A rant about friction factors called 'f'

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1. What's wrong with these equations?[1]

Many engineers reading this will be familiar with the following equation for frictional pressure drop in fluid flow,

$$\Delta P \quad = \quad \frac{fL}{D_h} 1/2 \rho U^2. \tag{1}$$

Unfortunately many other engineers reading this are familiar with a slightly different one that is easy to confuse with (1). It is

$$\Delta P \quad = \quad \frac{4fL}{D_h} 1/2 \rho U^2. \tag{2}$$

(1) and (2) both use f to represent their friction factor, but their values of f differ by a factor of four. Both give the same pressure drop ΔP given the same values of L, D_h, ρ and U.

These days, f in (1) is usually known as Darcy friction factor and f in (2) is usually known as Fanning friction factor. f in (1) and f in (2) both have long histories in the literature—the earliest instance of f in (1) I could find was 1942 (in the USA) and the earliest instance of f in (2) I could find was 1917 (also in the USA). But it is clear that friction factors that differ by a factor of four have been around a lot longer: in his 1878 book [1], T J Fanning takes Weisbach's data (he quotes Weisbach's data as Darcy friction factor ζ) and converts it to Fanning friction factor k. Engineers in the 19th century were used to handling friction factors that differed by a factor of four, but had enough sense to use different symbols for them.

Unlike us! For at least 80 years the engineering literature has been rife with papers using f to represent Fanning friction factor and other papers using f to represent Darcy friction factor. Every time I read a paper that uses friction factor f, I need to look closely at it to figure out whether Fanning or Darcy friction factor is meant. When I was a young engineer I frequently mistook one for the other. Now I'm (mostly) wise to it. And I've seen a **lot** of colleagues fall into the same trap over the years.

[1] with apologies to Prof. N D Mermin.

2. Some history, and a few things to be wary of

The factor of four problem originated in the difference between hydraulic mean depth in open-channel flow and hydraulic diameter in pipe flow.

Serious fluid friction research started in the 1700s with open-channel flow in rivers and canals. They used hydraulic mean depth (river flow area divided by riverbed perimeter). When pipe flow developed in the 1800s, they tried renaming hydraulic mean depth as hydraulic radius, but quickly realized it was too confusing (as the hydraulic radius of a circular pipe is half its physical radius). The pipe flow practitioners multiplied hydraulic mean depth by four and started calling it hydraulic diameter. Much saner, as the hydraulic diameter of a circular pipe is the same as its physical diameter.

Having created hydraulic diameter, it was an easy step to create a friction factor that was four times higher than the friction factor used in open channel flow. It let you remove that factor of four in the numerator of your friction equation. As long as everyone takes care to use different symbols, what could possibly go wrong?

So different symbols were used for the two friction factors (for example: the ζ and k that Fanning quotes in his 1878 book [1], where $\zeta = 4k$). But inevitably someone started using f for Fanning's k and someone else started using f for Weisbach's ζ. Both definitions of f gained popularity and from the early to mid 20th century, fluid mechanics literature became the minefield that we all love walking through whenever we pick up a paper and see friction factor f.

The current situation seems to be "Darcy friction factor f is used in north America and Fanning friction factor f is used in some professions in Europe and Japan". But there are counterexamples:

- Harding and Willard's 1917 HVAC book [2] and Singstad's 1929 paper on the Holland Tunnel [3] both use Fanning friction factor f in conjunction with hydraulic radius. Both were written in the US.
- The US-sponsored translation of Nikuradse's 1933 sand roughness experiments (by NACA in 1950 [4]) uses Darcy friction factor λ.
- Shapiro's 1950s books on compressible fluid flow [5] (written in the US) use f to represent Fanning friction factor.
- D. S. Miller's 1978 book on the resistance of ducts [6] uses f for Darcy friction factor despite having being written in the UK.

A subtle source of confusion is that in older papers when the word "Darcy" is used it may not match present-day terminology. For example, Fox's 1968 ICE paper on the method of characteristics [7] states "Darcy f", but a little investigation makes it clear that the friction factor he uses is the one that we nowadays call Fanning friction factor. So even if a paper states "Darcy friction factor f" in the text, it may not be the friction factor that you expect. Be on your guard!

Worse, some journals switched symbols over time. Compare the typesetting in these papers, all from the Proceedings of the Institution of Civil Engineers:

- Anderson's 1936 paper on the Mersey Tunnel [8] used f to represent Fanning friction factor,
- Colebrook's 1939 paper on pipe friction [9] used λ to represent Darcy friction factor,
- Colebrook's 1958 paper on hydropower tunnel friction [10] used f to represent Darcy friction factor.

I have no idea why the journal's editors thought that using f in [10] would not lead to confusion when reading [10] alongside [8]. I mean, tunnel friction is about as niche a field as you can get: both [8] and [10] have full-scale test data in them, they're going to be read side-by-side more than once!

I see the whole "*which friction factor is represented by f*" catching out a lot of engineers venturing into the high speed rail tunnel field for the first time. Japanese and European high speed rail tunnel papers mostly use f in the sense of (2). Most of the engineers I've worked with from North America use f in the sense of (1) and don't learn about (2) until a reviewer points out a mistake.

To suggest "OK, why don't we just agree to use a consistent set of symbols in future?" is *not* a solution. I reckon that is probably what started it: my guess is that some time between 1920 and 1940 a major engineering body decided "From now on let's use symbol f for Darcy friction factor and stop using Fanning friction factor entirely. Everyone else will be sure to follow our lead."

And it would not solve the problem if every journal in the world agreed to use a consistent symbol and a consistent equation from now on (good though that would be). All those older published papers—many of which have genuinely useful full-scale test data—are still out there, ready to be misinterpreted.

3. What am I going to do?

If you've managed to make it this far, you probably understand my dilemma. I'm writing ventilation software. If I make f in (1) or f in (2) the default, I run a risk of pushing unwary users into a trap that I've fallen into many times and that I've seen almost all my colleagues fall into as well.

My solution is this:

- I don't use the symbol f at all (except in this document).
- I don't set a default type of friction factor. Instead I force the user to make a choice between Fanning and Darcy friction factor.
- I prepared a Moody chart with both Fanning and Darcy friction factor on the ordinate axis.
- I wrote this (surprisingly cathartic!) rant about the difference between them, to help those unaware that there even **is** a problem.

Fanning friction factor is represented by c_f (a popular symbol for it in the aeronautics field) and Darcy friction factor is represented by λ (a popular symbol for it in the pipe flow field). The friction equations then become

$$\Delta P = \frac{c_f S L}{A} 1/2 \rho U^2, \tag{3}$$

$$= \frac{4 c_f L}{D_h} 1/2 \rho U^2, \tag{4}$$

$$= \frac{\lambda L}{D_h} 1/2 \rho U^2, \tag{5}$$

$$= \frac{c_f L}{R_h} 1/2 \rho U^2 \tag{6}$$

where

$$\begin{aligned}
\Delta P &= \text{pressure drop } (Pa), \\
c_f &= \text{Fanning friction factor } (-), \\
S &= \text{tunnel perimeter } (m), \\
A &= \text{tunnel area } (m^2), \\
L &= \text{tunnel length } (m), \\
\lambda &= \text{Darcy friction factor } (-), \\
D_h &= \text{hydraulic diameter } (m) \equiv \frac{4A}{S}, \\
R_h &= \text{hydraulic mean depth/hydraulic radius } (m) \equiv \frac{A}{S}, \\
\rho &= \text{fluid density } (kg/m^3), \\
U &= \text{fluid velocity } (m/s).
\end{aligned}$$

If you only know about one friction factor f and don't know if your f is c_f in (3) or λ in (5), then the Moody chart at the end of this document is probably the best place to start. That Moody chart has two ordinate axes; one for c_f, the other for λ. Compare them to the ordinate axis on a Moody chart in a textbook that you trust. You should be able to figure out if the f you were taught about is c_f or λ.

One final point. If you cast your net wide enough you will encounter other definitions of friction factor. I know of four:

- c_f, skin friction coefficient or Fanning friction factor (dimensionless) [11]
- λ, Darcy friction factor (dimensionless) [12], $\lambda = 4c_f$
- k, Atkinson friction factor (units kg/m^3) [13], [14], $k = \frac{1}{2}\rho c_f$
- α, friction factor from mine ventilation in the USSR (units kg-s^2/m^4) [12], [15], $\alpha = \frac{1}{2}\rho c_f/g$

where ρ is standard air density and g is the acceleration of gravity.

k and α are included here because I have often stumbled across useful experimental data in mine ventilation literature and this is a good place to store their definitions for future reference.

3. Friction factor approximations

When C F Colebrook defined the Colebrook–White function for friction factor in his 1939 ICE paper [9] he used an implicit approximation,

$$\frac{1}{\sqrt{\lambda}} = -2\log_{10}\left(\frac{\varepsilon}{3.7D_h} + \frac{2.51}{Re\sqrt{\lambda}}\right). \tag{7}$$

On the next page Colebrook noted that the implicit formulation was inconvenient and gave an explicit term that could replace the last term in (7) with little loss of accuracy. It was

$$-2\log_{10}\frac{2.51}{Re\sqrt{\lambda}} \approx -1.8\log_{10}\frac{Re}{7}. \tag{8}$$

When converted to a form that is more common today, Colebrook's explicit approximation is

$$\lambda \approx \frac{0.25}{\left\{\log_{10}\left[\frac{\varepsilon}{3.7D_h} + \left(\frac{7}{Re}\right)^{0.9}\right]\right\}^2} \approx \frac{0.25}{\left[\log_{10}\left(\frac{\varepsilon}{3.7D_h} + \frac{5.76}{Re^{0.9}}\right)\right]^2}. \tag{9}$$

Colebrook's 1939 explicit approximation was pretty accurate. And constants of 0.9 & 7 (or 0.9 & 5.76 if you prefer) are easier to remember than some of the horrors I've seen in more recent papers. Alas, the presence of an adequate explicit approximation in the original paper did not stop everyone and their dog piling in to write another over the years. I'd guess there must be well over a hundred explicit approximations by now.

One that has been popular in the tunnel vent field was produced by L F Moody in 1947 [16]. It is

$$\lambda = 0.0055\left\{1 + \left(20,000\frac{\varepsilon}{D_h} + \frac{10^6}{Re}\right)^{1/3}\right\}. \tag{10}$$

Moody's approximation couldn't have been popular for its accuracy—it is woeful at high ε/D_h.[2] Perhaps it was easier to work out on a slide rule than Colebrook's?

Various researchers have developed more complex explicit expressions that correlate ever better to the exact values. I suspect that few would have bothered if they had read Colebrook's paper thoroughly, as that paper has a *lot* of rounding. Early in the paper Colebrook derived the constant 3.7 in the denominator of the Colebrook–White function. He took 0.113 (a value that Colebrook and White had derived from experiments) and multiplied it by 33 (a value that Nikuradse had derived from experiments) to get 3.7.

The true product of those two numbers is 3.729. I did a quick test in a spreadsheet: if the 3.7 in the denominator of (7) is changed to 3.729 the calculated values of friction change by about a quarter of a percent at $\varepsilon/D_h = 0.008$. If that level of imprecision was accepted in the original paper then developing an

[2] It is 5% too low at $\varepsilon/D_h = 0.02$ and 26% too low at $\varepsilon/D_h = 0.1$.

explicit approximation that matches (7) to better than a quarter of a percent seems a bit pointless.

Developers of tunnel ventilation software have, by and large, chosen an approximation from the popular literature and stuck with it over the years.

Henson and Fox [17] used Moody's approximation (10) when developing the 1D method of characteristics that became the Mott MacDonald Aero program.

Those who have looked into the SES v4.1 source code (`Omega5.for`) will know that it uses a modified version of Moody's approximation that improves the accuracy at high ε/D_h,

$$\lambda = 0.0055\left\{1 + \left(\frac{\varepsilon}{D_h}\left[19000 + 16000\left(\frac{\varepsilon}{0.05D_h}\right)^{1.5}\right] + \frac{10^6}{Re}\right)^{1/3}\right\}. \tag{11}$$

I don't know what is in SVS v6, but I have seen no indication that WSP have moved away from (11).

Two commonly-used tunnel vent programs can't be directly compared, as I don't know the details of how they calculate friction. They are

1. **Thermotun**. I'd guess that Thermotun started off using Moody's approximation (both Aero and Thermotun originated in Prof. Fox's research group at the University of Leeds). But Prof. Vardy mentioned approximations attributed to Schlichting and to Jain in his 1996 IMechE paper [18]. And he has published a number of papers in the field of transient friction factors, including an interesting one co-authored with Fox about transient friction factors in the annulus around trains [19]. So Thermotun probably has a sensible default and some esoteric options.

2. **IDA**. The friction factor calculation in IDA Tunnel/IDA RTV is unclear. IDA users will know that you set a Darcy friction factor λ for each section. But if you delve into the subsegment properties after a calculation it is clear that each subsegment's friction factor is being adjusted for Reynolds number somehow. But as far as I'm aware the formula of the adjustment isn't in IDA's Theoretical Reference.

4. How the program handles friction

Hobyah has no default friction factor. You must include `frictiontype Darcy` or `frictiontype Fanning` in your input file's `options` block to tell it which type you want it to use. It forces you to look into the difference between Darcy and Fanning friction factor.

I did seriously consider making Darcy friction factor the default. It would be consistent with two widely used programs (SES and IDA Tunnel). But when I thought back on the engineers I've worked with who only knew about one friction factor, the majority of them only knew about Darcy friction factor f. This makes them most at risk of falling into the trap when they encounter a source that uses Fanning friction factor f. Not setting a default forces them to stop, think, and (hopefully) avoid the trap.

In places where you input friction factors, if you set `frictiontype Darcy` it will interpret your values as λ. If you had set `frictiontype Fanning` instead, it will interpret your values as c_f.

In places where you input roughness heights, if you set `frictiontype Darcy` the printed or plotted output will be λ. if you had set `frictiontype Fanning`instead, the printed or plotted output will be c_f.

When converting roughness heights into friction factors you can select a friction factor approximation by an optional setting, `frictionapprox`. It has the following options:

- `frictionapprox Colebrook`—you set a roughness height, it uses Colebrook's 1939 explicit approximation (9).
- `frictionapprox Moody`—you set a roughness height, it uses Moody's 1947 explicit approximation (10).
- `frictionapprox SES`—you set a roughness height, it uses the explicit approximation from the SES source code (11).
- `frictionapprox fixed`—you set a friction factor, not a roughness height. Your friction factors do not change with Reynolds number.
- `frictionapprox Colebrook-White`—you set a roughness height, the program makes up to fifty iterations of the Colebrook-White function, stopping when the difference between successive estimates is less than 0.1%. I reckon that level of precision is pointless for engineering software, given how low the accuracy of all our other inputs are (e.g. train cross-sectional areas, draught relief shaft pressure loss factors). But it is there if you think you need it.

For what it is worth, the heatmaps on the next page identify the absolute percentage difference between the three inbuilt explicit approximations (`colebrook`, `moody`, `ses`)[3] and the exact (iterated) value of the Colebrook-White function for a range of relative roughnesses and Reynolds numbers.

The inbuilt approximations are all steady-state friction factor approximations. I have no plans to look into transient friction factors.

The default is Colebrook's 1939 explicit approximation (9), mostly on the basis of the heatmaps. The other options are included so that runs can be set up that can be compared to the output of other tunnel ventilation programs (for validation) and to let you assess what the effects of the different approximations are.

I have heard plausible arguments that Moody's approximation ought to be preferred because it was the approximation used to back-calculate train and tunnel roughnesses in many full-scale tests in the UK and Europe in the 1970s and 1980s. I have a lot of sympathy for that view, as I have test reports/papers that give both friction factors and roughness heights, all of which are related by Moody's approximation. On the other hand, I know of at least one project in which a (very rough) roughness height was based on test data using Moody's approximation (10). The roughness height was given to a firm doing a third party check, who converted it into a friction factor using (11), the approximation in SES. The tested friction factor and the friction factor calculated by (11) were quite different.

In the end I decided to make the default the explicit approximation in Colebrook's original 1939 paper—it is easy to remember and more accurate than many that came after it. I've provided alternate approximations that I know are used in other tunnel ventilation software. Competent programmers who want a specific approximation should have no trouble adding it.

There are three other behaviours worth mentioning about the default friction factor routines in the program, all based on Fox's recommendations [20].

- At Reynolds number of 2300 and above the turbulent friction factor approximation is used.
- When the Reynolds number falls below 2300, the friction factor is estimated by the expression for laminar flow in a circular pipe ($c_f = 16/Re$, $\lambda = 64/Re$).
- If Reynolds number falls below 0.1 a fixed value of $c_f = 160$ is used.

References

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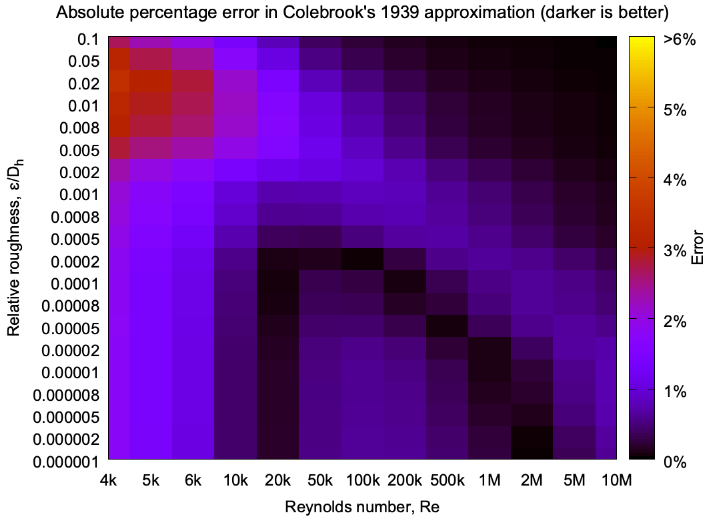
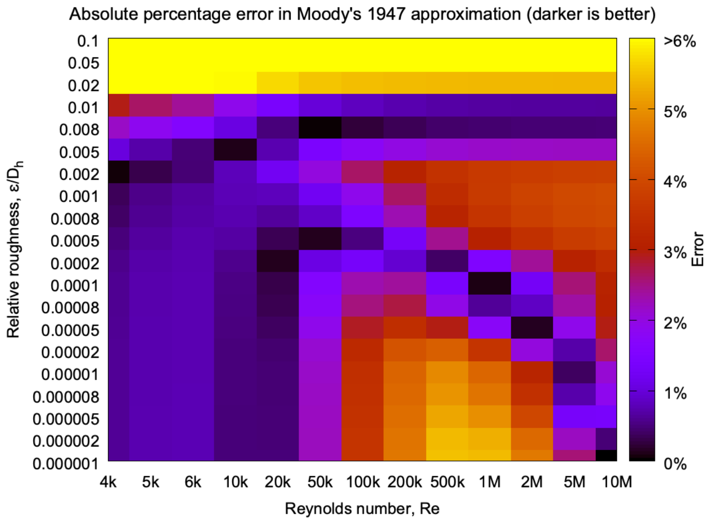
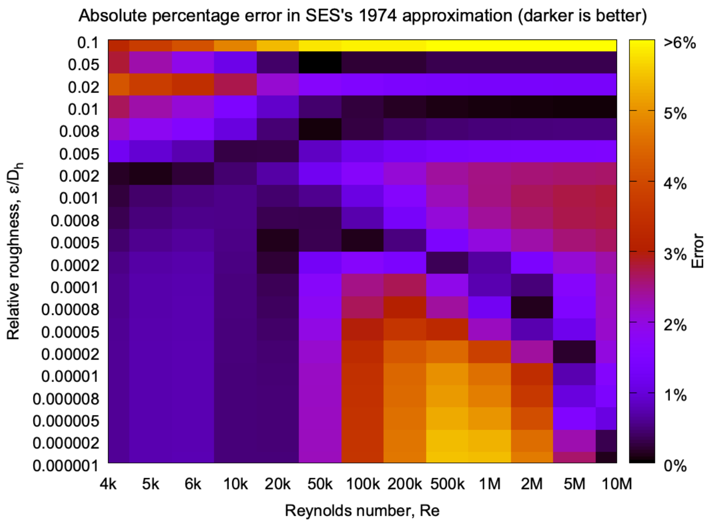
[3] I put these terms in lower case here just so I could include a footnote emphasizing that the input files are not case sensitive.


Absolute percentage error in Colebrook's 1939 approximation (darker is better)
Relative roughness, ε/D_h
0.1
0.05
0.02
0.01
0.008
0.005
0.002
0.001
0.0008
0.0005
0.0002
0.0001
0.00008
0.00005
0.00002
0.00001
0.000008
0.000005
0.000002
0.000001
4k
5k
6k
10k
20k
50k
100k
200k
500k
1M
2M
5M
10M
Reynolds number, Re
>6%
5%
4%
3%
2%
1%
0%
Error


Absolute percentage error in Moody's 1947 approximation (darker is better)
Relative roughness, ε/D_h
0.1
0.05
0.02
0.01
0.008
0.005
0.002
0.001
0.0008
0.0005
0.0002
0.0001
0.00008
0.00005
0.00002
0.00001
0.000008
0.000005
0.000002
0.000001
4k
5k
6k
10k
20k
50k
100k
200k
500k
1M
2M
5M
10M
Reynolds number, Re
>6%
5%
4%
3%
2%
1%
0%
Error


Absolute percentage error in SES's 1974 approximation (darker is better)
Relative roughness, ε/D_h
0.1
0.05
0.02
0.01
0.008
0.005
0.002
0.001
0.0008
0.0005
0.0002
0.0001
0.00008
0.00005
0.00002
0.00001
0.000008
0.000005
0.000002
0.000001
4k
5k
6k
10k
20k
50k
100k
200k
500k
1M
2M
5M
10M
Reynolds number, Re
>6%
5%
4%
3%
2%
1%
0%
Error

2. Harding, L A & Willard, A C, "Mechanical equipment of buildings", Wiley, 1917
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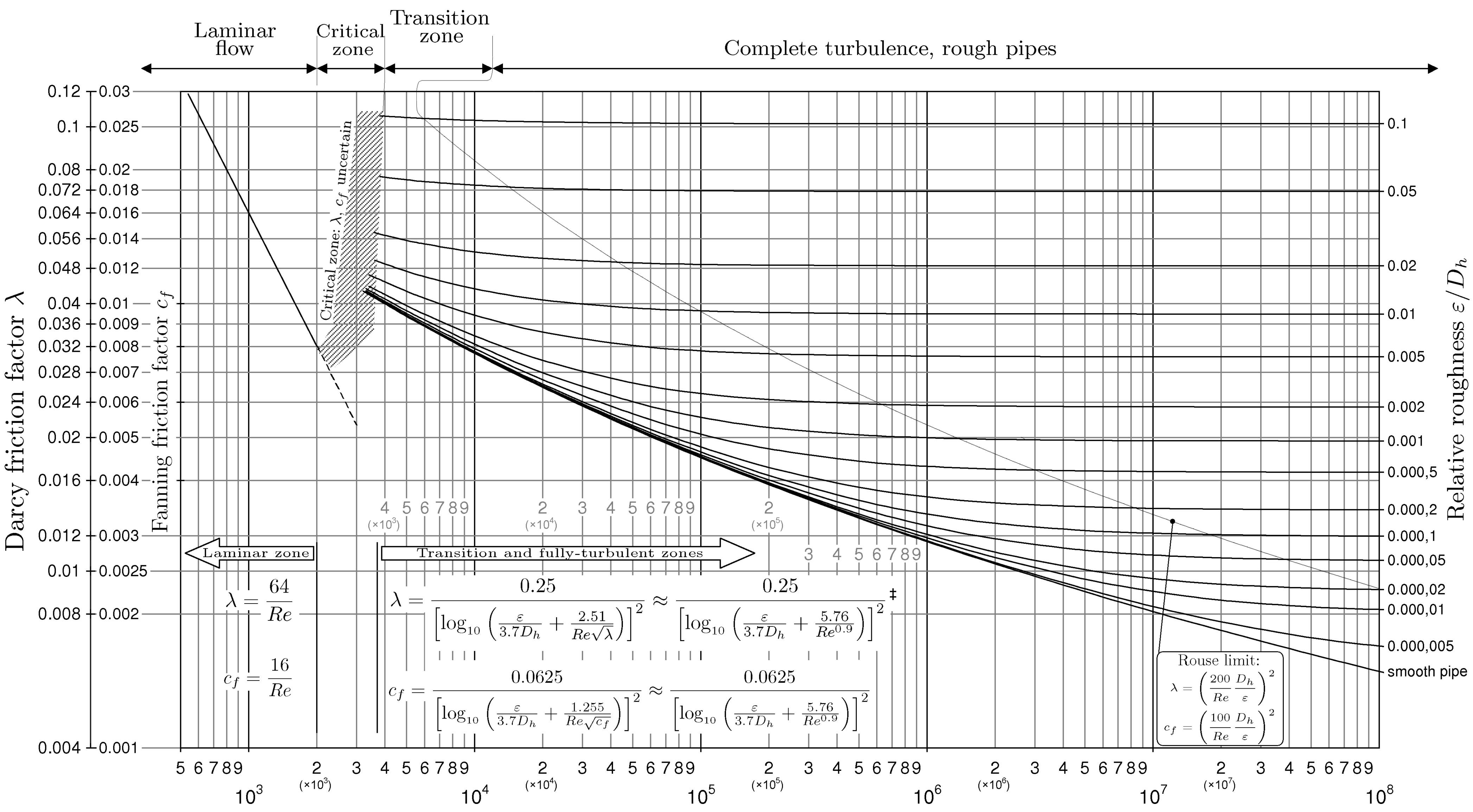
Moody chart for Darcy friction factor λ, used in $\Delta P = \frac{\lambda L}{D_h}\frac{1}{2}\rho U^2$

(also shows Fanning friction factor c_f, used in $\Delta P = \frac{4c_f L}{D_h}\frac{1}{2}\rho U^2$)



‡ This approximation to the Colebrook-White function is taken from equation 14 in: Colebrook, C F, "Turbulent Flow in Pipes, with particular reference to the Transition Region between the Smooth and Rough Pipe Laws," Paper 5204, *J. Instn. Civ. Engrs.* 1939, no. 4, vol 11, pp. 133–156.

Moody-chart.pdf