A plodding derivation of the equations for compressible air flow in tunnels

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This document is a derivation of the equations of the homentropic method of characteristics (MoC) that is often used for calculating air flow in the tunnel ventilation field.

Many researchers have written papers that summarized the derivation of characteristic forms of equations. Until I started a sabbatical I never had the time or patience to sit down and figure out how those researchers got from A to B. I wrote this document to help me figure it all out.

There are four sections to this document.

- Derivation of the method of characteristics (MoC) for two independent variables.
- Derivation of the relationship between speed of sound, pressure and density in an ideal gas.
- Derivation of the differential forms of the isentropic flow equations.
- Conversion of the mass and momentum equations for isentropic compressible flow at low Mach number into a form suitable for solving by MoC.

This document might be expanded to cover non-homentropic flow if I ever get manage my head around the papers by Woods, Gawthorpe and Pope. But that seems unlikely.

No doubt there are errors in this document. I welcome corrections.

1 Method of characteristics

Sources:

- Lister, M, *"The numerical solution of hyperbolic partial differential equations by the method of characteristics"*, in Ralston & Wilf, "Mathematical Methods for Digital Computers", Wiley, 1960
- Fox, J A, *"The use of the digital computer in the solution of waterhammer problems"*, Paper 7020, J. Instn. Civ. Engnrs, 1968
- Fox, J A, and Henson, D A, *"The prediction of the magnitudes of pressure transients generated by a train entering a single tunnel*, Paper 7635, J. Instn. Civ. Engnrs, 1971
- Fox, J A, *"Hydraulic analysis of unsteady flow in pipe networks"*, Macmillan Press, 1977

In the tunnel vent field, Lister's 1960 paper seems to have been the main influence. Fox used her derivation to write waterhammer programs in Algol 60 at the University of Leeds in the 1960s, then supervised the Ph.Ds of three of the early players in the compressible airflow in tunnels field. In 1977 Fox wrote an excellent book about how to calculate unsteady flows in a practical manner.

The derivation in this section is mostly an expansion of the summary in chapter 4 of Fox's book.

Consider a pair of partial differential equations with two independent variables x and y and two dependent variables u and v:

$$L_1 = A_1\frac{\partial u}{\partial x} + B_1\frac{\partial u}{\partial y} + C_1\frac{\partial v}{\partial x} + D_1\frac{\partial v}{\partial y} + E_1 = 0 \tag{1}$$

$$L_2 = A_2\frac{\partial u}{\partial x} + B_2\frac{\partial u}{\partial y} + C_2\frac{\partial v}{\partial x} + D_2\frac{\partial v}{\partial y} + E_2 = 0 \tag{2}$$

where A_1, A_2, ... are known functions of x, y, u and v.

In terms of the software, independent variable x will be time and independent variable y will be distance along a tunnel (or vice-versa - doesn't matter yet). They are independent because when we write the software we choose our timestep and our grid length. As long as the dependent variables u and v define the state of the fluid it doesn't appear to matter what they are. Fox used water head and water velocity, Lister used gas velocity and speed of sound.

Consider a linear combination of L_1 and L_2 using an arbitrary function ϑ:

$$\begin{aligned} L &= L_1 + \vartheta L_2 & (3) \\ &= (A_1 + \vartheta A_2)\frac{\partial u}{\partial x} + (B_1 + \vartheta B_2)\frac{\partial u}{\partial y} + \end{aligned}$$

$$
\begin{aligned}
&(C_1 + \vartheta C_2)\frac{\partial v}{\partial x} + (D_1 + \vartheta D_2)\frac{\partial v}{\partial y} + \\
&(E_1 + \vartheta E_2) \qquad (4)
\end{aligned}
$$

Let $y = y(x)$, a curve. Its tangent slope is dy/dx.

If $u = u(x, y)$ and $v = v(x, y)$ and u and v are valid solutions of L_1 and L_2 then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \qquad (5)$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy. \qquad (6)$$

Take the u terms in (4),

$$(A_1 + \vartheta A_2)\frac{\partial u}{\partial x} + (B_1 + \vartheta B_2)\frac{\partial u}{\partial y} : \qquad (7)$$

this can be rewritten as

$$(A_1 + \vartheta A_2)\left(\frac{\partial u}{\partial x} + \frac{(B_1 + \vartheta B_2)}{(A_1 + \vartheta A_2)}\frac{\partial u}{\partial y}\right). \qquad (8)$$

Now take the v terms in (4),

$$(C_1 + \vartheta C_2)\frac{\partial v}{\partial x} + (D_1 + \vartheta D_2)\frac{\partial v}{\partial y} : \qquad (9)$$

this can be rewritten as

$$(C_1 + \vartheta C_2)\left(\frac{\partial v}{\partial x} + \frac{(D_1 + \vartheta D_2)}{(C_1 + \vartheta C_2)}\frac{\partial v}{\partial y}\right). \qquad (10)$$

At this point both Fox and Lister do something that I just took on trust, as I'm too stupid to understand it. We choose our value of ϑ such that

$$\frac{(B_1 + \vartheta B_2)}{(A_1 + \vartheta A_2)} = \frac{(D_1 + \vartheta D_2)}{(C_1 + \vartheta C_2)} = \frac{dy}{dx}. \qquad (11)$$

We can now replace $\frac{(B_1+\vartheta B_2)}{(A_1+\vartheta A_2)}$ in (8) with $\frac{dy}{dx}$:

$$(A_1 + \vartheta A_2)\left(\frac{\partial u}{\partial x} + \frac{dy}{dx}\frac{\partial u}{\partial y}\right). \qquad (12)$$

Likewise, we can replace $\frac{(D_1+\vartheta D_2)}{(C_1+\vartheta C_2)}$ in (10) with $\frac{dy}{dx}$:

$$(C_1 + \vartheta C_2)\left(\frac{\partial v}{\partial x} + \frac{dy}{dx}\frac{\partial v}{\partial y}\right). \qquad (13)$$

Now $\frac{dy}{dx}\frac{\partial u}{\partial y} = 0$ and $\frac{dy}{dx}\frac{\partial v}{\partial y} = 0$ so those terms vanish. We get

$$(A_1 + \vartheta A_2)\frac{du}{dx} \text{ and} \qquad (14)$$

$$(C_1 + \vartheta C_2)\frac{dv}{dx}. \qquad (15)$$

Recall that these are equivalent to the u and v terms in (4), so

$$(A_1 + \vartheta A_2)\frac{\partial u}{\partial x} + (B_1 + \vartheta B_2)\frac{\partial u}{\partial y} = (A_1 + \vartheta A_2)\frac{du}{dx} \tag{16}$$

and

$$(C_1 + \vartheta C_2)\frac{\partial v}{\partial x} + (D_1 + \vartheta D_2)\frac{\partial v}{\partial y} = (C_1 + \vartheta C_2)\frac{dv}{dx}. \tag{17}$$

The careful selection of the value of ϑ allows us to hide the four terms B_1, B_2, D_1 and D_2 in (4) and eliminate the partial derivatives. We substitute the right hand sides of (16) and (17) back into (4) to get

$$(A_1 + \vartheta A_2)\frac{du}{dx} + (C_1 + \vartheta C_2)\frac{dv}{dx} + E_1 + \vartheta E_2. \tag{18}$$

Both L_1 and L_2 equal zero, so their sum must also equal zero i.e.,

$$(A_1 + \vartheta A_2)\frac{du}{dx} + (C_1 + \vartheta C_2)\frac{dv}{dx} + E_1 + \vartheta E_2 = 0. \tag{19}$$

Now we put (19) to one side for a while. We want to express ϑ in terms of the A_1, A_2, ... terms, because those are known functions of x, y, u and v. We rearrange one version of (11) to place ϑ on its own:

$$\begin{aligned}
\frac{(B_1 + \vartheta B_2)}{(A_1 + \vartheta A_2)} &= \frac{dy}{dx} \\
\Rightarrow (B_1 + \vartheta B_2)dx &= (A_1 + \vartheta A_2)dy && (20)\\
\Rightarrow B_1 dx + \vartheta B_2 dx &= A_1 dy + \vartheta A_2 dy && (21)\\
\Rightarrow \vartheta B_2 dx - \vartheta A_2 dy &= A_1 dy - B_1 dx && (22)\\
\Rightarrow \vartheta(B_2 dx - A_2 dy) &= A_1 dy - B_1 dx && (23)\\
\Rightarrow \vartheta &= \frac{A_1 dy - B_1 dx}{B_2 dx - A_2 dy} && (24)
\end{aligned}$$

We rearrange the other version of (11) to place ϑ on its own in the same way and end up with

$$\begin{aligned}
\frac{(B_1 + \vartheta B_2)}{(A_1 + \vartheta A_2)} &= \frac{dy}{dx} \\
\Rightarrow \vartheta &= \frac{C_1 dy - D_1 dx}{D_2 dx - C_2 dy}. && (25)
\end{aligned}$$

(24) and (25) both give ϑ so they are equal to each other.

$$\frac{A_1 dy - B_1 dx}{B_2 dx - A_2 dy} = \frac{C_1 dy - D_1 dx}{D_2 dx - C_2 dy} \tag{26}$$

We can multiply them out and collect terms associated with dx^2, $dxdy$ and dy^2.

$$(A_1 dy - B_1 dx)(D_2 dx - C_2 dy) = (C_1 dy - D_1 dx)(B_2 dx - A_2 dy) \tag{27}$$

$$\begin{aligned}
\Rightarrow \quad & A_1D_2dydx - A_1C_2dy^2 - B_1D_2dx^2 + B_1C_2dxdy = \\
& \qquad C_1B_2dydx - C_1A_2dy^2 - D_1B_2dx^2 + D_1A_2dxdy \qquad (28) \\
\Rightarrow \quad & A_1D_2dxdy - A_1C_2dy^2 - B_1D_2dx^2 + B_1C_2dxdy - \\
& \qquad C_1B_2dxdy + C_1A_2dy^2 + D_1B_2dx^2 - D_1A_2dxdy = 0 \qquad (29) \\
\Rightarrow \quad & (D_1B_2 - B_1D_2)dx^2 + \qquad (30) \\
& (A_1D_2 + B_1C_2 - C_1B_2 - D_1A_2)dxdy + \\
& (C_1A_2 - A_1C_2)dy^2 = 0 \qquad (31)
\end{aligned}$$

Then we divide both sides by dx^2 to generate a quadratic in $\frac{dy}{dx}$.

$$\begin{aligned}
& (D_1B_2 - B_1D_2)\frac{dx^2}{dx^2} + \\
& (A_1D_2 + B_1C_2 - C_1B_2 - D_1A_2)\frac{dxdy}{dx^2} + \\
& (C_1A_2 - A_1C_2)\frac{dy^2}{dx^2} = 0 \qquad (32)
\end{aligned}$$

In the process both Lister and Fox reversed the sign of the coefficients, presumably for a good reason (the sum is still zero). We'll do the same.

$$\begin{aligned}
(B_1D_2 - D_1B_2) & \quad + \\
(C_1B_2 + D_1A_2 - A_1D_2 - B_1C_2)\frac{dy}{dx} & \quad + \\
(A_1C_2 - C_1A_2)\frac{dy^2}{dx^2} & \quad = \quad 0 \qquad (33)
\end{aligned}$$

We define the following collective terms for the coefficients:

$$\begin{aligned}
p &= A_1C_2 - C_1A_2, \qquad (34) \\
q &= C_1B_2 + D_1A_2 - A_1D_2 - B_1C_2, \qquad (35) \\
r &= B_1D_2 - D_1B_2. \qquad (36)
\end{aligned}$$

We can now re-write (33) as

$$p\frac{dy^2}{dx^2} + q\frac{dy}{dx} + r = 0 \qquad (37)$$

The solution of this quadratic can have one of three types:

$$\begin{aligned}
q^2 - 4pr &> 0\text{: two different real roots, the PDEs are hyperbolic,} \qquad (38) \\
q^2 - 4pr &= 0\text{: one real roots, the PDEs are parabolic,} \qquad (39) \\
q^2 - 4pr &< 0\text{: two complex roots, the PDEs are elliptical.} \qquad (40)
\end{aligned}$$

In this case the PDEs are hyperbolic. There are two roots, which we will call ζ_+ and ζ_-, defined by

$$\zeta_+ = \frac{-q + \sqrt{q^2 - 4pr}}{2p}, \qquad (41)$$

$$\zeta_- = \frac{-q - \sqrt{q^2 - 4pr}}{2p}. \qquad (42)$$

These two solutions represent values of dy/dx and are thus the slopes of lines in the $x-y$ plane (characteristic lines). At each point in the $x-y$ plane two characteristic lines pass through, one with slope ζ_+ the other with slope ζ_-. For convenience we will represent them both by one symbol, $\zeta_\pm$ so that we only have to do the derivation once. We can go back to (24) and (25) and determine values of ϑ in terms of ζ_+. We start with (24):

$$\vartheta = \frac{A_1 dy - B_1 dx}{B_2 dx - A_2 dy} \tag{43}$$

We divide both top and bottom by dx.

$$\begin{aligned} \vartheta &= \frac{(A_1 dy - B_1 dx)/dx}{(B_2 dx - A_2 dy)/dx} \\ \Rightarrow \vartheta &= \frac{A_1 \frac{dy}{dx} - B_1 \frac{dx}{dx}}{B_2 \frac{dx}{dx} - A_2 \frac{dy}{dx}}. \end{aligned} \tag{44}$$

We cancel $\frac{dx}{dx}$:

$$\vartheta = \frac{A_1 \frac{dy}{dx} - B_1}{B_2 - A_2 \frac{dy}{dx}}. \tag{45}$$

We replace $\frac{dy}{dx}$ with the slopes of our characteristic lines, $\zeta_\pm$.

$$\vartheta = \frac{A_1 \zeta_\pm - B_1}{B_2 - A_2 \zeta_\pm}. \tag{46}$$

The same process is then applied to (25):

$$\begin{aligned} \vartheta &= \frac{C_1 dy - D_1 dx}{D_2 dx - C_2 dy} \\ \Rightarrow \vartheta &= \frac{C_1 \zeta_\pm - D_1}{D_2 - C_2 \zeta_\pm}. \end{aligned} \tag{47}$$

Now we go back to (19) and substitute for ϑ:

$$(A_1 + \vartheta A_2)\frac{du}{dx} + (C_1 + \vartheta C_2)\frac{dv}{dx} + E_1 + \vartheta E_2 = 0 \tag{48}$$

$$\begin{aligned} \Rightarrow (A_1 + \frac{A_1 \zeta_\pm - B_1}{B_2 - A_2 \zeta_\pm} A_2)\frac{du}{dx} &+ \\ (C_1 + \frac{A_1 \zeta_\pm - B_1}{B_2 - A_2 \zeta_\pm} C_2)\frac{dv}{dx} &+ \\ E_1 + \frac{A_1 \zeta_\pm - B_1}{B_2 - A_2 \zeta_\pm} E_2 &= 0 \end{aligned} \tag{49}$$

If we multiply both sides by dx we get

$$(A_1 + \frac{A_1 \zeta_\pm - B_1}{B_2 - A_2 \zeta_\pm} A_2)du \quad +$$

$$
\begin{aligned}
(C_1 + \frac{A_1\zeta_\pm - B_1}{B_2 - A_2\zeta_\pm}C_2)dv \quad &+ \\
(E_1 + \frac{A_1\zeta_\pm - B_1}{B_2 - A_2\zeta_\pm}E_2)dx \quad &= \quad 0. \qquad (50)
\end{aligned}
$$

Next we multiply both sides by $B_2 - A_2\zeta_\pm$ and collect common terms of $\zeta_\pm$:

$$
\begin{aligned}
& [A_1(B_2 - A_2\zeta_\pm) + (A_1\zeta_\pm - B_1)A_2]\, du + \\
& [C_1(B_2 - A_2\zeta_\pm) + (A_1\zeta_\pm - B_1)C_2]\, dv + \\
& [E_1(B_2 - A_2\zeta_\pm) + (A_1\zeta_\pm - B_1)E_2]\, dx = 0 \qquad (51) \\
\Rightarrow\; & [A_1B_2 - A_1A_2\zeta_\pm + A_1\zeta_\pm A_2 - A_2B_1]\, du + \\
& [B_2C_1 - A_2C_1\zeta_\pm + A_1\zeta_\pm C_2 - B_1C_2]\, dv + \\
& [B_2E_1 - A_2E_1\zeta_\pm + A_1\zeta_\pm E_2 - B_1E_2]\, dx = 0 \qquad (52) \\
\Rightarrow\; & [A_1B_2 - A_2B_1]\, du + \\
& [B_2C_1 - A_2C_1\zeta_\pm + A_1\zeta_\pm C_2 - B_1C_2]\, dv + \\
& [B_2E_1 - A_2E_1\zeta_\pm + A_1\zeta_\pm E_2 - B_1E_2]\, dx = 0 \qquad (53) \\
\Rightarrow\; & [A_1B_2 - A_2B_1]\, du + \\
& [B_2C_1 - B_1C_2 - A_2C_1\zeta_\pm + A_1C_2\zeta_\pm]\, dv + \\
& [B_2E_1 - B_1E_2 - A_2E_1\zeta_\pm + A_1E_2\zeta_\pm]\, dx = 0 \qquad (54) \\
\Rightarrow\; & [A_1B_2 - A_2B_1]\, du + \\
& [B_2C_1 - B_1C_2 + (A_1C_2 - A_2C_1)\zeta_\pm]\, dv + \\
& [B_2E_1 - B_1E_2 + (A_1E_2 - A_2E_1)\zeta_\pm]\, dx = 0. \qquad (55)
\end{aligned}
$$

Recall that A_1, A_2, ... are known functions of x, y, u and v. We define five new terms of their combinations:

$$
\begin{aligned}
N &= A_1B_2 - A_2B_1, & (56)\\
O &= B_1C_2 - B_2C_1, & (57)\\
P &= A_1C_2 - A_2C_1, & (58)\\
Q &= B_1E_2 - B_2E_1, & (59)\\
R &= A_1E_2 - A_2E_1. & (60)
\end{aligned}
$$

We can rewrite (55) as

$$N du + (O - P\zeta_\pm)dv + (Q - R\zeta_\pm)dx = 0 \qquad (61)$$

and know that it applies along the characteristic lines we also have the characterstic slope equations $\zeta_\pm = \frac{dy}{dx}$.

We have four unknowns du, dv, ζ_+ and ζ_-. We have four equations

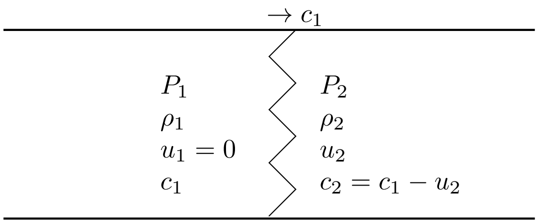
$$
\begin{aligned}
N du + (O - P\zeta_+)dv + (Q - R\zeta_+)dx &= 0, & (62)\\
dy - \zeta_+ dx &= 0, & (63)\\
N du + (O - P\zeta_-)dv + (Q - R\zeta_-)dx &= 0 \;\& & (64)\\
dy - \zeta_- dx &= 0 & (65)
\end{aligned}
$$

and can solve these by finite difference methods.

2 Speed of sound in a perfect gas

The main source of this derivation is chapter 6 of *"Intermediate Fluid Mechanics"* by R H Nunn (Hemisphere Publishing, 1989).

First we look for a relationship between the conditions on either side of a weak pressure wave in a rigid duct, travelling at the speed of the wave (which is c_1). The control volume is around the wavefront and we assume it is short enough to allow us to discount changes in area. The figure below shows conditions on either side of the control volume as it travels through the duct.



On the left-hand side we have pressure P_1, density ρ_1, velocity zero (because the control volume is travelling at the same speed as the wave) and speed of sound c_1. On the right-hand side we have pressure P_2, density ρ_2, velocity (relative to the wavefront) u_2 and speed of sound $c_1 - u_2$.

First we look at conservation of mass and rearrange it to make u_2 the subject.

$$\rho_1 c_1 = \rho_2(c_1 - u_2) \tag{66}$$
$$\Rightarrow \rho_1 c_1 = \rho_2 c_1 - \rho_2 u_2 \tag{67}$$
$$\Rightarrow \frac{\rho_1}{\rho_2} c_1 = c_1 - u_2 \tag{68}$$
$$\Rightarrow u_2 = c_1 - \frac{\rho_1}{\rho_2} c_1 \tag{69}$$
$$\Rightarrow u_2 = c_1 \left(1 - \frac{\rho_1}{\rho_2}\right) \tag{70}$$

Next we take the momentum equation across the pressure wave and rearrange it so that the pressure terms are on the left-hand side.

$$P_1 + \rho_1 c_1^2 = P_2 + \rho_2 (c_1 - u_2)^2 \tag{71}$$
$$\Rightarrow P_1 - P_2 = \rho_2 (c_1 - u_2)^2 - \rho_1 c_1^2 \tag{72}$$

Now we substitute for u_2 and collect common terms.

$$P_1 - P_2 = \rho_2 \left[c_1 - c_1 (1 - \frac{\rho_1}{\rho_2})^2\right] - \rho_1 c_1^2 \tag{73}$$
$$\Rightarrow P_1 - P_2 = \rho_2 \left(c_1 - c_1 + c_1 \frac{\rho_1^2}{\rho_2^2}\right) - \rho_1 c_1^2 \tag{74}$$
$$\Rightarrow P_1 - P_2 = \rho_2 \left(c_1 \frac{\rho_1^2}{\rho_2^2}\right) - \rho_1 c_1^2 \tag{75}$$

$$\Rightarrow P_1 - P_2 = c_1^2 \left(\rho_2 \frac{\rho_1^2}{\rho_2^2}\right) - c_1^2 \rho_1 \tag{76}$$

$$\Rightarrow P_1 - P_2 = c_1^2 \left(\frac{\rho_1^2}{\rho_2}\right) - c_1^2 \rho_1 \tag{77}$$

$$\Rightarrow P_1 - P_2 = c_1^2 \rho_1 \left(\frac{\rho_1}{\rho_2}\right) - c_1^2 \rho_1 \tag{78}$$

$$\Rightarrow P_1 - P_2 = c_1^2 \rho_1 \left(\frac{\rho_1}{\rho_2} - 1\right) \tag{79}$$

$$\Rightarrow P_1 - P_2 = c_1^2 \frac{\rho_1}{\rho_2} (\rho_1 - \rho_2) \tag{80}$$

$$\Rightarrow \frac{P_1 - P_2}{\rho_1 - \rho_2} = c_1^2 \frac{\rho_1}{\rho_2} \tag{81}$$

Finally, we make c_1^2 the subject of the equation.

$$c_1^2 = \frac{P_1 - P_2}{\rho_1 - \rho_2} \frac{\rho_2}{\rho_1} \tag{82}$$

Now we define $P_2 = P_1 + \delta P$ and $\rho_2 = \rho_1 + \delta\rho$. This lets us simplify it further.

$$c_1^2 = \frac{P_1 - (P_1 + \delta P)}{\rho_1 - (\rho_1 + \delta\rho)} \frac{\rho_1 + \delta\rho}{\rho_1} \tag{83}$$

$$\Rightarrow c_1^2 = \frac{P_1 - P_1 - \delta P}{\rho_1 - \rho_1 - \delta\rho} \frac{\rho_1 + \delta\rho}{\rho_1} \tag{84}$$

$$\Rightarrow c_1^2 = \frac{-\delta P}{-\delta\rho} \frac{\rho_1 + \delta\rho}{\rho_1} \tag{85}$$

$$\Rightarrow c_1^2 = \frac{\delta P}{\delta\rho} \frac{\rho_1 + \delta\rho}{\rho_1}. \tag{86}$$

Now when δP and $\delta\rho$ tend to small values, $\frac{\delta P}{\delta\rho} \to \frac{dP}{d\rho}$ and $\frac{\rho_1 + \delta\rho}{\rho_1} \to \frac{\rho_1}{\rho_1}$. So

$$c_1^2 = \frac{dP}{d\rho} \frac{\rho_1}{\rho_1} \tag{87}$$

$$\Rightarrow c_1^2 = \frac{dP}{d\rho}(1) \tag{88}$$

$$\Rightarrow c_1^2 = \frac{dP}{d\rho}. \tag{89}$$

3 Differential forms of the isentropic flow relationships

Most papers in the field express the isentropic flow relationships in differential form without giving a derivation. Even Shapiro's book *"The Dynamics and Thermodynamics of Compressible Fluid Flow"* (rightly regarded as the most comprehensive treatment of the subject) takes the isentropic flow relationship, the equation of state for a perfect gas and the expression for sound velocity

$$\frac{P}{\rho^\gamma} = \text{constant}, \tag{90}$$

$$\frac{P}{\rho T} = \text{constant and} \tag{91}$$

$$c^2 = \gamma R T \tag{92}$$

and states that they can be rearranged into their differential forms (eqn. 23.7, vol. 2, page 910),

$$\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dP}{P} = \frac{1}{\gamma - 1}\frac{dT}{T} = \frac{2}{\gamma - 1}\frac{dc}{c}. \tag{93}$$

Those differential forms were not at all obvious to me. This section is just a derivation of the four terms in (93) from (90)–(92). The main source I used for this was a NASA web page on isentropic fluid flow relationships.[1]

We start by stating a couple of additional thermodynamic relationships. The first relates the two specific heats of a gas (specific heat at constant pressure c_p and at constant volume c_v) to the gas constant R.

$$R = c_p - c_v. \tag{94}$$

The second is the definition of the ratio of specific heats, γ. It is

$$\gamma = \frac{c_p}{c_v}. \tag{95}$$

We rearrange (94) and put (95) into it to get $\frac{c_p}{R}$ in terms of γ;

$$\frac{R}{c_p} = \frac{c_p - c_v}{c_p} \tag{96}$$

$$\Rightarrow = \frac{c_p}{c_p} - \frac{c_v}{c_p} \tag{97}$$

$$\Rightarrow = 1 - \frac{1}{\gamma} \tag{98}$$

We take the reciprocal of $\frac{R}{c_p}$ and simplify it

$$\frac{c_p}{R} = \frac{1}{1 - \frac{1}{\gamma}} \tag{99}$$

[1] https://www.grc.nasa.gov/WWW/BGH/isndrv.html (accessed 29 July 2020).

$$\Rightarrow \quad = \quad \left(\frac{1}{1-\frac{1}{\gamma}}\right)\left(\frac{\gamma}{\gamma}\right) \tag{100}$$

$$\Rightarrow \quad = \quad \frac{\gamma}{(1-\frac{1}{\gamma})\gamma} \tag{101}$$

$$\Rightarrow \frac{c_p}{R} \quad = \quad \frac{\gamma}{\gamma-1} \tag{102}$$

Next we take the isentropic flow relationship, the perfect gas equation (in slightly different form to (91)), the expression for speed of sound and the relationship for entropy change ds. These are:

$$\frac{P}{\rho^{\gamma}} = \text{constant}, \tag{103}$$

$$P = \rho R T, \tag{104}$$

$$c^2 = \gamma R T, \tag{105}$$

$$ds = c_p \frac{dT}{T} - R\frac{dP}{P} \tag{106}$$

For an isentropic process $ds = 0$, so

$$c_p \frac{dT}{T} - R\frac{dP}{P} = 0 \tag{107}$$

$$\Rightarrow c_p \frac{dT}{T} = R\frac{dP}{P} \tag{108}$$

$$\Rightarrow \frac{c_p}{R}\frac{dT}{T} = \frac{dP}{P} \tag{109}$$

We can substitute (102) into (109) to yield one of the relationships in Shapiro's equation 23.7,

$$\frac{\gamma}{\gamma-1}\frac{dT}{T} = \frac{dP}{P}, \tag{110}$$

$$\Rightarrow \frac{1}{\gamma-1}\frac{dT}{T} = \frac{1}{\gamma}\frac{dP}{P}. \tag{111}$$

Next we take the equation of state (104) and substitute it for P in the entropy equation (108).

$$c_p \frac{dT}{T} = R\frac{dP}{\rho R T}. \tag{112}$$

$$\tag{113}$$

We can now cancel R and T,

$$\Rightarrow c_p dT = \frac{dP}{\rho}. \tag{114}$$

Now we take the equation of state (104) again, make T the subject and differentiate it.

$$T = \frac{P}{\rho R}, \tag{115}$$

$$\Rightarrow \partial T = \partial\left(\frac{P}{\rho R}\right), \tag{116}$$

$$\Rightarrow \partial T = \frac{1}{R}\partial\left(\frac{P}{\rho}\right). \tag{117}$$

Replace ∂T in (114) with the expression for ∂T in (117) and put it into differential form:

$$\frac{c_p}{R}\partial\left(\frac{P}{\rho}\right) = \frac{dP}{\rho}, \tag{118}$$

$$\Rightarrow \frac{c_p}{R}\left[\frac{dP}{\rho} + Pd\left(\frac{1}{\rho}\right)\right] = \frac{dP}{\rho}, \tag{119}$$

$$\Rightarrow \frac{c_p}{R}\left[\frac{dP}{\rho} - \frac{P}{\rho^2}d\rho\right] = \frac{dP}{\rho}. \tag{120}$$

Collect common terms of dP on the left hand side, multiply by $\frac{P}{\rho}$ and simplify:

$$\frac{c_p}{R}\frac{dP}{\rho} - \frac{dP}{\rho} = \frac{c_p}{R}\frac{P}{\rho^2}d\rho, \tag{121}$$

$$\Rightarrow \left(\frac{c_p}{R} - 1\right)\frac{dP}{\rho} = \frac{c_p}{R}\frac{P}{\rho^2}d\rho, \tag{122}$$

$$\Rightarrow \left(\frac{c_p}{R} - 1\right)\frac{dP}{P} = \frac{c_p}{R}\frac{\rho}{\rho^2}d\rho, \tag{123}$$

$$\Rightarrow \left(\frac{c_p}{R} - 1\right)\frac{dP}{P} = \frac{c_p}{R}\frac{d\rho}{\rho}. \tag{124}$$

(102) gives $\frac{c_p}{R}$ in terms of γ and we substitute that into (124) and simplify.

$$\left[\frac{\gamma}{\gamma-1} - 1\right]\frac{dP}{P} = \frac{\gamma}{\gamma-1}\frac{d\rho}{\rho}, \tag{125}$$

$$\Rightarrow (\gamma-1)\left[\frac{\gamma}{\gamma-1} - 1\right]\frac{dP}{P} = (\gamma-1)\frac{\gamma}{\gamma-1}\frac{d\rho}{\rho}, \tag{126}$$

$$\Rightarrow (\gamma-1)\left[\frac{\gamma}{\gamma-1} - 1\right]\frac{dP}{P} = \gamma\frac{d\rho}{\rho}, \tag{127}$$

$$\Rightarrow [\gamma - (\gamma-1)]\frac{dP}{P} = \gamma\frac{d\rho}{\rho}, \tag{128}$$

$$\Rightarrow [1]\frac{dP}{P} = \gamma\frac{d\rho}{\rho}, \tag{129}$$

$$\Rightarrow \frac{dP}{P} = \gamma\frac{d\rho}{\rho}, \tag{130}$$

$$\Rightarrow \frac{1}{\gamma}\frac{dP}{P} = \frac{d\rho}{\rho}. \tag{131}$$

The end result (131) is another identity in Shapiro's equation 23.7. It may be worth stating the three relationships together as we will use them to get to the fourth. They are

$$\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dP}{P} = \frac{1}{\gamma-1}\frac{dT}{T}. \tag{132}$$

Now re-arrange the speed of sound relationship (105) to make T the subject and differentiate it;

$$
\begin{aligned}
T &= \frac{c^2}{\gamma R}, && (133)\\
\partial T &= \partial\left(\frac{c^2}{\gamma R}\right) && (134)\\
\Rightarrow dT &= \frac{1}{\gamma R} d(c^2), && (135)\\
\Rightarrow \frac{dT}{T} &= \frac{1}{\gamma R T} d(c^2). && (136)
\end{aligned}
$$

Now $\gamma RT = c^2$, from (105). We also multiply by $\frac{1}{\gamma - 1}$ to give a result that is compatible with (132),

$$
\begin{aligned}
\frac{dT}{T} &= \frac{1}{c^2}\partial(c^2), && (137)\\
\Rightarrow \frac{1}{\gamma-1}\frac{dT}{T} &= \frac{1}{\gamma-1}\frac{d(c^2)}{c^2}. && (138)
\end{aligned}
$$

Finally—and I only write this out in full because I know I'll forget it—we take a new variable to represent c^2, say $X = c^2$ and use it to express $d(c^2)$ in terms of dc;

$$
\begin{aligned}
X &= c^2, && (139)\\
\Rightarrow \partial X &= \partial(c^2), && (140)\\
\frac{dX}{dc} &= \frac{d}{dc}(c^2), && (141)\\
\Rightarrow \frac{dX}{dc} &= 2c, && (142)\\
\Rightarrow dX &= 2c dc, && (143)\\
\Rightarrow d(c^2) &= 2c dc. && (144)
\end{aligned}
$$

If we substitute (144) into (138) and cancel $\frac{c}{c}$ we get the last of the terms in Shapiro's equation 23.7,

$$
\begin{aligned}
\frac{1}{\gamma-1}\frac{dT}{T} &= \frac{2c}{\gamma-1}\frac{dc}{c^2}, && (145)\\
\Rightarrow \frac{1}{\gamma-1}\frac{dT}{T} &= \frac{2}{\gamma-1}\frac{dc}{c}. && (146)
\end{aligned}
$$

Putting them all together, the relationships for isentropic flow in differential form in Shapiro equation 23.7 are

$$
\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dP}{P} = \frac{1}{\gamma-1}\frac{dT}{T} = \frac{2}{\gamma-1}\frac{dc}{c}. \qquad (147)
$$

4 1D compressible flow equations

We consider unsteady, one-dimensional, compressible fluid flow in a rigid pipe of constant area (e.g. air in a tunnel).

We take the equation of conservation of mass

$$\frac{\partial u}{\partial x} + \frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{1}{\rho}\frac{\partial \rho}{\partial t} = 0 \tag{148}$$

and the equation of momentum

$$\frac{1}{\rho}\frac{\partial P}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + E = 0. \tag{149}$$

We have two independent variables: x, distance along the tunnel and t, time.

We have three dependent variables: P, air pressure; ρ, air density; and v, air velocity.

E represents body forces on the fluid, such as wall friction (we ignore smooth changes in area, porous walls and changes in elevation).

We are interested in isentropic flow. In sections 2 and 3 we showed that when we have isentropic flow,

$$c^2 = \frac{dP}{d\rho}, \tag{89}$$

$$\frac{d\rho}{\rho} = \frac{2}{\gamma - 1}\frac{dc}{c}. \tag{147}$$

We can use these relationships to make both P and ρ dependent variables of c. Then we can use the method of characteristics to make a set of equations that can be solved simultaneously for the two dependendent variables u and c, use u and c in the mass equation to get ρ, then use those three to determine P.

First we make P the subject and differentiate P with respect to ρ to get

$$\partial P = c^2 \partial \rho. \tag{150}$$

This allows us to replace $\frac{1}{\rho}\frac{\partial P}{\partial x}$ with $\frac{c^2}{\rho}\frac{\partial \rho}{\partial x}$ in the momentum equation, giving

$$\frac{c^2}{\rho}\frac{\partial \rho}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + E = 0. \tag{151}$$

It follows from (147) that

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = \frac{2}{(\gamma - 1)c}\frac{\partial c}{\partial t} \text{ and} \tag{152}$$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial x} = \frac{2}{(\gamma - 1)c}\frac{\partial c}{\partial x}. \tag{153}$$

We substitute the $\partial\rho$ terms with ∂c terms in the equation of mass (150)

$$\frac{\partial u}{\partial x} + \frac{2u}{(\gamma-1)c}\frac{\partial c}{\partial x} + \frac{2}{(\gamma-1)c}\frac{\partial c}{\partial t} = 0 \tag{154}$$

For convenience we multiply by $c\frac{\gamma-1}{2}$ to get

$$c\frac{\gamma-1}{2}\frac{\partial u}{\partial x} + u\frac{\partial c}{\partial x} + \frac{\partial c}{\partial t} = 0. \tag{155}$$

Now we make the same substitution in the modified equation of momentum (153) to get

$$\frac{2c^2}{(\gamma-1)c}\frac{\partial c}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + E = 0, \tag{156}$$

$$\Rightarrow \frac{2c}{\gamma-1}\frac{\partial c}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + E = 0. \tag{157}$$

As an aside, equations (155) and (157) are generally what all the "it can be shown that..." papers jump to when starting from (148) and (149). It's taken me five pages to get here; no wonder they all made the jump.