Pre-lecture brain teaser

What is the running time of the following algorithm:

Consider computing $f(x, y)$ by recursive function + memoization.

$$f(x, y) = \sum_{i=1}^{min(x,y)} x * f(x + y - i, i - 1),$$

$$f(0, y) = y \qquad f(x, 0) = x.$$

The resulting algorithm when computing $f(n, n)$ would take:

(a) $O(n^2)$
(b) $O(n^3)$
(c) $O(2^n)$
(d) $O(n^n)$
(e) The function is ill defined - it can not be computed.

ECE-374-B: Lecture 13 - Dynamic Programming II

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October 10, 2023

University of Illinois at Urbana-Champaign

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Recipe for Dynamic Programming

1. Develop a recursive backtracking style algorithm $\mathcal{A}$ for given problem.
2. Identify structure of subproblems generated by $\mathcal{A}$ on an instance I of size n
   2.1 Estimate number of different subproblems generated as a function of n. Is it polynomial or exponential in n?
   2.2 If the number of problems is "small" (polynomial) then they typically have some "clean" structure.
3. Rewrite subproblems in a compact fashion.
4. Rewrite recursive algorithm in terms of notation for subproblems.
5. Convert to iterative algorithm by bottom up evaluation in an appropriate order.
6. Optimize further with data structures and/or additional ideas.

Edit Distance and Sequence Alignment

Spell Checking Problem

Given a string "exponen" that is not in the dictionary, how should a spell checker suggest a nearby string?

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What does nearness mean?

Question: Given two strings $x_1 x_2 \ldots x_n$ and $y_1 y_2 \ldots y_m$ what is a distance between them?

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What does nearness mean?

Question: Given two strings $x_1 x_2 \ldots x_n$ and $y_1 y_2 \ldots y_m$ what is a distance between them?

Edit Distance: minimum number of "edits" to transform x into y.

Edit Distance

Definition

Edit distance between two words X and Y is the number of letter insertions, letter deletions and letter substitutions required to obtain Y from X.

Example

The edit distance between FOOD and MONEY is at least 4:

$$\underline{\mathrm{F}}\mathrm{OOD} \rightarrow \mathrm{MO}\underline{\mathrm{O}}\mathrm{D} \rightarrow \mathrm{MON}\underline{\mathrm{O}}\mathrm{D} \rightarrow \mathrm{MONE}\underline{\mathrm{D}} \rightarrow \mathrm{MONEY}$$

Edit Distance: Alternate View

Alignment
Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

$$\begin{array}{ccccc} \mathrm{F} & \mathrm{O} & \mathrm{O} & & \mathrm{D} \\ \mathrm{M} & \mathrm{O} & \mathrm{N} & \mathrm{E} & \mathrm{Y} \end{array}$$

Edit Distance: Alternate View

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$$\begin{array}{ccccc} \mathrm{F} & \mathrm{O} & \mathrm{O} & & \mathrm{D} \\ \mathrm{M} & \mathrm{O} & \mathrm{N} & \mathrm{E} & \mathrm{Y} \end{array}$$

Formally, an alignment is a set M of pairs (i, j) such that each index appears at most once, and there is no "crossing": $i < i'$ and i is matched to j implies i' is matched to $j' > j$. In the above example, this is $M = \{(1,1),(2,2),(3,3),(4,5)\}$.

Edit Distance: Alternate View

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Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

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Edit Distance Problem

Problem
Given two words, find the edit distance between them, i.e., an alignment of smallest cost.

Applications

- Spell-checkers and Dictionaries
- Unix `diff`
- DNA sequence alignment . . . but, we need a new metric

Sequence alignment problem - Similarity Metric

Definition
For two strings X and Y, the cost of alignment M is

- [Gap penalty] For each gap in the alignment, we incur a cost δ.
- [Mismatch cost] For each pair p and q that have been matched in M, we incur cost α_{pq}; typically $\alpha_{pp} = 0$.

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Edit distance is special case when $\delta = \alpha_{pq} = 1$.

Edit distance as alignment

An Example

Example

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|}
o & & c & u & r & r & a & n & c & e \\
o & c & c & u & r & r & e & n & c & e
\end{array} \qquad \text{Cost } = \delta + \alpha_{ae}$$

Alternative:

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
o & & c & u & r & r & & a & n & c & e \\
o & c & c & u & r & r & e & & n & c & e
\end{array} \qquad \text{Cost } = 3\delta$$

Or a really stupid solution (delete string, insert other string):

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
o & c & u & r & r & a & n & c & e & & & & & & & & & & \\
 & & & & & & & & & o & c & c & u & r & r & e & n & c & e
\end{array}$$

Cost $= 19\delta$.

What is the edit distance between...

What is the minimum edit distance for the following two strings, if insertion/deletion/change of a single character cost 1 unit?

$\boxed{374}$

$\boxed{473}$

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

What is the edit distance between...

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What is the edit distance between...

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$\boxed{37}$

$\boxed{473}$

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(b) 2
(c) 3
(d) 4
(e) 5

Sequence Alignment

Input Given two words X and Y, and gap penalty δ and mismatch costs α_{pq}

Goal Find alignment of minimum cost

Edit distance: The algorithm

Edit distance - Basic observation

Let $X = \alpha x$ and $Y = \beta y$

α, β: strings.

x and y single characters.

Think about optimal edit distance between X and Y as alignment, and consider last column of alignment of the two strings:

α	x
β	y

or

α	x
βy	

or

αx	
β	y

Prefixes must have optimal alignment!

Problem Structure

Let $X = x_1x_2 \cdots x_m$ and $Y = y_1y_2 \cdots y_n$. If (m, n) are not matched then either the m^{th} position of X remains unmatched or the n^{th} position of Y remains unmatched.

- Case x_m and y_n are matched.
  - Pay mismatch cost $\alpha_{x_m y_n}$ plus cost of aligning strings $x_1 \cdots x_{m-1}$ and $y_1 \cdots y_{n-1}$
- Case x_m is unmatched.
  - Pay gap penalty plus cost of aligning $x_1 \cdots x_{m-1}$ and $y_1 \cdots y_n$
- Case y_n is unmatched.
  - Pay gap penalty plus cost of aligning $x_1 \cdots x_m$ and $y_1 \cdots y_{n-1}$

Subproblems and Recurrence

$x_1 \ldots x_{i-1}$	x_i
$y_1 \ldots y_{j-1}$	y_j

or

$x_1 \ldots x_{i-1}$	x
$y_1 \ldots y_{j-1} y_j$	

or

$x_1 \ldots x_{i-1} x_i$	
$y_1 \ldots y_{j-1}$	y_j

Optimal Costs

Let $\mathrm{Opt}(i, j)$ be optimal cost of aligning $x_1 \cdots x_i$ and $y_1 \cdots y_j$.

Then

$$\mathrm{Opt}(i, j) = \min \begin{cases} \alpha_{x_i y_j} + \mathrm{Opt}(i-1, j-1), \\ \delta + \mathrm{Opt}(i-1, j), \\ \delta + \mathrm{Opt}(i, j-1) \end{cases}$$

Subproblems and Recurrence

$x_1 \dots x_{i-1}$	x_i
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or

$x_1 \dots x_{i-1}$	x
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$x_1 \dots x_{i-1} x_i$	
$y_1 \dots y_{j-1}$	y_j

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$$\mathrm{Opt}(i,j) = \min \begin{cases} \alpha_{x_i y_j} + \mathrm{Opt}(i-1, j-1), \\ \delta + \mathrm{Opt}(i-1, j), \\ \delta + \mathrm{Opt}(i, j-1) \end{cases}$$

Base Cases: $\mathrm{Opt}(i, 0) = \delta \cdot i$ and $\mathrm{Opt}(0, j) = \delta \cdot j$

Recursive Algorithm

Assume X is stored in array $A[1..m]$ and Y is stored in $B[1..n]$
Array $COST$ stores cost of matching two chars. Thus $COST[a, b]$ give the cost of matching character a to character b.

$EDIST(A[1..m], B[1..n])$
  `If` $(m = 0)$ `return` $n\delta$
  `If` $(n = 0)$ `return` $m\delta$
  $m_1 = \delta + EDIST(A[1..(m-1)], B[1..n])$
  $m_2 = \delta + EDIST(A[1..m], B[1..(n-1)]))$
  $m_3 = COST[A[m], B[n]] + EDIST(A[1..(m-1)], B[1..(n-1)])$
  `return` $\min(m_1, m_2, m_3)$

Example: DEED and DREAD

	ε	*D*	*R*	*E*	*A*	*D*
ε						
D						
E						
E						
D						

$\mathrm{Opt}(i, j) =$

$$\min \begin{cases} \alpha_{x_i y_j} + \mathrm{Opt}(i-1, j-1), \\ \delta + \mathrm{Opt}(i-1, j), \\ \delta + \mathrm{Opt}(i, j-1) \end{cases}$$

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D	4					

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E	3	2	2	1	2	3
D	4	3	3	2	2	2

D	R	E	A	D
D	E	E		D

Example: DEED and DREAD

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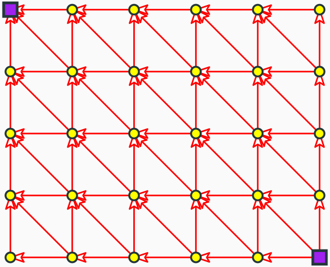
E	2	1	1	1	2	3
E	3	2	2	1	2	3
D	4	3	3	2	2	2

D	R	E	A	D
D	E	E		D



Example: DEED and DREAD

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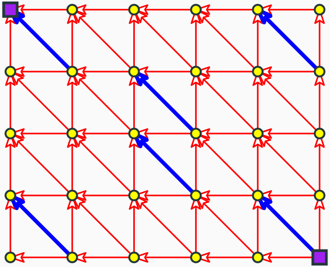
D	1	0	1	2	3	4
E	2	1	1	1	2	3
E	3	2	2	1	2	3
D	4	3	3	2	2	2

D	R	E	A	D
D	E	E		D



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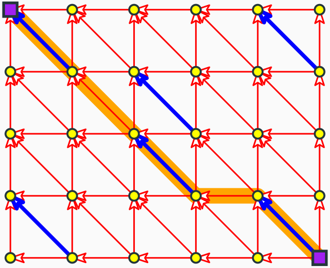

Dynamic programming algorithm for edit-distance

As part of the input...

The cost of aligning a character against another character

Σ: Alphabet

We are given a <u>cost</u> function (in a table):

$$\forall b, c \in \Sigma \qquad COST[b][c] = \text{ cost of aligning } b \text{ with } c.$$
$$\forall b \in \Sigma \qquad COST[b][b] = 0$$

δ : price of deletion of insertion of a single character

Dynamic program for edit distance

$EDIST(A[1..m], B[1..n])$

$\quad int \quad M[0..m][0..n]$

$\quad$ **for** $i = 1$ `to` m **do** $M[i,0] = i\delta$

$\quad$ **for** $j = 1$ `to` n **do** $M[0,j] = j\delta$

$\quad$ **for** $i = 1$ `to` m **do**

$\quad\quad$ **for** $j = 1$ `to` n **do**

$$M[i][j] = \min \begin{cases} COST\left[A[i]\right]\left[B[j]\right] + M[i-1][j-1], \\ \delta + M[i-1][j], \\ \delta + M[i][j-1] \end{cases}$$

Dynamic program for edit distance

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$\quad$ *int* $M[0..m][0..n]$

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$$M[i][j] = \min \begin{cases} COST\left[A[i]\right]\left[B[j]\right] + M[i-1][j-1], \\ \delta + M[i-1][j], \\ \delta + M[i][j-1] \end{cases}$$

Analysis

- Running time is $O(mn)$.
- Space used is $O(mn)$.

Reducing space for edit distance

Matrix and DAG of computation of edit distance

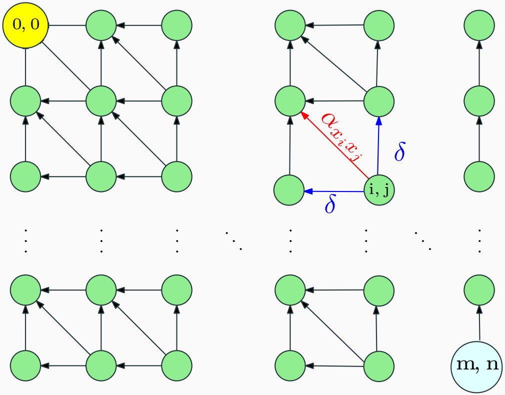

Figure 1: Iterative algorithm in previous slide computes values in row order.

Optimizing Space

- Recall

$$M(i,j) = \min \begin{cases} \alpha_{x_i y_j} + M(i-1, j-1), \\ \delta + M(i-1, j), \\ \delta + M(i, j-1) \end{cases}$$

- Entries in j^{th} column only depend on $(j-1)^{st}$ column and earlier entries in j^{th} column
- Only store the current column and the previous column reusing space; $N(i,0)$ stores $M(i,j-1)$ and $N(i,1)$ stores $M(i,j)$

Example: DEED vs. DREAD filled by column

	ε	D	R	E	A	D
ε						
D						
E						
E						
D						

Example: DEED vs. DREAD filled by column

	ε	D	R	E	A	D
ε	0	1	2	3	4	5
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Example: DEED vs. DREAD filled by column

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D	1	0				
E	2	1				
E	3	2				
D	3	3				

Example: DEED vs. DREAD filled by column

	ε	*D*	*R*	*E*	*A*	*D*
ε	0	1	2	3	4	5
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E	2	1	1			
E	3	2	2			
D	3	3	3			

Example: DEED vs. DREAD filled by column

	ε	*D*	*R*	*E*	*A*	*D*
ε	0	1	2	3	4	5
D	1	0	1	2		
E	2	1	1	1		
E	3	2	2	1		
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Example: DEED vs. DREAD filled by column

	ε	D	R	E	A	D
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E	2	1	1	1	2	
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D	3	3	3	2	2	

Example: DEED vs. DREAD filled by column

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D	3	3	3	2	2	2

Computing in column order to save space

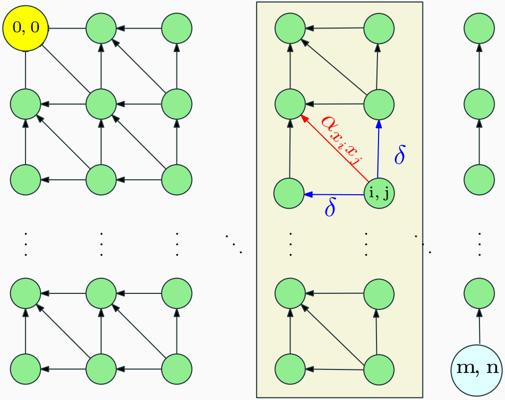

Figure 2: $M(i, j)$ only depends on previous column values. Keep only two columns and compute in column order.

Space Efficient Algorithm

```
for all i do N[i,0] = iδ
for j = 1 to n do
    N[0,1] = jδ (* corresponds to M(0,j) *)
    for i = 1 to m do
```

$$N[i,1] = \min \begin{cases} \alpha_{x_i y_j} + N[i-1,0] \\ \delta + N[i-1,1] \\ \delta + N[i,0] \end{cases}$$

```
    for i = 1 to m do
        Copy N[i,0] = N[i,1]
```

Analysis
Running time is $O(mn)$ and space used is $O(2m) = O(m)$

Analyzing Space Efficiency

- From the $m \times n$ matrix M we can construct the actual alignment (exercise)
- Matrix N computes cost of optimal alignment but no way to construct the actual alignment
- Space efficient computation of alignment? More complicated algorithm — see notes and Kleinberg-Tardos book.

Longest Common Subsequence Problem

LCS Problem

Definition

LCS between two strings X and Y is the length of longest common subsequence between X and Y.

$ABAZDC$	$ABAZDC$
$BACBAD$	$BACBAD$

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Example

LCS between ABAZDC and BACBAD is 4 via ABAD

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$ABAZDC$ $\quad$ $ABAZDC$

$BACBAD$ $\quad$ $BACBAD$

Example

LCS between ABAZDC and BACBAD is 4 via ABAD

Derive a dynamic programming algorithm for the problem.

How do we plan out the recursion?

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- Assuming $A[m] \neq B[n]$
  - The one or neither of the end characters are in the LCS. Therefore becomes:

    $$\max\left(LCS(A[1...m-1], B[1...n]), LCS(A[1...m], B[1...n-1])\right)$$

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- Assuming $A[m] = B[n]$
  - Either $A[m]$ and $B[n]$ are both in the LCS. Therefore:

    $$LCS(A[1...m], B[1...n]) = 1 + LCS(A[1...m-1], B[1...n-1])$$

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- Assuming $A[m] = B[n]$
  - Either $A[m]$ and $B[n]$ are both in the LCS. Therefore:

    $$LCS(A[1...m], B[1...n]) = 1 + LCS(A[1...m-1], B[1...n-1])$$

  - Or $A[m]$ and $B[n]$ is not in the LCS. Therefore the LCS is either:

    $$LCS(A[1...m-1], B[1...n])$$
    $$LCS(A[1...m], B[1...n-1])$$

How do we plan out the recursion?

Start off with $A[1...m]$ and $B[1...n]$ and reason the following:

- Assuming $A[m] \neq B[n]$
  - The one or neither of the end characters are in the LCS. Therefore becomes:

    $$\max\left(LCS(A[1...m-1], B[1...n]), LCS(A[1...m], B[1...n-1])\right)$$

- Assuming $A[m] = B[n]$
  - Either $A[m]$ and $B[n]$ are both in the LCS. Therefore:

    $$LCS(A[1...m], B[1...n]) = 1 + LCS(A[1...m-1], B[1...n-1])$$

  - Or $A[m]$ and $B[n]$ is not in the LCS. Therefore the LCS is either:

    $$LCS(A[1...m-1], B[1...n])$$
    $$LCS(A[1...m], B[1...n-1])$$

- Base Case: A is empty or B is empty

LCS recursive definition

$A[1..n], B[1..m]$: Input strings.

$$LCS(i,j) = \begin{cases} 0 & i = 0 \text{ or } j = 0 \\ \max\begin{pmatrix} LCS(i-1,j), \\ LCS(i,j-1) \end{pmatrix} & A[i] \neq B[j] \\ \max\begin{pmatrix} LCS(i-1,j), \\ LCS(i,j-1), \\ 1 + LCS(i-1,j-1) \end{pmatrix} & A[i] = B[j] \end{cases}$$

LCS recursive definition

$A[1..n], B[1..m]$: Input strings.

$$LCS(i,j) = \begin{cases} 0 & i = 0 \text{ or } j = 0 \\ \max\begin{pmatrix} LCS(i-1,j), \\ LCS(i,j-1) \end{pmatrix} & A[i] \neq B[j] \\ \max\begin{pmatrix} LCS(i-1,j), \\ LCS(i,j-1), \\ 1 + LCS(i-1,j-1) \end{pmatrix} & A[i] = B[j] \end{cases}$$

Running time: Similar to edit distance... $O(nm)$

Space: $O(m)$ space.

Longest common subsequence is just edit distance for the two sequences...

A, B: input sequences, Σ: "alphabet" all the different values in A and B

$$\forall b, c \in \Sigma : b \neq c \qquad COST[b][c] = +\infty.$$
$$\forall b \in \Sigma \qquad COST[b][b] = 1$$

$1:$ price of deletion of insertion of a single character

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$$1: \text{ price of deletion of insertion of a single character}$$

										ED	LCS
Maximum ED	D	R	E	A	D					9	0
Min LCS						D	E	E	D		
Sub-opt ED	D	R	E	A	D					8	1
Sub-opt LCS					D	E	E	D			
Min ED	D	R	E	A		D				6	3
Max LCS	D		E		E	D					

Longest common subsequence is just edit distance for the two sequences...

A, B: input sequences, Σ: "alphabet" all the different values in A and B

$$\begin{aligned} &\forall b, c \in \Sigma : b \neq c \qquad && COST[b][c] = +\infty. \\ &\forall b \in \Sigma && COST[b][b] = 1 \end{aligned}$$

$$1: \text{ price of deletion of insertion of a single character}$$

Length of longest common sub-sequence $= m + n - \mathrm{ed}(A, B)$

Is in L^k?

A variation

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsStringinL**(*string* x) that decides whether x is in L, and non-negative integer k

Goal Decide if $w \in L^k$ using **IsStringinL**(*string* x) as a black box sub-routine

Example
Suppose L is *English* and we have a procedure to check whether a string/word is in the *English* dictionary.

- Is the string "isthisanenglishsentence" in $English^5$?
- Is the string "isthisanenglishsentence" in $English^4$?
- Is "asinineat" in $English^2$?
- Is "asinineat" in $English^4$?
- Is "zibzzzad" in $English^1$?

Recursive Solution

When is $w \in L^k$?

Recursive Solution

When is $w \in L^k$?

$$IsStringinLk(A[1 \ldots i], k) = \begin{cases} w \in L^k \text{ iff } w = \epsilon & k = 0 \\ w \in L^k \text{ iff } w \in L & k = 1 \\ w \in L^k \text{ iff } \begin{array}{l} u \in L^{k-1}, \\ v \in L \end{array} \bigg| w = uv & k > 1 \end{cases}$$

```
IsStringinLk(A[1...i], k):
    if k = 0 and i = 0 then return YES
    if k = 0 then return NO   // i > 0
    if k = 1 then
        return IsStringinL(A[1...i])

    for ℓ = 1...i − 1 do
        if IsStringinLk(A[1...ℓ], k − 1) and IsStringinL(A[ℓ + 1...i]) then
            return YES

    return NO
```

Analysis

```
IsStringinLk(A[1 . . . i], k):
    if k = 0 and i = 0 then return YES
    if k = 0 then return NO   // i > 0
    if k = 1 then
        return IsStringinL(A[1 . . . i])

    for ℓ = 1 . . . i − 1 do
        if IsStringinLk(A[1 . . . ℓ], k − 1) and IsStringinL(A[ℓ + 1 . . . i]) then
            return YES

    return NO
```

- How many distinct sub-problems are generated by **IsStringinLk**$(A[1..n], k)$?

Analysis

```
IsStringinLk(A[1 ... i], k):
    if k = 0 and i = 0 then return YES
    if k = 0 then return NO   // i > 0
    if k = 1 then
        return IsStringinL(A[1 ... i])

    for ℓ = 1 ... i − 1 do
        if IsStringinLk(A[1 ... ℓ], k − 1) and IsStringinL(A[ℓ + 1 ... i]) then
            return YES

    return NO
```

- How many distinct sub-problems are generated by **IsStringinLk**$(A[1..n], k)$? $O(nk)$

Analysis

```
IsStringinLk(A[1...i], k):
    if k = 0 and i = 0 then return YES
    if k = 0 then return NO   // i > 0
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        return IsStringinL(A[1...i])

    for ℓ = 1...i − 1 do
        if IsStringinLk(A[1...ℓ], k − 1) and IsStringinL(A[ℓ + 1...i]) then
            return YES

    return NO
```

- How many distinct sub-problems are generated by **IsStringinLk**$(A[1..n], k)$? $O(nk)$
- How much space?

Analysis

```
IsStringinLk(A[1 ... i], k):
    if k = 0 and i = 0 then return YES
    if k = 0 then return NO   // i > 0
    if k = 1 then
        return IsStringinL(A[1 ... i])

    for ℓ = 1 ... i − 1 do
        if IsStringinLk(A[1 ... ℓ], k − 1) and IsStringinL(A[ℓ + 1 ... i]) then
            return YES

    return NO
```

- How many distinct sub-problems are generated by **IsStringinLk**$(A[1..n], k)$? $O(nk)$
- How much space? $O(nk)$

Analysis

```
IsStringinLk(A[1 ... i], k):
    if k = 0 and i = 0 then return YES
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        if IsStringinLk(A[1 ... ℓ], k − 1) and IsStringinL(A[ℓ + 1 ... i]) then
            return YES

    return NO
```

- How many distinct sub-problems are generated by **IsStringinLk**$(A[1..n], k)$? $O(nk)$
- How much space? $O(nk)$
- Running time if we use memoization?

Analysis

IsStringinLk$(A[1\ldots i], k)$:
    if $k = 0$ `and` $i = 0$ **then return** `YES`
    if $k = 0$ **then return** `NO` // $i > 0$
    if $k = 1$ **then**
        return IsStringinL$(A[1\ldots i])$

    for $\ell = 1 \ldots i-1$ **do**
        if IsStringinLk$(A[1\ldots\ell], k-1)$ `and` **IsStringinL**$(A[\ell+1\ldots i])$ **then**
            return `YES`

    return `NO`

- How many distinct sub-problems are generated by **IsStringinLk**$(A[1..n], k)$? $O(nk)$
- How much space? $O(nk)$
- Running time if we use memoization? $O(n^2k)$