Multiple Linear Regression: practical aspects

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Relevance of a theory in econometric models

The Basic Theory of Production (Sadoulet & de Janvry, 1995, Chapter 3)

Farm's production function

$$h(q, x, z) = 0,$$

where:

- q - output quantities (agricultural produce)
- x - variable input quantities (fertilizers, labor, seed, water, …)
- z - fixed factors (private: land, equipment; public: infrastructure, extension; exogenous features …)

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Farm's optimization problem

Prices of **inputs** w' and **outputs** p' affect farmers decisions of:

- what and how much to **produce** q, and
- what and how much to use as the **input** x.

Farm's objective function is **to maximize own profit**:

$$\max_{x,q} \quad p'q - w'x,$$
$$\text{s.t.} \;\; h(q, x, z) = 0$$

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Solution to the producer's optimization problem

$$\max_{x,q} \quad p^{'}q - w^{'}x,$$
$$\text{s.t.} \;\; h(q,x,z) = 0$$

Is a set of **input demand** and **output supply** functions:

$$\begin{cases} x = x(p,w,z), & \text{factor demand function} \\ q = q(p,w,z), & \text{supply function} \end{cases}$$

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Farm's optimization problem is

Structural form.

$$\max_{x,q} \quad p^{'}q - w^{'}x,$$
$$\text{s.t.} \;\; h(q,x,z) = 0$$

- Models that we can simulate with linear programming, structural equation modelling

Reduced forms.

$$\begin{cases} x = x(p,w,z), & \text{factor demand} \\ q = q(p,w,z), & \text{supply function} \end{cases}$$

- Models that we estimate with regressions

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The Production Function (Sadoulet & de Janvry, 1995, Appendix A.4)

Mathematical function that transforms inputs quantities x into output q.

Taylor expansion (Sadoulet & de Janvry, 1995, Appendix A.2)

Introduced by Brook Taylor in 1715 …

Value of any function could be approximated at an arbitrary point x_0 with a Taylor expansion:

$$f(x) \approx f(x_0) + f^{'}(x_0)(x - x_0) + \frac{f^{''}(x_0)}{2!}(x - x_0)^2 + \cdots.$$

Taylor expansion example:

What is the value of $e^{0.4}$?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{0.4} \approx 1 + .4 + \frac{.4^2}{2!} + \frac{.4^3}{3!} + \frac{.4^4}{4!} + \frac{.4^5}{5!}$$

$$\approx 1 + .4 + .16/2 + 0.064/6 + 0.0256/24 + 0.01024/120$$

$$\approx 1 + .4 + .08 + 0.0106(6) + 0.00106(6) + 0.000106(6)$$

$$\approx 1.491819$$

$$\exp(0.4) = 1.491825$$

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First order Taylor approximation

$q \approx f(x)$ at x_0.

$$
\begin{aligned}
f(x) &\approx f(x_0) + f^{'}(x_0)(x - x_0) \\
&\approx \underbrace{f(x_0) - f^{'}(x_0)x_0}_{\alpha} + \underbrace{f^{'}(x_0)}_{\beta} x
\end{aligned}
$$

Simple regression: $q \approx \alpha + \beta\, x$

$q \approx f(x_1, x_2)$ at a, b

$$
f(x_1, x_2) \approx \underbrace{f(a,b) - f^{'}_{x_1}(a,b)a - f^{'}_{x_2}(a,b)b}_{\alpha} + \underbrace{f^{'}_{x_1}(a,b)}_{\beta_1} x_1 + \underbrace{f^{'}_{x_2}(a,b)}_{\beta_2} x
$$

Multiple regression: $q \approx \alpha + \beta_1\, x_1 + \beta_2\, x_2$

Second and greater order Taylor approximation

Results in polynomial functions with interaction terms:

$$q \approx \alpha + \beta_1\, x_1 + \beta_2\, x_2 + \beta_3\, x_1^2 + \beta_4\, x_2^2 + \beta_5 x_1 x_2$$

Production functions

Consider

$$q = f(x_i)$$

Production function of several inputs x_i, where

$$x_i = e^{\ln x_i}$$

and q is the quantity of goods produced.

Then,

$$\ln q = \ln f(e^{\ln x_i}) \quad \text{or} \quad \ln q = g(\ln x_i)$$

Let us approximate $\ln q = g(\ln x_i)$ using Taylor expansion.

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Fist order Taylor approximation

$$\ln q \approx \alpha + \sum_i \beta_i \ln x_i$$

This is a Cobb-Douglas production function: $q \approx e^{\alpha} \prod_i x_i^{\beta_i}$

Second order Taylor approximation

of $\ln q = g(\ln x_i)$ around an arbitrary point $x_{i0} \neq 0$

$$\ln q \approx \alpha + \sum_i \beta_i \ln x_i + \frac{1}{2} \sum_{ij} \gamma_{ij} \ln x_i \ln x_j$$

This is a trans-log production function

Example 1. production function

Problem and research question

- Policy makers are considering to abolish the share-cropping land tenure system.
- As an alternative either owner-cultivation with hired labor or pure rental relationship should be established.
- Main rationale is in the claim that sharecropping is an inefficient farm structure.

- **Theoretical debate**
- Contract theory (Dasgupta, Knight, & Love, 1999) sees sharecropping as a form of the optimal risks and incentives sharing contract.
- It is a **second best contract** in agriculture after pure rental of hired labor contract.
- Sharecropping can be as efficient as rental/labor contracts, when interlinked labor, resources, credit and insurance markets fail.

Research question: Is there any negative effect of sharecropping on farms productivity?

How to answer this question? (Any ideas)

Data that we have covers a sample of the rice-cultivating farmers in India (Feng & Horrace, 2010):

- output is gross output of rice in kg
- land the total area cultivated with rice, measured in hectares
- labor total labor inputs (excluding harvest labor) in hours
- status land tenure system on the farm: owner-operated, Share-cropping and mixed

Let us estimate a simple production function of a rice-producing farm using linear form and Cobb-Douglas function.

$$\mathrm{output} = \alpha + \beta_1 \mathrm{status} : \mathrm{share} + \beta_2 \mathrm{status} : \mathrm{mixed} + \beta_3 \mathrm{land} + \beta_4 \mathrm{labo}$$

$$\ln \mathrm{output} = \alpha + \beta_1 \mathrm{status} : \mathrm{share} + \beta_2 \mathrm{status} : \mathrm{mixed} + \beta_3 \ln \mathrm{land} + \beta_4$$

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Data (1)

```r
library(tidyverse)
library(GGally)
library(modelsummary)
glimpse(farm_dta)
```

```
Rows: 1,026
Columns: 4
$ output <int> 7980, 4083, 2650, 4500, 16300, 17424, 3840, 2800, 950, 240, 150…
$ land   <dbl> 3.000, 2.000, 1.000, 2.000, 3.572, 3.572, 1.420, 1.420, 0.428, …
$ labor  <int> 2915, 2155, 1075, 2091, 3889, 3519, 810, 855, 460, 109, 230, 18…
$ status <fct> owner, owner, owner, owner, share, share, mixed, mixed, mixed, …
```


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Exploratory statistics (1)

```
datasummary_skim(farm_dta, output = "markdown")
```

	Unique (#)	Missing (%)	Mean	SD	Min	Median	Max
output	482	0	1405.2	1921.8	42.0	886.5	20960.0
land	300	0	0.4	0.5	0.0	0.3	5.3
labor	544	0	388.4	484.2	17.0	252.0	4774.0

Exploratory statistics (2)

```
datasummary_skim(farm_dta, output = "markdown", "categorical")
```

status	N	%
owner	736	71.7
share	79	7.7
mixed	211	20.6

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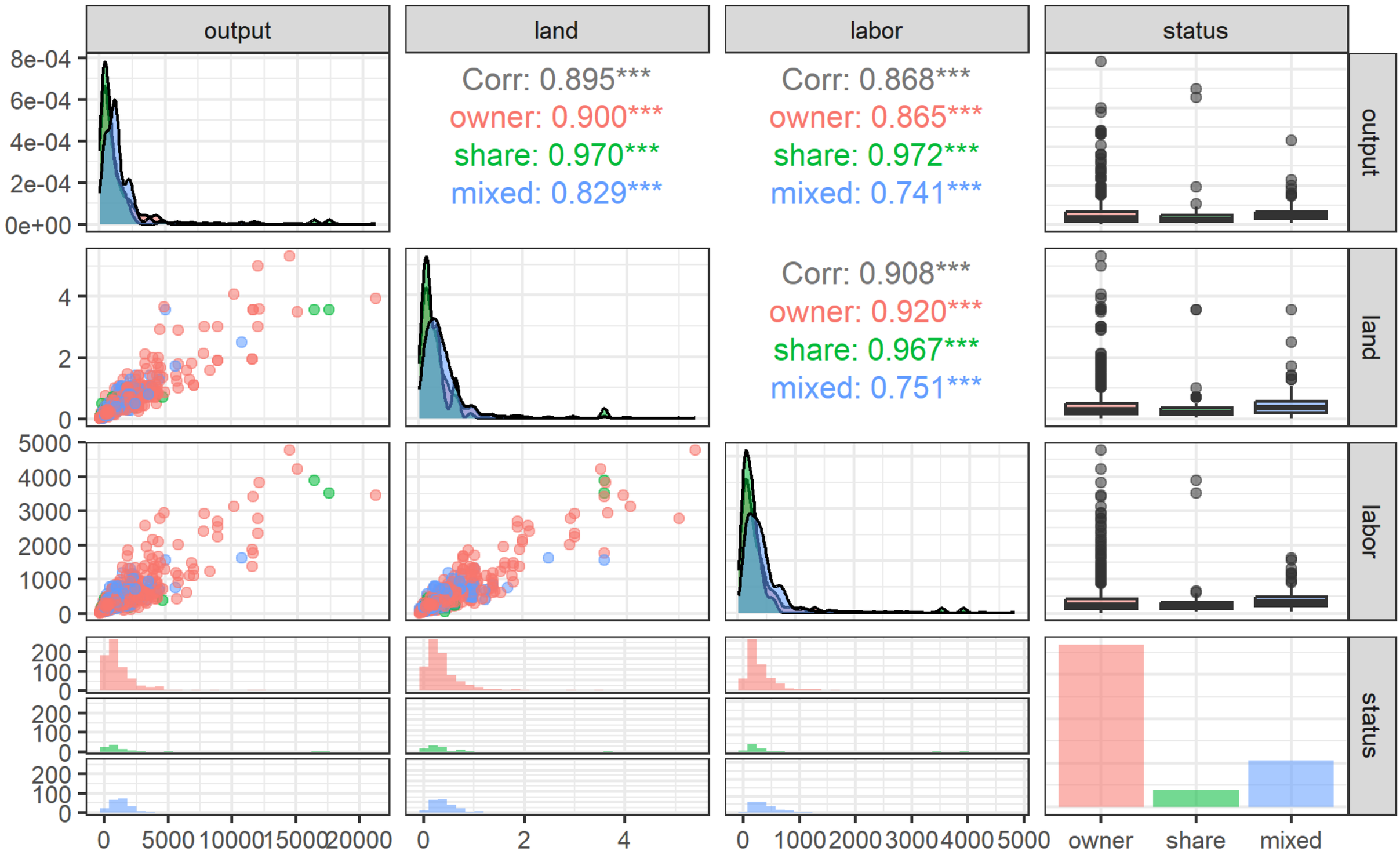
Exploratory statistics (3)

```
datasummary( (land  + labor  + output) ~ status * (mean + sd), farm_dta, output = "markdown")
```

	owner / mean	owner / sd	share / mean	share / sd	mixed / mean	mixed / sd
land	0.44	0.58	0.36	0.56	0.44	0.39
labor	395.42	522.12	315.51	566.17	391.44	262.63
output	1435.68	2014.60	1250.65	2639.25	1356.59	1104.10

Exploratory statistics (3)

```
ggpairs(farm_dta, aes(colour = status, alpha = 0.2))
```



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Regression and interpretation

```r
fit1 <- lm(output ~ status + land + labor,
           data = farm_dta)

fit2_cd <- lm(output ~ status + log(land) + log(labor),
           data = farm_dta)

modelsummary(
  list(`Linear prod.fn. (level-level)` = fit1,
       `CD prod.fn. (log-log)` = fit2_cd),
  estimate = "{estimate} {stars} ({std.error})",
  statistic = NULL
)
```

	Linear prod.fn. (level-level)	CD prod.fn. (log-log)
(Intercept)	6.203 (36.845)	-2549.390 *** (666.234)
Sharecropping	87.849 (96.953)	132.761 (154.889)
Mixed	-90.608 (63.901)	-480.042 *** (102.642)
Land, ha	2145.659 *** (111.521)	743.287 *** (89.588)
Labor, hours	1.248 *** (0.126)	901.025 *** (100.797)
Num.Obs.	1026	1026
R2	0.819	0.540
R2 Adj.	0.819	0.538
F	1158.652	299.783

Conclusion

	Linear prod. fn. (level-level)	CD prod. fn. (log-log)
(Intercept)	6.203	-2549.390 ***
	(36.845)	(666.234)
Sharecropping	87.849	132.761
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- Interpret coefficients in both models.
- What is the estimated effect of share-cropping?
- Is this model shows causal effects of x on y?
- Is there a ceteris paribus?

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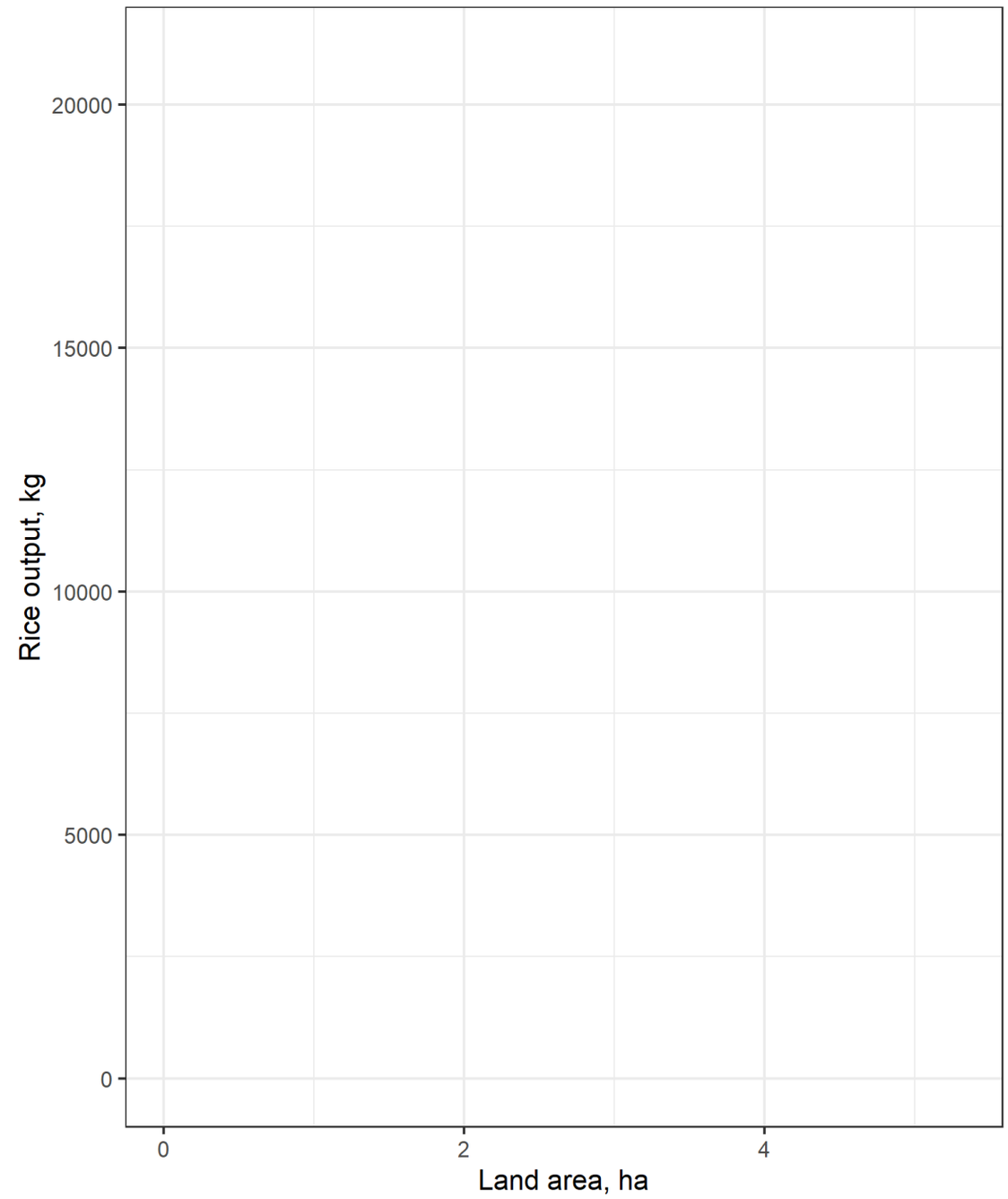
Entertainment: Fitted values (1)

```r
farm_dta %>%
  mutate(fitted = fitted(fit1)) %>%
  ggplot() +
  aes(x = land,
      y = output,
      colour = "Observed") +
  xlab("Land area, ha") +
  ylab("Rice output, kg")
```



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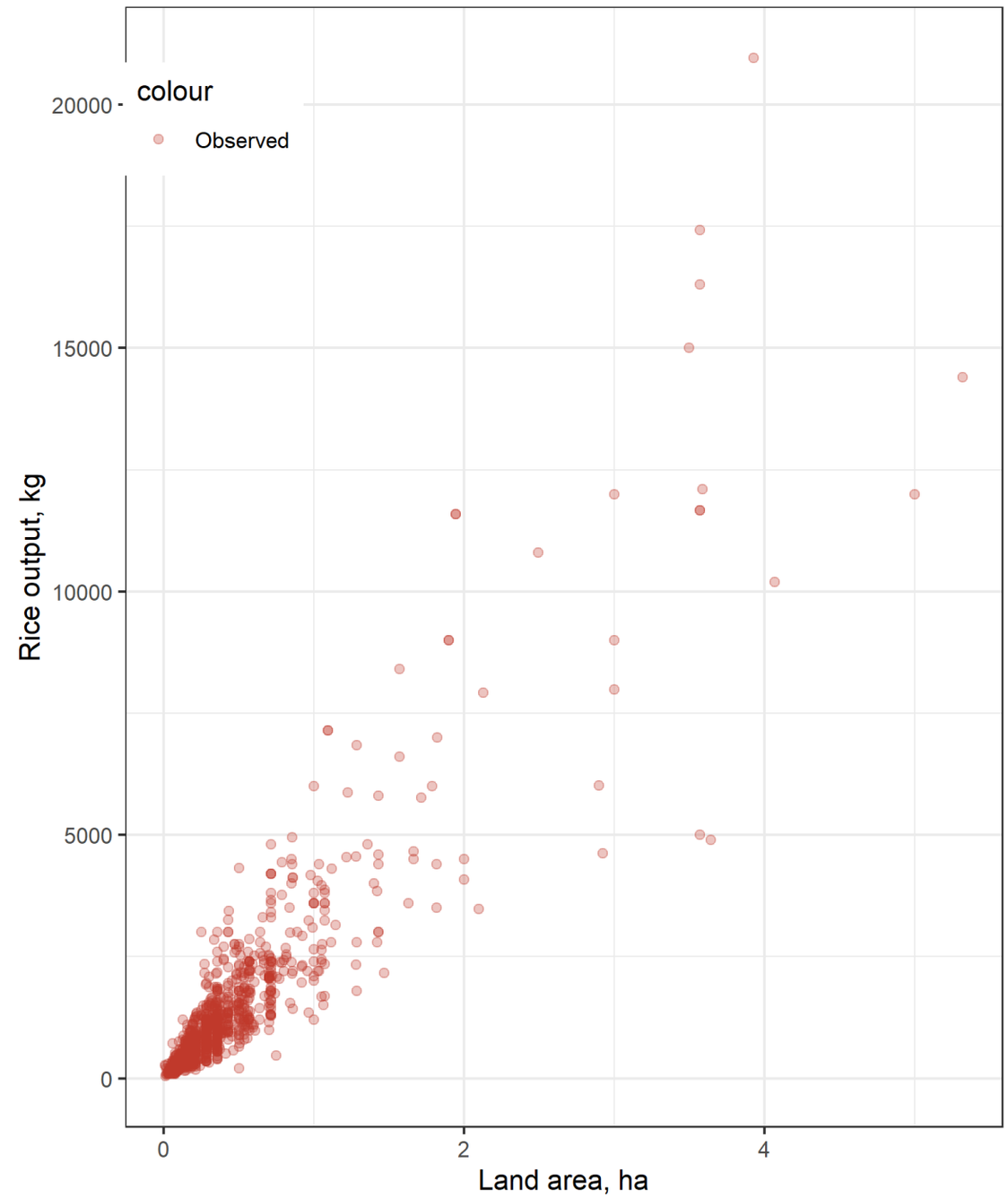
Entertainment: Fitted values (1)

```r
farm_dta %>%
  mutate(fitted = fitted(fit1)) %>%
  ggplot() +
  aes(x = land,
      y = output,
      colour = "Observed") +
  xlab("Land area, ha") +
  ylab("Rice output, kg") +
  geom_point(alpha = 0.3) +
  theme(legend.position = c(0.1, 0.9)) +
  scale_color_manual(
    values = c('#BF382A', '#0C4B8E'))
```



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Entertainment: Fitted values (1)

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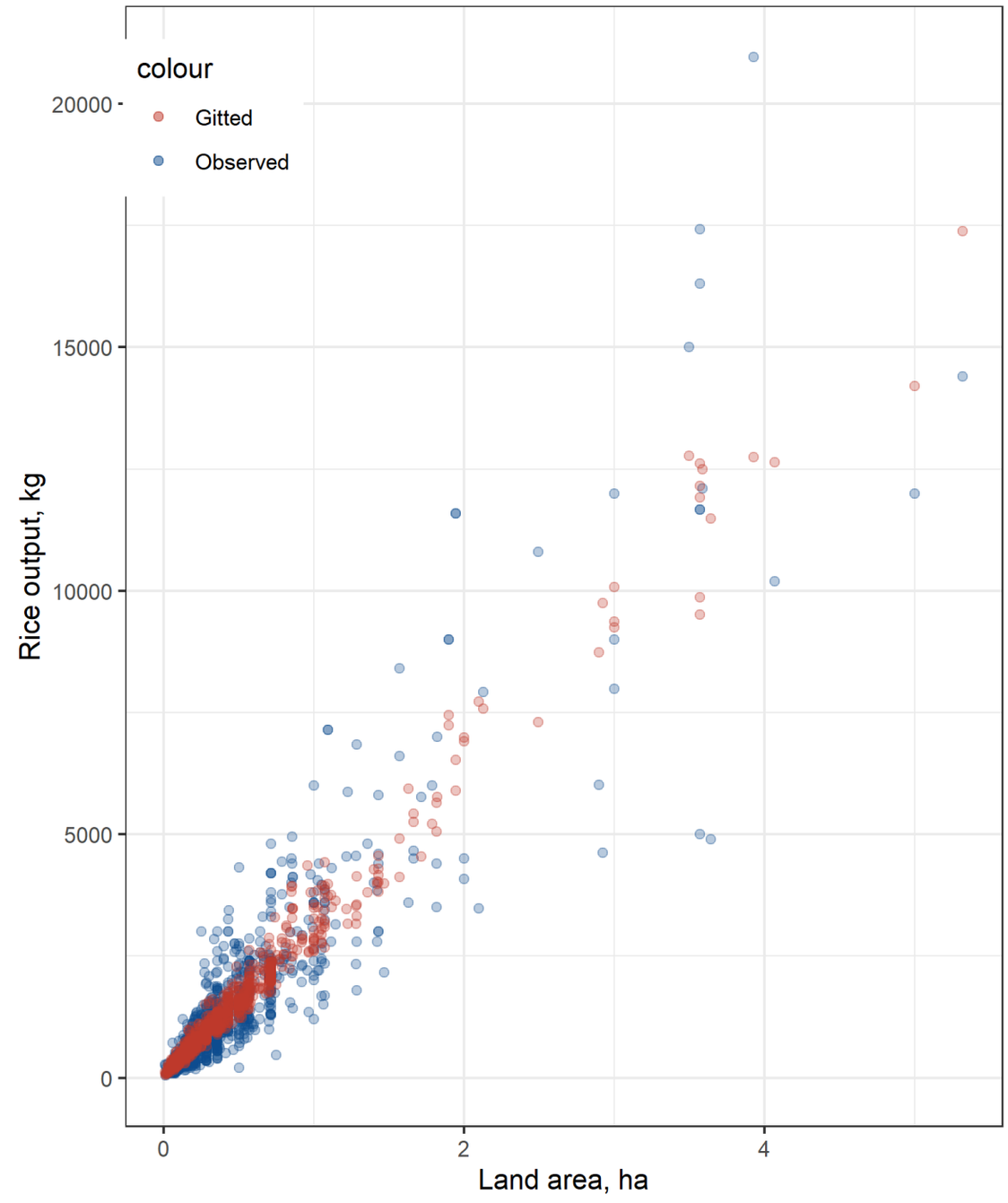
  theme(legend.position = c(0.1, 0.9)) +
  scale_color_manual(
    values = c('#BF382A', '#0C4B8E')) +
  geom_point(
    aes(y = fitted, colour = "Gitted"),
    alpha = 0.3)
```



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Entertainment: Fitted values (1)

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  mutate(fitted = fitted(fit1)) %>%
  ggplot() +
  aes(x = land,
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  xlab("Land area, ha") +
  ylab("Rice output, kg") +
  geom_point(alpha = 0.3) +
  theme(legend.position = c(0.1, 0.9)) +
  scale_color_manual(
    values = c('#BF382A', '#0C4B8E')) +
  geom_point(
    aes(y = fitted, colour = "Gitted"),
    alpha = 0.3) +
  scale_x_log10() +
  scale_y_log10()
```

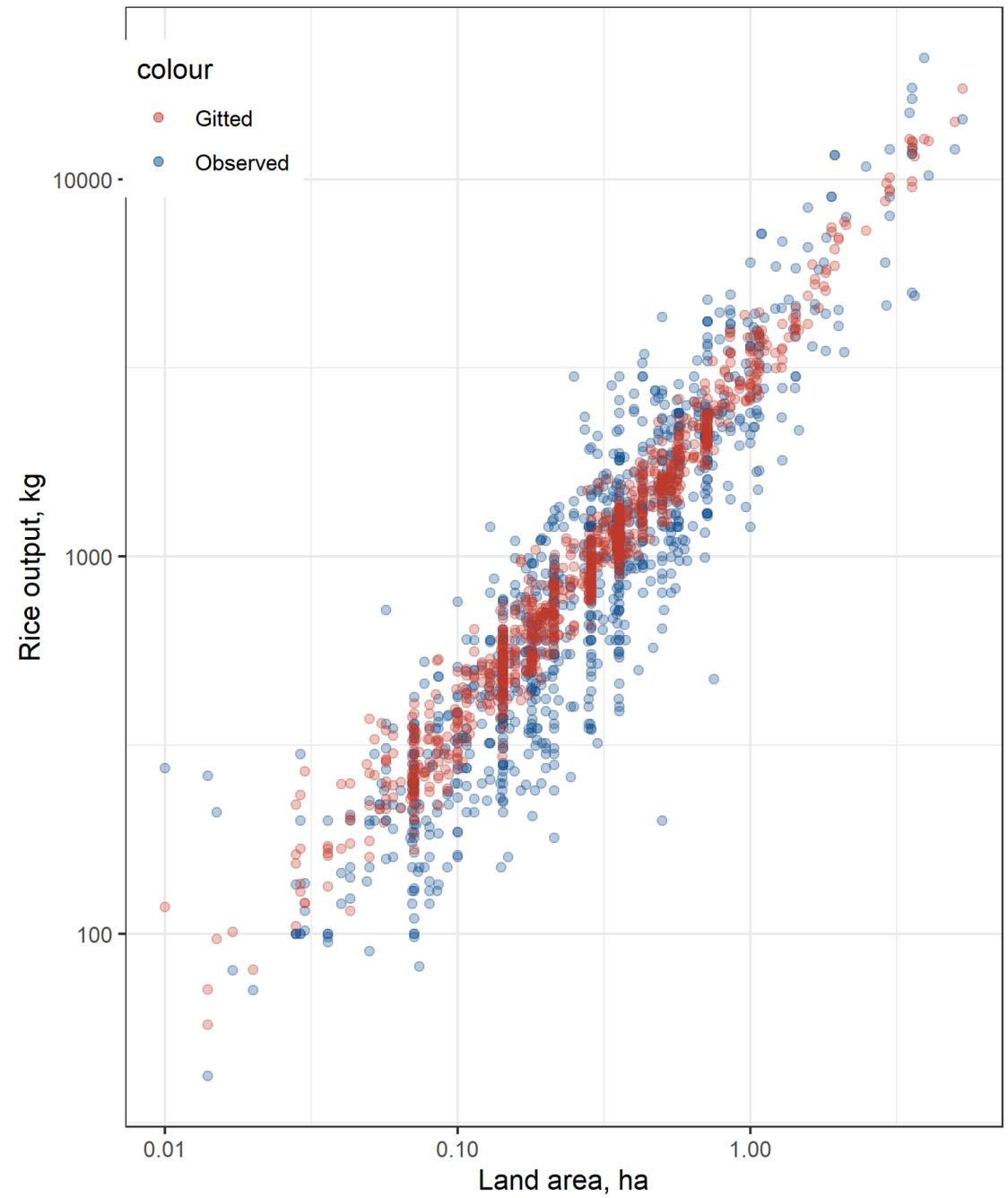

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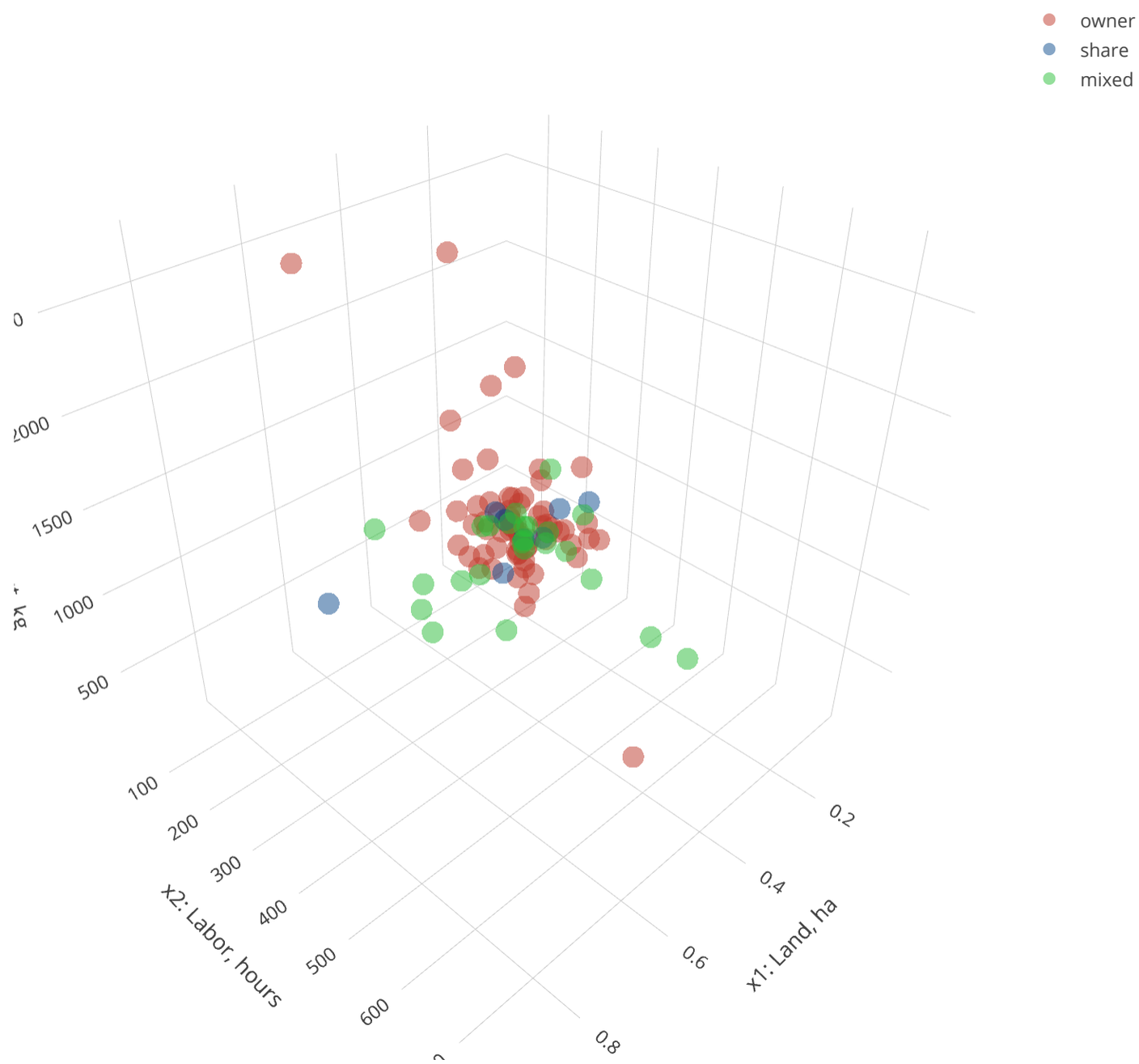
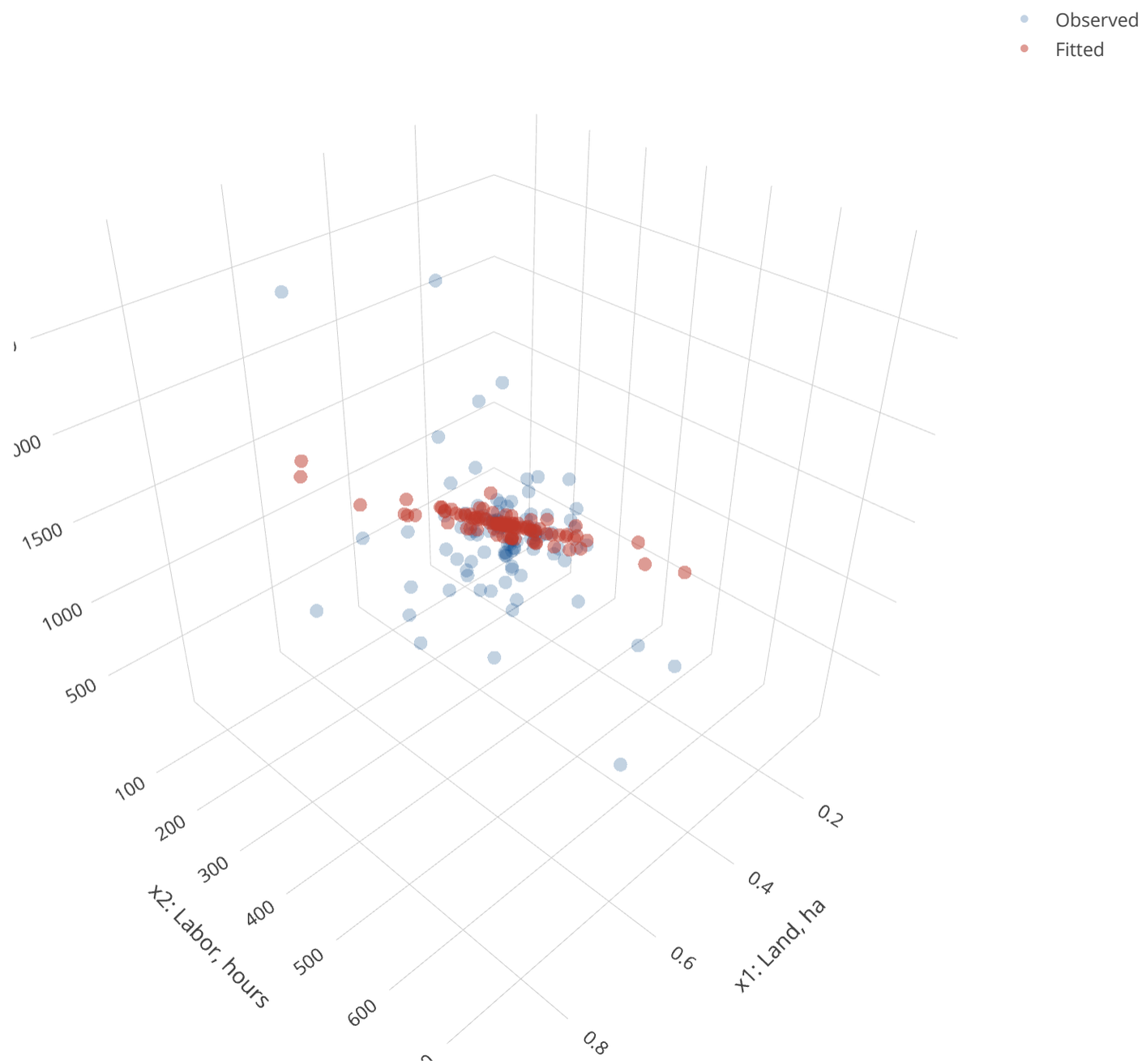
Entertainment: Actual values 3D



Entertainment: Actual + Fitted values 3D



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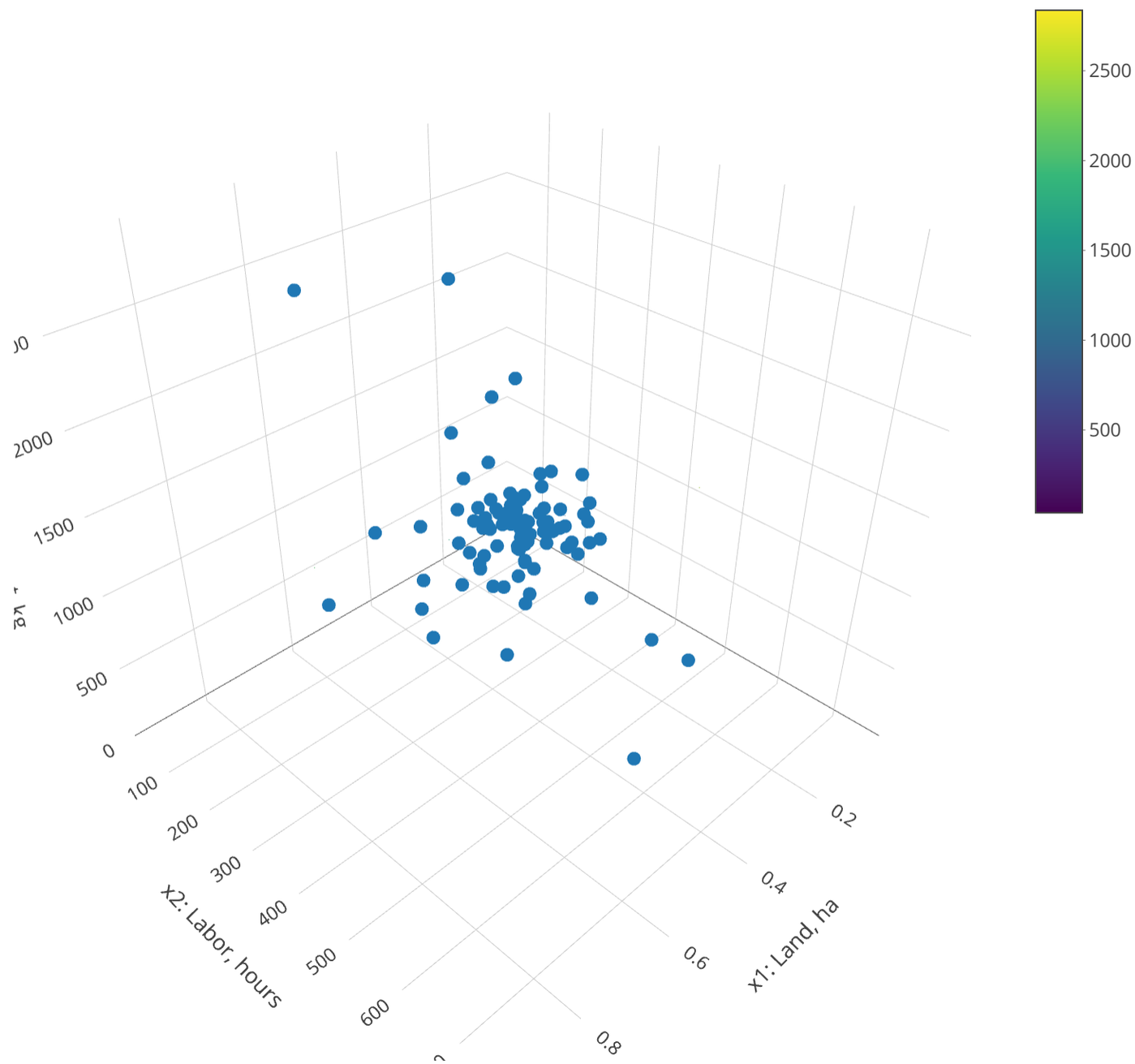
Entertainment: Fitted values 3D v.2



Assumptions

1. Linearity - **validated using data**
2. Random sampling - “accepted based on faith in data collectors”
3. No (Perfect) Collinearity - **validated using data**
4. No Endogeneity - OVB
5. Homoscedasticity (No heteroscedasticity) - **validated using data**
6. Normality - **validated using data**

Linearity

Means: in $y = \alpha + \beta x$; y is a linear function of x.

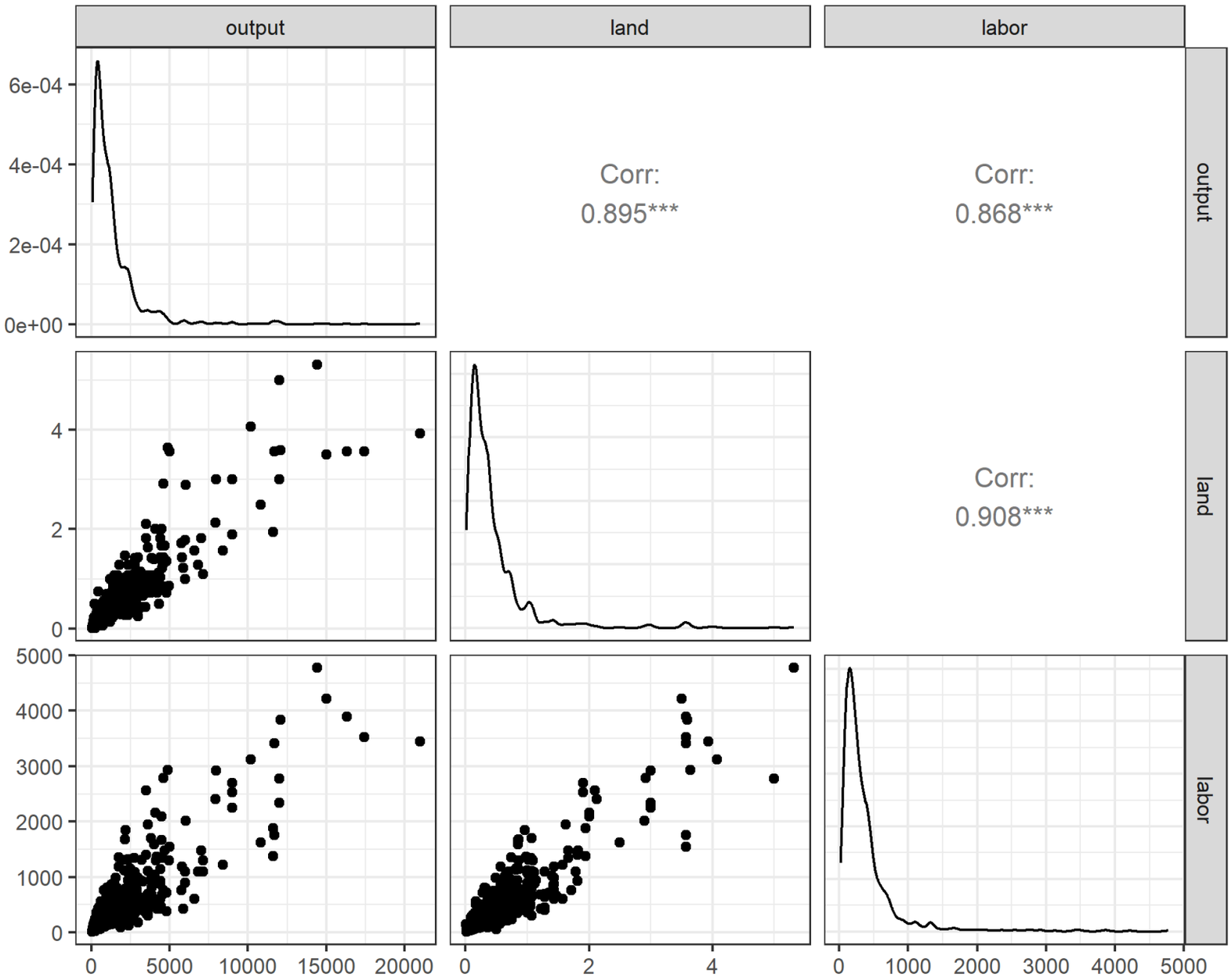
If violated: α and β are biased and inefficient.

Detected: with scatter plots: y against x, observed/residuals against fitted).

Resolved: with a linear transformation.

Linearity detection: scatter plots

```
farm_dta %>% select(-status) %>% ggpairs()
```



- Note the magnitude of x and y
- From min to max y increases about 5000 times.
- If a variable increases more than 10 times between min and max, log transformation may be needed.

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Linearity detection: fitted values vs residuals

```
library(performance)
check_model(fit1, check = "linearity", panel = FALSE)
```

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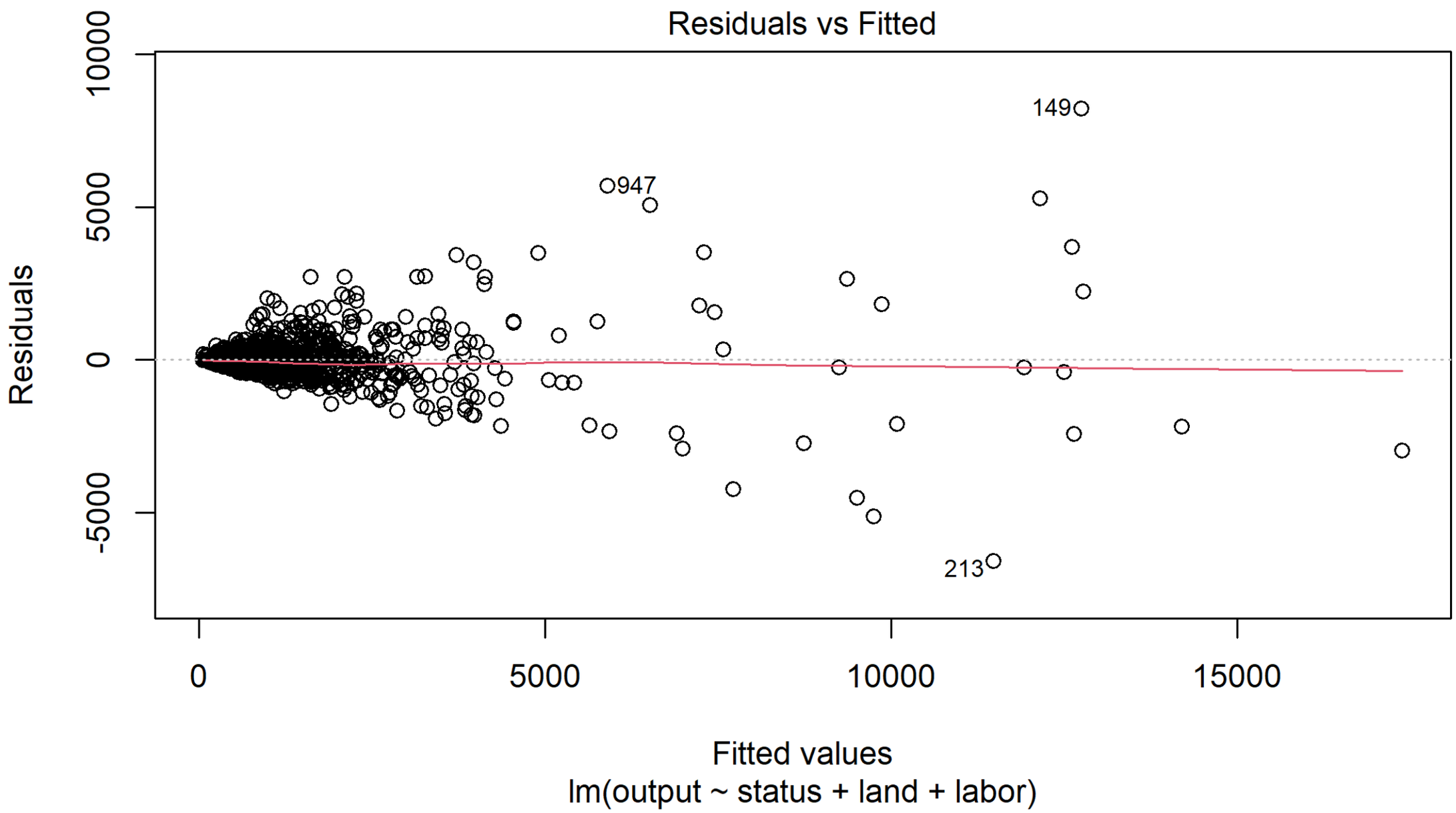
Linearity detection: fitted values vs residuals (v2)

```
plot(fit1, which = 1)
```



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Linearity Conclusion

Model:

```
lm(formula = output ~ status + land + labor, data = farm_dta)
```

Is the linearity assumption fulfilled?

What could be done to improve the model?

Linear transformation with log: partial (level-log or log-level) or full (log-log).

$$\ln(\text{output}) = \alpha + \beta_1 \text{status} : \text{share} + \beta_2 \text{status} : \text{mixed}$$
$$+ \beta_3 \ln(\text{land}) + \beta_4 \ln(\text{labor}) + e$$

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Linear transformation of the model

```r
fit1 <- lm(output ~ status + land + labor,
           data = farm_dta)

fit2_cd <- lm(output ~ status + log(land) + log(labor),
           data = farm_dta)

modelsummary(
  list(`Linear prod.fn. (level-level)` = fit1,
       `CD prod.fn. (log-log)` = fit2_cd),
  estimate = "{estimate} {stars} ({std.error})",
  statistic = NULL
)
```


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Linear transformation of the model

	Linear prod.fn. (level-level)	CD prod.fn. (log-log)
(Intercept)	6.203 (36.845)	5.703 *** (0.202)
Sharecropping	87.849 (96.953)	0.004 (0.047)
Mixed	-90.608 (63.901)	0.017 (0.031)
Land, ha	2145.659 *** (111.521)	0.679 *** (0.027)
Labor, hours	1.248 *** (0.126)	0.344 *** (0.030)
Num.Obs.	1026	1026
R2	0.819	0.843
R2 Adj.	0.819	0.842
F	1158.652	1369.377

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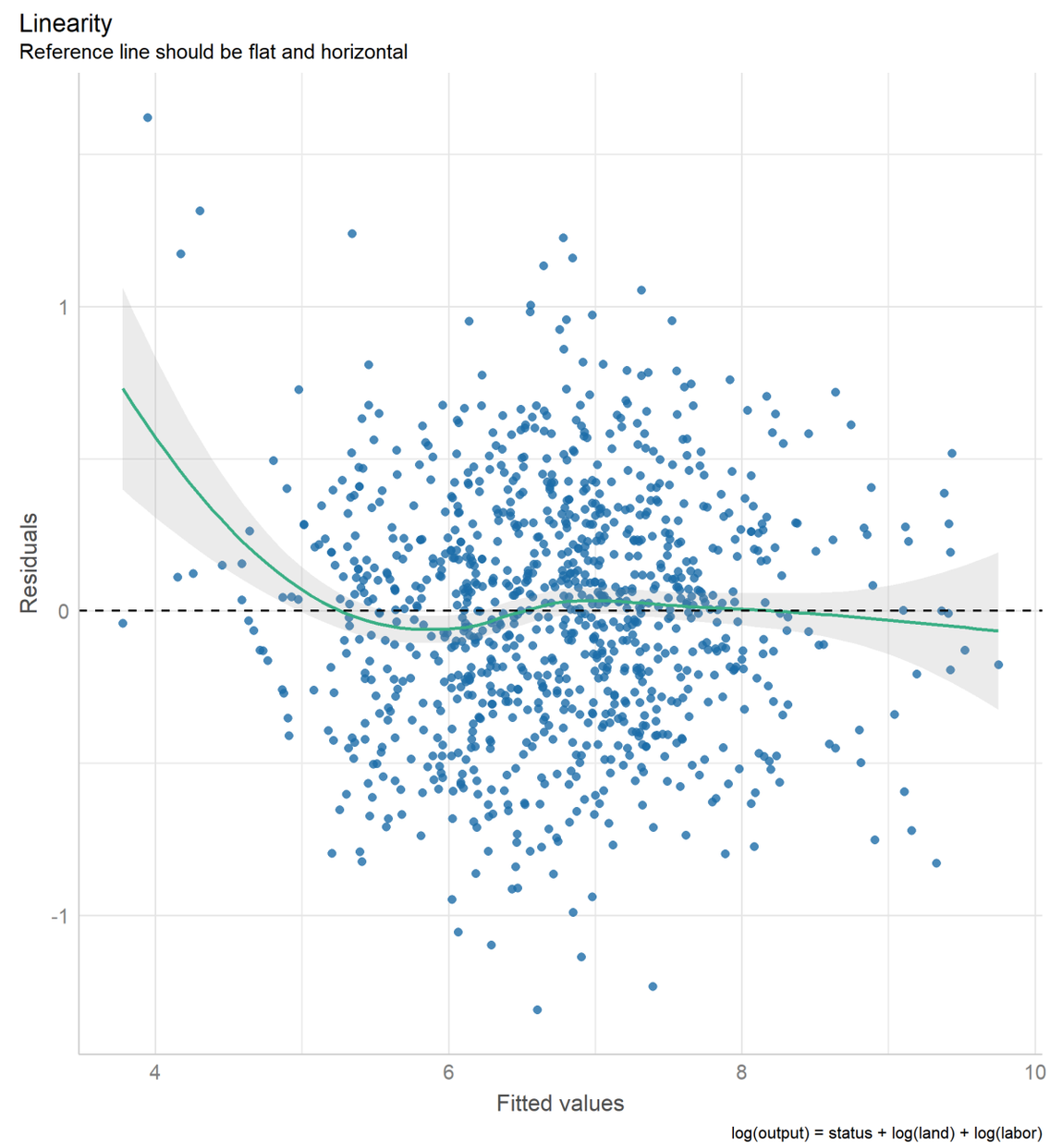
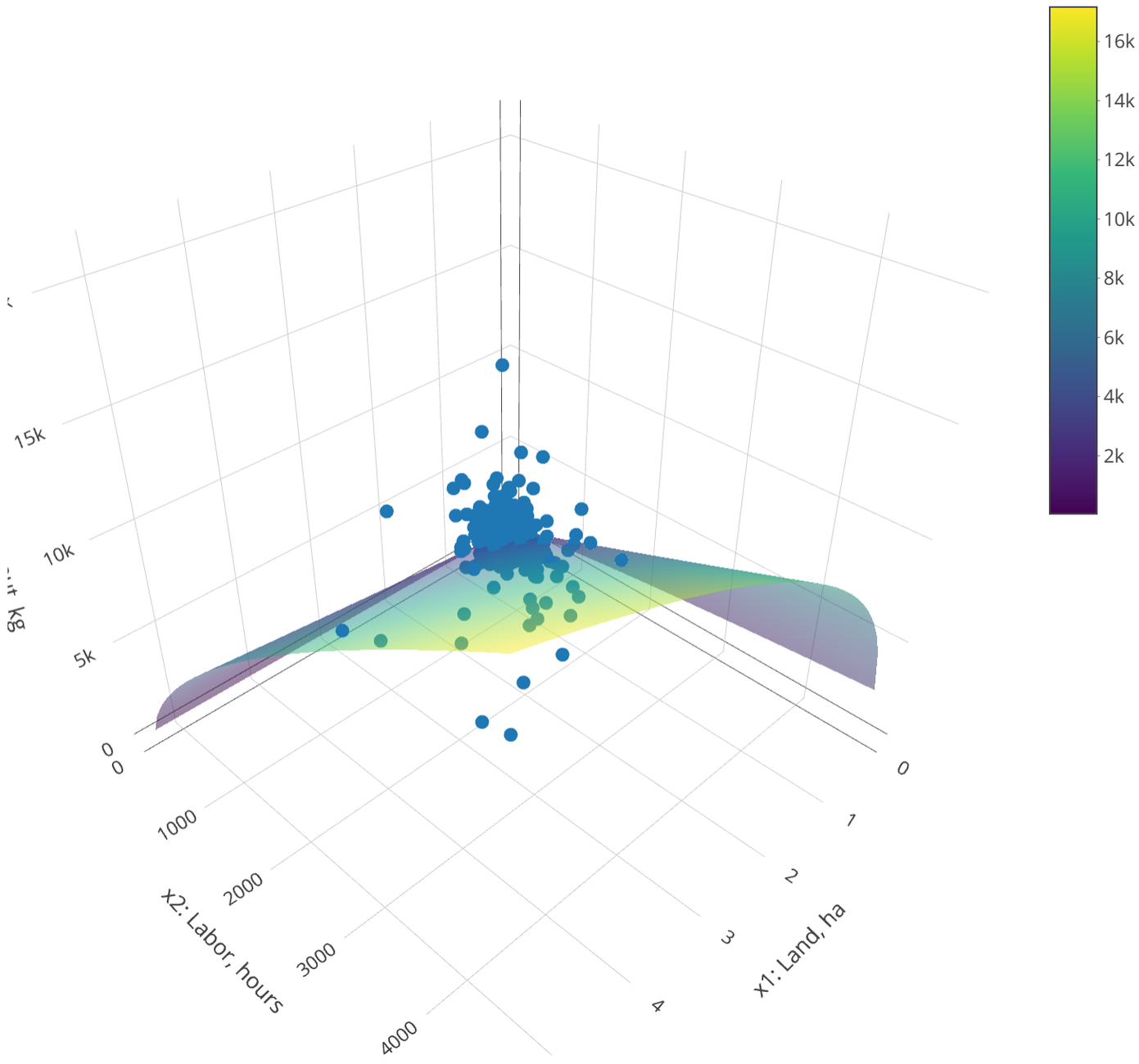
Change in the linearity assumption

Without log-log

With log-log



Change in Fitted values 3D v.2



Partial transformation (1)

```
fit3_level_log <- lm(output ~ status + log(land) + log(labor), data = farm_dta)
```

	Linear prod.fn. (level-level)	CD prod.fn. (log-log)	Level-log prod.fn. (level-log)
(Intercept)	6.203 (36.845)	5.703 *** (0.202)	-2549.390 *** (666.234)
Sharecropping	87.849 (96.953)	0.004 (0.047)	132.761 (154.889)
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R2	0.819	0.843	0.540
R2 Adj.	0.819	0.842	0.538
F	1158.652	1369.377	299.783

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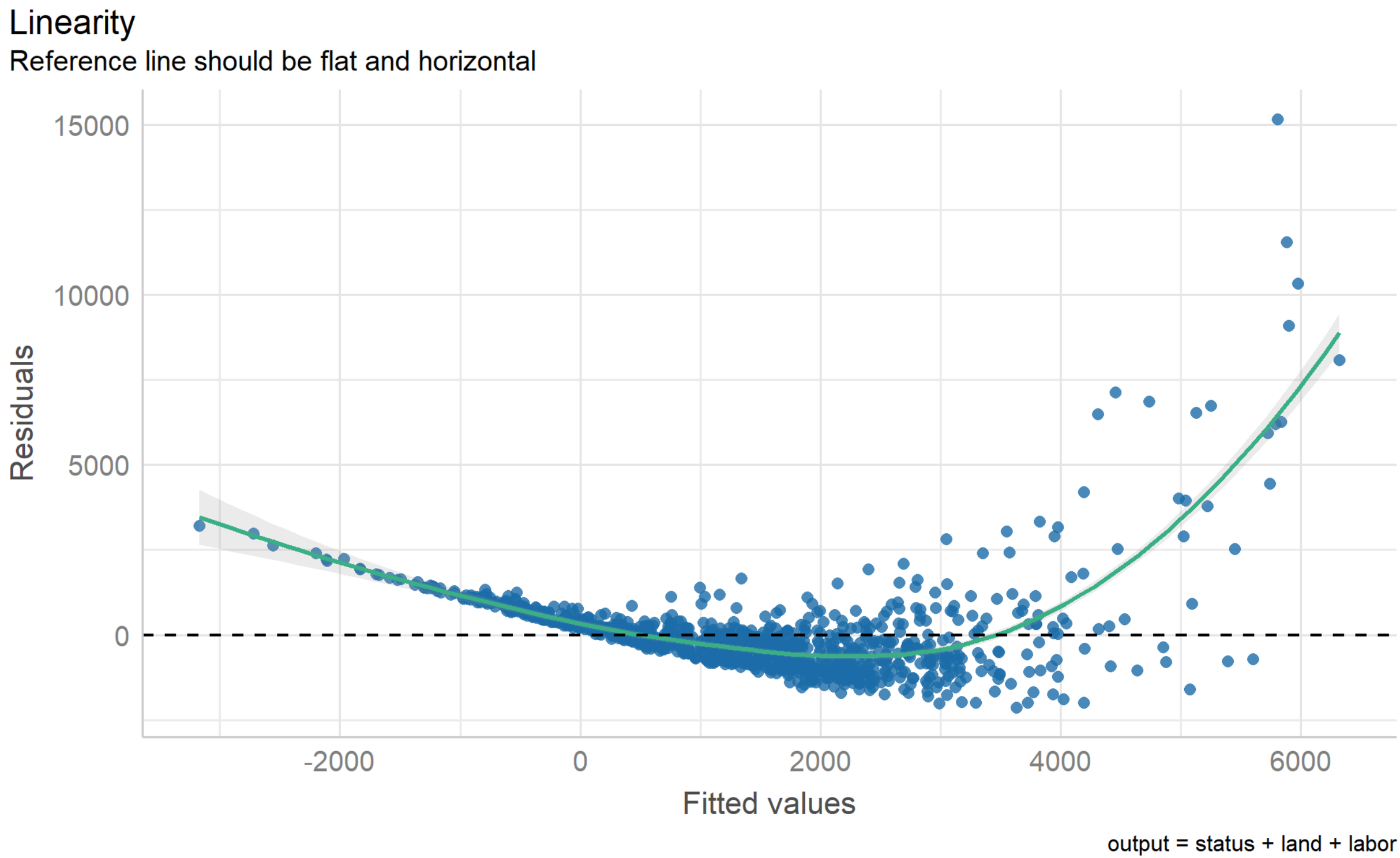
Partial transformation (2)

```
check_model(fit3_level_log, check = "linearity", panel = FALSE)
```



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No (perfect) collinearity (multicollinearity)

Two sub-problems:

1. Perfect collinearity: when present regression will not be estimated.
2. Near-perfect collinearity: regression works, but inference is misleading.

No Perfect Collinearity

Problem:

- none of the regressors can be written as an **exact linear combinations** of some other regressors in the model.
- For example:
  - in $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$,
  - where $x_3 = x_2 + x_1$,
  - all x_i are collinear.

Detection: Through a regression equation

Solution: model re-specification.

Perfect Collinearity Example 1

$$\hat{wage} = \alpha + \hat{\beta}_1 education + \hat{\beta}_2 experience + \hat{\beta}_3 male + \hat{\beta}_4 female$$

- $male + female = 1$
- Is there a multicollinearity problem here?

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Perfect Collinearity Example 2

$$\hat{output} = \alpha + \hat{\beta}_1 land + \hat{\beta}_2 seeds + \hat{\beta}_3 fertilizers + \hat{\beta}_4 others + \hat{\beta}_5 total$$

- where $total = seeds + fertilizers + others$
- Is there a multicollinearity problem here?
- Coefficient $\hat{\beta}_5$ is aliased and wont be estimated

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Perfect Collinearity Example 3

Same model but in log:

$$\ln(\hat{output}) = \hat{\beta}_1 \ln(land) + \hat{\beta}_2 \ln(seeds) + \hat{\beta}_3 \ln(fertilizers) + \hat{\beta}_4 \ln(others) + \hat{\beta}_5 \ln(total)$$

where $total = seeds + fertilizers + others$

- Is there a multicollinearity problem here?
- Think!
- $log(a) + log(b) = log(a * b)$
- 🚧 **Danger**

  No, there is no collinearity because $\ln(seeds) + \ln(fertilizers) + \ln(others) \neq \ln(total)$. This is not a linear combination! $\ln(seeds \cdot fertilizers \cdot others) \neq \ln(total)$

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Perfect Collinearity Example 4

Using the production function estimated above, let us convert tenure type into a set of dummy variables:

```r
farm_dta_coll <-
  farm_dta %>%
  mutate(
    status_owner = ifelse(status == "owner", 1, 0),
    status_share = as.integer(status == "share"),
    status_mixed = as.integer(status == "mixed")
    )
glimpse(farm_dta_coll)
```

```
Rows: 1,026
Columns: 7
$ output       <int> 7980, 4083, 2650, 4500, 16300, 17424, 3840, 2800, 950, 24…
$ land         <dbl> 3.000, 2.000, 1.000, 2.000, 3.572, 3.572, 1.420, 1.420, 0…
$ labor        <int> 2915, 2155, 1075, 2091, 3889, 3519, 810, 855, 460, 109, 2…
$ status       <fct> owner, owner, owner, owner, share, share, mixed, mixed, m…
$ status_owner <dbl> 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 1, …
$ status_share <int> 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, …
$ status_mixed <int> 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, …
```


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Perfect Collinearity Example 4

```
Call:
lm(formula = output ~ status_owner + status_share + status_mixed +
    land + labor, data = farm_dta_coll)

Coefficients:
 (Intercept)  status_owner  status_share  status_mixed          land
     -84.405        90.608       178.457            NA      2145.659
       labor
       1.248


Call:
lm(formula = output ~ status_mixed + status_owner + status_share +
    land + labor, data = farm_dta_coll)

Coefficients:
 (Intercept)  status_mixed  status_owner  status_share          land
      94.052      -178.457       -87.849            NA      2145.659
       labor
       1.248
```

Perfect Collinearity Example 4

	Linear prod.fn. (level-level)	Linear prod.fn. with dummies 1	Linear prod.fn. with dummies 2
(Intercept)	6.203 (36.845)	-84.405 (60.186)	94.052 (93.612)
statusshare	87.849 (96.953)		
statusmixed	-90.608 (63.901)		
land	2145.659 *** (111.521)	2145.659 *** (111.521)	2145.659 *** (111.521)
labor	1.248 *** (0.126)	1.248 *** (0.126)	1.248 *** (0.126)
status_owner		90.608 (63.901)	-87.849 (96.953)
status_share		178.457 + (107.992)	
status_mixed			-178.457 + (107.992)
Num.Obs.	1026	1026	1026
R2	0.819	0.819	0.819
R2 Adj.	0.819	0.819	0.819
F	1158.652		

Multicollinearity: Near-perfect collinearity

Problem

- When two variables x_1 and x_2 are highly correlated.
- And both (x_1 and x_2) affect y significantly.
- x_1 and x_2 are collinear
- Estimated Standard Errors are inflated (larger then they could be).
- Inference about x_1 and x_2 (only) is misleading.

Detection

1. Variance Inflation Factor
   - Not a statistical test, only an informative number.
   - This shows by how much, the SE of the collinear variables are larger than what they could have been if there was no collinearity.
2. Step-wise regression approach:
   - Include collinear variables one by one and together, and
   - observe how the R^2, and variables significance changes

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Solutions

1. Find different independent variables.
2. Drop one if it is redundant and repeating.
3. Keep both and ignore if x_1 and x_2 are just control variables.
4. Keep is removing may cause the OVB.

Multicollinearity Example 1. Production Function

```r
library(performance)
check_collinearity(fit2_cd)
```

```
# Check for Multicollinearity

Low Correlation

       Term  VIF   VIF 95% CI Increased SE Tolerance Tolerance 95% CI
     status 1.02 [1.00, 1.49]         1.01      0.98     [0.67, 1.00]
  log(land) 4.36 [3.93, 4.87]         2.09      0.23     [0.21, 0.25]
 log(labor) 4.38 [3.94, 4.89]         2.09      0.23     [0.20, 0.25]
```

If VIF > 10, we may suspect multicollinearity.

Removing land or labor will cause the OVB (omitted variable bias), we must keep both variables.

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Multicollinearity Example 2

$$\hat{output} = \hat{\beta}_1 land + \hat{\beta}_2 seeds + \hat{\beta}_3 fertilizers + \hat{\beta}_4 others$$

- where $seeds$ and $fertilizers$ highly correlate between each other (r=0.9),
- VIF of $seeds$ and $fertilizers$ is > 12.2
- our key-interest variable is $land$.
- As $fertilizers$ is a control variable and we may have OVB if we remove it,
- If we really want to reduce VIF…:
  - Dis-aggregate fertilizers into mineral and organic, for example.
  - Aggregate fertilizers and seeds

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Multicollinearity Example 3

Same model but with a quadratic term:

$$\hat{output} = \hat{\beta}_1 land + \hat{\beta}_2 land^2 + \hat{\beta}_3 seeds + \hat{\beta}_4 fertilizers + \hat{\beta}_5 others$$

- VIF of $land$ and $land^2$ is > 25
- Is there a multicollinearity problem here?
- Think!
- Not really
- $land^2$ is not a linear combination of $land$;
- Linear combination is when $land + land$ not when $land \times land$;

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Multicollinearity Example 4

Step-wise approach to regression analysis:

	Model 1	Model 2	Model 3	Model 4
(Intercept)	4.997*** (0.098)	-20.158*** (2.726)	-10.254*** (1.553)	-1.278 (11.509)
muniPrecip	-0.009+ (0.005)	-0.010*** (0.002)	-0.010*** (0.002)	-0.011*** (0.002)
year		0.013*** (0.001)		-0.011 (0.014)
log(muniPop)			1.025*** (0.104)	1.917 (1.138)
Num.Obs.	24	24	24	24
R2	0.148	0.832	0.848	0.852
R2 Adj.	0.110	0.815	0.833	0.830

Note: ^^ Model 1: `log(muniUse)` ~ muniPrecip
Model 2: `log(muniUse)` ~ muniPrecip + year
Model 3: `log(muniUse)` ~ muniPrecip + `log(muniPop)`
Model 4: `log(muniUse)` ~ muniPrecip + year + `log(muniPop)`

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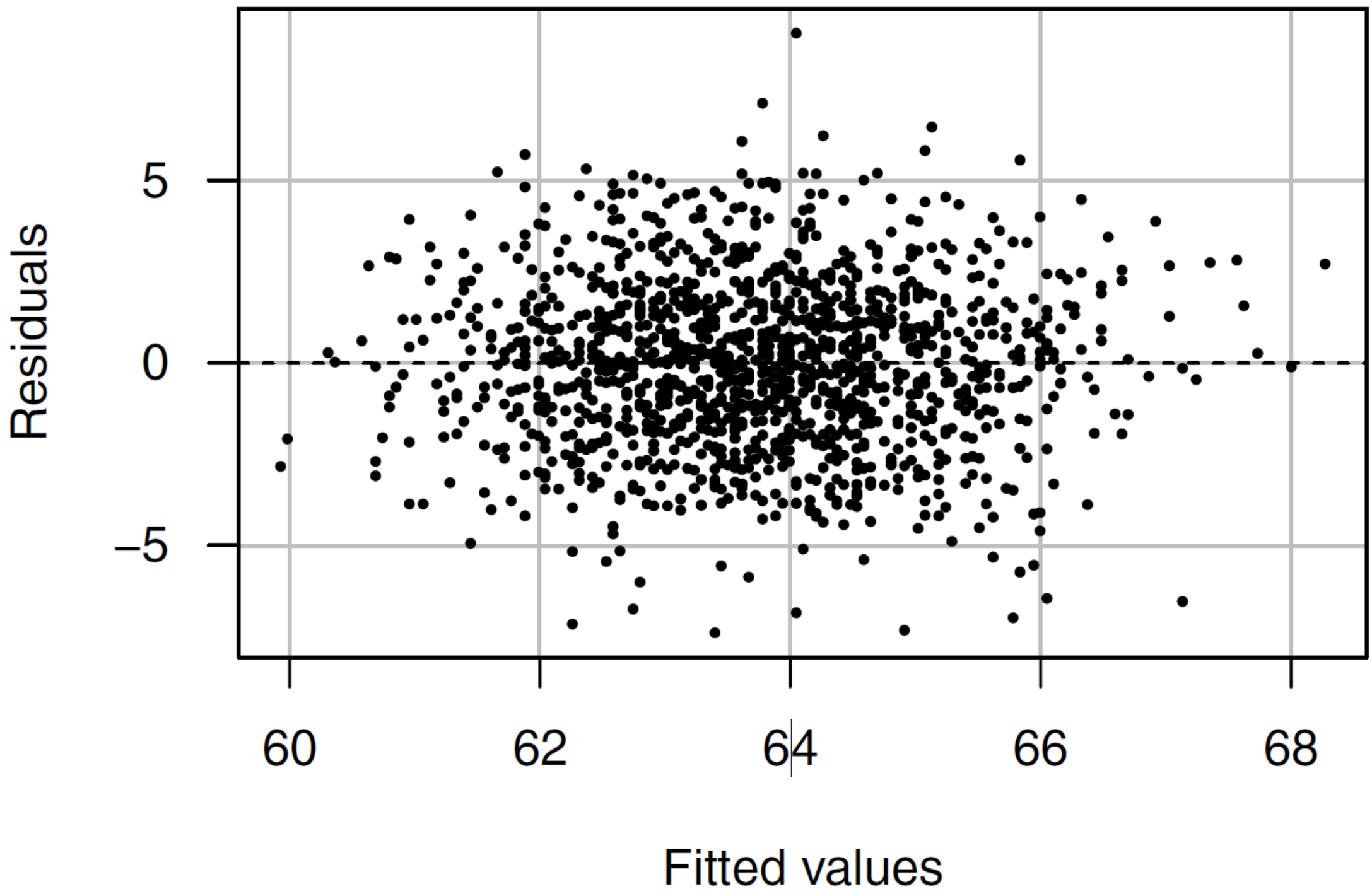
Homoscedasticity (No heteroscedasticity)

Homoscedasticity

$$Var(u|x_{i1}, x_{i2}, \cdots, x_{ik}) = \sigma^2$$



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Problem and detection

- **Problem:** when $Var(u|x_{i1}, x_{i2}, \cdots, x_{ik}) \neq \sigma^2$
  - Estimates are not biased, but;
  - Standard errors are inefficient;
  - Inference is misleading and false.
- **Detection:**
  - Graphical: residuals vs regressors and fitted values plots;
  - Statistical tests: Breusch-Pagan, White test, Goldfeld-Quandt.

Solutions

- Re-specify the model/ choose different regressors;
- Correct SE with alternative estimation procedures
  - robust: use heteroskedasticity-consistent (robust) standard errors.
  - cluster: use clustered standard errors.
- Use a weighted regression estimated with Generalized Linear Model (GLM) estimator instead of OLS;
  - Generally discouraged for small samples (n < 5000) as GLM have weaker asymptotic properties as compared to the OLS.
- See also: (Angrist & Pischke, 2009, Ch. 3.4), “Assumption AMLR.5” in (Wooldridge, 2020)

Heteroscedasticity example (1)

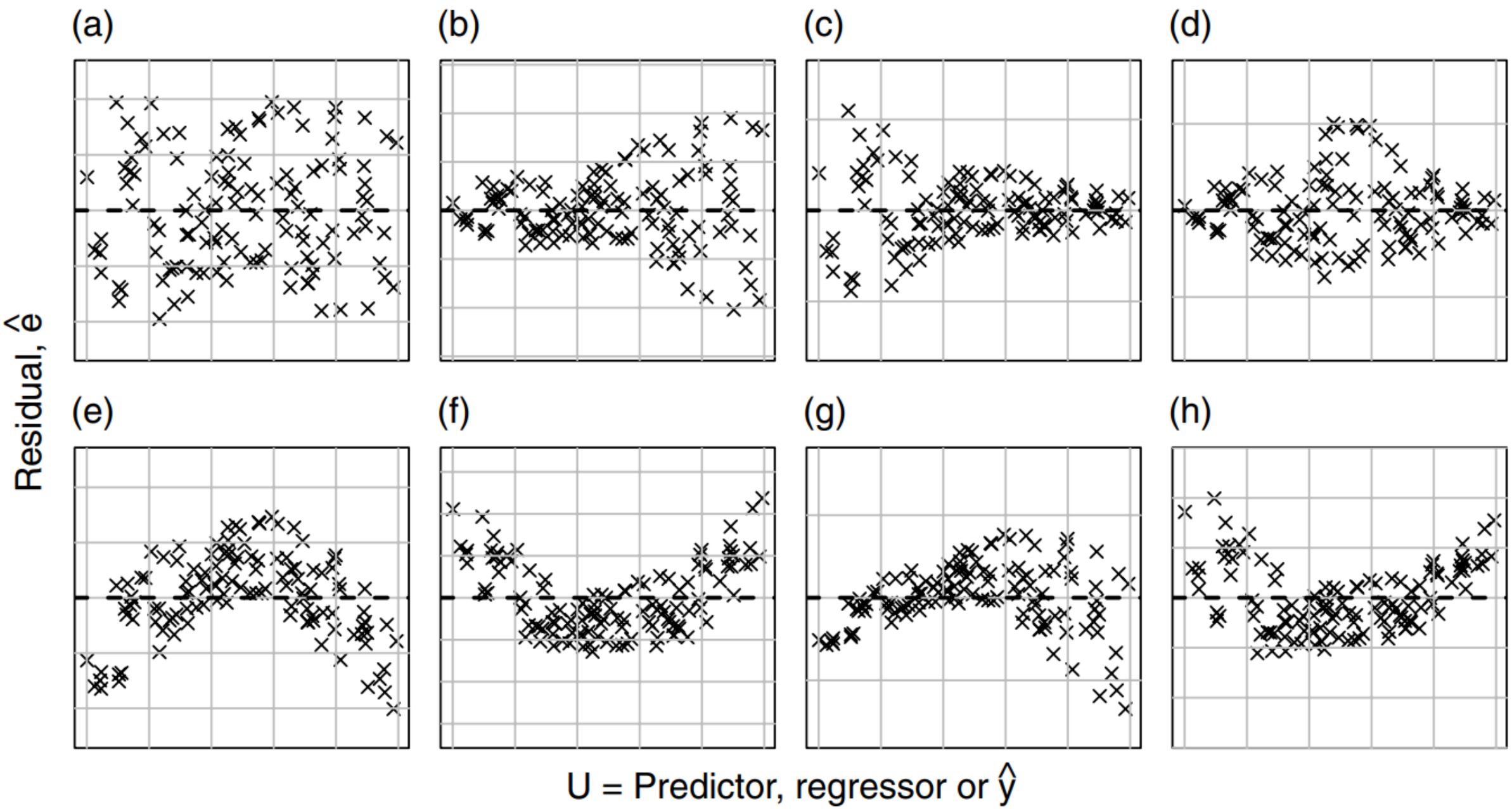

Figure 9.2 Residual plots: (a) null plot; (b) right-opening megaphone; (c) left-opening megaphone; (d) double outward bow; (e) and (f) nonlinearity; (g) and (h) combinations of nonlinearity and nonconstant variance function.

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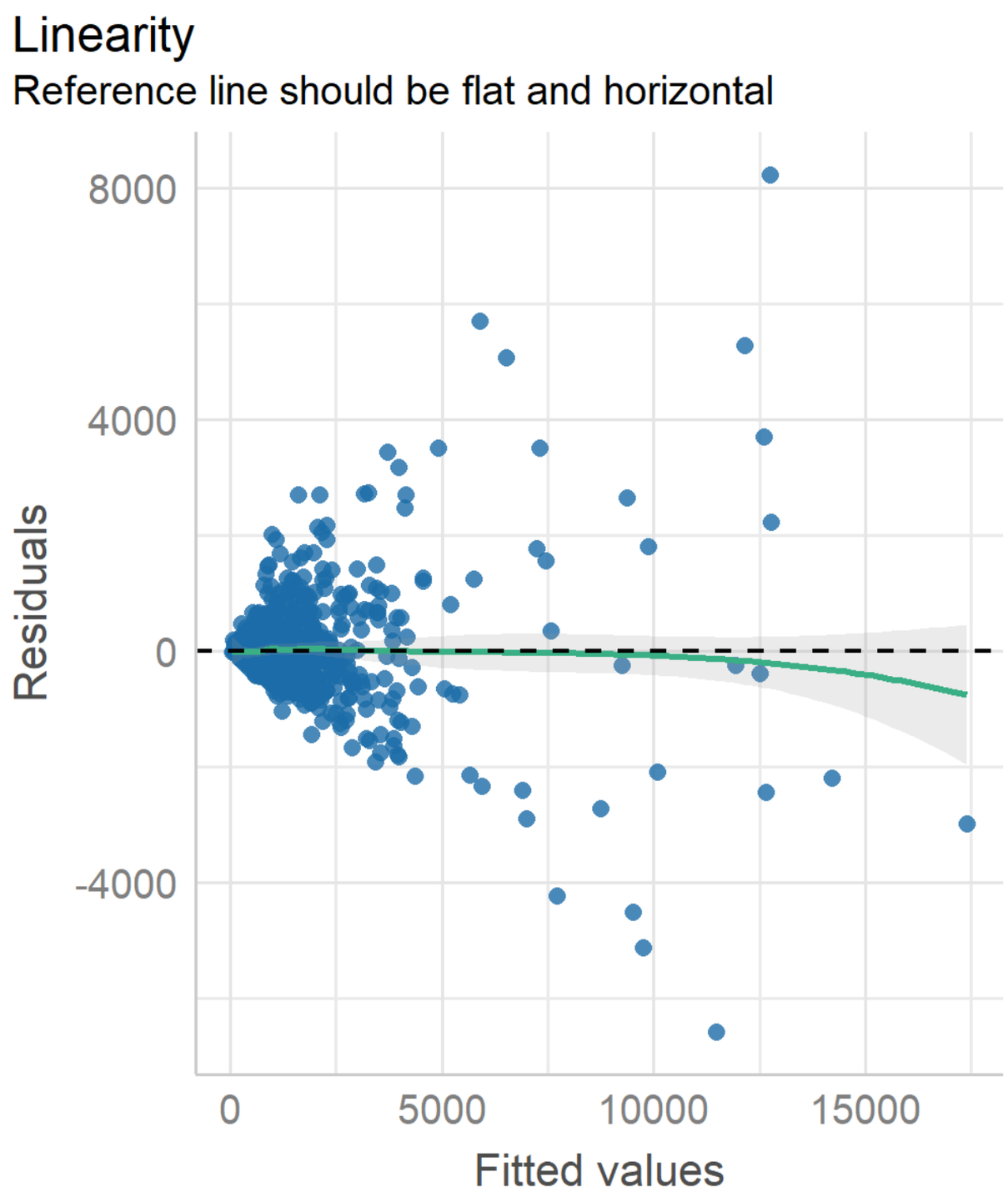
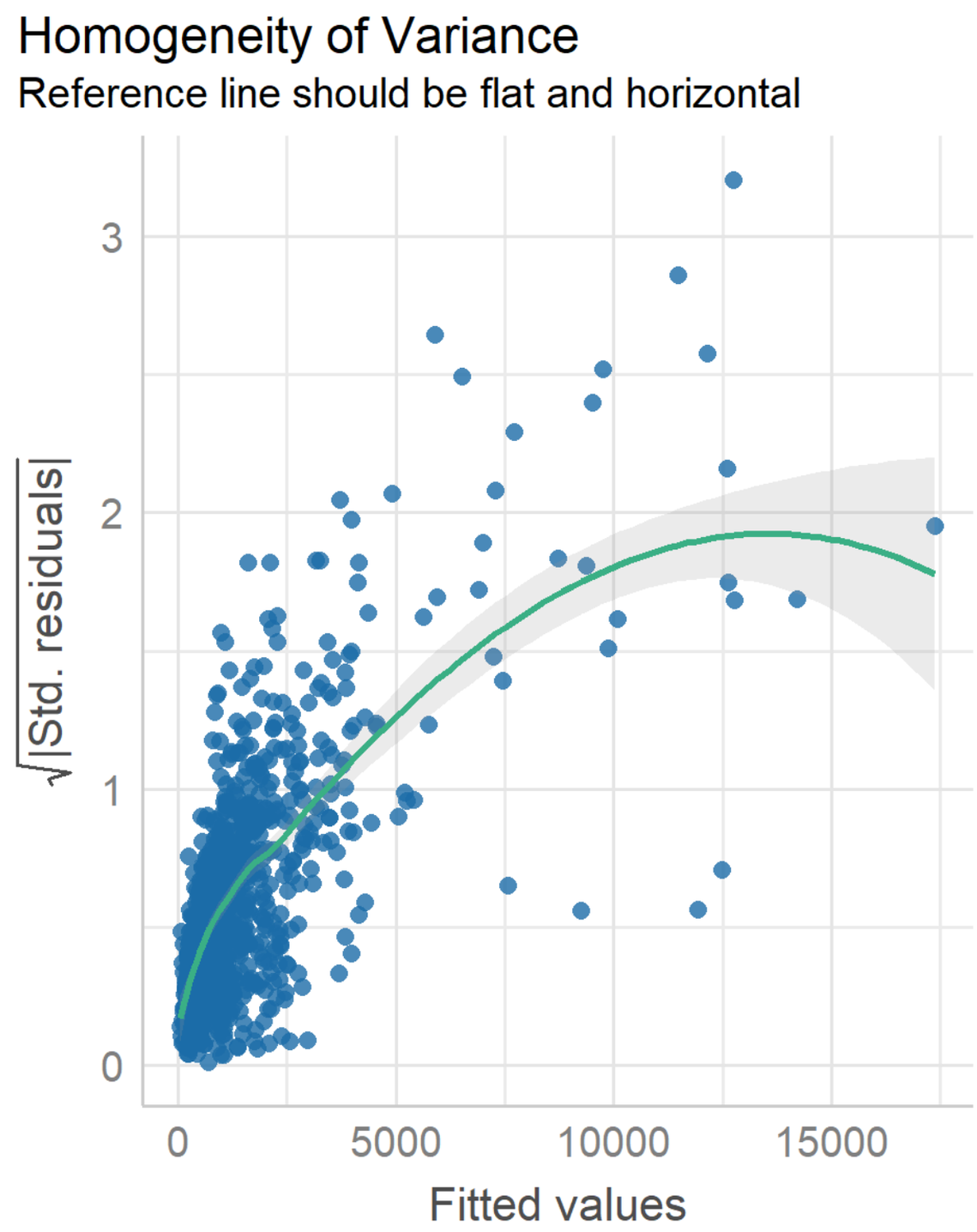
Heteroscedasticity (2). Production function

```
lm(formula = output ~ status + land + labor, data = farm_dta)
```

```
check_model(fit1, check = c("linearity", "homogeneity"))
```





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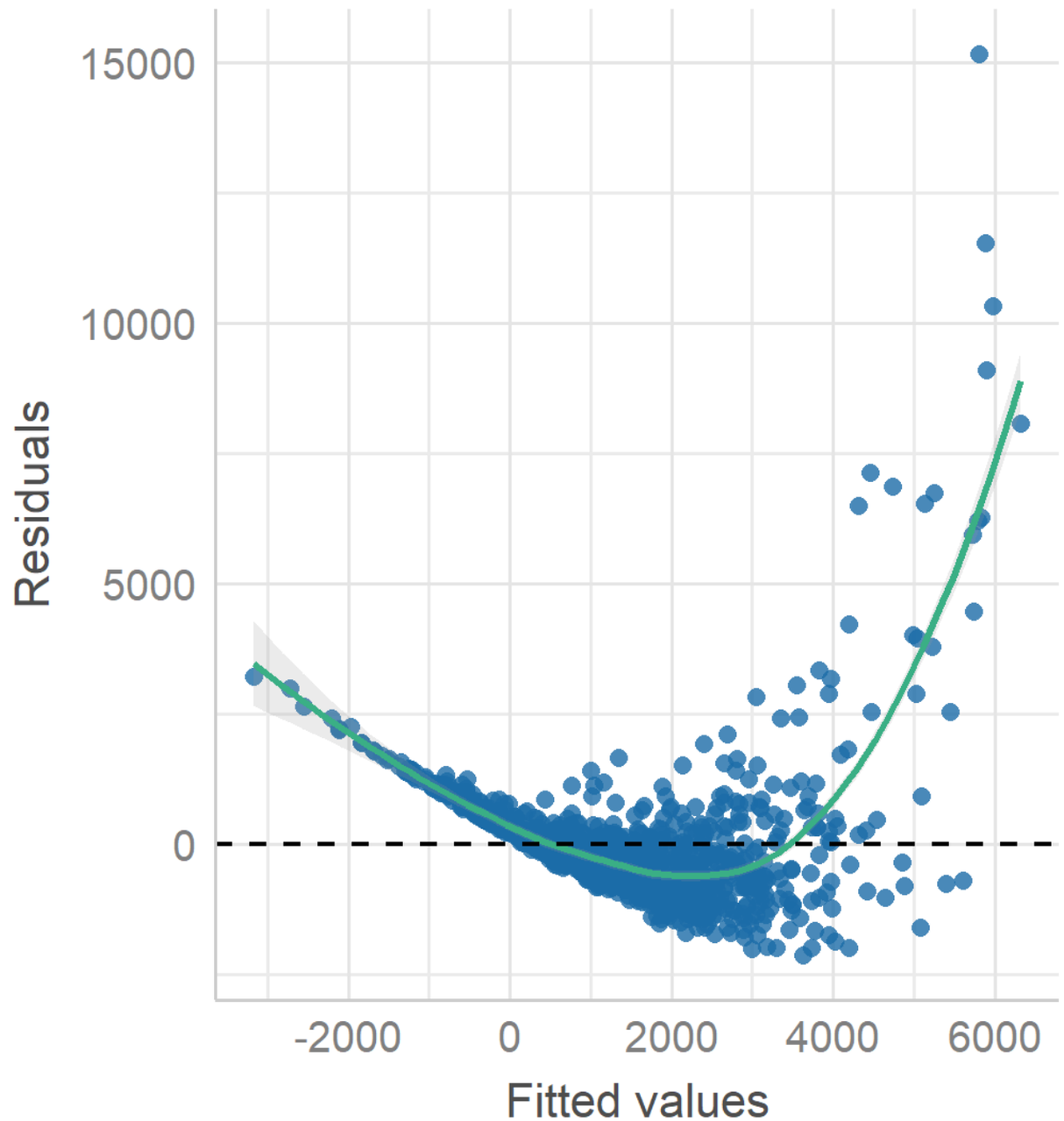
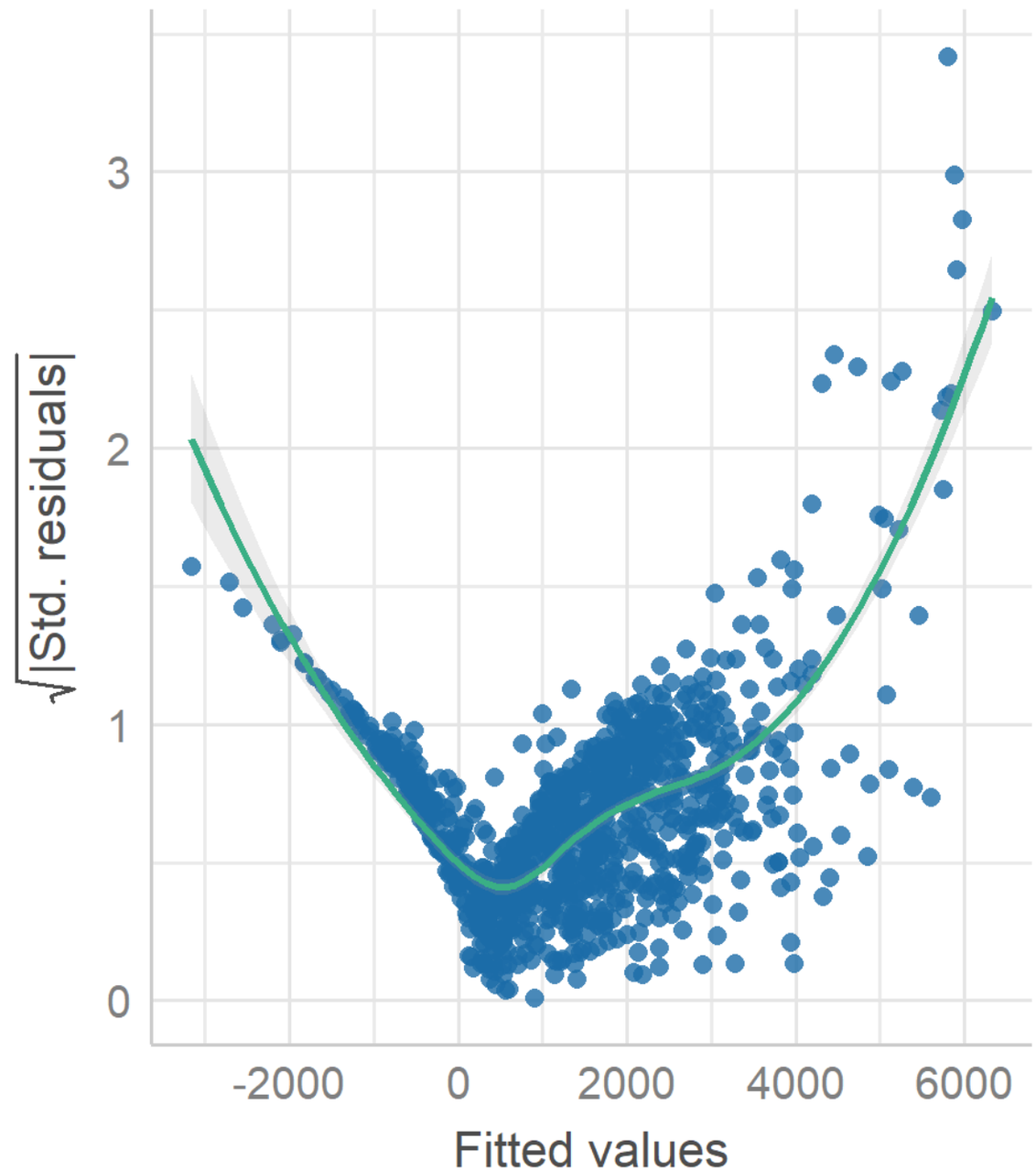
Heteroscedasticity (2). Partial transformation

```
lm(formula = output ~ status + log(land) + log(labor), data = farm_dta)
```

```
check_model(fit3_level_log, check = c("linearity", "homogeneity"))
```





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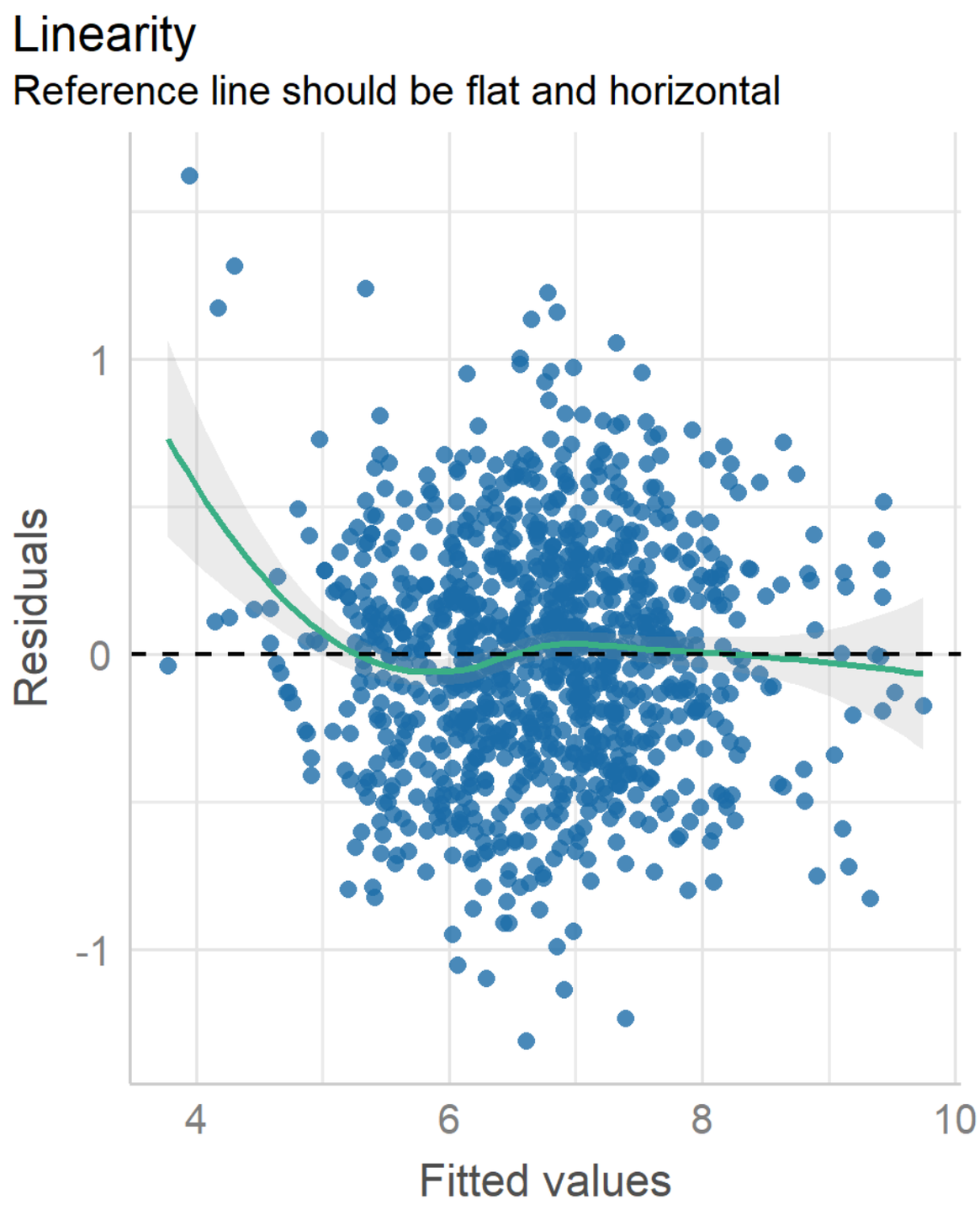
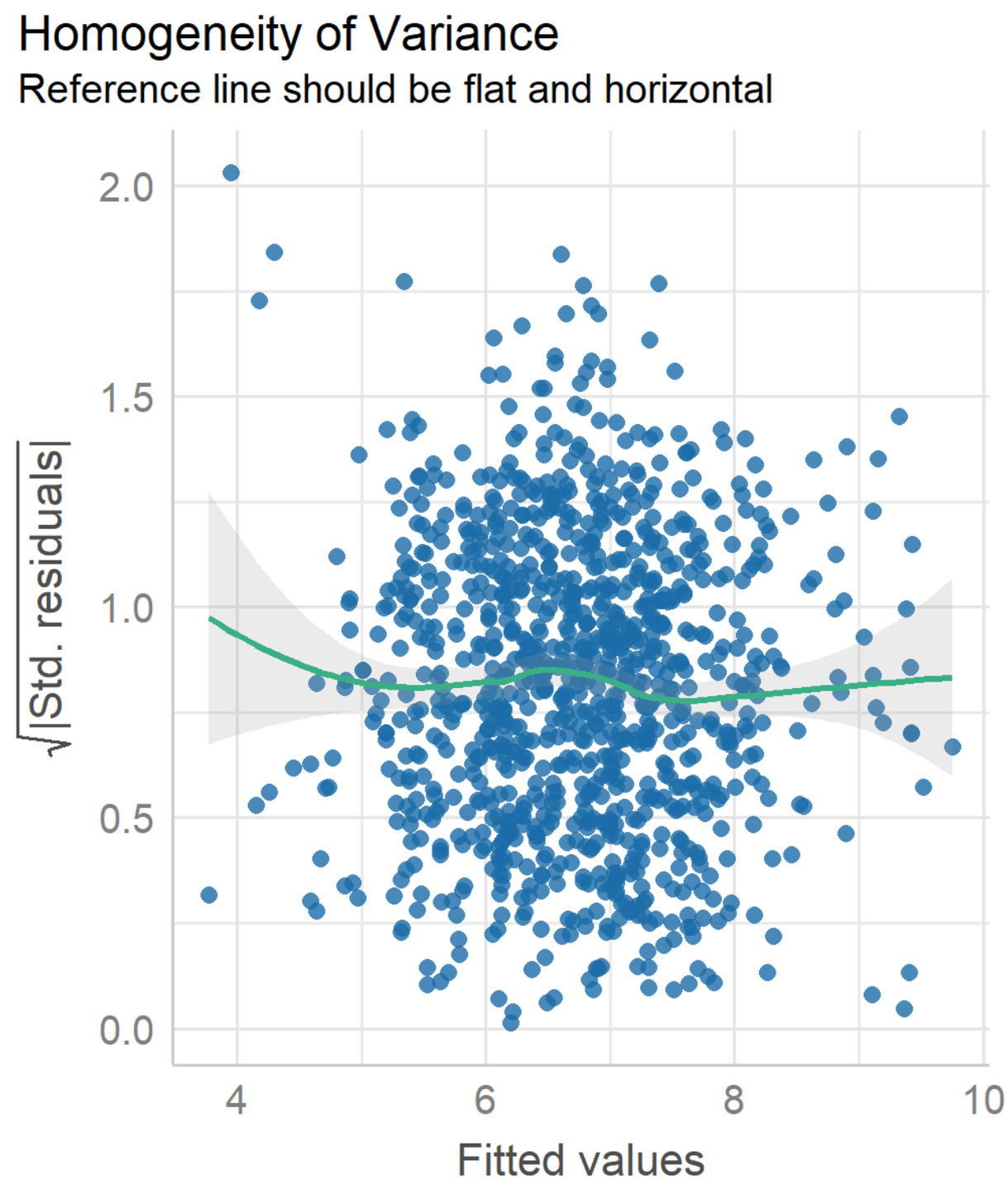
Heteroscedasticity (2). CD production function

```
lm(formula = log(output) ~ status + log(land) + log(labor), data = farm_dta)
```

```
check_model(fit2_cd, check = c("linearity", "homogeneity"))
```





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Statistical tests (1): Breusch-Pagan

- H_0 : variance is constant; H_1 non constant variance.
- If H_0 is rejected, we have a Heteroscedasticity.

```
lm(formula = output ~ status + land + labor,
data = farm_dta)
```

```r
library(lmtest)
bptest(fit1)
```

```
    studentized Breusch-Pagan test

data:  fit1
BP = 348.57, df = 4, p-value < 2.2e-16
```

```
lm(formula = log(output) ~ status + log(land)
+ log(labor), data = farm_dta)
```

```r
library(lmtest)
bptest(fit2_cd)
```

```
    studentized Breusch-Pagan test

data:  fit2_cd
BP = 19.776, df = 4, p-value = 0.0005528
```


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Statistical tests (2): White test

Same idea as Breusch-Pagan test. Assumes that residuals variance is a function of polynomial of regressors:

$$\text{residuals} = \log(\text{land}) + \log(\text{labor}) + \log(\text{land})^2 + \log(\text{labor})^2$$

```
lm(formula = output ~ status + land + labor,
data = farm_dta)
```

```
bptest(fit1,
        ~ land * labor + I(labor^2) + I(labo
        data = farm_dta)
```

```
    studentized Breusch-Pagan test

data:  fit1
BP = 364.4, df = 4, p-value < 2.2e-16
```

```
lm(formula = log(output) ~ status + log(land)
+ log(labor), data = farm_dta)
```

```
bptest(fit1,
        ~ log(land) * log(labor) +
          I(log(labor)^2) + I(log(labor)^2)
        data = farm_dta)
```

```
    studentized Breusch-Pagan test

data:  fit1
BP = 309.24, df = 4, p-value < 2.2e-16
```


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Robust standard errors (1)

Methods of correcting the variance-covariance matrix.

```
vcov(fit1)
```

	(Intercept)	statusshare	statusmixed	land	labor
(Intercept)	1357.56	-997.12	-909.30	-383.44	-0.71
statusshare	-997.12	9399.81	910.48	-15.30	0.24
statusmixed	-909.30	910.48	4083.37	-146.28	0.16
land	-383.44	-15.30	-146.28	12437.02	-12.75
labor	-0.71	0.24	0.16	-12.75	0.02

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Common methods of robust standard errors

Robust estimators for variance covariance matrix:

- `sandwich::vcovHC` - heteroskedasticity consistent
- `sandwich::vcovCL` - clustered SE
- `clubSandwich::vcovCR` - clustered heteroskedasticity consistent SE
- `sandwich::vcovHAC` - heteroskedasticity and autocorrelation consistent
- Estimation methods:
  - `HC3` - optimal one as per (Long & Ervin, 2000)
  - `HC1` - default in `Stata`

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An example of robust vcov

```
lm(formula = output ~ status + land + labor, data = farm_dta)
```

Regular vcov

```
vcov(fit1)
```

	(Intercept)	statusshare	statusmixed	land	labor
(Intercept)	1357.56	-997.12	-909.30	-383.44	-0.71
statusshare	-997.12	9399.81	910.48	-15.30	0.24
statusmixed	-909.30	910.48	4083.37	-146.28	0.16
land	-383.44	-15.30	-146.28	12437.02	-12.75
labor	-0.71	0.24	0.16	-12.75	0.02

Robust vcov

```
library(sandwich)
vcovHC(fit1, type = "HC3")
```

	(Intercept)	statusshare	statusmixed	land	labor
(Intercept)	4235.06	-3084.98	249.07	-6038.87	-5.91
statusshare	-3084.98	11437.98	799.53	-2421.05	8.76
statusmixed	249.07	799.53	2692.17	1349.86	-4.71
land	-6038.87	-2421.05	1349.86	75901.71	-62.24
labor	-5.91	8.76	-4.71	-62.24	0.09

Effect of robust SE on inference

```
lm(formula = output ~ status + land + labor, data = farm_dta)
```

	Linear prod.fn. (level-level) regular SE
(Intercept)	6.203
	(36.845)
Sharecropping	87.849
	(96.953)
Mixed	-90.608
	(63.901)
Land, ha	2145.659 ***
	(111.521)
Labor, hours	1.248 ***
	(0.126)
Num.Obs.	1026
R2 Adj.	0.819

	Linear prod.fn. (level-level) robust SE
(Intercept)	6.203
	(68.312)
Sharecropping	87.849
	(111.881)
Mixed	-90.608 +
	(53.100)
Land, ha	2145.659 ***
	(291.480)
Labor, hours	1.248 ***
	(0.312)
Num.Obs.	1026
R2 Adj.	0.819
Std.Errors	HC3

Effect of robust SE on inference

```
lm(formula = log(output) ~ status + log(land) + log(labor), data = farm_dta)
```

	CD prod.fn. (level-level) regular SE
(Intercept)	5.703 ***
	(0.202)
Sharecropping	0.004
	(0.047)
Mixed	0.017
	(0.031)
Land, ha	0.679 ***
	(0.027)
Labor, hours	0.344 ***
	(0.030)
Num.Obs.	1026
R2 Adj.	0.842

	CD prod.fn. (level-level) robust SE
(Intercept)	5.703 ***
	(0.263)
Sharecropping	0.004
	(0.048)
Mixed	0.017
	(0.028)
Land, ha	0.679 ***
	(0.036)
Labor, hours	0.344 ***
	(0.039)
Num.Obs.	1026
R2 Adj.	0.842
Std.Errors	HC3

Homework

Watch:

- Regression Part I: Call in the CIA (or here: https://youtu.be/OwNxEaOF8yY)
- Regression Part II: Theory (or here: https://youtu.be/KKJ-2_UUhnc)

Reproduce analysis from the slides in R.

- Make complex production function exercise.

Takeaways

1. Assumptions and how to validated them.
2. Statistical plots, measures (VIF), statistical tests and how to interpret them.

References

References

Angrist, J. D., & Pischke, J.-S. (2009). *Mostly harmless econometrics*. Princeton University Press. http://doi.org/10.1515/9781400829828

Dasgupta, S., Knight, T. O., & Love, H. A. (1999). Evolution of Agricultural Land Leasing Models: A Survey of the Literature. *Applied Economic Perspectives and Policy*, *21*(1), 148–176. http://doi.org/10.2307/1349978

Feng, Q., & Horrace, W. C. (2010). Alternative technical efficiency measures: Skew, bias and scale. *Journal of Applied Econometrics*, *27*(2), 253–268. http://doi.org/10.1002/jae.1190

Long, J. S., & Ervin, L. H. (2000). Using heteroscedasticity consistent standard errors in the linear regression model. *The American Statistician*, *54*(3), 217–224. http://doi.org/10.1080/00031305.2000.10474549

Sadoulet, E., & de Janvry, A. (1995). *Quantitative Development Policy Analysis*. THE JOHNS HOPKINS UNIVERSITY PRESS BALTIMORE AND LONDON.

Wooldridge, J. M. (2020). *Introductory econometrics: A modern approach*. South-Western. Retrieved from https://www.cengage.uk/shop/isbn/9781337558860