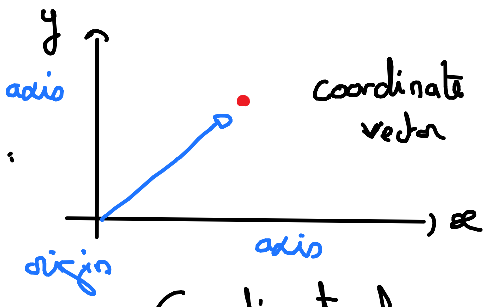
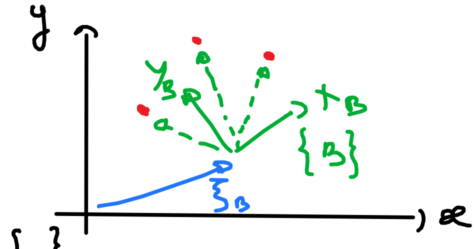
Kinematics : Representing position et orientation.

Exemple : 2D



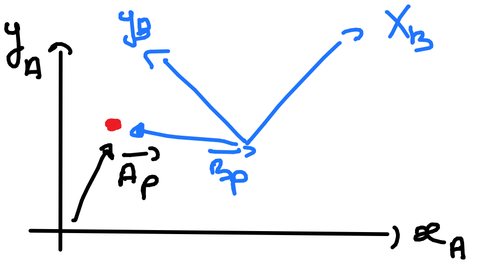
Coordinate frame

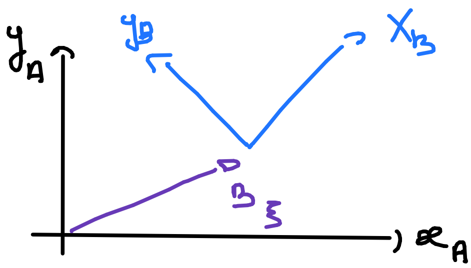


{o}
{A} ξ_B : pose of the coordinate frame B

A coordinate frame is a set of orthogonal axis which interact at the origin

A point p can be described by coordinate vectors with respect to {A} or {B}

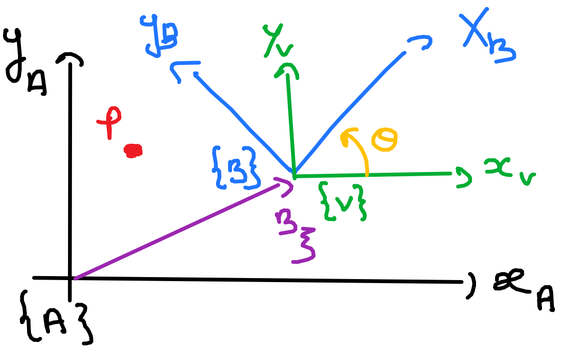




Representing Pose in 2D

$$\vec{X_B} = \cos\theta \cdot \vec{x_v} + \sin\theta \cdot \vec{y_v}$$

$$\vec{Y_B} = -\sin\theta \cdot \vec{x_v} + \cos\theta \, \vec{y_v}$$



$$\begin{pmatrix} X_B \\ Y_B \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{x_v} \\ \vec{y_v} \end{pmatrix} \xrightarrow[\text{Transposition}]{} (X_B, Y_B) = (\vec{x_v}, \vec{y_v}) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$P = B_x \cdot \vec{x}_B + B_y \cdot \vec{y}_B = \begin{pmatrix} \vec{x}_B & \vec{y}_B \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix}$$

$$= \begin{pmatrix} x_v & y_v \end{pmatrix} \boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix}}$$

$$= \begin{pmatrix} x_v & y_v \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \vec{R}_{B \to V} \begin{pmatrix} B_x \\ B_y \end{pmatrix}$$

Rotation Matrix : Rotate a vector in {B} to {V}

Properties of rotation matrix R in 2D

1) R is orthonormal $\Leftrightarrow$ - each column is a unit vector
- the column are orthogonal to each other

2) Its determinant is 1 (det $R = 1$)

3) $\boxed{R^{-1} = R^T}$ $\qquad R \in SO(2) \subset \mathbb{R}^{2\times 2}$

↪ special orthogonal Group of dim 2.

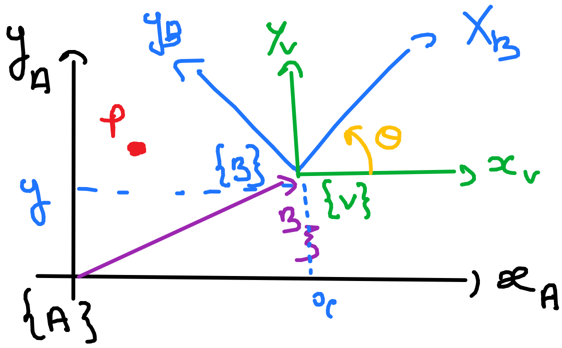
$$\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \left(R_{B \to V}\right)^{-1} \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

We now need to consider the translation between the origin.

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} V_x \\ V_y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ 1 \end{pmatrix}$$



$$\begin{pmatrix} A_x \\ A_y \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{R}_{B\to A} & t \\ \vec{0}_{1\times 2} & 1 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ 1 \end{pmatrix} \Rightarrow \vec{{}^{A}\tilde{p}} = \vec{T}_{B\to A} \cdot {}^{B}\tilde{p}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \begin{pmatrix} x \\ y \end{pmatrix}$$

Homogeneous transformation matrix

$T \in SE(2) \subset \mathbb{R}^{3\times 3}$

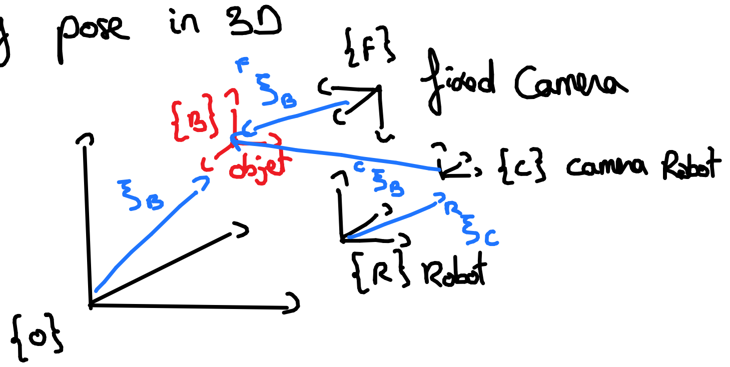
↳ special euclidian groupe of dimension 2

In summary Pose $\triangleq \xi_B(x, y, \theta) \sim T$

$$T = \begin{pmatrix} \vec{R}(\theta) & \vec{t} \\ \vec{0} & 1 \end{pmatrix} \text{ where } \vec{t} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Representing pose in 3D

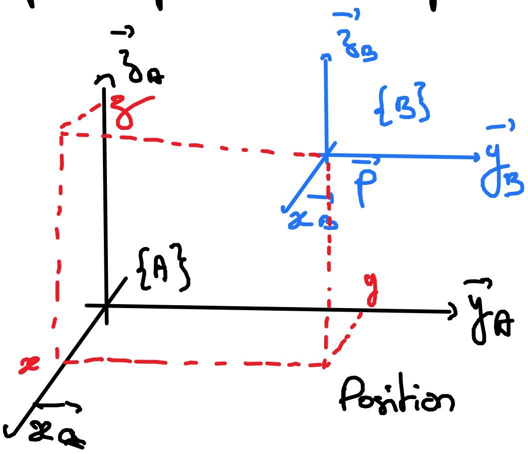
Exemple :



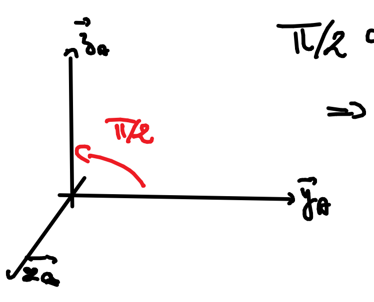
In 3D, we add one extra coordinate axis to the 2D case, denotate by z which is orthogonal to both the x and y axis

The direction of z-axis obey the right hand rule: $\vec{z} = \vec{x} \wedge \vec{y}$
$\vec{x} = \vec{y} \wedge \vec{z}$
$\vec{y} = \vec{z} \wedge \vec{x}$.

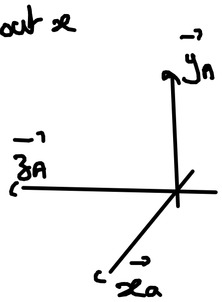
A point $\vec{p}$ can be represented as: $\vec{p} = x\vec{x} + y\vec{y} + z\vec{z}$



Position

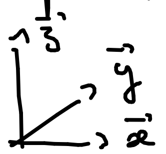


$\pi/2$ about x



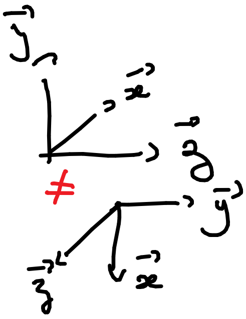
Rotation

Exemple: No commutativity of the rotation matrices



1) Rotate $\pi/2$ about $\vec{x}$ and $\pi/2$ about $\vec{y}$

2) Rotate $\pi/2$ about $\vec{y}$ and $\pi/2$ about $\vec{x}$

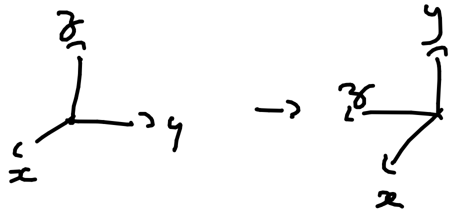


orthonormal rotation matrix in 3D

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exemple : $\pi/2$ about x-axis



$$\left(\vec{x}_B \;\; \vec{y}_B \;\; \vec{z}_B\right) = \left(\vec{x}_0 \;\; \vec{y}_0 \;\; \vec{z}_0\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad I_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{array}{ccc} \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 0 \\ -1 \\ 0 \end{matrix}} \end{pmatrix} \\ \vec{x}_B \quad \vec{y}_B \quad \vec{z}_B \end{array}$$

$\vec{x}_B = \vec{x}_0$

$\vec{y}_B = \vec{z}_0$

$\vec{z}_B = -\vec{y}_0$

Three angles representations (Eulers rotation theorem)

There are two classes of rotation sequence.

1) Eurelian : XYX, XZX, YXY, YZY, ZXZ, ZYZ.

2) Cardanian : XYZ, XZY, YZX, YXZ, ZXY, ZYX.

↳ roll-pitch-yaw

$$R = R_x(\theta_r) \cdot R_y(\theta_p) \cdot R_z(\theta_y)$$

Combining translation and orientation (homogeneous transformation matrix)

$$\begin{pmatrix} x_a \\ y_a \\ z_a \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{R}_{B \to A} & \vec{t} \\ \vec{O}_{1\times 4} & 1 \end{pmatrix} \begin{pmatrix} x_B \\ y_B \\ z_B \\ 1 \end{pmatrix}$$

If we concatenate two transformation

$$\vec{T}_1 \cdot \vec{T}_2 = \begin{pmatrix} \vec{R}_1 & \vec{t}_1 \\ \vec{0} & 1 \end{pmatrix} \begin{pmatrix} \vec{R}_2 & \vec{t}_2 \\ \vec{0} & 1 \end{pmatrix} = \begin{pmatrix} \vec{R}_1 \cdot \vec{R}_2 & \vec{t}_1 + \vec{R}_1 \vec{t}_2 \\ \vec{0} & 1 \end{pmatrix}$$

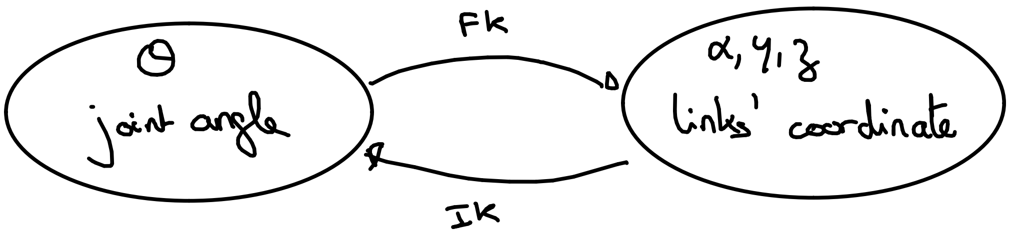
we can invert transformation.

$$T^{-1} = \begin{pmatrix} \vec{R} & \vec{t} \\ \vec{0} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \vec{R}^{-1} & -\vec{R}^{T} \vec{t} \\ \vec{0} & 1 \end{pmatrix}$$

Denavit-Hartemberg (DH) parameters.

Def: DH parameters are four parameters associated with a particular convention for attaching reference frames to a robot manipulator.

Forward kinematics (FK) vs. Inverse kinematic (IK)



DH convention: $A_i = Rot_{z,\theta_i} \cdot Trans_{z,d_i} \cdot Trans_{x,a_i} \cdot Rot_{x,\alpha_i}$