Theta and Elliptic Functions

The study of theta and elliptic functions starts as a story in complex analysis—engineering functions that satisfy a certain relation—and connects to complex manifolds, elliptic curves, integration, and quadratic forms.

We'll start out with the question: what is the complex analog of a periodic function? First, we'll have to review a bit of complex analysis (Section 1). First, as practice for what we will do with theta and elliptic functions, we will "derive" expressions for the sine and cosine, using our complex analytic machinery (Section **??**).

Unfortunately, we find that the only entire functions that are *doubly* periodic are constant, so we have to relax one of these conditions.

- Relaxing periodicity gives the theory of **theta functions** (Section **??**). We use complex analysis to construct a "basic" theta function θ in 3 different ways, and show that they are all equal up to a constant. Then we characterize all theta functions, by showing that *every* theta function can be written using a product of translates of θ, in the spirit of the Fundamental Theorem of Algebra.

- Relaxing entirety gives the theory of **elliptic functions** (Section **??**). We construct a "basic" elliptic function, the Weierstrass $\wp$-function, in 2 ways and show all elliptic functions can be built out of $\wp$ (and $\wp'$). We then give a list of basic properties of elliptic functions.

  Knowing the theory of theta functions makes our job easier, as elliptic functions can be written in terms of theta functions.

Now comes the applications. In much the same way that ($\sin x$, $\frac{d}{dx}\sin x = \cos x$) parameterize the circle, we find that $(\wp, \wp')$ also parameterize an algebraic variety—an *elliptic curve* (Section **??**). This is the key application of elliptic functions to number theory. Following history, we'll use elliptic functions to calculate the arc length of an ellipse in Section **??** (this is where the name came from).

(See Ahlfors pg. 238 for this?)

(I might put in the application to the 4-square problem... but this is more easily done with modular forms.)

It is important to keep in mind that the functions here depend on *two* parameters. For instance, θ is a function in z and τ:

$$\theta(z, \tau)$$

As we will see, τ specifies a lattice Λ.

- In this article, we are mostly concerned with θ as a function of z. (But it will be helpful to change our view sometimes!) θ is a theta function in the parameter z (with certain transformation properties).

- However, we can also think of θ as a function in τ. Then θ is (sort-of) a modular function in the parameter τ (with certain transformation properties, coming from the fact that θ is really a function of the lattice $\langle 1, \tau\rangle$).

Thus, theta functions and modular functions are intricately related! But the story of modular functions is a story for another day and article.

1 Some complex analysis

sec:complex-analysis In this section we review some facts from complex analysis. (For a longer "crash course," see §30 in the number theory text.)

A complex analytic function is a function that is complex differentiable. This is a much stronger condition than real differentiability, and gives complex analytic functions very nice properties—for instance, they are equal to their power series expansions (barring any poles). An entire function is a complex analytic function defined on all of $\mathbb{C}$. A meromorphic function is allowed to have poles.

Theorem 1.1 (Liouville)**:** thm:liouville A bounded entire function is constant.

1.1 Series and product developments

We know that locally, we can write a meromorphic function f as a Laurent series $\sum_{n=-\infty}^{\infty} a_n x^n$. There are two other representations that are useful, depending on what information we have about the function f.

1. If we know the *poles* of f, we can write f as a sum of rational functions

$$f(z) = \sum_{n=1}^{\infty} \left[P_n\left(\frac{1}{z - z_n}\right) - Q_n(z) \right] + g(z).$$

2. If f is entire and we know the *zeros* of f, we can write f as an infinite product

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{P_n\left(\frac{z}{z_n}\right)}.$$

(Think of this as "factoring" f, much like a polynomial can be factored as in the fundamental theorem of algebra.) These representations come about from convergence properties of holomorphic functions (which tell us that infinite sums and products converge to holomorphic functions) and by Liouville's theorem—if we engineer a function that is close enough to f then it must be equal to f.

Theorem 1.2 (Mittag-Leffler)**:** Let z_n be a sequence with $\lim_{n\to\infty} |z_n| = \infty$ (or a finite sequence), and P_n polynomials without constant term.

1. (Existence) There is a meromorphic function f with poles exactly at z_n, with Laurent expansion $P_n\left(\frac{1}{z-z_n}\right) + \cdots$ at z_n.

2. (Uniqueness) All such functions f are in the form

$$\sum_{n=1}^{\infty}\left(P_n\left(\frac{1}{z-z_n}\right)-Q_n(z)\right)+g(z)$$

where Q_n is a polynomial and $g(z)$ is analytic.

Proof. See Ahlfors [1, p. 187]. □

Warning: this does not converge for all P_n. Typically we take Q_n to the the first terms of the Laurent expansion of $P_n\left(\frac{1}{z-z_n}\right)$, to ensure cancellation of high-order terms.

Definition 1.3: The **order** of an entire function f is the smallest $\alpha \in [0,\infty]$ such that

$$|f(z)| \precsim_{\varepsilon} e^{|z|^{\alpha+\varepsilon}}$$

for all $\varepsilon > 0$.

Theorem 1.4 (Hadamard)**:** product-development Let z_n be a sequence with $\lim_{n\to\infty}|z_n| = \infty$. If f is entire with order $\alpha < \infty$ with zeros $z_1, z_2, \dots$ (with multiplicity, not including 0), then it has a product formula

$$product-formula f(z) = z^r e^{g(z)} \prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right) e^{\frac{z}{z_n}+\frac{1}{2}\left(\frac{z}{z_n}\right)^2+\cdots+\frac{1}{m}\left(\frac{z}{z_m}\right)^m}, \tag{1}$$

where

- $m = \lfloor \alpha \rfloor$,
- r is the order of vanishing of f at 0, and
- g is a polynomial of degree at most a.

The product converges uniformly locally. Moreover,

$$num-zeros|\{k : z_k < R\}| \precsim_{\varepsilon} R^{\alpha+\varepsilon}. \tag{2}$$

Conversely, if $a = \lfloor \alpha \rfloor$ and z_k is a sequence satisfying (2), then the RHS of (1) defines an entire function of order at most α.

Proof. See Ahlfors [1, p. 195]. □

Hence the order of a entire function gives an asymptotic bound for the number of zeros.[1]

[1] A function which grows faster is allowed to have more zeros—much like a polynomial with lots of zeros grows fast simply because it has higher degree.

2 Periodic functions

We now use the product expansion to come up with expressions for sin and cos.

Theorem 2.1: We have that

$$\sin \pi z = z e^{g(z)} \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

Proof. We know that $\sin \pi z$ has zeros at the integers $\mathbb{Z}$. The number of integers with absolute value at most R is $O(R)$, so by the product development 1.4, we know that

$$f(z) = z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{a}}$$

is an entire function with zeros at exactly the integers.

Now we just need to show $\sin \pi z = \pi f(z)$.

First note that $\mathbb{Z}$ are the *only* zeros for sin: We have $\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2}$, and this is 0 iff $e^{i\pi z} = e^{-i\pi z}$, iff $i\pi z = 2\pi i n - i\pi z$ for some n, iff $z \in \mathbb{Z}$. The zeros of sin and $f(z)$ are both simple, so $\frac{\sin \pi z}{f(z)}$ is an entire function.

Second, note that $\frac{\sin \pi z}{f(z)}$ is *bounded*. take log derivative, see Ahlfors p. 197. Blah blah. $\square$

Now let's take a step back. Suppose we didn't know the existence of the function sin, and someone told us to construct a nontrivial complex analytic function with period 1. What would we do? We might look for a function which has zeros at exactly the integers, and hope that it is periodic. In this way we could have "come up with" sin using the product expansion.

Now let's take an algebraic viewpoint. Note that $(\sin z, \cos z)$, $z \in \mathbb{R}$ parameterizes the circle $x^2 + y^2 = 1$. If we were studying the curve $x^2 + y^2 = 1$; we might think: how could we parameterize it using nice analytic functions? The answer is quite obvious, but if we didn't know, we might think: the circle is homeomorphic to $\mathbb{R}/\mathbb{Z}$ so we would try to pick functions that are periodic with period 1... Again, we're led back to sin.

Another question we could ask is, what about other functions that are periodic with period 1? Can we always write them in terms of sin and cos? Unfortunately, no. Counterexample? However, for many functions we can: for instance, $\sin(n\pi z)$ can be expanded in terms of $\sin(\pi z)$ and $\cos(\pi z)$. We have a lot of trig identities.

In a way, although having a single period was easy to analyze, we are somewhat out of things to do at this point. If we have a random function with period 1, to characterize it we just need to know how it behaves along a strip $a + bi$, $a \in [k, k+1)$. This strip is infinite, however—$\mathbb{Z}$ is a 1-D lattice in a 2-D space, $\mathbb{C}$.

Things are much nicer if we have a function that is periodic in *two* directions, i.e., is periodic with respect to a 2-D lattice in $\mathbb{C}$. Then this function is determined by its values on a parallelogram, which is bounded. Here we have a hope of a simple (non-infinite) representation for such a function—if we can get two periodic functions like this, f and

g, and know either $\frac{f}{g}$ or $f - g$ is bounded, then the values it attains are just those on a parallelogram, and by Liouville's constant we know f and g just differ by a constant (multiplied or added, respectively).

$x^2 + y^2 = 1$ is a simple curve, though it's not very interesting. When we're working over $\mathbb{C}$ we want to consider the surface over $\mathbb{C}$ and our parametrization doesn't really make $x^2 + y^2 = 1$ into anything compact (over $\mathbb{C}$). *Elliptic curves* are where the idea of using analytic functions to parameterize algebraic varieties really shines. Elliptic curves are the simplest kind of *abelian variety* (1-dimensional), and abelian varieties allow nice analytic parameterizations. ($x^2 + y^2 = 1$ is not an abelian variety.)

3 Theta functions

We ask: what is the complex analytic analogue of a periodic function? Since $\mathbb{C}$ is 2-dimensional, a natural analogue would be a function that is

1. entire, and
2. has 2 $\mathbb{R}$-independent periods, i.e., there are ω_1 and ω_2 with $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$, such that
$$f(z + \omega_1) = f(z + \omega_2) = f(z).$$

For f satisfying the second condition, we say that f is doubly-periodic, or f is periodic with lattice $\langle \omega_1, \omega_2 \rangle$. We call $\Pi = \{a\omega_1 + b\omega_2 : a, b \in [0, 1)\}$ the fundamental parallelogram.

The bad news is that any such function must be a constant.

Proposition 3.1: An entire doubly-periodic function is constant.

Proof. If f is periodic with lattice $\langle \omega_1, \omega_2 \rangle$, it is determined by its values in the fundamental parallelogram Π. Now $\overline{\Pi}$ is closed and bounded, hence compact, so f is bounded on $\overline{\Pi}$. But this means f is bounded, and by Liouville's Theorem 1.1, f is constant. □

Hence to develop an interesting theory of complex periodic functions, we have to relax one of our conditions:

1. double periodicity,
2. entirety.

In this section we relax double periodicity.

Definition 3.2: A **theta function** of degree n with parameter $b \neq 0$ with respect to (ω_1, ω_2) is an entire function $f(z)$ such that

$$f(z + \omega_1) = f(z), \qquad f(z + \omega_2) = be^{-\frac{2\pi i n z}{\omega_1}} f(z).$$

The reason for the multiplier $be^{-\frac{2\pi i n z}{\omega_1}} f(z)$ will be apparent during our attempt to construct a theta function.

3.1 Take 1: The Weierstrass σ-function

For simplicity, let's take $\omega_1 = 1$ and $\omega_2 = \tau$, $\Im\tau > 0$. This can always be achieved by rescaling z and permuting ω_1, ω_2.

In trying to make a doubly periodic function, our first try will be to emulate what we did for sin: let's try to make a function with zeros at exactly the points of $\Lambda = \langle 1, \tau\rangle$. We use the product development 1.4. This time, the number of points of Λ inside a ball of radius R is $O(R^2)$, so we construct the following function.

Definition 3.3: Define the **Weierstrass σ-function** with respect to the lattice $\Lambda = \langle 1, \tau\rangle$ by

$$\sigma(z) := z \prod_{\lambda\in\Lambda^*} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2}$$

where $\Lambda^* = \Lambda \backslash \{0\}$.

By the product development, σ has zeros exactly at the points of Λ. Note that σ implicitly depends on τ as well, but we are thinking of τ as being fixed. If we want to think of it as a function of 2 variables we will write $\sigma(z, \tau)$.

In this form, it is not clear that σ satisfies any periodicity condition. In fact, it will not be doubly periodic. To see how it behaves with respect to $z \hookleftarrow z + \omega_1, z + \omega_2$, we take a page from our development with sin and take the derivative.

Proposition 3.4: σ is a theta function.

Proof. To deal with this product we take the *logarithmic derivative*:

$$\begin{aligned}\zeta(z) := \frac{\sigma'(z)}{\sigma(z)} &= \frac{1}{z} + \sum_{\lambda\in\Lambda^*} \left(\frac{-\frac{1}{\lambda}}{1 - \frac{z}{\lambda}} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right) \\ &= \frac{1}{z} + \sum_{\lambda\in\Lambda^*} \left(\frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right).\end{aligned}$$

We'd like to know how this transforms with respect to $z \hookleftarrow z + \lambda$, but the fact that the sum is over Λ^* instead of Λ causes problems. We take another derivative:

$$\wp(z) := -\zeta'(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2}\right).$$

Taking yet another derivative kills the constant and gives us a sum over Λ.

$$\wp'(z) = -\zeta''(z) = -\frac{2}{z^3} - \sum_{\lambda\in\Lambda^*} \frac{2}{(z - \lambda)^3} = \sum_{\lambda\in\Lambda} \frac{2}{(z - \lambda)^3}.$$

Because $\wp'(z)$ is a sum over Λ, it is periodic with respect to Λ:

$$\wp'(z + \lambda) = \wp'(z) \implies (\wp(z + \lambda) - \wp(z))' = 0$$

Now we start integrating. This gives

$$\wp(z+\lambda) = \wp(z) + \eta(\lambda)$$

for some constant $\eta(\lambda)$ depending only on λ. ...

□

3.2 Take 2: A q-product expansion

To write down a theta function using the product development, we took a product over *all* nonzero points in the lattice Λ. However, there's something different and clever we can do: we can use the fact that Λ is a lattice to write down a theta function using an infinite product over integers insted.

We want our function to have zeros exactly when

$$z = m + n\tau$$

for some $m, n \in \mathbb{Z}$, i.e., exactly when

$$2\pi i z = 2\pi i m + 2\pi i n\tau \text{ for some } m, n \in \mathbb{Z} \qquad \Longleftrightarrow \qquad e^{2\pi i z} = e^{2\pi i n\tau}$$

for some n. For ease of notation, we'll write from now on

$$eq : qq = e^{2\pi i\tau}, \tag{3}$$

so our condition is $q^n = e^{2\pi i z}$ for some $n \in \mathbb{Z}$. We now write down the following function:

$$\phi(z) := \prod_{n=0}^{\infty} (1 - q^n e^{2\pi i z})(1 - q^{n+1} e^{-2\pi i z}).$$

By our above considerations, ϕ has zeros at exactly the points of Λ. (We do have to check convergence.) (Note our notation ϕ is not standard; this is actually ad-hoc because we will want to multiply $\phi(z)$ by a constant depending on q later on.)

Let's verify that ϕ is elliptic. We have

$$\begin{aligned}
\phi(z+\tau) &= \prod_{n=0}^{\infty} (1 - q^n e^{2\pi i(z+\tau)})(1 - q^{n+1} e^{-2\pi i(z+\tau)}) \\
&= \prod_{n=0}^{\infty} (1 - q^{n+1} e^{2\pi i})(1 - q^n e^{2\pi i z}) \\
&= \frac{1 - e^{-2\pi i z}}{1 - e^{2\pi i z}} \prod_{n \in \mathbb{Z}} (1 - q^n e^{2\pi i z})(1 - q^{n+1} e^{-2\pi i z}) \\
&= -e^{-2\pi i z} \phi(z).
\end{aligned}$$

Thus ϕ is elliptic with parameter $b = -1$.

The kinds of products in the definition of ϕ will come up so often that we will establish some shorthand notation.

Definition 3.5: Define the **q-Pochhammer symbol** by

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

and let

$$(q)_n = (q;q)_n = \prod_{k=1}^{n}(1-q^k).$$

(We allow $n=\infty$.)

Thus ϕ can be written

$$\phi(z) = (e^{2\pi iz};q)_\infty(e^{-2\pi iz};-q)_\infty.$$

3.3 Take 3: Fourier expansion

This time we forgo products entirely, and work with the Fourier expansion. A theta function is periodic with period 1, so it has a Fourier expansion by the following theorem.

Proposition 3.6: Let f be a complex analytic function defined on $\Im z \ge k$, and suppose f is holomorphic at infinity, that is, $\ln f$ has a removable discontinuity at 0. Then f has a Fourier expansion

$$f(z) = \sum_{k=0}^{\infty} a_k e^{2\pi iz}.$$

picture here!

We now characterize all theta functions by their power series expansions.

Proposition 3.7: The theta functions with degree...

- $n<0$ is 0.
- $n=0$ are the functions $Ce^{2\pi ikz}$. They form a 1-dimensional space.
- $n>0$ are the functions $\sum_{k=0}^{\infty} a_k e^{2\pi iz}$, where $a_0,\ldots,a_{n-1}$ can be freely chosen and the a_k satisfy

  the recursive relation

  $$a_{m+pn} = b^{-p}q_0^{mp+\frac{np(p-1)}{2}}a_m, \quad q_0 = e^{-2\pi i\tau}.$$

  They form a n-dimensional space.

In particular, the following is a theta function of degree 1 and parameter b:

$$\theta(z) = \sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{k(k-1)}{2}} e^{2\pi ikz}$$

Proof. We consider the 3 cases separately. ∎

3.4 Factoring theta functions

We have the following analogue of the fundamental theorem of algebra.

Theorem 3.8: Any theta function of degree n is in the form

$$f(z) = K\theta(z - z_1) \cdots \theta(z - z_n)q^r$$

for some $z_1, \ldots, z_n \in \mathbb{C}$ and $r \in \mathbb{Z}$.

3.5 (2)=(3): Jacobi Triple Product Identity

References

[1] L. Ahlfors. *Complex Analysis*. McGraw-Hill, 1979.