A plodding derivation of the equations for compressible air flow in tunnels

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This document is a derivation of the equations of the homentropic method of characteristics (MoC) that is often used for calculating 1D air flow in the tunnel ventilation field.

Many researchers have written papers that summarized the derivation of the characteristic forms of the equations. Until I started a sabbatical I never had the time or patience to sit down and figure out how those researchers got from A to B and turn it into code. I wrote this document to help me figure it all out.

Part of the work involved delving into technical papers written in the 1970s and 1980s then showing how their content relates to how things are done today. This may prove useful to others even if they never use the program.

This document has eight sections:

- Derivation of the method of characteristics (MoC) for partial differential equations with two independent variables.
- Derivation of the relationship between speed of sound, pressure and density in an ideal gas.
- Derivation of the differential forms of the isentropic flow equations.
- Derivation of the relationship between density and speed of sound in the mass continuity equation.
- Conversion of the mass and momentum equations for isentropic compressible flow at low Mach number into a form suitable for solving by MoC.
- A summary of the different ways of calculating pressure losses at train noses and train tails.
- Derivation of the equations used to model body forces (such as jet fans and road traffic drag), prime movers (such as fans, portal doors and dampers) and boundary conditions.
- Some notes on the momentum coefficient.

This document might be expanded to cover non-homentropic flow if I ever get my head around the papers by Woods, Gawthorpe and Pope. But that is unlikely.

I have no doubt that there are errors in this document that I haven't spotted. I welcome corrections.

1 Method of characteristics for two variables

Sources:

- Lister, M, *"The numerical solution of hyperbolic partial differential equations by the method of characteristics"*, in Ralston & Wilf, "Mathematical Methods for Digital Computers", Wiley, 1960
- Fox, J A, *"The use of the digital computer in the solution of waterhammer problems"*, Paper 7020, J. Instn. Civ. Engnrs, 1968
- Fox, J A, and Henson, D A, *"The prediction of the magnitudes of pressure transients generated by a train entering a single tunnel"*, Paper 7635, J. Instn. Civ. Engnrs, 1971
- Fox, J A, *"Hydraulic analysis of unsteady flow in pipe networks"*, Macmillan Press, ISBN 0333 19142 0, 1977

In the tunnel vent field, Lister's 1960 paper seems to have been the main influence. Fox used her derivation to write waterhammer programs in Algol 60 at the University of Leeds in the 1960s, then supervised the Ph.Ds of three of the early players in the compressible airflow in tunnels field. In 1977 Fox wrote an excellent book about how to calculate unsteady flows in pipes in a practical manner.

The derivation in this section is mostly an expansion of the summary in chapter 4 of Fox's book.

Consider a pair of partial differential equations (PDEs) with two independent variables x and y and two dependent variables u and v:

$$L_1 = A_1 \frac{\partial u}{\partial x} + B_1 \frac{\partial u}{\partial y} + C_1 \frac{\partial v}{\partial x} + D_1 \frac{\partial v}{\partial y} + E_1 = 0 \tag{1}$$

$$L_2 = A_2 \frac{\partial u}{\partial x} + B_2 \frac{\partial u}{\partial y} + C_2 \frac{\partial v}{\partial x} + D_2 \frac{\partial v}{\partial y} + E_2 = 0 \tag{2}$$

where A_1, A_2, ... E_1 and E_2 are known functions of x, y, u and v.

In terms of one dimensional tunnel ventilation software, independent variable x will be time and independent variable y will be distance along a tunnel (or vice-versa—doesn't matter yet). x and y are independent variables because when we write the software we choose our timestep and our grid length.

As long as the dependent variables u and v define the state of the fluid it doesn't appear to matter what they are. Fox used water head and water velocity in his waterhammer papers. Lister used gas velocity and speed of sound in her (somewhat more theoretical) paper. Fox, Henson, Vardy and Higton used air velocity and speed of sound in their tunnel ventilation papers.

OK, let's get into the gory mathematical detail. Consider a linear combination of L_1 and L_2 using an arbitrary function ϑ:

$$\begin{aligned} L &= L_1 + \vartheta L_2 && (3) \\ &= (A_1 + \vartheta A_2)\frac{\partial u}{\partial x} + (B_1 + \vartheta B_2)\frac{\partial u}{\partial y} + \\ &\quad (C_1 + \vartheta C_2)\frac{\partial v}{\partial x} + (D_1 + \vartheta D_2)\frac{\partial v}{\partial y} + \\ &\quad (E_1 + \vartheta E_2) && (4) \end{aligned}$$

Let $y = y(x)$, a curve. Its tangent slope is dy/dx.

If $u = u(x, y)$ and $v = v(x, y)$ and u and v are valid solutions of L_1 and L_2 then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \tag{5}$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy. \tag{6}$$

Take the u terms in (4),

$$(A_1 + \vartheta A_2)\frac{\partial u}{\partial x} + (B_1 + \vartheta B_2)\frac{\partial u}{\partial y} : \tag{7}$$

this can be rewritten as

$$(A_1 + \vartheta A_2)\left(\frac{\partial u}{\partial x} + \frac{(B_1 + \vartheta B_2)}{(A_1 + \vartheta A_2)}\frac{\partial u}{\partial y}\right). \tag{8}$$

Now take the v terms in (4),

$$(C_1 + \vartheta C_2)\frac{\partial v}{\partial x} + (D_1 + \vartheta D_2)\frac{\partial v}{\partial y} : \tag{9}$$

this can be rewritten as

$$(C_1 + \vartheta C_2)\left(\frac{\partial v}{\partial x} + \frac{(D_1 + \vartheta D_2)}{(C_1 + \vartheta C_2)}\frac{\partial v}{\partial y}\right). \tag{10}$$

At this point both Fox and Lister do something that I just took on trust, as I'm too stupid to understand it. We choose our value of the arbitrary function ϑ so that

$$\frac{(B_1 + \vartheta B_2)}{(A_1 + \vartheta A_2)} = \frac{(D_1 + \vartheta D_2)}{(C_1 + \vartheta C_2)} = \frac{dy}{dx}. \tag{11}$$

We can now replace $\frac{(B_1+\vartheta B_2)}{(A_1+\vartheta A_2)}$ in (8) with $\frac{dy}{dx}$:

$$(A_1 + \vartheta A_2)\left(\frac{\partial u}{\partial x} + \frac{dy}{dx}\frac{\partial u}{\partial y}\right). \tag{12}$$

Likewise, we can replace $\frac{(D_1+\vartheta D_2)}{(C_1+\vartheta C_2)}$ in (10) with $\frac{dy}{dx}$:

$$(C_1 + \vartheta C_2)\left(\frac{\partial v}{\partial x} + \frac{dy}{dx}\frac{\partial v}{\partial y}\right). \tag{13}$$

Now $\frac{dy}{dx}\frac{\partial u}{\partial y} = 0$ and $\frac{dy}{dx}\frac{\partial v}{\partial y} = 0$, so those terms vanish. We get

$$(A_1 + \vartheta A_2)\frac{du}{dx} \text{ and} \tag{14}$$

$$(C_1 + \vartheta C_2)\frac{dv}{dx}. \tag{15}$$

Recall that these are equivalent to the u and v terms in (4). We can now replace (7) and (9) (which have partial derivatives) with two terms that only use total derivatives. They are the right-hand sides of

$$(A_1 + \vartheta A_2)\frac{\partial u}{\partial x} + (B_1 + \vartheta B_2)\frac{\partial u}{\partial y} = (A_1 + \vartheta A_2)\frac{du}{dx} \tag{16}$$

and

$$(C_1 + \vartheta C_2)\frac{\partial v}{\partial x} + (D_1 + \vartheta D_2)\frac{\partial v}{\partial y} = (C_1 + \vartheta C_2)\frac{dv}{dx}. \tag{17}$$

The careful selection of the value of ϑ allows us to hide the four terms B_1, B_2, D_1 and D_2 in (4) and eliminate the partial derivatives.

We substitute the right hand sides of (16) and (17) back into (4) to get

$$(A_1 + \vartheta A_2)\frac{du}{dx} + (C_1 + \vartheta C_2)\frac{dv}{dx} + E_1 + \vartheta E_2. \tag{18}$$

Both L_1 and L_2 equal zero, so $L_1 + \vartheta L_2$ must also equal zero:

$$(A_1 + \vartheta A_2)\frac{du}{dx} + (C_1 + \vartheta C_2)\frac{dv}{dx} + E_1 + \vartheta E_2 = 0. \tag{19}$$

Now we put (19) to one side for a while. We want to express ϑ in terms of the A_1, A_2, ... terms, because those are known functions of x, y, u and v. We rearrange one version of (11) to place ϑ on its own:

$$\begin{aligned}
\frac{(B_1 + \vartheta B_2)}{(A_1 + \vartheta A_2)} &= \frac{dy}{dx} \\
\Rightarrow (B_1 + \vartheta B_2)dx &= (A_1 + \vartheta A_2)dy && (20) \\
\Rightarrow B_1 dx + \vartheta B_2 dx &= A_1 dy + \vartheta A_2 dy && (21) \\
\Rightarrow \vartheta B_2 dx - \vartheta A_2 dy &= A_1 dy - B_1 dx && (22) \\
\Rightarrow \vartheta(B_2 dx - A_2 dy) &= A_1 dy - B_1 dx && (23) \\
\Rightarrow \vartheta &= \frac{A_1 dy - B_1 dx}{B_2 dx - A_2 dy} && (24)
\end{aligned}$$

We rearrange the other version of (11) to place ϑ on its own in the same way and end up with

$$\begin{aligned}\frac{(D_1+\vartheta D_2)}{(C_1+\vartheta C_2)} &= \frac{dy}{dx} \\ \Rightarrow \vartheta &= \frac{C_1dy - D_1dx}{D_2dx - C_2dy}. \end{aligned} \tag{25}$$

Both (24) and (25) define ϑ, so their right-hand sides are equal to each other:

$$\frac{A_1dy - B_1dx}{B_2dx - A_2dy} = \frac{C_1dy - D_1dx}{D_2dx - C_2dy}. \tag{26}$$

We can multiply this out and collect terms associated with dx^2, $dxdy$ and dy^2.

$$(A_1dy - B_1dx)(D_2dx - C_2dy) = (C_1dy - D_1dx)(B_2dx - A_2dy) \tag{27}$$

$$\begin{aligned}\Rightarrow \quad & A_1D_2dydx - A_1C_2dy^2 - B_1D_2dx^2 + B_1C_2dxdy = \\ & \qquad C_1B_2dydx - C_1A_2dy^2 - D_1B_2dx^2 + D_1A_2dxdy && (28)\\ \Rightarrow \quad & A_1D_2dxdy - A_1C_2dy^2 - B_1D_2dx^2 + B_1C_2dxdy - \\ & \qquad C_1B_2dxdy + C_1A_2dy^2 + D_1B_2dx^2 - D_1A_2dxdy = 0 && (29)\\ \Rightarrow \quad & (D_1B_2 - B_1D_2)dx^2 + \\ & (A_1D_2 + B_1C_2 - C_1B_2 - D_1A_2)dxdy + \\ & (C_1A_2 - A_1C_2)dy^2 = 0 && (30)\end{aligned}$$

Then we divide both sides by dx^2 to generate a quadratic in $\frac{dy}{dx}$.

$$\begin{aligned}& (D_1B_2 - B_1D_2)\frac{dx^2}{dx^2} + \\ & (A_1D_2 + B_1C_2 - C_1B_2 - D_1A_2)\frac{dxdy}{dx^2} + \\ & (C_1A_2 - A_1C_2)\frac{dy^2}{dx^2} = 0 \end{aligned} \tag{31}$$

In the process both Lister and Fox reversed the sign of the coefficients, presumably for a good reason (the sum is still zero). We'll do the same.

$$\begin{aligned}(B_1D_2 - D_1B_2) &\quad + \\ (C_1B_2 + D_1A_2 - A_1D_2 - B_1C_2)\frac{dy}{dx} &\quad + \\ (A_1C_2 - C_1A_2)\frac{dy^2}{dx^2} &\quad = \quad 0 \end{aligned} \tag{32}$$

We define the following collective terms for the coefficients:

$$\begin{aligned} p &= A_1C_2 - C_1A_2, && (33)\\ q &= C_1B_2 + D_1A_2 - A_1D_2 - B_1C_2, && (34)\\ r &= B_1D_2 - D_1B_2. && (35)\end{aligned}$$

We can now re-write (32) as

$$p\frac{dy^2}{dx^2} + q\frac{dy}{dx} + r = 0 \tag{36}$$

The solution of this quadratic equation has one of three types:

$$\begin{aligned} q^2 - 4pr &> 0\text{: two different real roots, the PDEs are hyperbolic,} && (37)\\ q^2 - 4pr &= 0\text{: one real root, the PDEs are parabolic,} && (38)\\ q^2 - 4pr &< 0\text{: two complex roots, the PDEs are elliptical.} && (39) \end{aligned}$$

In this case the PDEs are hyperbolic (don't ask me why, I don't understand the math that well!). The quadratic has two roots, which we will call κ_+ and κ_-. They are defined by the two forms of the quadratic formula:

$$\kappa_+ = \frac{-q + \sqrt{q^2 - 4pr}}{2p}, \tag{40}$$

$$\kappa_- = \frac{-q - \sqrt{q^2 - 4pr}}{2p}. \tag{41}$$

These two solutions represent values of dy/dx and are thus the slopes of lines in the $x - y$ plane (characteristic lines). At each point in the $x - y$ plane two characteristic lines pass through, one with slope κ_+ the other with slope κ_-. For convenience we will represent them both by one symbol, $\kappa_\pm$ so that we only have to do the following derivation once. We can go back to (24) and (25) and determine values of ϑ in terms of $\kappa_\pm$. We start with (24):

$$\vartheta = \frac{A_1 dy - B_1 dx}{B_2 dx - A_2 dy} \tag{24}$$

We divide both top and bottom by dx.

$$\begin{aligned} \vartheta &= \frac{(A_1 dy - B_1 dx)/dx}{(B_2 dx - A_2 dy)/dx} \\ \Rightarrow \vartheta &= \frac{A_1 \frac{dy}{dx} - B_1 \frac{dx}{dx}}{B_2 \frac{dx}{dx} - A_2 \frac{dy}{dx}}. \end{aligned} \tag{42}$$

We cancel $\frac{dx}{dx}$:

$$\vartheta = \frac{A_1 \frac{dy}{dx} - B_1}{B_2 - A_2 \frac{dy}{dx}}. \tag{43}$$

We replace $\frac{dy}{dx}$ with the slopes of our characteristic lines, $\kappa_\pm$.

$$\vartheta = \frac{A_1 \kappa_\pm - B_1}{B_2 - A_2 \kappa_\pm}. \tag{44}$$

The same process is then applied to (25):

$$\begin{aligned} \vartheta &= \frac{C_1 dy - D_1 dx}{D_2 dx - C_2 dy} \\ \Rightarrow \vartheta &= \frac{C_1 \kappa_\pm - D_1}{D_2 - C_2 \kappa_\pm}. \end{aligned} \tag{45}$$

Now we go back to (19) and substitute the right hand side of (44) for ϑ:

$$(A_1 + \vartheta A_2)\frac{du}{dx} + (C_1 + \vartheta C_2)\frac{dv}{dx} + E_1 + \vartheta E_2 = 0 \tag{46}$$

$$\begin{aligned} \Rightarrow (A_1 + \frac{A_1\kappa_\pm - B_1}{B_2 - A_2\kappa_\pm}A_2)\frac{du}{dx} &+ \\ (C_1 + \frac{A_1\kappa_\pm - B_1}{B_2 - A_2\kappa_\pm}C_2)\frac{dv}{dx} &+ \\ E_1 + \frac{A_1\kappa_\pm - B_1}{B_2 - A_2\kappa_\pm}E_2 &= 0 \end{aligned} \tag{47}$$

If we multiply both sides by dx we get

$$\begin{aligned} (A_1 + \frac{A_1\kappa_\pm - B_1}{B_2 - A_2\kappa_\pm}A_2)du &+ \\ (C_1 + \frac{A_1\kappa_\pm - B_1}{B_2 - A_2\kappa_\pm}C_2)dv &+ \\ (E_1 + \frac{A_1\kappa_\pm - B_1}{B_2 - A_2\kappa_\pm}E_2)dx &= 0. \end{aligned} \tag{48}$$

Next we multiply both sides by $(B_2 - A_2\kappa_\pm)$ and collect common terms of $\kappa_\pm$:

$$\begin{aligned} &[A_1(B_2 - A_2\kappa_\pm) + (A_1\kappa_\pm - B_1)A_2]\,du + \\ &[C_1(B_2 - A_2\kappa_\pm) + (A_1\kappa_\pm - B_1)C_2]\,dv + \\ &[E_1(B_2 - A_2\kappa_\pm) + (A_1\kappa_\pm - B_1)E_2]\,dx = 0 \end{aligned} \tag{49}$$

$$\begin{aligned} \Rightarrow\ &[A_1B_2 - A_1A_2\kappa_\pm + A_1\kappa_\pm A_2 - A_2B_1]\,du + \\ &[B_2C_1 - A_2C_1\kappa_\pm + A_1\kappa_\pm C_2 - B_1C_2]\,dv + \\ &[B_2E_1 - A_2E_1\kappa_\pm + A_1\kappa_\pm E_2 - B_1E_2]\,dx = 0 \end{aligned} \tag{50}$$

$$\begin{aligned} \Rightarrow\ &[A_1B_2 - A_2B_1]\,du + \\ &[B_2C_1 - A_2C_1\kappa_\pm + A_1\kappa_\pm C_2 - B_1C_2]\,dv + \\ &[B_2E_1 - A_2E_1\kappa_\pm + A_1\kappa_\pm E_2 - B_1E_2]\,dx = 0 \end{aligned} \tag{51}$$

$$\begin{aligned} \Rightarrow\ &[A_1B_2 - A_2B_1]\,du + \\ &[B_2C_1 - B_1C_2 - A_2C_1\kappa_\pm + A_1C_2\kappa_\pm]\,dv + \\ &[B_2E_1 - B_1E_2 - A_2E_1\kappa_\pm + A_1E_2\kappa_\pm]\,dx = 0 \end{aligned} \tag{52}$$

$$\begin{aligned} \Rightarrow\ &[A_1B_2 - A_2B_1]\,du + \\ &[B_2C_1 - B_1C_2 + (A_1C_2 - A_2C_1)\kappa_\pm]\,dv + \\ &[B_2E_1 - B_1E_2 + (A_1E_2 - A_2E_1)\kappa_\pm]\,dx = 0. \end{aligned} \tag{53}$$

Recall that A_1, A_2, $\ldots$ are known functions of x, y, u and v. We define five new terms of their combinations:

$$N = A_1B_2 - A_2B_1, \tag{54}$$
$$O = A_1C_2 - A_2C_1, \tag{55}$$
$$P = B_1C_2 - B_2C_1, \tag{56}$$
$$Q = A_1E_2 - A_2E_1, \tag{57}$$
$$R = B_1E_2 - B_2E_1. \tag{58}$$

We can rewrite (53) as

$$N du + (O\kappa_{\pm} - P)dv + (Q\kappa_{\pm} - R)dx = 0 \tag{59}$$

and know that it applies along the characteristic lines. We also have the characteristic slope equations $\kappa_{\pm} = \frac{dy}{dx}$.

We have four unknowns du, dv, κ_+ and κ_-. We have four equations

$$\begin{aligned} N du + (O\kappa_+ - P)dv + (Q\kappa_+ - R)dx &= 0, & (60)\\ dy - \kappa_+ dx &= 0, & (61)\\ N du + (O\kappa_- - P)dv + (Q\kappa_- - R)dx &= 0 \ \& & (62)\\ dy - \kappa_- dx &= 0 & (63) \end{aligned}$$

and can solve these by finite difference methods, as shown in Section 4 of this document.

2 Speed of sound in a perfect gas

The main source of this derivation is chapter 6 of *"Intermediate Fluid Mechanics"* by R H Nunn (Hemisphere Publishing, 1989).

First we look for a relationship between the conditions on either side of a weak pressure wave whose left side is moving at speed of sound c_1 in a rigid duct. The control volume around the wavefront is travelling at the speed of sound on the left side of the wave (c_1) and we assume it is short enough to allow us to discount changes in area. Figure 1 shows conditions on either side of the control volume as it travels through the duct.

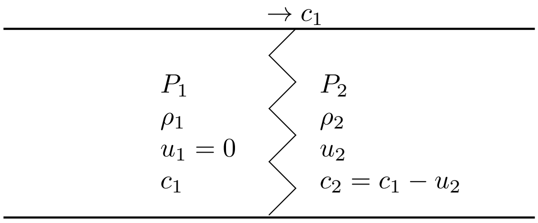

Figure 1: Weak pressure wave

On the left-hand side we have pressure P_1, density ρ_1, velocity zero (because the control volume is travelling at the same speed as the fluid on the left side of the wave) and speed of sound c_1. On the right-hand side we have pressure P_2, density ρ_2, velocity (relative to the wavefront) u_2 and speed of sound $c_1 - u_2$.

We take conservation of mass and rearrange it to make u_2 the subject.

$$\rho_1 c_1 = \rho_2(c_1 - u_2) \tag{64}$$
$$\Rightarrow \rho_1 c_1 = \rho_2 c_1 - \rho_2 u_2 \tag{65}$$
$$\Rightarrow \frac{\rho_1}{\rho_2} c_1 = c_1 - u_2 \tag{66}$$
$$\Rightarrow u_2 = c_1 - \frac{\rho_1}{\rho_2} c_1 \tag{67}$$
$$\Rightarrow u_2 = c_1 \left(1 - \frac{\rho_1}{\rho_2}\right) \tag{68}$$

Next we take the momentum equation across the pressure wave and rearrange it so that the pressure terms are on the left-hand side.

$$P_1 + \rho_1 c_1^2 = P_2 + \rho_2(c_1 - u_2)^2 \tag{69}$$
$$\Rightarrow P_1 - P_2 = \rho_2(c_1 - u_2)^2 - \rho_1 c_1^2 \tag{70}$$

Now we substitute the right-hand side of (68) for u_2 and collect common terms.

$$P_1 - P_2 = \rho_2 \left[c_1 - c_1(1 - \frac{\rho_1}{\rho_2})^2\right] - \rho_1 c_1^2 \tag{71}$$
$$\Rightarrow P_1 - P_2 = \rho_2 \left(c_1 - c_1 + c_1 \frac{\rho_1^2}{\rho_2^2}\right) - \rho_1 c_1^2 \tag{72}$$

$$\Rightarrow P_1 - P_2 = \rho_2 \left(c_1 \frac{\rho_1^2}{\rho_2^2} \right) - \rho_1 c_1^2 \tag{73}$$

$$\Rightarrow P_1 - P_2 = c_1^2 \left(\rho_2 \frac{\rho_1^2}{\rho_2^2} \right) - c_1^2 \rho_1 \tag{74}$$

$$\Rightarrow P_1 - P_2 = c_1^2 \left(\frac{\rho_1^2}{\rho_2} \right) - c_1^2 \rho_1 \tag{75}$$

$$\Rightarrow P_1 - P_2 = c_1^2 \rho_1 \left(\frac{\rho_1}{\rho_2} \right) - c_1^2 \rho_1 \tag{76}$$

$$\Rightarrow P_1 - P_2 = c_1^2 \rho_1 \left(\frac{\rho_1}{\rho_2} - 1 \right) \tag{77}$$

$$\Rightarrow P_1 - P_2 = c_1^2 \frac{\rho_1}{\rho_2} (\rho_1 - \rho_2) \tag{78}$$

$$\Rightarrow \frac{P_1 - P_2}{\rho_1 - \rho_2} = c_1^2 \frac{\rho_1}{\rho_2} \tag{79}$$

Finally, we make c_1^2 the subject of the equation.

$$c_1^2 = \frac{P_1 - P_2}{\rho_1 - \rho_2} \frac{\rho_2}{\rho_1} \tag{80}$$

Now we define $P_2 = P_1 + \delta P$ and $\rho_2 = \rho_1 + \delta\rho$. This lets us simplify it further.

$$c_1^2 = \frac{P_1 - (P_1 + \delta P)}{\rho_1 - (\rho_1 + \delta\rho)} \frac{\rho_1 + \delta\rho}{\rho_1} \tag{81}$$

$$\Rightarrow c_1^2 = \frac{P_1 - P_1 - \delta P}{\rho_1 - \rho_1 - \delta\rho} \frac{\rho_1 + \delta\rho}{\rho_1} \tag{82}$$

$$\Rightarrow c_1^2 = \frac{-\delta P}{-\delta\rho} \frac{\rho_1 + \delta\rho}{\rho_1} \tag{83}$$

$$\Rightarrow c_1^2 = \frac{\delta P}{\delta\rho} \frac{\rho_1 + \delta\rho}{\rho_1}. \tag{84}$$

Now when δP and $\delta\rho$ tend to small values, $\frac{\delta P}{\delta\rho} \to \frac{dP}{d\rho}$ and $\frac{\rho_1+\delta\rho}{\rho_1} \to \frac{\rho_1}{\rho_1}$. So

$$c_1^2 = \frac{dP}{d\rho} \frac{\rho_1}{\rho_1} \tag{85}$$

$$\Rightarrow c_1^2 = \frac{dP}{d\rho}(1) \tag{86}$$

$$\Rightarrow c_1^2 = \frac{dP}{d\rho}. \tag{87}$$

We will use this relationship to turn functions of P into functions of c and ρ.

3 Differential forms of the isentropic flow relationships

Most papers in the field express the isentropic flow relationships in differential form without giving a derivation. Even Shapiro's book *"The Dynamics and Thermodynamics of Compressible Fluid Flow"* (rightly regarded as the most comprehensive treatment of the subject) takes the isentropic flow relationship, the equation of state for a perfect gas and the expression for sound velocity and states that they can be rearranged into their differential forms. These are, respectively

$$\frac{P}{\rho^\gamma} = \text{constant}, \tag{88}$$
$$\frac{P}{\rho T} = \text{constant } \textit{and} \tag{89}$$
$$c^2 = \gamma R T. \tag{90}$$

The differential forms of the isentropic flow relationships (Shapiro vol. 2, eqn. 23.7, page 910) are

$$\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dP}{P} = \frac{1}{\gamma - 1}\frac{dT}{T} = \frac{2}{\gamma - 1}\frac{dc}{c}. \tag{91}$$

Those differential forms were not obvious to me.

This section is just a derivation of the four terms in (91) from (88)–(90). The main source I used for this was a NASA web page on isentropic fluid flow relationships.[1]

We start by stating a couple of additional thermodynamic relationships that we will need. The first relates the two specific heats of a gas (specific heat at constant pressure c_p, specific heat at constant volume c_v) to the gas constant R:

$$R = c_p - c_v. \tag{92}$$

R is a genuine constant for any mixture of gases. Don't ask me why; it's just how our universe works.
The second thermodynamic relationship is the definition of the ratio of specific heats, γ. It is

$$\gamma = \frac{c_p}{c_v}. \tag{93}$$

We treat c_p and c_v as constants too, even though it is not strictly true (treating them as constants is good enough for engineering work). So in the derivations that follows, each time we find c_p, c_v, R or γ inside a derivative term we can move them outside the derivatives and treat them as constants.

[1] https://www.grc.nasa.gov/WWW/BGH/isndrv.html (accessed 29 July 2020).

If we rearrange (92) and put (93) into it, we can get $\frac{c_p}{R}$ in terms of γ;

$$\frac{R}{c_p} = \frac{c_p - c_v}{c_p} \tag{94}$$

$$\Rightarrow \frac{R}{c_p} = \frac{c_p}{c_p} - \frac{c_v}{c_p} \tag{95}$$

$$\Rightarrow \frac{R}{c_p} = 1 - \frac{1}{\gamma}. \tag{96}$$

We take the reciprocal of $\frac{R}{c_p}$ and simplify it;

$$\frac{c_p}{R} = \frac{1}{1 - \frac{1}{\gamma}} \tag{97}$$

$$= \left(\frac{1}{1 - \frac{1}{\gamma}}\right)\left(\frac{\gamma}{\gamma}\right) \tag{98}$$

$$= \frac{\gamma}{(1 - \frac{1}{\gamma})\gamma} \tag{99}$$

$$\Rightarrow \frac{c_p}{R} = \frac{\gamma}{\gamma - 1}. \tag{100}$$

Next we take the isentropic flow relationship, the perfect gas equation, the expression for speed of sound and the relationship for entropy change ds. These are:

$$\frac{P}{\rho^\gamma} = \text{constant}, \tag{101}$$

$$P = \rho R T, \tag{102}$$

$$c^2 = \gamma R T, \tag{103}$$

$$ds = c_p \frac{dT}{T} - R\frac{dP}{P} \tag{104}$$

For an isentropic process $ds = 0$, so

$$c_p \frac{dT}{T} - R\frac{dP}{P} = 0 \tag{105}$$

$$\Rightarrow c_p \frac{dT}{T} = R\frac{dP}{P} \tag{106}$$

$$\Rightarrow \frac{c_p}{R}\frac{dT}{T} = \frac{dP}{P} \tag{107}$$

We can substitute (100) into (107) to yield one of the relationships in Shapiro's equation 23.7,

$$\frac{\gamma}{\gamma - 1}\frac{dT}{T} = \frac{dP}{P} \tag{108}$$

$$\Rightarrow \frac{1}{\gamma - 1}\frac{dT}{T} = \frac{1}{\gamma}\frac{dP}{P}. \tag{109}$$

Next we take the equation of state (102) and substitute it for P in the right-hand side of the entropy equation (106):

$$c_p \frac{dT}{T} = R\frac{dP}{\rho R T}. \tag{110}$$

This step lets us cancel out both R and T:

$$\Rightarrow c_p dT = \frac{dP}{\rho}. \tag{111}$$

Now we take the equation of state (102) again, make T the subject, differentiate it and multiply by c_p.

$$\begin{aligned} T &= \frac{P}{\rho R}, & (112)\\ \Rightarrow \partial T &= \partial\left(\frac{P}{\rho R}\right), & (113)\\ \Rightarrow c_p \partial T &= \frac{c_p}{R}\partial\left(\frac{P}{\rho}\right). & (114) \end{aligned}$$

The left-hand sides of (111) and (114) are the same, so we can equate their right-hand side terms and simplify:

$$\begin{aligned} \frac{c_p}{R}\partial\left(\frac{P}{\rho}\right) &= \frac{dP}{\rho}, & (115)\\ \Rightarrow \frac{c_p}{R}\left[\frac{dP}{\rho} + Pd\left(\frac{1}{\rho}\right)\right] &= \frac{dP}{\rho}, & (116)\\ \Rightarrow \frac{c_p}{R}\left[\frac{dP}{\rho} - \frac{P}{\rho^2}d\rho\right] &= \frac{dP}{\rho}. & (117) \end{aligned}$$

Collect common terms of dP on the left hand side and simplify:

$$\begin{aligned} \frac{c_p}{R}\frac{dP}{\rho} - \frac{dP}{\rho} &= \frac{c_p}{R}\frac{P}{\rho^2}d\rho & (118)\\ \Rightarrow \left(\frac{c_p}{R} - 1\right)\frac{dP}{\rho} &= \frac{c_p}{R}\frac{P}{\rho^2}d\rho & (119) \end{aligned}$$

Now multiply both sides by $\frac{\rho}{P}$ and simplify again:

$$\begin{aligned} \left(\frac{c_p}{R} - 1\right)\frac{dP}{\rho}\frac{\rho}{P} &= \frac{c_p}{R}\frac{P}{\rho^2}d\rho\frac{\rho}{P} & (120)\\ \Rightarrow \left(\frac{c_p}{R} - 1\right)\frac{dP}{P} &= \frac{c_p}{R}\frac{\rho}{\rho^2}d\rho, & (121)\\ \Rightarrow \left(\frac{c_p}{R} - 1\right)\frac{dP}{P} &= \frac{c_p}{R}\frac{d\rho}{\rho}. & (122) \end{aligned}$$

(100) gives $\frac{c_p}{R}$ in terms of γ. We substitute it into (122) and simplify:

$$\frac{c_p}{R} = \frac{\gamma}{\gamma - 1} \tag{100}$$

$$\left[\frac{\gamma}{\gamma - 1} - 1\right]\frac{dP}{P} = \frac{\gamma}{\gamma - 1}\frac{d\rho}{\rho}, \tag{123}$$

$$\Rightarrow (\gamma-1)\left[\frac{\gamma}{\gamma-1}-1\right]\frac{dP}{P} = (\gamma-1)\frac{\gamma}{\gamma-1}\frac{d\rho}{\rho}, \tag{124}$$

$$\Rightarrow (\gamma-1)\left[\frac{\gamma}{\gamma-1}-1\right]\frac{dP}{P} = \gamma\frac{d\rho}{\rho}, \tag{125}$$

$$\Rightarrow [\gamma-(\gamma-1)]\frac{dP}{P} = \gamma\frac{d\rho}{\rho}, \tag{126}$$

$$\Rightarrow [1]\frac{dP}{P} = \gamma\frac{d\rho}{\rho}, \tag{127}$$

$$\Rightarrow \frac{dP}{P} = \gamma\frac{d\rho}{\rho}, \tag{128}$$

$$\Rightarrow \frac{1}{\gamma}\frac{dP}{P} = \frac{d\rho}{\rho}. \tag{129}$$

The end result (129) is another identity in Shapiro's equation 23.7. It may be worth stating the three relationships together before we get to the fourth. The ones we have are:

$$\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dP}{P} = \frac{1}{\gamma-1}\frac{dT}{T}. \tag{130}$$

To get the last, we rearrange the speed of sound relationship (103) to make T the subject and differentiate it;

$$T = \frac{c^2}{\gamma R}, \tag{131}$$

$$\partial T = \partial\left(\frac{c^2}{\gamma R}\right) \tag{132}$$

$$\Rightarrow dT = \frac{1}{\gamma R}d(c^2), \tag{133}$$

$$\Rightarrow \frac{dT}{T} = \frac{1}{\gamma RT}d(c^2). \tag{134}$$

Now $\gamma RT = c^2$, from (103). We also multiply by $\frac{1}{\gamma-1}$ to give a result that is compatible with (130),

$$\frac{dT}{T} = \frac{1}{c^2}d(c^2), \tag{135}$$

$$\Rightarrow \frac{1}{\gamma-1}\frac{dT}{T} = \frac{1}{\gamma-1}\frac{d(c^2)}{c^2}. \tag{136}$$

Finally—and I only write this out in full because I know I'll forget it otherwise—we take a new variable to represent c^2, say $X = c^2$ and use it to express $d(c^2)$ in terms of dc.

$$X = c^2, \tag{137}$$

$$\Rightarrow dX = d(c^2), \tag{138}$$

$$\Rightarrow \frac{dX}{dc} = \frac{d}{dc}(c^2), \tag{139}$$

$$\Rightarrow \frac{dX}{dc} = 2c, \tag{140}$$

$$\Rightarrow dX = 2cdc, \tag{141}$$
$$\Rightarrow d(c^2) = 2cdc. \tag{142}$$

If we substitute (142) into (136) we can cancel $\frac{c}{c}$ and get the last of the terms in Shapiro's equation 23.7,

$$\frac{1}{\gamma - 1}\frac{dT}{T} = \frac{2c}{\gamma - 1}\frac{dc}{c^2}, \tag{143}$$
$$\Rightarrow \frac{1}{\gamma - 1}\frac{dT}{T} = \frac{2}{\gamma - 1}\frac{dc}{c}. \tag{144}$$

Putting (130) and (144) together, the relationships for isentropic flow in differential form in Shapiro's equation 23.7 are

$$\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dP}{P} = \frac{1}{\gamma - 1}\frac{dT}{T} = \frac{2}{\gamma - 1}\frac{dc}{c}. \tag{145}$$

4 1D compressible flow equations

We consider unsteady, one-dimensional, compressible, isentropic fluid flow in a rigid pipe of constant area with changes of height (e.g. air in a tunnel).

The main sources for this section are

- Fox, J A, *"Hydraulic analysis of unsteady flow in pipe networks"*, Macmillan Press, 1977
- Kestin, J and Glass, J S, *"Application of the Method of Characteristics to the Transient Flow of Gases*, Proc. Instn. Mech. Engnrs vol. 161, 1949

Fox's book was written for engineers rather than mathematicians, so it was much easier for me to follow than many of the other sources I came across.

The 1949 IMechE paper by Kestin & Glass is a gem. They take time to explain some things in detail, no doubt because it was written in the early years of the field. For example, they explain why the best pair of parameters for isentropic gas flow are air velocity u and speed of sound c. More recent papers just take that as a given without explaining why.

We take the equation of conservation of mass

$$\frac{\partial u}{\partial x} + \frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{1}{\rho}\frac{\partial \rho}{\partial t} - = 0 \tag{146}$$

and the equation of conservation of momentum

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial x} + E = 0. \tag{147}$$

We have two independent variables: x, distance along the tunnel and t, time.

We have three dependent variables: P, static air pressure; ρ, air density; and u, air velocity.

And we have E, which represents body forces on the fluid, such as wall friction, jet fan thrust or traffic drag. We ignore temperature changes, gradual changes in area and airflow through porous walls.

We are interested in isentropic flow. In Sections 2 and 3 we showed that when we have isentropic flow,

$$c^2 = \frac{dP}{d\rho} \text{ and} \tag{87}$$

$$\frac{d\rho}{\rho} = \frac{2}{\gamma - 1}\frac{dc}{c}. \tag{145}$$

We can use these relationships to make P and ρ dependent variables of c.

Then we can use the method of characteristics to make a set of equations that can be solved simultaneously for the two dependent variables u and c, use u and c in the mass equation to get ρ, then use those three to determine P.

First we take (87) and rearrange it to make ∂P the subject:

$$\partial P = c^2 \partial \rho. \tag{148}$$

This allows us to replace $\frac{1}{\rho}\frac{\partial P}{\partial x}$ with $\frac{c^2}{\rho}\frac{\partial \rho}{\partial x}$ in the momentum equation (147), giving

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} + E = 0. \tag{149}$$

Next we substitute the ρ terms for c terms. From (145) it follows that

$$\frac{1}{\rho}\frac{\partial \rho}{\partial x} = \frac{2}{(\gamma - 1)c}\frac{\partial c}{\partial x} \text{ and} \tag{150}$$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = \frac{2}{(\gamma - 1)c}\frac{\partial c}{\partial t}. \tag{151}$$

(150) lets us substitute the ρ terms with c term in the modified equation of momentum (149). We get

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \frac{2c^2}{(\gamma - 1)c}\frac{\partial c}{\partial x} + E = 0 \tag{152}$$

$$\Rightarrow u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \frac{2c}{\gamma - 1}\frac{\partial c}{\partial x} + E = 0. \tag{153}$$

Next we substitute ∂c terms (150 and 151) for the two $\partial \rho$ terms in the equation of mass (146);

$$\frac{\partial u}{\partial x} + \frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{1}{\rho}\frac{\partial \rho}{\partial t} = 0 \tag{146}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{2u}{(\gamma - 1)c}\frac{\partial c}{\partial x} + \frac{2}{(\gamma - 1)c}\frac{\partial c}{\partial t} = 0 \tag{154}$$

$$\Rightarrow c\frac{\partial u}{\partial x} + \frac{2u}{\gamma - 1}\frac{\partial c}{\partial x} + \frac{2}{\gamma - 1}\frac{\partial c}{\partial t} = 0. \tag{155}$$

(153) and (155) are the equations of momentum and mass with two independent variables (x as distance along the tunnel and t as time). Applying the constraint that the flow is isentropic allows us to use two dependent variables (celerity c and velocity u) instead of three. Pressure P and density ρ are not present.

We treat the ratio of specific heats γ as a constant.

To make the momentum and mass equations easier to typeset we will define a constant ψ:

$$\psi = \frac{2}{\gamma - 1}. \tag{156}$$

Substituting ψ into the momentum (153) and mass (155) equations gives us

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \psi c\frac{\partial c}{\partial x} + E = 0 \quad \text{(momentum) and} \tag{157}$$
$$c\frac{\partial u}{\partial x} + \psi u\frac{\partial c}{\partial x} + \psi\frac{\partial c}{\partial t} = 0 \quad \text{(mass).} \tag{158}$$

Next we put (157) and (158) in the same form as (1) and (2). It seems to makes things easier if we treat time t as the independent variable x in (1) and distance x as the other independent variable y in (2). That might seem counterintuitive: just let it go. Expressed in the form of the two partial differential equations at the start of this document [(1) and (2)] we get

$$1\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + 0\frac{\partial c}{\partial t} + \psi c\frac{\partial c}{\partial x} + E = 0 \text{ (momentum)} \tag{159}$$
$$0\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \psi\frac{\partial c}{\partial t} + \psi u\frac{\partial c}{\partial x} + 0 = 0 \text{ (mass).} \tag{160}$$

The ten coefficient terms in (1) and (2) that relate to these are

$$A_1 = 1 \tag{161}$$
$$B_1 = u \tag{162}$$
$$C_1 = 0 \tag{163}$$
$$D_1 = \psi c \tag{164}$$
$$E_1 = -E \tag{165}$$
$$A_2 = 0 \tag{166}$$
$$B_2 = c \tag{167}$$
$$C_2 = \psi \tag{168}$$
$$D_2 = \psi u \tag{169}$$
$$E_2 = 0 \tag{170}$$

It may be worth explaining why we set $E_1 = -E$. E_1 is a body force term which could be positive (impelling airflow) or negative (retarding airflow). The simplest form of E_1 is wall friction, which I usually think of as $\lambda L/D_h \times 1/2\rho v^2$ and which we give a negative sign to so that it drains energy from the flow. After a bit of reflection I decided to follow the same convention as most other users rather than buck the trend.

We can use these to find the slopes of the characteristic lines, which are the roots calculated from p, q and r in the quadratic (36).

$$\begin{aligned} p &= A_1C_2 - C_1A_2 \\ &= 1 \times \psi - 0 \times 0 = \psi, \end{aligned} \tag{171}$$
$$\begin{aligned} q &= C_1B_2 + D_1A_2 - A_1D_2 - B_1C_2 \\ &= 0 \times c + \psi c \times 0 - 1 \times \psi u - u \times \psi = -2\psi u, \end{aligned} \tag{172}$$
$$\begin{aligned} r &= B_1D_2 - D_1B_2 \\ &= u \times \psi u - \psi c \times c = \psi(u^2 - c^2). \end{aligned} \tag{173}$$

So $p = \psi$; $q = -2\psi u$; and $r = \psi(u^2 - c^2)$.

The two roots of the quadratic ($\kappa_\pm$ in Section 1) are

$$\begin{aligned}
\kappa_\pm &= \frac{-q \pm \sqrt{q^2 - 4pr}}{2p} & (174)\\
&= \frac{+2\psi u \pm \sqrt{(-2\psi u)^2 - 4 \times \psi \times \psi(u^2 - c^2)}}{2 \times \psi} & (175)\\
&= \frac{2\psi u \pm \sqrt{4\psi^2 u^2 - 4\psi^2(u^2 - c^2)}}{2\psi} & (176)\\
&= u \pm \sqrt{u^2 - (u^2 - c^2)} & (177)\\
&= u \pm \sqrt{u^2 - u^2 + c^2} & (178)\\
&= u \pm \sqrt{c^2} & (179)\\
\kappa_\pm &= u \pm c. & (180)
\end{aligned}$$

So

$$\begin{aligned}
\kappa_+ &= u + c \text{ and} & (181)\\
\kappa_- &= u - c. & (182)
\end{aligned}$$

This is handy, because we calculate c and u at every timestep.

Next we use the ten coefficient terms to get the values of N–R in the simplified linear combination equation (59):

$$\begin{aligned}
N &= A_1B_2 - A_2B_1\\
&= 1 \times c - 0 \times u = c, & (183)\\
O &= A_1C_2 - A_2C_1\\
&= 1 \times \psi - 0 \times 0 = \psi, & (184)\\
P &= B_1C_2 - B_2C_1\\
&= u \times \psi - u \times 0 = \psi u, & (185)\\
Q &= A_1E_2 - A_2E_1\\
&= 1 \times 0 - 0 \times (-E) = 0, & (186)\\
R &= B_1E_2 - B_2E_1\\
&= u \times 0 - c \times (-E) = cE. & (187)
\end{aligned}$$

If we substitute these into (59) we get

$$cdu + [\psi\kappa_\pm - \psi u]\, dc + cEdt = 0 \qquad (188)$$

Now we can substitute κ_+ and κ_- into this and simplify them to get the equations that hold along the characteristic lines. First we do κ_+ ($\kappa_+ = u + c$):

$$\begin{aligned}
cdu + [\psi(u + c) - \psi u]dc + Ecdt &= 0 & (189)\\
\Rightarrow cdu + (\psi u + \psi c - \psi u)dc + Ecdt &= 0 & (190)\\
\Rightarrow cdu + \psi cdc + Ecdt &= 0 & (191)\\
\Rightarrow du + \psi dc + Edt &= 0. & (192)
\end{aligned}$$

Next we do κ_- ($\kappa_- = u - c$) in the same manner:

$$cdu + [\psi(u - c) - \psi u]dc + Ecdt = 0 \tag{193}$$
$$\Rightarrow cdu + (\psi u - \psi c - \psi u)dc + Ecdt = 0 \tag{194}$$
$$\Rightarrow cdu - \psi cdc + Ecdt = 0 \tag{195}$$
$$\Rightarrow du - \psi dc + Edt = 0. \tag{196}$$

We now have two equations that can be solved simultaneously to determine 1D compressible homentropic flow in tunnels. They are

$$du + \psi dc + Edt = 0 \tag{197}$$

which holds along a characteristic line of slope $u + c$ and

$$du - \psi dc + Edt = 0 \tag{198}$$

which holds along a characteristic line of slope $u - c$.

Most researchers in the field express (197) and (198) in a form where the dc term is always positive, so we'll rearrange them to follow that convention. We get

$$\psi dc + du + Edt = 0 \text{ and} \tag{199}$$
$$\psi dc - du - Edt = 0. \tag{200}$$

To get into the computational domain we have to put (199) and (200) into differential form and start wrangling with them in the distance–time domain. Take two cells of length Δx with three gridpoints (L, M and R) at which values of c and u are known at time t (Figure 2).

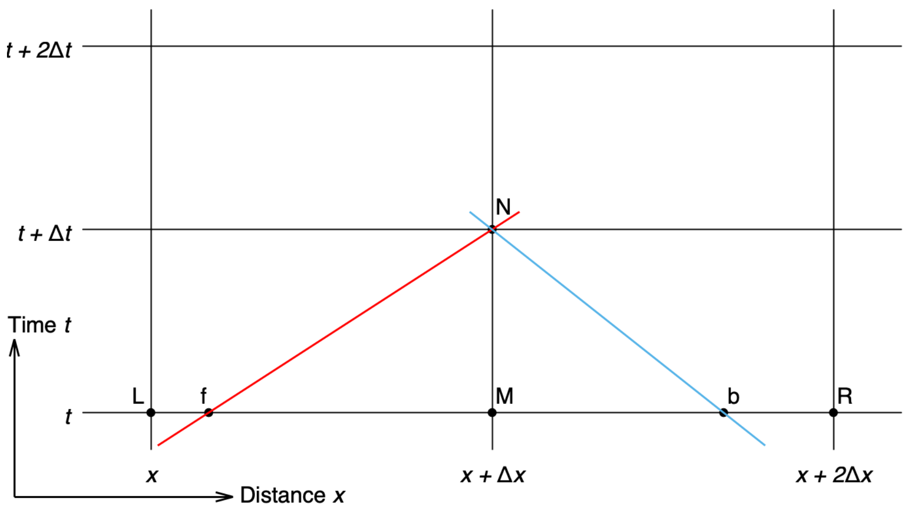

Figure 2: Gridpoints and characteristics

The cells span three equally-spaced grid points at which we know the values of u and c at time t (u and c at points L, M and R in Figure 2). We know how far apart they are (Δx) because we decide where the gridpoints go. We want to figure out the conditions at N (the same x location as gridpoint M, but in the next timestep). The characteristics for the middle gridpoint meet in the next timestep at N and intersect the values on the current timestep at f (on the forward characteristic with slope $u_N + c_N$) and at b (on the backward characteristic with slope $u_N - c_N$).

We have to choose a timestep Δt. We set Δt such that the forward and backward characteristics intersect the plane of known values near the outer grid points L and R (to minimise interpolation errors), but are not so close that there is a danger that the value of $u + c$ puts f to the left of L (which may cause the calculation to blow up).

We also add subscripts to the body force terms E along the backward characteristic and the forward characteristic to distinguish between them (E_b, E_f).

In differential form (199) and (200) are

$$\begin{aligned} \psi(c_N - c_f) + (u_N - u_f) + E_f dt &= 0 \text{ and} && (201) \\ \psi(c_N - c_b) - (u_N - u_b) - E_b dt &= 0. && (202) \end{aligned}$$

We can get the values of u_f and c_f by linear interpolation between the values at L and M using the slope of the forward characteristic $u_N + c_N$. We do not yet know the values of u_N and c_N; but if our cell sizes and timesteps are small, the characteristic lines should be short enough that u_M and c_M (which we know) are acceptable first approximations to u_N and c_N. So

$$\begin{aligned} x_{fM} &= \frac{\Delta x}{u_N + c_N}, && (203) \\ &\approx \frac{\Delta x}{u_M + c_M}, && (204) \\ u_f &= u_M + \frac{x_{fM}}{\Delta x}(u_L - u_M), && (205) \\ c_f &= c_M + \frac{x_{fM}}{\Delta x}(c_L - c_M). && (206) \end{aligned}$$

We get the values of u_b and c_b in a similar way, but using a negative sign instead of a positive sign when calculating x_{bM}:

$$\begin{aligned} x_{bM} &= \frac{-\Delta x}{u_N - c_N}, && (207) \\ &\approx \frac{-\Delta x}{u_M - c_M}, && (208) \\ u_b &= u_M + \frac{x_{bM}}{\Delta x}(u_R - u_M), && (209) \\ c_b &= c_M + \frac{x_{bM}}{\Delta x}(c_R - c_M). && (210) \end{aligned}$$

Now that we have estimates for u_b, c_b, u_f and c_f the only unknowns in the two differential forms (201) and (202) are c_N and u_N. Both equations hold at point N. We can solve them simultaneously to eliminate c_N and determine u_N:

$$\psi(c_N - c_f) + (u_N - u_f) + E_f dt = \psi(c_N - c_b) - (u_N - u_b) - E_b dt \tag{211}$$

$$\begin{aligned}
(u_N - u_f) + (u_N - u_b) &= \psi(c_N - c_b) - \psi(c_N - c_f) - (E_f + E_b)dt & (212)\\
\Rightarrow\ 2u_N - u_f - u_b &= \psi(c_f - c_b) - (E_f + E_b)dt & (213)\\
\Rightarrow\ 2u_N &= u_f + u_b + \psi(c_f - c_b) - (E_f + E_b)dt & (214)\\
\Rightarrow\ u_N &= \frac{1}{2}\left[u_f + u_b + \psi(c_f - c_b) - (E_f + E_b)dt\right] & (215)
\end{aligned}$$

Now that we have u_N we can back-substitute u_N into (201) and (202) to get c_N,

$$\begin{aligned}
c_N &= c_f + \frac{u_f - u_N - E_f dt}{\psi} & (216)\\
c_N &= c_b + \frac{u_N - u_b + E_b dt}{\psi}. & (217)
\end{aligned}$$

Doing both is a useful sanity check: if they don't give the same answer then either something is wrong in the code or the calculation has blown up. Hobyah has a set of error messages dedicated to giving useful information when calculations blow up.

Having evaluated u_N and c_N we can re-evaluate x_{fM} and x_{bM} with them and go round again as many times as is necessary (Fox states that few or no iterations are typically needed and I found that to be the case most of the time too).

After calculating u and c at every gridpoint at every timestep, we use the differential form of (145) and the knowledge that any change in the conditions is an isentropic change to determine ρ and P at the timesteps we want to print at. If our tunnel is at sea level, then the base atmospheric conditions are $P_0 = 101325\ Pa$ and $\rho_0 = 1.225\ kg/m^3$ (the International Standard Atmosphere at sea level). First we calculate the celerity of that air,

$$\begin{aligned}
c_0 &= \sqrt{\frac{\gamma P_0}{\rho_0}} & (218)\\
&= \sqrt{\frac{1.4 \times 101325}{1.225}} & (219)\\
&= 340.294 \text{ m/s.}
\end{aligned}$$

Next we cast (145) in differential form, use ρ_0 and c_0 as our reference conditions and solve for ρ_N;

$$\begin{aligned}
\frac{d\rho}{\rho} &= \frac{2}{\gamma - 1}\frac{dc}{c} & (145)\\
\Rightarrow \frac{\Delta\rho}{\rho_0} &= \frac{2}{\gamma - 1}\frac{\Delta c}{c_0} & (220)\\
\Rightarrow \frac{\rho_N - \rho_0}{\rho_0} &= \frac{2}{\gamma - 1}\frac{c_N - c_0}{c_0} & (221)
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \frac{\rho_N - \rho_0}{\rho_0} &= \frac{2}{\gamma - 1}\frac{c_N - c_0}{c_0} && (222)\\
\Rightarrow \rho_N &= \rho_0 + \rho_0 \frac{2}{\gamma - 1}\frac{c_N - c_0}{c_0} && (223)\\
\Rightarrow \rho_N &= \rho_0 \left(1 + \frac{2}{\gamma - 1}\frac{c_N - c_0}{c_0}\right). && (224)
\end{aligned}
$$

Now that we have both c_N and ρ_N we can determine P_N by an equivalent of (218),

$$
\begin{aligned}
c_N &= \sqrt{\frac{\gamma P_N}{\rho_N}} && (225)\\
\Rightarrow P_N &= \rho_N \frac{c_N^2}{\gamma} && (226)
\end{aligned}
$$

5 Mass continuity at boundaries

Mass continuity needs to be satisfied across boundaries. For the purpose of this discussion a "boundary" is any point at which we have two gridpoints. It might be a change of area in a tunnel, it might be tee junction, it might be a manifold with six air paths attached, it might be at the end of a train.

Wherever we have n gridpoints at the same location we need to satisfy

$$\sum_{i=1}^{i=n} (\rho_i A_i u_i) = 0. \tag{227}$$

The characteristic calculations work in terms of u and c. It is inconvenient to have to figure out ρ at each gridpoint at each timestep. If we can work out how to relate c to ρ we can reduce the time needed to calculate at each timestep.

In isentropic flow there is a simple relationship between ρ and c that can be used to eliminate ρ in (227). It is derived from

$$c^2 = \frac{\gamma P}{\rho} \text{ and} \tag{228}$$

$$\frac{P}{\rho^\gamma} = \text{constant}. \tag{229}$$

These can be substituted into (227) to express mass continuity in terms of c and u instead of ρ and u.

Take fluid at two different conditions in isentropic flow. One has pressure P_1, density ρ_1 and speed of sound c_1. The other has pressure P_2, density ρ_2 and speed of sound c_2.

There are two expressions for the ratio P_1/P_2.
From (228) we have

$$\frac{P_1}{P_2} = \frac{c_1^2 \rho_1/\gamma}{c_2^2 \rho_2/\gamma} \tag{230}$$

$$\Rightarrow \frac{P_1}{P_2} = \frac{c_1^2}{c_2^2}\frac{\rho_1}{\rho_2}. \tag{231}$$

From (229) we have

$$\frac{P_1}{P_2} = \frac{\text{constant } \rho_1^\gamma}{\text{constant } \rho_2^\gamma} \tag{232}$$

$$\Rightarrow \frac{P_1}{P_2} = \frac{\rho_1^\gamma}{\rho_2^\gamma} \tag{233}$$

The left hand sides of (231) and (233) are identical so their right-hand sides must be equal:

$$\frac{c_1^2}{c_2^2}\frac{\rho_1}{\rho_2} = \frac{\rho_1^\gamma}{\rho_2^\gamma}. \tag{234}$$

We can rearrange the terms to give us a ratio of densities:

$$\frac{c_1^2}{c_2^2} = \frac{\rho_1^\gamma \rho_2}{\rho_2^\gamma \rho_1} \quad (235)$$

$$\Rightarrow \frac{c_1^2}{c_2^2} = \frac{\rho_1^\gamma \rho_1^{-1}}{\rho_2^\gamma \rho_2^{-1}} \quad (236)$$

$$\Rightarrow \frac{c_1^2}{c_2^2} = \frac{\rho_1^{\gamma-1}}{\rho_2^{\gamma-1}} \quad (237)$$

$$\Rightarrow \frac{c_1}{c_2} = \sqrt{\frac{\rho_1^{\gamma-1}}{\rho_2^{\gamma-1}}} \quad (238)$$

$$\Rightarrow \frac{c_1}{c_2} = \frac{\rho_1^{\frac{\gamma-1}{2}}}{\rho_2^{\frac{\gamma-1}{2}}} \quad (239)$$

$$\Rightarrow \frac{c_1}{c_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\frac{\gamma-1}{2}} \quad (240)$$

$$\Rightarrow \frac{\rho_1}{\rho_2} = \left(\frac{c_1}{c_2}\right)^{\frac{2}{\gamma-1}} \quad (241)$$

$$\Rightarrow \frac{\rho_1}{\rho_2} = \left(\frac{c_1}{c_2}\right)^{\psi}. \quad (242)$$

So the mass continuity equation at boundaries and junctions (227) can be expressed in terms of c^ψ instead of ρ. We have

$$\sum(\rho A u) = 0 \quad (227)$$

$$\Rightarrow \sum(c^\psi A u) = 0. \quad (243)$$

It is worth taking this a bit further. Many papers in the field use powers of 5, powers of 7, powers of 1/5 and powers of 1/7 without ever explaining where those numbers came from. This is perfectly acceptable when your readers already know a lot about compressible flow, no help to readers who don't (like me before I wrote this document).

My document is a plodding derivation of the method of characteristics. So a bit of bloviating on my part might be useful to others interested in it. So here is a derivation that explains why 5, 7, 1/5 and 1/7 keep turning up in the engineering literature about compressible airflow.

First express the isentropic pressure–density relationship (233) in a form that is compatible with the density–celerity relationship (242) and get relationships between all three properties:

$$\frac{P_1}{P_2} = \frac{\rho_1^\gamma}{\rho_2^\gamma} \quad (233)$$

$$\Rightarrow \left(\frac{P_1}{P_2}\right)^{1/\gamma} = \frac{\rho_1}{\rho_2} \quad (244)$$

$$\Rightarrow \frac{\rho_1}{\rho_2} = \left(\frac{c_1}{c_2}\right)^{\psi} = \left(\frac{P_1}{P_2}\right)^{1/\gamma}, \tag{245}$$

$$\Rightarrow \frac{c_1}{c_2} = \left(\frac{\rho_1}{\rho_2}\right)^{1/\psi} = \left(\frac{P_1}{P_2}\right)^{1/\gamma\psi} \quad \text{and} \tag{246}$$

$$\Rightarrow \frac{P_1}{P_2} = \frac{\rho_1^{\gamma}}{\rho_2^{\gamma}} = \left(\frac{c_1}{c_2}\right)^{\psi\gamma} \tag{247}$$

γ for air varies with temperature, but not by much: it changes from 1.401 to 1.398 over the range −25 °C to +75 °C. In most papers about tunnel ventilation, γ is taken to be a constant (1.4). In some of those papers the exponents above are shown as numbers, all based on γ being equal to 1.4:

$$\psi = \frac{2}{\gamma - 1} = \frac{2}{1.4 - 1} = 5, \tag{248}$$

$$\psi\gamma = \frac{2\gamma}{\gamma - 1} = \frac{2 \times 1.4}{1.4 - 1} = 7, \tag{249}$$

$$1/\psi = 1/5 \text{ and} \tag{250}$$

$$1/\psi\gamma = 1/7. \tag{251}$$

6 Body forces, prime movers and boundary conditions

The main sources for this section are

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- Kestin, J and Glass, J S, *"Application of the Method of Characteristics to the Transient Flow of Gases"*, Proc. Instn. Mech. Engnrs vol. 161, 1949
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- N Kato, N, Ito, S, Mizuno, A, Chihara, T and Hashimoto, S, *"Boosting pressure generated by jet-fans when operated against the longitudinal airflow in road tunnels*, Proc. 18th ISAVFT, BHR Group 2019

6.1 Introduction

This section was quite difficult to write. I'm a mechanical engineer who has designed a lot of tunnel ventilation systems and written a lot of incompressible flow spreadsheets and programs. I have no training in compressible flow and no intuitive understanding of it. The closest we got to it in my degree course was simple waterhammer calculations (not even characteristics).

But I have been using and writing tunnel ventilation calculation programs and spreadsheets for about three decades. In this section I take a lot of the guidance I use when I have to design a tunnel (PIARC publications, EN 14067, BHR ISAVVT conference papers) and try to hammer the square pegs I have in my (incompressible flow) toolbox into the round holes of the compressible flow calculations developed in earlier sections of this document.

6.2 Friction

Friction is a body force that reduces pressure in the direction of air flow. In traditional (incompressible) fluid mechanics it is

$$\Delta P = \frac{1}{2}\rho\frac{\lambda L}{D_h}v|v|. \tag{252}$$

In differential form in the characteristic equations, friction appears as

$$E = \frac{1}{2}\frac{\lambda}{D_h}v|v|. \tag{253}$$

This transformation is based on the treatment of friction in equation 8 of Fox & Henson 1971. E has dimensions m^2/s^2 per metre. E is multiplied by the timestep dt when used in equations (201) and (202).

I could probably waste a good deal of sabbatical time deriving (252) from (253). Instead, I'm going to take the rule that switch between them as a given and move on. The rule is: you remove the density term ρ and the length term L and multiply the result by the timestep dt.

If I take those rules as a given for friction, then the rules can be used for other body forces such as road traffic drag and jet fans.

One final note: I find it easier to work in terms of Fanning friction factor c_f when considering ducts in which parts of the perimeter travel at different speeds (i.e. the annulus around moving trains in tunnels). The actual expression used in the code is

$$E = \frac{1}{2}\frac{c_f S}{A} v|v|. \tag{254}$$

6.3 Road traffic drag

The body force of road traffic drag in tunnel ventilation design programs is usually stated as

$$\Delta P = -\frac{1}{2}\frac{\rho L_t}{A_t}\Sigma_{i=1}^{N}\left[D_{v_i} c_{d_i} A_{v_i} (v_{v_i} - v_t)|v_{v_i} - v_t|\right] \tag{255}$$

where

$$\begin{aligned}
\Delta P &= \text{Pressure difference } (Pa) \\
\rho &= \text{Air density } (kg/m^3) \\
D_{v_i} &= \text{Density of vehicles of type } i \text{ in the tunnel } (veh/m) \\
L_t &= \text{Tunnel length } (m) \\
c_{d_i} &= \text{Drag coefficient of vehicles of type } i\ (-) \\
A_{v_i} &= \text{Cross-sectional area of vehicles of type } i\ (m^2) \\
A_t &= \text{Tunnel area } (m^2) \\
v_{v_i} &= \text{Speed of vehicles of type } i\ (m/s) \\
v_t &= \text{Tunnel air velocity } (m/s) \\
N &= \text{Count of types of vehicle being modelled.}
\end{aligned}$$

Putting some of the terms inside a bounded sum term ($\Sigma_{i=1}^{N}$ term) is a nod to the fact that we have to consider vehicles of different areas, different drag factors, different vehicle densities and different speeds in the same tunnel. Typical engineering practice today is to have 3–6 different vehicle types in design calculations, some of which (heavy goods vehicles, a.k.a. HGVs) move slower than lighter vehicles on steep gradients. The absolute value term $|v_v - v_t|$ is there to ensure that traffic drag acts to slow or accelerate the air depending on whether traffic is slower than the air in the tunnel or faster than it.

In the context of looking at body forces acting between two gridpoints, tunnel length L_t is equivalent to Δx in section 4 of this document. If we apply the rule derived from the tunnel friction relationship we can remove ρL_t to get

$$E = -\frac{1}{2A_t}\Sigma_{i=1}^{N}\left[D_{v_i} c_{d_i} A_{v_i} (v_{v_i} - v_t)|v_{v_i} - v_t|\right] \tag{256}$$

which can be added to E_f and E_b in (201) and (202) as just another body force like friction.

We have to do some low-level arithmetic on vehicle densities D_v where a queue of stopped traffic ends partway between two gridpoints, but that's just arithmetic in the code.

6.4 Change of elevation

Increases in elevation appear as a decrease in static air pressure and a decrease in density (both represented by changes in c and v in the characteristic equations). The easiest way to think of it is to start from stationary flow.

$$\Delta P = -\rho\, g\, dz \tag{257}$$

$$\Rightarrow = -\rho\, g\, \mathcal{I}\, L_t \tag{258}$$

where

$$\begin{aligned} \Delta P &= \text{Pressure difference due to a change in height } (Pa) \\ \rho &= \text{Air density } (kg/m^3) \\ g &= \text{Acceleration of gravity } (m/s^2) \\ dz &= \text{Change in height over distance } L_t\ (m) \\ L_t &= \text{Tunnel length } (m) \\ \mathcal{I} &= \text{Tunnel gradient (fraction, 0 to 1)} \end{aligned}$$

We use a negative sign here because an increase in elevation causes a decrease in air pressure.

We use the now-familiar party trick of removing ρL_t and adding dt:

$$E = -g\, \mathcal{I}\, dt \tag{259}$$

This turns the pressure change due to elevation into a form we can use as part of the E terms in the characteristic equations, just like friction and traffic drag.

Including pressure change due to height complicates matters. Most tunnel ventilation engineers prefer to think about airflow in terms in which in which a zero change in pressure between two points leads to no airflow between those two points. This is not the case when the two points are different elevations.

This leads to programmers of tunnel ventilation software to have two sets of account books when it comes to pressures:

- one set to show the engineers (in which the effects of $\rho\, g\, dz$ are removed so that we all keep our sanity) and
- another set to use for internal calculations that include the effects of $\rho\, g\, dz$.

Things can get complicated if you don't carefully distinguish between the two when plotting pressures.

6.5 Jet fans

Jet fans are axial fans hanging from a tunnel roof. They provide a high flowrate at low static pressure and high dynamic pressure. The fast jet they discharge decays downwind of the fan and transfers its momentum to the slower air in the tunnel, leading to a rise in pressure. A commonly-used equation for the pressure rise of a jet fan is

$$\Delta P = -T_f \frac{\rho}{\rho_0} \frac{N_f \eta_f}{A_t} \frac{(v_f - v_t)}{|v_f|} \tag{260}$$

where

$$\begin{aligned}
T_f &= \text{Static thrust of the jet fans at } \rho_0 \ (N) \\
\rho &= \text{Density of air entering the jet fan } (kg/m^3) \\
\rho_0 &= \text{Air density the static thrust is valid at } (1.2 \ kg/m^3) \\
N_f &= \text{Number of jet fans in the tunnel} \\
\eta_f &= \text{Installation efficiency of the jet fans } (-) \\
v_f &= \text{Discharge velocity of the jet fans } (m/s) \\
A_t &= \text{Tunnel area } (m^2) \\
v_t &= \text{Tunnel air velocity } (m/s)
\end{aligned}$$

This is a simplification of an expression derived by Eck & Meidinger in the early 1960s for jet fans and it seems to have become accepted for engineering work. The ratio ρ/ρ_0 is needed for compressible flow, as the density of air passing through the jet fan in the tunnel (ρ) may not be the same as the density at which the fan's static thrust was measured (ρ_0, which is almost certainly 1.2 kg/m^3).

Most tunnel ventilation programs that I've used (Aero, IDA, SES) treat jet fans as a jump in pressure at a grid point, and in pressure profiles plotted from such programs the jet fans show up as a series of step changes in pressure. In recent years this started to concern me, because the pressure profiles produced by these programs are now routinely used to determine what jet fans to run to pressurise the non-incident tunnel (to keep smoke out of escape cross-passages) and prevent smoke recirculation at the portals.

The pressure rise due to jet fans actually occurs over a long length of tunnel downstream of the site of the bank of fans. In my library I have five papers with experimental data on the shape of the pressure distribution/velocity distribution downstream of jet fans/Saccardo nozzles. Here they are, with a brief summary of the evidence they contain:

- Baba, T, Ohashi, H and Uesaki, K, *"Aerodynamic specialities in connection with the Tsuruga tunnel"*, Proc. 6th ISAVVT, BHRA 1988. Figure 7 of the paper shows velocity distributions downstream of a pair of Saccardo nozzles during full-scale testing: the plumes end between 70 m and 80 m downwind of the nozzle outlets in the Tsuruga tunnel.
- Yoshizawa, I, Komatsu, K, Ohta, Y and Hagiwara, K, *"Characteristics of blowing and exhaust nozzles used in a longitudinal ventilation system"*, Proc. 7th ISAVVT, BHRA 1991. Figure 5 of the paper shows the change in static pressure along a 1:50th scale model; the pressure rise caused by the Saccardo nozzle stops approximately 7 hydraulic diameters downwind of the outlet.

- Yano, R, Mizuno, A and Sato, J, *"An experimental and numerical investigation of the duct flow with an inclined jet injection"*, Proc. 9th ISAVVT, BHR Group 1997. Figures 4 & 5 show experimental data (dimensionless pressure coefficients derived from pressure measurements in a square duct of size 0.23 m on each side) showing that the thrust of the jets in their experiments took 13 hydraulic diameters to decay.[2]

- Saika, T, Nakajima, N, Setoyama, S, Haraguchi, Miyake, M and Kanoh, T, *"Jet fan thrust performance evaluation installed in road tunnels"*, Proc. 10th ISAVVT, BHR Group 2000. Figure 5 of the paper shows the change in static pressure with distance in one of the full-scale tests; the pressure rise caused by the jet fan stops approximately 80 m downwind of the jet fan (the jet fan is at 180 m, the friction gradient takes over at 260 m).

- Mizuno, A, Azuma, T, Ichikawa, A and Kanoh, T, *"Appropriate JF installation interval obtained by booster capacity measurements"*, Proceedings of the 10th ISAVVT, BHR Group, 2000. This is the most comprehensive set of full-scale test data I know of. Figures 7–9 show static pressure vs. distance with different jet fan spacings and different tunnel air velocities: most show the jet fans transferring thrust to the air over distances of around 80 m; only the site tests with tunnel air velocities approaching the jet fans of 12 m/s show it occurring over longer distances.

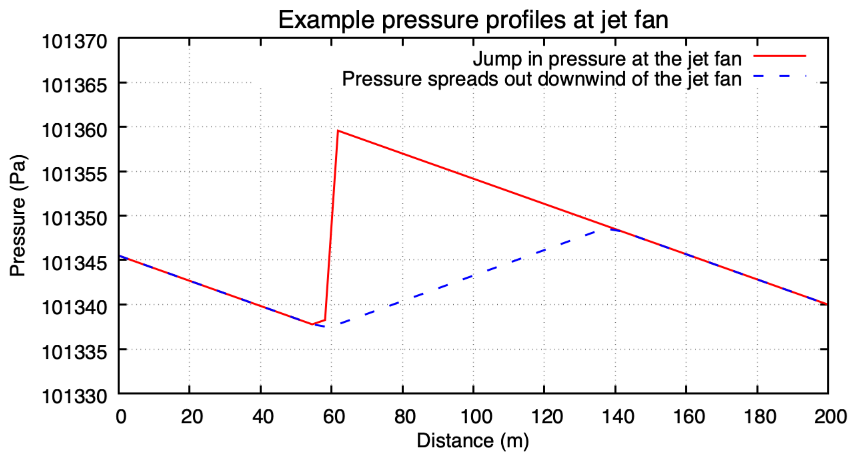

Figure 3: Profile of pressures past a jet fan

An example of the two types of profiles (calculated, not measured) is given in Figure 3. The first curve is from a calculation in which the jet fan pressure rise is applied over a distance of 1 m (effectively a step change and thus unrealistic) and the second is from a calculation in which the jet fan pressure rise is equally spread over the 80 m downwind of the fans (more realistic).

[2]This paper was one of five presented at the 9th ISAVVT that were not bound in to the conference Proceedings. They were provided as stapled A4 documents in the conference attendees' goodie bags. I have a scan of it. If you want a copy of the scan, please get in touch.

The problem with taking a step change in pressure is this: if this jet fan is acting to pressurise the non-incident tunnel and there is a cross-passage 20 m downwind of the bank of jet fans, the pressure in the non-incident tunnel at the cross-passage is too high, by about 15 Pascals. Having 15 Pa higher pressure in an incorrect calculation is a big help when you are dealing with escape route pressurisation systems in tunnels.

The approach I've used is to treat the pressure rise of the jet fan as a body force extending some distance downwind of the location of the jet fans. The distance will be user-settable but I reckon I'll set a default of 80 m, as in the figure above. I'll probably stick to spreading the pressure change equally over the distance: I'm sure that assumption isn't true, but all the experimental data I have indicate that an equally-spread pressure rise is good enough for engineering work.

One advantage of this approach is that the thrust of jet fans (like traffic drag) are independent of the grid points in the calculation—we do not need to fix a gridpoint at the chainage of the jet fans. There are a few disadvantages, though:

- Additional calculations are needed to figure out how much of the plume from a jet fan overlaps each cell[3]. This will slow the program, especially in runs where jet fans turn off and on during the run and we have to check if a jet fan is active in every timestep.
- In complex road tunnels with underground junctions, we will need to specify whether the plume from a bank of jet fans in a turnout cavern blows into the mainline tunnel or into an off-ramp tunnel (this may have happened in one of the Westconnex contracts in Sydney, but I can't check at the moment). Now that I think about it, this isn't a disadvantage in reality. It makes the programming more complex, but it makes it easier for the users.

Going back to (260), we will do some arithmetic beforehand to tell us what proportion of a cell contains the plume from a bank of jet fans. We'll give this term the arbitrary symbol D_{pl} and it will be in the range 0–1: 0 if no part of a plume is present, 1 if a plume crosses the entire length of the cell.

Using the relationship derived from the friction terms, we will divide by the distance between our gridpoints and multiply by our timestep. We include D_{pl} in the numerator so that where a plume ends in a cell, the calculation at the gridpoints at the end of that cell are accurate.

$$E = -\frac{T_f}{\rho_0}\frac{N_f \eta_f}{A_t} D_{pl} \frac{(v_f - v_t)}{|v_f|} \tag{261}$$

6.6 Alternative jet fan equations

Although equation (260) has become accepted as a suitable way to calculate the effect of jet fans, there are a few variants in the literature.

[3]Recall that a cell is one of the distances Δx between two gridpoints.

One that has crept up my radar in the last few years is an expression that treats jet fans blowing against an adverse airflow differently from jet fans blowing in the direction of airflow. I have seen a few colleagues use it over the years but never found a source for it. It has five parts:

$$v_f = 0 \quad : \quad \Delta P = 0 \tag{262}$$
$$v_t >= 0, v_f > 0 \quad : \quad \Delta P = -T_f \frac{\rho}{\rho_0} \frac{N_f \eta_f}{A_t} \frac{(v_f - v_t)}{v_f} \tag{263}$$
$$v_t >= 0, v_f < 0 \quad : \quad \Delta P = +T_f \frac{\rho}{\rho_0} \frac{N_f \eta_f}{A_t} \tag{264}$$
$$v_t <= 0, v_f < 0 \quad : \quad \Delta P = +T_f \frac{\rho}{\rho_0} \frac{N_f \eta_f}{A_t} \frac{(v_f - v_t)}{v_f} \tag{265}$$
$$v_t <= 0, v_f > 0 \quad : \quad \Delta P = -T_f \frac{\rho}{\rho_0} \frac{N_f \eta_f}{A_t} \tag{266}$$

There is a technical paper in the BHR Group's 2019 ISAVFT (Greece) that takes this a step further, (Kato *et al* 2019). They assessed the performance of jet fans blowing in reverse with CFD and found that the simple relationship we all use (260) did not hold when the fans were run in reverse in their CFD runs. I have been keeping an eye out for papers from this group at BHR tunnel ventilation conferences for supporting evidence from experimental work.

6.7 Discrete pressure losses

Where a discrete pressure loss occurs (such as a series of fittings in a vent shaft) an empirically-derived pressure loss factor ζ will have been calculated using fitting losses in Idelchik, Miller, ASHRAE or CIBSE. Such pressure losses often cause most of the pressure rise that main ventilation fans must deal with.

Take the case of a fixed pressure loss factor ζ applied to an arbitrary velocity at a point in the tunnel. I'm more comfortable working in incompressible flow, so I usually think of it as

$$\Delta P = \frac{1}{2}\rho\zeta v^2 \tag{267}$$
$$\Rightarrow P_u + \frac{1}{2}\rho v_u^2 = P_d + \frac{1}{2}\rho v_d^2 + \frac{1}{2}\rho\zeta v_d^2 \tag{268}$$
$$\Rightarrow P_u + \frac{1}{2}\rho v_u^2 = P_d + \frac{1}{2}\rho(1+\zeta)v_d^2 8 \tag{269}$$

where values with the subscript $_u$ are on the upwind side and values with the subscript $_d$ are on the downwind side. When expressed in terms of c and u in isentropic compressible flow it becomes

$$\psi c_u^2 + v_u^2 = \psi c_d^2 + v_d^2 + \zeta v_d^2 \tag{270}$$
$$\Rightarrow \psi c_u^2 + v_u^2 = \psi c_d^2 + (1+\zeta)v_d^2 \tag{271}$$

The energy represented by ζv_d^2 is removed from the system. In the real world the energy turns into heat (in a non-homentropic calculation there would be a slight rise in temperature at the downwind side) but as the calculations here are homentropic, the energy absorbed by the pressure loss factor is discarded.

6.8 Portals to atmosphere

Inflow at portals is treated as having a drop in pressure due to a loss factor ζ (in tunnels, ζ is often taken to be 0.5). Outflow at portals is usually taken to mean the loss of one dynamic head of air as it dissipates into the atmosphere ($\zeta = 1$).

Outside the portal there is a vast reservoir of unchanging air pressure (represented in the calculation by a constant celerity c_{atm}), zero velocity and no characteristic. Fox (1977) states that the velocity inside a portal can be determined by the celerity after the inflow loss has been applied (c_N) and a backward characteristic in the tunnel.

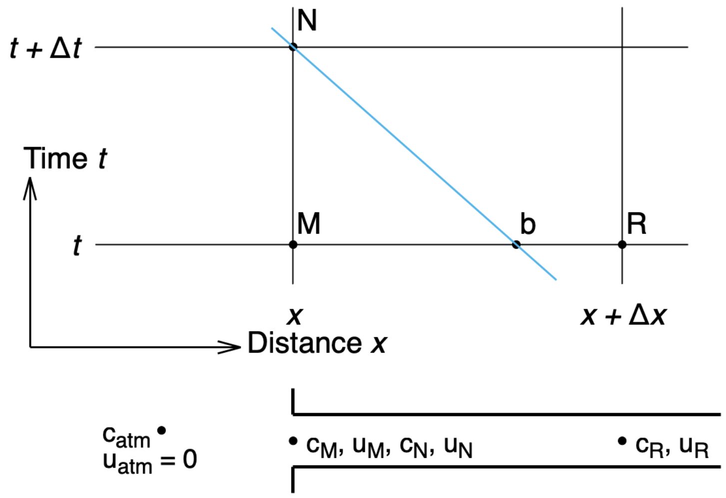

Figure 4: Flow at portals to atmosphere

Consider the portal in Figure 4. Just inside the tunnel there are conditions c_N and u_N at time $t + \Delta t$. Outside the tunnel there are fixed conditions c_{atm} and $u_{atm} = 0$. There may be a loss factor applied at the portal (ζ_{in} or ζ_{out} depending on the direction of flow). The energy equation between the atmosphere and the gridpoint inside the portal (271) may be used to determine c_N, but we have to consider the cases of inflow and outflow separately.

For outflow, between point N and atmosphere we have:

$$\psi c_N^2 + u_N^2 = \psi c_{atm}^2 + u_{atm}^2 + \zeta u_N^2 \tag{272}$$
$$\Rightarrow \psi c_N^2 + u_N^2 = \psi c_{atm}^2 + 0 + \zeta u_N^2 \tag{273}$$
$$\Rightarrow \psi c_N^2 + u_N^2 = \psi c_{atm}^2 + \zeta u_N^2 \tag{274}$$

$$
\begin{aligned}
\Rightarrow \psi c_N^2 + u_N^2 - \zeta u_N^2 &= \psi c_{atm}^2 && (275)\\
\Rightarrow \psi c_N^2 + (1-\zeta) u_N^2 &= \psi c_{atm}^2 && (276)\\
\Rightarrow \psi c_N^2 &= \psi c_{atm}^2 - (1-\zeta) u_N^2 && (277)\\
\Rightarrow \psi c_N^2 &= \psi c_{atm}^2 + (\zeta-1) u_N^2 && (278)\\
\Rightarrow c_N^2 &= c_{atm}^2 + \frac{(\zeta-1)u_N^2}{\psi} && (279)\\
\Rightarrow c_N &= \sqrt{c_{atm}^2 + \frac{(\zeta-1)u_N^2}{\psi}} && (280)
\end{aligned}
$$

Note that we must apply the pressure loss factor ζ to u_N because u_{atm} is zero.

For inflow, between atmosphere and point N we have

$$
\begin{aligned}
\psi c_{atm}^2 + u_{atm}^2 &= \psi c_N^2 + (\zeta+1) u_N^2 && (271)\\
\Rightarrow c_N^2 &= c_{atm}^2 - \frac{(\zeta+1)u_N^2}{\psi} && (281)\\
\Rightarrow c_N &= \sqrt{c_{atm}^2 - \frac{(\zeta+1)u_N^2}{\psi}} && (282)
\end{aligned}
$$

Between the two gridpoints inside the tunnel we can use a backward characteristic to the portal (Figure 4):

$$
\begin{aligned}
\psi(c_N - c_b) - (u_N - u_b) - E_b dt &= 0 && (283)\\
\Rightarrow u_N = u_b - \psi(c_N - c_b) - E_b dt. & && (284)
\end{aligned}
$$

For outflow, we can solve (280) and (284) iteratively to determine c_N and u_N. The approach used is to take u_M as the first estimate of u_N, calculate c_N from (282) then calculate a new value of u_N from (284). We iterate (282) and (284) until the values converge. The process of iteration is quick: when I set up a tunnel full of still air and set a fixed velocity of 5 m/s inflow at a portal, it took about six iterations to arrive at a value of c_N that forced air to flow in at that speed.

For inflow, we can solve (282) and (284) in the same way.

6.9 Junctions

Tunnels are divided up into segments of constant area and constant mass flow. From this it follows that segments start and end at portals, dead ends, changes of area and junctions. This section is mostly based on Section 7.1 of Fox's book.

Figure 5 shows a two-way junction (shown as an area change but it could also be a damper or axial fan). The linked characteristics used in Figure 2 don't hold here, because the two tunnels have different areas. Instead we have two gridpoints on either side of a discontinuity.

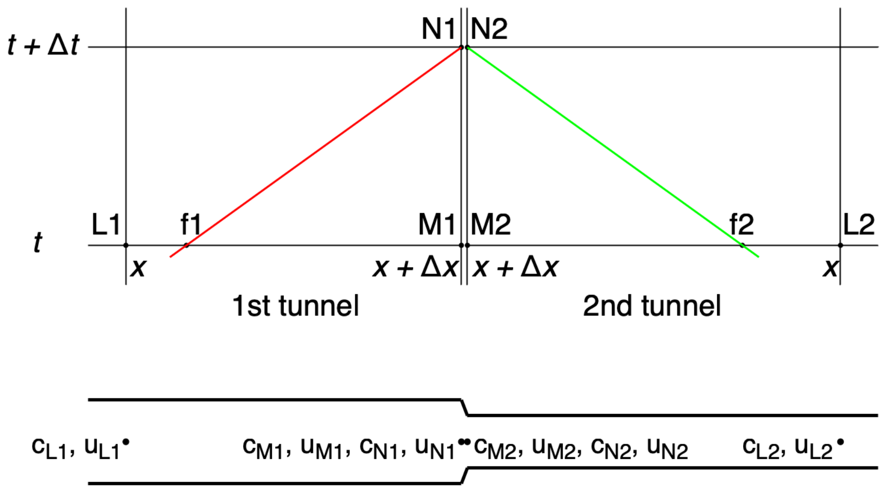

Figure 5: Characteristics at a two-way junction (area change)

We set up the co-ordinate system so that there are forwards characteristics in both cells and the join is at their respective $x + \Delta x$ in Figure 5 (this is not necessary but makes it easier to code).

We have two gridpoints on either side of the area change. At time t the gridpoint on the left side of the change has properties c_{M1} and u_{M1} and the gridpoint on the right side of the change has properties c_{M2} and u_{M2}.

To solve the flow split we need to satisfy mass continuity and pressure balance, calculating with celerity and velocity on both sides of the discontinuity.

We have an area change that has conditions on its left hand side (`M1` at the current timestep, `N1` at the next timestep) and a different set of conditions (`M2` and `N2`) at its right-hand side.

We have eight known values (velocities and celerities at `L1`, `M1`, `M2` and `L2` in Figure 5).

We have four unknown values, the velocities and celerities at `N1` and `N2`. We need four equations to solve for these. One is mass flow: the mass flow at `N1` must equal the mass flow at `N2`. We can use the mass continuity equation (243):

$$\sum (c^{\psi} A u) = 0 \qquad (243)$$

$$\Rightarrow c^{\psi}_{N1} A_1 u_{N1} + c^{\psi}_{N2} A_2 u_{N2} = 0 \qquad (285)$$

We have a plus sign in (285) because we set up the two cells such that positive values of air velocity always flow towards the junction.

The second is pressures: the stagnation pressure at `N1` must be the same as the stagnation pressure at `N2`. There may be pressure loss factors applied to the air velocities at both sides of the junction, which we will term ζ_1 (applied to u_{N1}) and ζ_2 (applied to u_{N2}).

$$P_{stag} = \psi c_{N1}^2 + u_{N1}^2 - \zeta_1 u_{N1}^2 = \psi c_{N2}^2 + u_{N2}^2 - \zeta_2 u_{N2}^2 \tag{286}$$

The other two equations are the forward characteristic in the 1st cell and the forward characteristic in the 2nd cell.

$$\begin{aligned} \psi(c_{N1} - c_{f1}) + (u_{N1} - u_{f1}) + E_{f1}dt &= 0 \quad (287) \\ \psi(c_{N2} - c_{f2}) + (u_{N2} - u_{f2}) + E_{f2}dt &= 0 \quad (288) \end{aligned}$$

The mass continuity equation (285) and the stagnation pressure equation (286) are nonlinear, so we need a nonlinear solver. As it happens the Python library `scipy` has a function designed for solving sets of nonlinear equations, `scipy.optimize.fsolve` (it calls the Fortran library `minpack` to solve mulitple nonlinear equations that all should return zero values—it is better than anything an amateur programmer like me could write). You give it a set of values to vary (in this case c_{N1}, c_{N2}, u_{N1} and u_{N2}) and a function that contains a set of expressions that evaluate to zero. In the case of a two-branch junction there are four values to vary (c_{N1}, c_{N2}, u_{N1} and u_{N2}) and four expressions that you want to evaluate to zero:

$$\begin{aligned} c_{N1}^{\psi} A_1 u_{N1} + c_{N2}^{\psi} A_2 u_{N2} &= 0 \quad \text{(continuity)} \quad (289) \\ \psi c_{N1}^2 + (1 - \zeta_1) u_{N1}^2 - \psi c_{N2}^2 - (1 + \zeta_2) u_{N2}^2 &= 0 \quad \text{(difference in } P_{stag}) \quad (290) \\ \psi(c_{N1} - c_{f1}) + (u_{N1} - u_{f1}) + E_{f1}dt &= 0 \quad \text{(characteristic 1)} \quad (291) \\ \psi(c_{N2} - c_{f2}) + (u_{N2} - u_{f2}) + E_{f2}dt &= 0 \quad \text{(characteristic 2)} \quad (292) \end{aligned}$$

`Fsolve` is fed initial guesses for the values to vary (in this case the initial guesses are c_{M1}, c_{M2}, u_{M1} and u_{M2}), evaluates the four expressions and modifies the guesses until all four expressions return zero.

This is coded as function `celvel2` in the Hobyah source code. Similar functions exist for other situations where we have two branches and want to see a step-change in pressure when we plot a pressure profile, such as at fans and dampers.

This process scales up easily to more branches. In the case of a three-way junction there are six values to vary (c_{N1}, c_{N2}, c_{N3}, u_{N1}, u_{N2} and u_{N3}) and six equations:

- Mass continuity across the junction,
- Two calculations of the differences between the stagnation pressures at the junction, and
- Three forwards characteristics in the three cells.

These are:

$$c_{N1}^{\psi} A_1 u_{N1} + c_{N2}^{\psi} A_2 u_{N2} + c_{N3}^{\psi} A_3 u_{N3} = 0 \tag{293}$$

$$\psi c_{N1}^2 + (1-\zeta_1)u_{N1}^2 - \psi c_{N2}^2 - (1+\zeta_2)u_{N2}^2 = 0 \quad (294)$$
$$\psi c_{N1}^2 + (1-\zeta_1)u_{N1}^2 - \psi c_{N3}^2 -)1 + \zeta_3)u_{N3}^2 = 0 \quad (295)$$
$$\psi(c_{N1} - c_{f1}) + (u_{N1} - u_{f1}) + E_{f1}dt = 0 \quad (296)$$
$$\psi(c_{N2} - c_{f2}) + (u_{N2} - u_{f2}) + E_{f2}dt = 0 \quad (297)$$
$$\psi(c_{N3} - c_{f3}) + (u_{N3} - u_{f3}) + E_{f3}dt = 0 \quad (298)$$

This is coded as function `celvel3` in the source code.

And so on. I stopped coding at six branches `celvel6`, though I'm sure I could have gone on indefinitely. I stopped at six for two reasons:

- First, I believe that's a reasonable maximum: six branches at the one node can represent a station concourse tunnel with two cross-passages to train platforms (four branches so far) right next to an escape stair (branch 5) and an intervention stair (branch 6). We had those all the time in Heathrow Express stations.
- Second, I have to write test files. The more branches there are, the more test files are needed. A six-way junction needs 64 test files, a seven-way needs 128, a ten-way junction needs 1024, and so on. Even if I automate the generation of the test files, I have to verify them.

There will be situations in which engineers want more than six air paths connecting at a node. For example, one of the vent stations in the Burnley road tunnel in Melbourne has six fans in parallel: to model it in Hobyah we would need eight air paths (six for the fans, one for the extract from the traffic space and one from the extract from the tunnel's smoke duct). A colleague of mine who worked on Sydney Westconnex 3B has mentioned running seven fans in parallel on that project, which would also need eight air paths to connect at a node.

In such circumstances, it may be OK to model these systems by adding a couple of air paths with minimal losses (tiny friction factors, short lengths, no fixed losses) to use as intermediate ducts between the junctions so that the rule of "maximum six air paths at a node" is adhered to. An example of how to do this can be found in the test file `ok-0?.?-76-trombones.txt`

6.10 Fan characteristics

Axial and centrifugal fans have flow-pressure characteristics that impel or retard airflow. The method of characteristics seems to be particularly suitable for calculating flow through fans because it is so stable: it can calculate flows through fans running in the freewheeling region and the reverse flow region fairly easily and only hunts when running through the stall hump.

Let's get one thing out in the open first: I always use fan total pressure and I **never** use fan static pressure.

I regard fan static pressure rise as a tool of the Devil, invented specifically to confuse all decent, God-fearing engineers and cause problems for us on our projects (by tempting us to treat fan static pressure rise as if it were fan total pressure rise, somehow totally beef all the checks and balances that every project should have, and end up buying fans that are underpowered).

For those readers familiar with ISO 5801; fan total pressure p_F is the fan pressure rise I like and fan static pressure p_{sF} is the one I think is cursed.

OK, that's enough dissing fan static pressure rise from me. Let's talk technical. Fans are relatively easy to implement in the method of characteristics: you take as input a fan total pressure characteristic (fan total pressure rise versus fan volume flow) and a fan base density. Convert the list of flow values into a list of air velocities in the tunnel the fan happens to be in. Use the base density to convert the pressure values into values of c^2 (square of celerity, $\mathrm{m}^2/\mathrm{s}^2$). Each time the velocity at the location of the fan changes, you interpolate (or extrapolate) on the characteristic for c^2_{fan} based on u_{N1} if the fan rotational speed is positive and on u_{N2} if it is negative.

Then you treat the fan as a two-way junction with no loss factors and a term for fan pressure rise (ψc^2_{fan}) in the stagnation pressure equation (300):

$$\begin{aligned}
c^{\psi}_{N1} A_1 u_{N1} + c^{\psi}_{N2} A_2 u_{N2} &= 0 \quad \text{(continuity)} && (299)\\
\psi c^2_{fan} + \psi c^2_{N1} + u^2_{N1} - \psi c^2_{N2} - u^2_{N2} &= 0 \quad (P_{stag}) && (300)\\
\psi(c_{N1} - c_{f1}) + (u_{N1} - u_{f1}) + E_{f1} dt &= 0 \quad \text{(characteristic 1)} && (301)\\
\psi(c_{N2} - c_{f2}) + (u_{N2} - u_{f2}) + E_{f2} dt &= 0 \quad \text{(characteristic 2)} && (302)
\end{aligned}$$