partial x} \cdot \frac{x}{y}$
Level - level	y	y	$y = \beta_0 + \beta_1 x$	β_1	$\Delta y = \beta_1 \Delta x$	$\beta_1 \frac{x}{y}$
Level - log	y	$\log x$	$\log y = \beta_0 + \beta_1 x$	$\beta_1 y$	$\Delta y = (\beta_1/100)\%\Delta x$	$\beta_1 x$
Log - level	$\log y$	x	$y = \beta_0 + \beta_1 \log x$	$\beta_1 \frac{1}{x}$	$\%\Delta y = 100\beta_1 \Delta x$	$\beta_1 \frac{1}{y}$
Log - log	$\log y$	$\log x$	$\log y = \beta_0 + \beta_1 \log x$	$\beta_1 \frac{y}{x}$	$\%\Delta y = \%\beta_1 \Delta x$	β_1

Log: Key limitations

- $\log(0) = -\infty$;
- what is the $\log(x)$, when $x < 0$?

Variables standardiztion to the standard normal distribution

Model	Dep. var.	Indep. var.	Equation	Slope	Interpretation	Elasticity
				$\frac{\partial y}{\partial x}$		$\frac{\partial y}{\partial x} \cdot \frac{x}{y}$
Standardized variables	$y^* = \frac{y-\bar{y}}{S_y}$	$x^* = \frac{x-\bar{x}}{S_x}$	$y^* = \beta_0 + \beta_1 x^*$	$\frac{\partial y^*}{\partial x^*}$	$\text{SD}\Delta y = \text{SD}\beta_1 \Delta x$	DIY

Key limitations:

- Not intuitive interpretation

Reciprocal

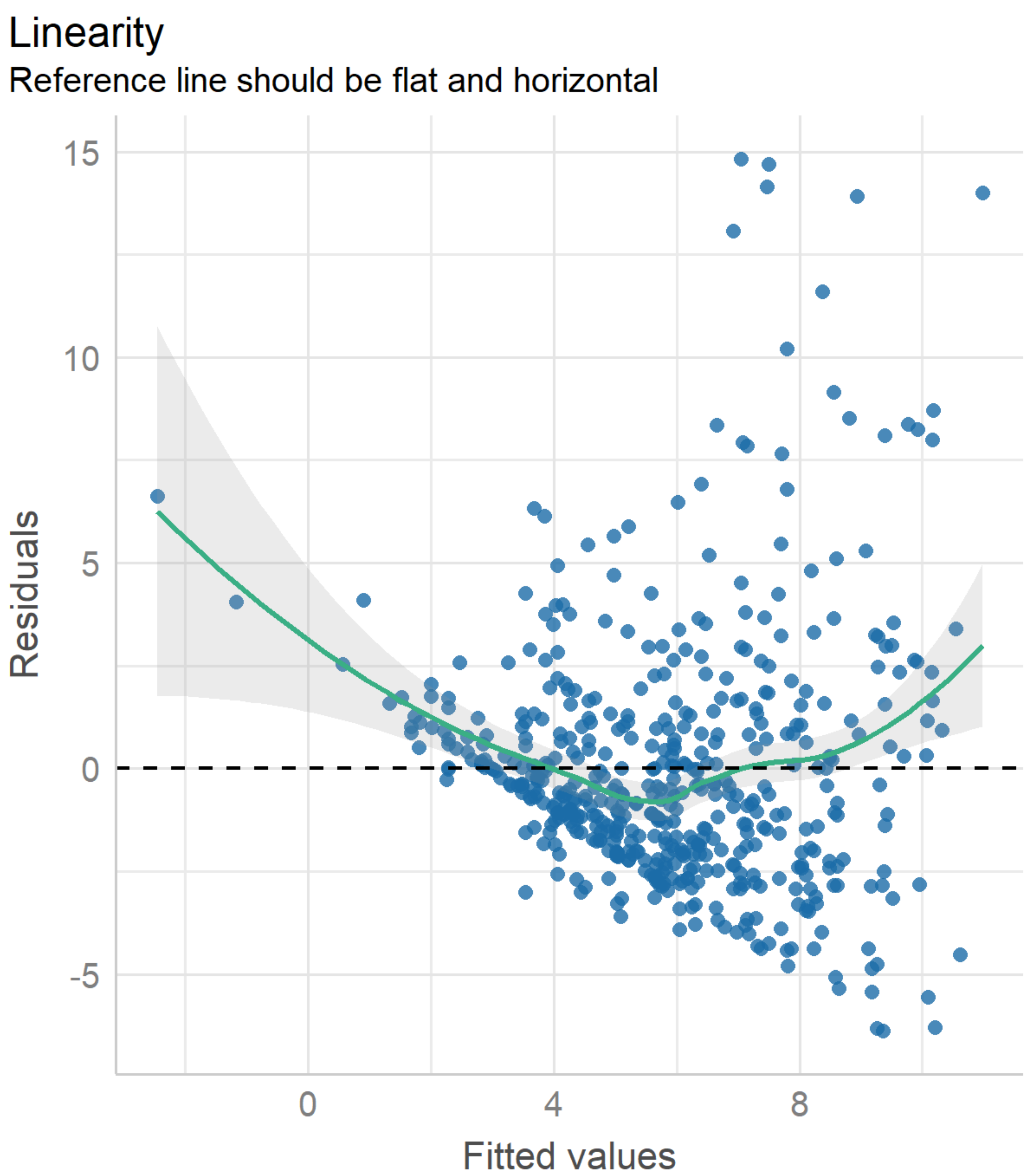
Model	Dep. var.	Indep. var.	Equation	Slope	Elasticity
				$\frac{\partial y}{\partial x}$	$\frac{\partial y}{\partial x} \cdot \frac{x}{y}$
Reciprocal	y	$\frac{1}{x}$	$y = \beta_0 + \beta_1 \frac{1}{x}$	$-\beta_1 \frac{1}{x^2}$	$-\beta_1 \frac{1}{xy}$

Interpretation:

- When x increases to infinity, y reaches asymptotically β_0

See Gujarati (2004) Chapter 6.7 for more details on interpreting the reciprocal relationship.

Linearity in the wage equation



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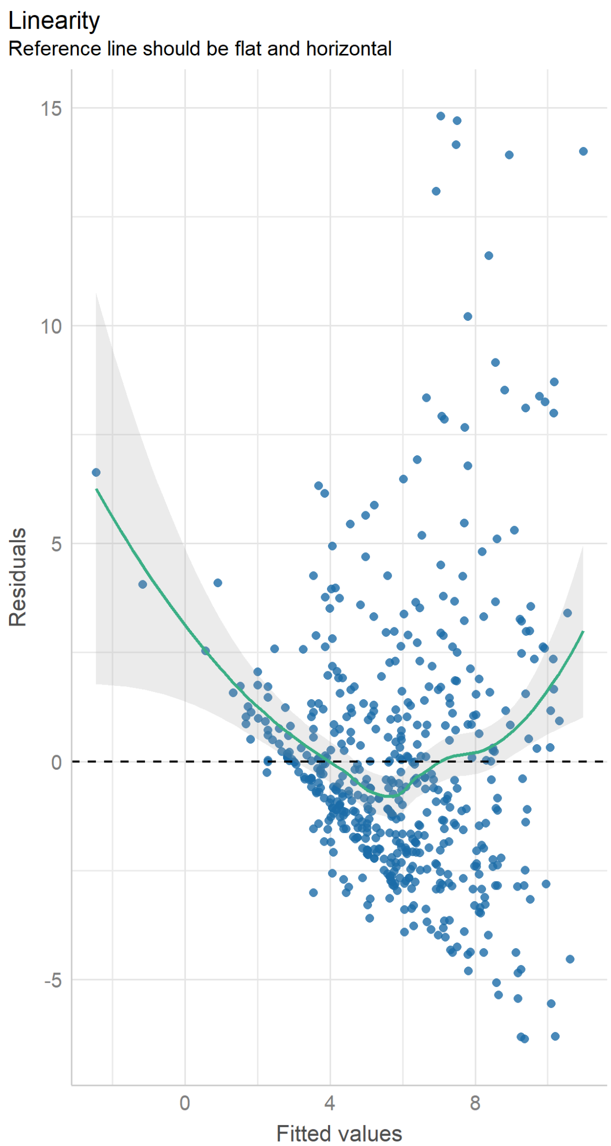
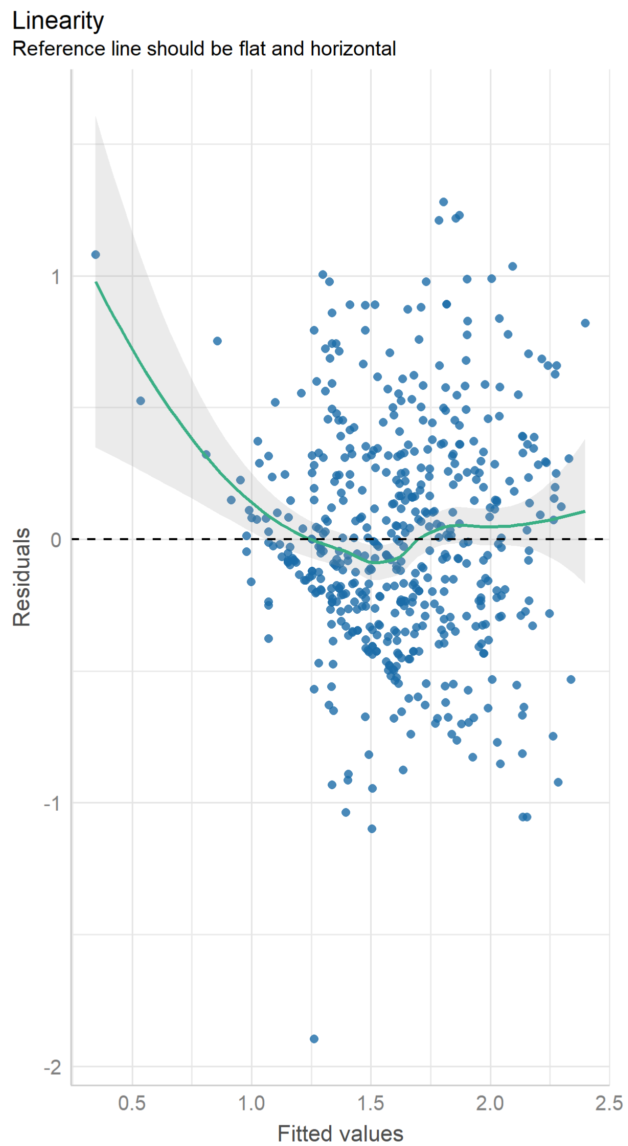
Wage equaition update

```
Call:
lm(formula = log(wage) ~ educ + exper + black + female, data = wage_dta)

Coefficients:
(Intercept)         educ         exper         black        female  
   0.483188     0.091192     0.009411     -0.009889     -0.343712  


Call:
lm(formula = log(wage) ~ educ + exper + black + female, data = wage_dta)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.89689 -0.26333 -0.03394  0.26654  1.28131 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.483188   0.106141   4.552 6.61e-06 ***
educ         0.091192   0.007156  12.743  < 2e-16 ***
exper        0.009411   0.001451   6.487 2.04e-10 ***
black       -0.009889   0.061913  -0.160    0.873    
female      -0.343712   0.037709  -9.115  < 2e-16 ***
```

Non-linearity change





Interpretation

	Model 1 (level-level)	Model 2 (log(wage)-level)
(Intercept)	-1.715 (0.762)*	0.483 (0.106)***
educ	0.602 (0.051)***	0.091 (0.007)***
exper	0.064 (0.010)***	0.009 (0.001)***
black	-0.084 (0.444)	-0.010 (0.062)
female	-2.156 (0.271)***	-0.344 (0.038)***
Num.Obs.	526	526
R2 Adj.	0.304	0.348
F	58.341	70.934

Perfect Collinearity

Collinearity or Muticollinearity

- No collinearity means
  - none of the regressors can be written as an exact linear combinations of some other regressors in the model.
- For example:
  - in $Y = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$,
  - where $X_3 = X_2 + X_1$,
  - all X are collinear.

Consequence of collinearity:

- biased estimates of the collinear variables
- over-significant results;

Detection of collinearity:

- Scatter plot; Correlation matrix;
- Model specification;
- Step-wise regression approach;
- Variance Inflation Factor;

Solution to collinearity:

- Re specify the model;
- Choose different regressors;
- See also:
  - Overview: "Assumption AMLR.3 No Perfect Collinearity" in (Wooldridge, 2020) ;
  - Examples of causes in Chapter 9.5 (Wooldridge, 2020) ;
  - Chapter 9.4-9.5 in (Weisberg, 2005);

Perfect collinearity with dummy variables

- We want to build a naive regression, where the wage is a function of sex (female and male):
- $\text{wage} = \beta_0 + \beta_1 \cdot \text{female} + \beta_2 \cdot \text{male}$
- The data is fictional:

```
Rows: 14
Columns: 3
$ female <int> 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1
$ male   <int> 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0
$ wage   <dbl> 10.847522, 7.167989, 4.941890, 7.477957, 9.391538, 8.087289, 9.…
```

Perfect collinearity with dummy variable (2)

	Model 1	Model 2	Model 3	Model 4	Model 5
(Intercept)	8.747***	6.094***	6.094***		
	(0.468)	(0.628)	(0.628)		
male	-2.652**			6.094***	6.094***
	(0.784)			(0.628)	(0.628)
female		2.652**	2.652**	8.747***	8.747***
		(0.784)	(0.784)	(0.468)	(0.468)
Num.Obs.	14	14	14	14	14
R2	0.488	0.488	0.488	0.974	0.974
R2 Adj.	0.446	0.446	0.446	0.969	0.969

Note: ^^ Model 1: wage ~ male

Model 2: wage ~ female

Model 3: wage ~ female + male

Model 4: wage ~ 0 + female + male

Model 5: wage ~ 0 + male + female

Perfect collinearity with dummy variable (2)

	Model 1	Model 2	Model 3
(Intercept)	8.747*** (0.468)	6.094*** (0.628)	6.094*** (0.628)
male	-2.652** (0.784)		
female		2.652** (0.784)	2.652** (0.784)
Num.Obs.	14	14	14
R2	0.488	0.488	0.488
R2 Adj.	0.446	0.446	0.446

Note: ^^ Model 1: wage ~ male

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Perfect collinearity with dummy variable (2)

	Model 1	Model 2	Model 3	Model 4
(Intercept)	8.747***	6.094***	6.094***	
	(0.468)	(0.628)	(0.628)	
male	-2.652**			6.094***
	(0.784)			(0.628)
female		2.652**	2.652**	8.747***
		(0.784)	(0.784)	(0.468)
Num.Obs.	14	14	14	14
R2	0.488	0.488	0.488	0.974
R2 Adj.	0.446	0.446	0.446	0.969

Note: ^^ Model 1: wage ~ male

Model 2: wage ~ female

Model 3: wage ~ female + male

Model 4: wage ~ 0 + female + male

Homeworks:

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Watch these videos on youtube and read

Video 1: Ceteris Paribus: Public vs. Private University or this link: https://youtu.be/iPBV3BlV7jk

Re watch video 2: Selection Bias or this link: https://youtu.be/6YrIDhaUQOE

Read:

(Angrist & Pischke, 2014, Chapter 2; optional Angrist & Pischke, 2009, Chapter 3)

Do:

Follow pre-recorded videos in the order below. Please note that slides below supplement some of those practical works.

- Ex.03a Regression basics
- Ex.03b Wage education
- Ex.03c Hedonic Land Prices Model

HW Slides for: Ex.03a Regression basics

HW03a Regression basics

$$\boldsymbol{y} = \boldsymbol{x}\beta + \boldsymbol{e}$$

where

$$\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{x}\hat{\beta}$$

Dependent variable:

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$$

Independent variables:

$$\boldsymbol{x} = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1n} \\ 1 & x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{k1} & x_{k2} & \dots & x_{kn} \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 & \hat{\beta}_1 & \hat{\beta}_2 & \cdots & \hat{\beta}_n \end{bmatrix}$$

Where do β come from?

$$\boldsymbol{y} = \boldsymbol{x}\hat{\beta}$$

$$\boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{x}^T\boldsymbol{x_i}\hat{\beta}$$

$$\frac{1}{\boldsymbol{x}^T\boldsymbol{x_i}}\boldsymbol{x}^T\boldsymbol{y} = \frac{1}{\boldsymbol{x}^T\boldsymbol{x_i}}\boldsymbol{x}^T\boldsymbol{x_i}\hat{\beta}$$

$$(\boldsymbol{x}^T\boldsymbol{x_i})^{-1}\boldsymbol{x}^T\boldsymbol{y} = \hat{\beta}$$

where:

- $\boldsymbol{x}^T$ is the transposed matrix $\boldsymbol{x}$
- $(\cdot)^{-1}$ is the inverse of a matrix

Fitted values

$$\hat{\boldsymbol{y}} = \boldsymbol{x}\hat{\beta} = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1n} \\ 1 & x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{k1} & x_{k2} & \dots & x_{kn} \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{bmatrix} =$$

$$\begin{bmatrix} \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} + \dots + \beta_n x_{1n} \\ \beta_0 + \beta_1 x_{21} + \beta_2 x_{22} + \dots + \beta_n x_{2n} \\ \vdots \\ \beta_0 + \beta_1 x_{k1} + \beta_2 x_{k2} + \dots + \beta_n x_{kn} \end{bmatrix} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_k \end{bmatrix}$$

Error terms

$$\hat{e} = \boldsymbol{y} - \hat{\boldsymbol{y}} = \begin{bmatrix} y_1 - \hat{y}_1 \\ y_2 - \hat{y}_2 \\ \vdots \\ y_k - \hat{y}_k \end{bmatrix} = \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_k \end{bmatrix}$$

Standard Errors

Measure of variance in the estimated parameters β. Computed based on the **Variance Covariance** matrix

$$Var(\hat{\beta}) = (\boldsymbol{x}^T\boldsymbol{x})^{-1}\hat{\sigma}_e$$

where $\hat{\sigma}_e$ is the estimate of the variance in error terms:

$$\hat{\sigma}_e = \frac{\hat{\mathbf{e}}^T\hat{\mathbf{e}}}{\boldsymbol{n} - \boldsymbol{r}}$$

n - number of observations and r number of regressors including intercept.

Standard Errors:

$$\text{SE} = \sqrt{\text{diag}(Var(\hat{\beta}))}$$

Why do we need standard errors?

- SE are needed for the inference!
- To conclude about the population based on the sample regression results.

Takeaways

Takeaways

Get comfortable with the terminology:

- Control variables for creating Ceteris Paribus
- Selection Bias in Regression:
  - OVB;
  - Long and Short models;

Regression components and how do one produce them:

- x, y, β, standard errors.

Why assumptions are important?

Linearity and how to detect it?

- Log transformation and its interpretation.

What is perfect collinearity?

References

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