Theoretical Guarantees for Sparse Graph Signal Recovery

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***Abstract*—Sparse graph signals have recently been utilized in graph signal processing (GSP) for tasks such as graph signal reconstruction, blind deconvolution, and sampling. In addition, sparse graph signals can be used to model real-world network applications across various domains, such as social, biological, and power systems. Despite the extensive use of sparse graph signals, limited attention has been paid to the derivation of theoretical guarantees on their recovery. In this paper, we present a novel theoretical analysis of the problem of recovering a node-domain sparse graph signal from the output of a first-order graph filter. The graph filter we study is the Laplacian matrix, and we derive upper and lower bounds on its mutual coherence. Our results establish a connection between the recovery performance and the minimal graph nodal degree. The proposed bounds are evaluated via simulations on the Erdős-Rényi graph.**

***Index Terms*—Graph signal processing, graph signals, Laplacian matrix, sparse recovery, mutual coherence.**

I. Introduction

Graph signal processing (GSP) extends traditional signal processing theory and techniques to signals defined on irregular domains represented by graphs [1], [2]. GSP offers a wide range of tools for the analysis, sampling, and filtering of graph signals [3]–[12]. In particular, sparse graph signals play a crucial role in various GSP applications, such as signal reconstruction [13]–[15], blind deconvolution [16], [17], system identification [18], and sampling [19]. Compressed sensing (CS) methods are used in network science applications [20]–[24], such as the analysis of the spread of disease [25], malware allocation in computer networks [26], and anomaly detection in power systems [27]–[29]. Despite the growing interest in sparse methods in GSP, theoretical guarantees for sparse graph signal recovery have not been well investigated.

CS is a well-established framework that leverages sparsity for the recovery of signals from limited measurements, which is a fundamental problem in signal processing [30]–[34]. Theoretical guarantees for CS have been extensively studied in various signal processing settings, including structured, random, and application-specific dictionary matrices [32], [33], [35]–[38]. These guarantees often involve analyzing the sparse dictionary matrix using measures such as the restricted isometry constant (RIC) [36], the exact recovery coefficient (ERC) [39], the spark [40], and the mutual coherence [32].

In graphical settings, theoretical guarantees have appeared in a few specific problems. In [21], guarantees are provided for network tomography, where network link characteristics are modeled as a signal over graph edges, using the graph incidence matrix as the dictionary matrix. Theoretical guarantees have also been established for signals lying on the graph nodes. For instance, guarantees have been derived for the recovery of graph signals on perturbed graphs, where the sparsity is in the graph spectral domain [41]. For data gathering in wireless sensor networks (WSNs), guarantees cover cases where the dictionary matrix is the adjacency matrix of an expander graph [42]. Additional guarantees have been derived for Laplacian-regularized sparse coding [43] as a function of the sensing matrix mutual coherence. Mutual coherence, a key measure for analyzing dictionary matrices [32], is also used in [44] for depth map compression and in [45] for vertex-frequency analysis on graphs. In these papers, the mutual coherence is derived for the product between the basis of graph Laplacian eigenvectors and columns of a Hadamard sensing matrix [44] or the Kronecker deltas [45]. Despite these contributions, there remains a significant gap in the theoretical understanding of sparse recovery in GSP, especially for guarantees on the recovery of node-domain sparse graph signals from graph filter outputs, which is a common task in GSP.

In this letter, we present a theoretical analysis of the problem of sparse recovery of a node-domain sparse graph signal from the output of a first-order Laplacian-based graph filter. We analyze the inner product between the columns of the Laplacian matrix, which serves as the sparse dictionary atoms in this case, and relate it to the node degrees. Based on this analysis, we derive upper and lower bounds on the mutual coherence of the Laplacian matrix. Our results establish a connection between the recovery performance and the underlying properties of the graph, i.e. the minimal nodal degree. Simulations conducted with the considered model on an Erdős-Réyni graph demonstrate the influence of the graph sparsity on the mutual coherence and the proposed bounds.

II. Problem Formulation

Consider an undirected unweighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which consists of a set of N nodes (vertices) $\mathcal{V} = \{1, 2, \dots, N\}$, and a set of edges $\mathcal{E}$. For each node k, the one-hop neighborhood, defined by

$$\mathcal{N}(k) \stackrel{\triangle}{=} \{m \in \mathcal{V} : \ (k, m) \in \mathcal{E}\}, \tag{1}$$

includes all nodes directly connected to node k. The degree of node k is the cardinality of the one-hop neighborhood, i.e. $d(k) = |\mathcal{N}(k)|$. The Laplacian matrix associated with the

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graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ is defined elementwisely by

$$L_{k,m}=\begin{cases} d(k) & k=m \\ -1 & \delta(k,m)=1 \qquad k,m=1,\dots,N, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where the hop distance, $\delta(k,m)$, is the number of edges in the shortest path between nodes k and m.

In this letter, we consider the Laplacian matrix to be the dictionary matrix of the sparse recovery problem. The Laplacian is a fundamental graph shift operator (GSO) that can be used for graph filtering models [1]. Consequently, our results in the following lay the groundwork for broader applications in sparse recovery problems with higher-order graph filters. Here, we focus on deriving theoretical guarantees for the performance of Laplacian-based sparse recovery [32]:

$$\begin{aligned} \hat{\mathbf{x}} \;\; = \;\; & \arg\min_{\mathbf{x}\in\mathbb{R}^N} \|\mathbf{y}-\mathbf{L}\mathbf{x}\|^2 \\ & \text{such that:} \;\; \|\mathbf{x}\|_0 \le s, \end{aligned} \tag{3}$$

where $\|\cdot\|_0$ is the ℓ_0 semi-norm and $s\in\mathbb{N}^+$ is the sparsity level. The vector $\mathbf{y}\in\mathbb{R}^N$ contains the observations, where we assume full observability with a measurement from each node. Thus, both $\mathbf{y}$ and $\mathbf{x}$ can be considered to be graph signals.

III. Bounds on the mutual coherence

In this section, we analyze the Laplacian matrix (2), which is the sparse dictionary matrix in (3). In Subsection III-A, we analyze the inner product between two Laplacian matrix columns. In Subsection III-B, we use this analysis to derive bounds on the mutual coherence of the Laplacian matrix.

A. Inner Product between Dictionary Atoms

The following claim provides a closed-form expression for the inner product of two Laplacian dictionary atoms, which is essential for deriving bounds on the mutual coherence, and can be applied for other purposes, such as graph filter analysis.

Claim 1. *The inner product between two Laplacian matrix columns satisfies*

$$\mathbf{L}_k^T\mathbf{L}_m=\begin{cases} d^2(k)+d(k) & k=m \\ |\mathcal{N}(k)\cap\mathcal{N}(m)|-d(k)-d(m) & \delta(k,m)=1 \\ |\mathcal{N}(k)\cap\mathcal{N}(m)| & \delta(k,m)=2 \\ 0 & \delta(k,m)>2, \end{cases} \tag{4}$$

where $\mathbf{L}_k$ is the kth column of $\mathbf{L}$, and $\mathcal{N}(k)$ is defined in (1).

Proof: Substituting (2) in $\mathbf{L}_k^T\mathbf{L}_m$, results in

$$\begin{aligned} \mathbf{L}_k^T\mathbf{L}_m &= \sum_{i=1}^N (d(k)\mathbb{1}_{\{i=k\}}-\mathbb{1}_{\{\delta(k,i)=1\}}) \\ &\quad \times(d(m)\mathbb{1}_{\{i=m\}}-\mathbb{1}_{\{\delta(m,i)=1\}}) \\ &= d^2(k)\mathbb{1}_{\{k=m\}}-(d(k)+d(m))\mathbb{1}_{\{\delta(k,m)=1\}} \\ &\quad +|\mathcal{N}(k)\cap\mathcal{N}(m)|, \end{aligned} \tag{5}$$

where $L_{k,i}$ is the (k,i)th entry of $\mathbf{L}$, and $\mathbb{1}_\mathcal{A}$ is the indicator of the event $\mathcal{A}$. By substituting in (5), in turn, the four cases: 1) $k=m$; 2) $\delta(k,m)=1$; 3) $\delta(k,m)=2$; and 4) $\delta(k,m)>2$; and using for item 1 the result $|\mathcal{N}(k)\cap\mathcal{N}(k)|=|\mathcal{N}(k)|=d(k)$, we obtain the result in (4). ■

It can be seen that the inner product expression in (4) between two dictionary atoms depends on the number of shared neighbors between their associated nodes. In the following claim, we provide an upper bound on this number.

Claim 2. *The number of shared neighbors for each pair of nodes is bounded by their minimum nodal degree:*

$$|\mathcal{N}(k)\cap\mathcal{N}(m)|\le\min(d(k),d(m)). \tag{6}$$

Proof: From set theory, the intersection between two sets $\mathcal{N}(k)$ and $\mathcal{N}(m)$ is included in each of these sets. Hence, $\mathcal{N}(k)\cap\mathcal{N}(m)\subseteq\mathcal{N}(k)$ and $\mathcal{N}(k)\cap\mathcal{N}(m)\subseteq\mathcal{N}(m)$. Consequently, we obtain (6). ■

Based on Claims 1 and 2, we derive the following results. Claim 3 shows that the inner product between two dictionary atoms decreases as the hop distance between their associated nodes increases. Claim 4 establishes that the inner product between Laplacian columns is bounded by the sum of the nodal degrees of the associated nodes.

Claim 3. *If $k\neq m$ and $\delta(k,m)<\delta(k,j)$, then*

$$|\mathbf{L}_k^T\mathbf{L}_m|\ge|\mathbf{L}_k^T\mathbf{L}_j|.$$

Proof: Under the claim's conditions, if $\delta(k,m)\ge2$, then $\delta(k,j)>2$, and thus, according to the last row of (4), we have $0=|\mathbf{L}_k^T\mathbf{L}_j|\le|\mathbf{L}_k^T\mathbf{L}_m|$. If $\delta(k,m)=1$, (4) implies that 1) if $\delta(k,j)>2$, then $0=|\mathbf{L}_k^T\mathbf{L}_j|\le|\mathbf{L}_k^T\mathbf{L}_m|$;
2) if $\delta(k,j)=2$, then

$$\begin{aligned} \mathbf{L}_k^T\mathbf{L}_j &\underset{(a)}{=} |\mathcal{N}(k)\cap\mathcal{N}(j)| \underset{(b)}{\le} \min(d(k),d(j)) \\ &\underset{(c)}{\le} d(k)+d(m)-\min(d(k),d(m)) \\ &\underset{(d)}{\le} d(k)+d(m)-|\mathcal{N}(k)\cap\mathcal{N}(m)| \underset{(e)}{=} \mathbf{L}_k^T\mathbf{L}_m, \end{aligned} \tag{7}$$

where (a) is obtained by substituting $\delta(k,m)=2$ in (4) from Claim 1, (b) and (d) are obtained from (6) in Claim 2, and (e) is obtained from (4). ■

Claim 4. *For any two nodes k and m we have*

$$|\mathbf{L}_k^T\mathbf{L}_m|\le d(k)+d(m), \quad \forall k\neq m,\; k,m\in\{1,\dots,N\}. \tag{8}$$

Proof: If $\delta(k,m)>2$ then by using (4) from Claim 1, $|\mathbf{L}_k^T\mathbf{L}_m|=0$, which satisfies $|\mathbf{L}_k^T\mathbf{L}_m|\le d(k)+d(m)$, given that node degrees are inherently non-negative. Otherwise, from Claim 2, it can be verified that

$$|\mathcal{N}(k)\cap\mathcal{N}(m)|\le\min(d(k),d(m))\le d(k)+d(m). \tag{9}$$

Hence, if $\delta(k,m)=2$, the statement in (8) is true. Moreover, if $\delta(k,m)=1$, then by substituting (4), we obtain

$$|\mathbf{L}_k^T\mathbf{L}_m|=|d(k)+d(m)-|\mathcal{N}(k)\cap\mathcal{N}(m)||. \tag{10}$$

Thus, using (9) and (10), we obtain

$$|\mathbf{L}_k^T\mathbf{L}_m|\le d(k)+d(m), \tag{11}$$

where we used the fact that $|\mathcal{N}(k)\cap\mathcal{N}(m)|\ge0$. ■

B. Mutual Coherence

In this subsection, we leverage Claims 1-4 to derive the bounds on the mutual coherence of the Laplacian matrix. Mutual coherence is a key measure used to analyze CS recovery algorithms [32]. If the mutual coherence of the measurement matrix is sufficiently small, standard recovery methods like Lasso, orthogonal matching pursuit (OMP), and thresholding can estimate the signal with a squared error proportional to the sparsity level and the noise variance, times a factor that is logarithmic in the signal length [33], [36]. The mutual coherence of the Laplacian matrix is defined as [32]

$$\mu_{\mathbf{L}} = \max_{k\neq m} \frac{|\mathbf{L}_k^T \mathbf{L}_m|}{\|\mathbf{L}_k\|\|\mathbf{L}_m\|}. \tag{12}$$

We base the derivations of the upper and lower bounds on the mutual coherence of the findings in Subsection III-A.

The following theorem establishes an upper bound on the mutual coherence of the Laplacian matrix.

Theorem 1. *The mutual coherence of the Laplacian matrix is upper-bounded by $\mu_{\mathbf{L}} \leq \zeta_{ub}$, where*

$$\zeta_{ub} = \frac{d(i_1)+d(i_2)}{\sqrt{d^2(i_1)+d(i_1)}\sqrt{d^2(i_2)+d(i_2)}}, \tag{13}$$

and $i_1 \dots i_N$ is the degree-based ordering of the nodes such that $d(i_1) \leq d(i_2) \leq \ldots \leq d(i_N)$.

Proof: By substituting the bound in (8) from Claim 4 and (4) from Claim 1 for $k = m$ in (12), we obtain

$$\mu_{\mathbf{L}} \leq \max_{k\neq m} \frac{d(k)+d(m)}{\sqrt{d^2(k)+d(k)}\sqrt{d^2(m)+d(m)}}. \tag{14}$$

In the appendix, it is shown that the objective function in (14) is a monotonic decreasing function of $d(k)$ (and of $d(m)$). Thus, the maximum of the right hand side (r.h.s.) of (14) is obtained by selecting the two nodes with the lowest degrees. This maximum is set to be the proposed upper bound. ■

In the following theorem, we show that the minimal nodal degree $d(i_1)$ governs the mutual coherence upper bound ζ_{ub}.

Theorem 2. *The mutual coherence upper-bound in Theorem 1 is bonded by*

$$\zeta_{ub} \leq \frac{2}{d(i_1)+1}. \tag{15}$$

Proof: The upper bound, ζ_{ub}, is a function of $d(i_1)$ and $d(i_2)$. Since $d(i_1) \leq d(i_2)$, and the r.h.s. of (13) is a monotonically decreasing function of $d(i_2)$ (see the appendix), we obtain (15). ■

The following theorem establishes a lower bound on the mutual coherence of the Laplacian matrix.

Theorem 3. *The mutual coherence of the Laplacian matrix in* (2) *is lower-bounded by $\mu_{\mathbf{L}} \geq \zeta_{lb}$, where*

$$\zeta_{lb} = \frac{1}{\sqrt{1+\frac{1}{d(i_1^{\max})}}\sqrt{d^2(i_1)+d(i_1)}}, \tag{16}$$

in which $d(i_1^{\max}) \stackrel{\triangle}{=} \max_{k\in\mathcal{N}(i_1)} d(k)$ and $d(i_1) \leq d(k)$, $\forall k \in \mathcal{V}$.

Proof: By maximizing the r.h.s. of (12) only over the set of connected nodes, we obtain

$$\begin{aligned}\mu_{\mathbf{L}} &\geq \max_{(k,m)\in\mathcal{E}} \frac{|\mathbf{L}_k^T \mathbf{L}_m|}{\|\mathbf{L}_k\|\|\mathbf{L}_m\|} \\ &= \max_{(k,m)\in\mathcal{E}} \frac{d(k)+d(m)-|\mathcal{N}(k)\cap\mathcal{N}(m)|}{\sqrt{d^2(k)+d(k)}\sqrt{d^2(m)+d(m)}},\end{aligned} \tag{17}$$

where the last equality is obtained by substituting (4) from Claim 1 with $\delta(k,m)=1$ and $k=m$, respectively. Hence, by substituting the result in (6) from Claim 2 in (17), we obtain

$$\mu_{\mathbf{L}} \geq \max_{(k,m)\in\mathcal{E}} \frac{d(k)+d(m)-\min(d(k),d(m))}{\sqrt{d^2(k)+d(k)}\sqrt{d^2(m)+d(m)}}. \tag{18}$$

Without loss of generality, we assume that $d(k) \geq d(m)$. Thus,

$$\begin{aligned}\mu_{\mathbf{L}} &\geq \max_{(k,m)\in\mathcal{E}} \frac{d(k)}{\sqrt{d^2(k)+d(k)}\sqrt{d^2(m)+d(m)}} \\ &= \max_{(k,m)\in\mathcal{E}} \frac{1}{\sqrt{1+\frac{1}{d(k)}}\sqrt{d^2(m)+d(m)}}.\end{aligned} \tag{19}$$

Finally, we select the connected nodes $m = i_1$ and $k = i_1^{\max}$ (a feasible choice for (19)) to obtain the lower bound (16). ■

The proof of Theorem 3 involves two heuristic choices that are used to obtain (17) and the final result in (16). The rationale behind these selections is as follows.

1) In (17) the maximization is restricted to be conducted over connected nodes. This choice is guided by Claim 3, which shows that the inner product associated with two nodes, $|\mathbf{L}_k^T \mathbf{L}_m|$, decreases as their hop-distance decreases. However, there may be cases where the r.h.s. of (12) is maximized when the dictionary atoms correspond to second-neighbor nodes.

2) The denominator in (19) is the product of $\sqrt{1+\frac{1}{d(k)}}$ and $\sqrt{d^2(m)+d(m)}$. While it can be verified that the first component is bounded, as

$$1 < 1 + \frac{1}{d(k)} \leq 2, \tag{20}$$

the second component can vary significantly depending on the choice of node m. To maximize (19), it is intuitive to select m as the node with the minimal nodal degree, i.e. i_1. Once we set $m = i_1$, choosing node k as its neighbor with the highest nodal degree maximizes the objective on the r.h.s. of (19).

The proposed bounds on the mutual coherence are governed by the minimal nodal degree of the graph, where higher degrees lead to lower coherence. This indicates that mutual coherence is affected by the graph sparsity, with denser graphs generally exhibiting a higher minimal nodal degree. As a result, recovery performance is expected to improve as the number of edges increases. This relationship becomes apparent as follows: A sufficient condition for the recovery of s-sparse signals is given by (see, e.g. Section 4.4 in [32]):

$$s < \frac{1}{2}\left(1+\frac{1}{\zeta_{ub}}\right). \tag{21}$$

By substituting the upper bound from (15) in (21), one obtains

$$s < \frac{1}{2}\left(1+\frac{d(i_1)+1}{2}\right) = \frac{d(i_1)+3}{4}. \tag{22}$$

Thus, the maximum recoverable sparsity level s increases as the minimal nodal degree, $d(i_1)$, increases, and consequently, also as the graph density increases. The above results align with findings from network science, where greater connectivity between nodes generally supports more robust and efficient operation by enabling, e.g., the redirecting of information, resources, or power through alternative connections [46]–[48].

Remark 1. *For the special case of d-regular graphs, where $d(k) = d$, $\forall k \in \mathcal{V}$, and a given $d \in \mathbb{N}^+$, the bounds in* (14) *and* (16) *from Theorems 1 and 3, respectively, yield*

$$\frac{1}{d+1} \leq \mu_{\mathbf{L}} \leq \frac{2}{d+1}. \tag{23}$$

Consequently, a larger value of d, which corresponds to a denser graph, results in a lower value of $\mu_{\mathbf{L}}$.

IV. Simulations

In this section, we evaluate the performance of the proposed bounds on the mutual coherence of the Laplacian matrix from Section III-B. To this end, the underlying graph is modeled by the Erdős-Réyni $\mathcal{G}(N,p)$ random graph [49], with N nodes and where p is the edge-presence probability. The evaluation is conducted over 10^5 Monte Carlo simulations. The proposed bounds were also evaluated on the distance-based graph [2], and sparse recovery performance was tested for both graphs. These results are in the supplementary material.

The upper plot in Fig. 1 presents the averaged mutual coherence and the corresponding averaged upper and lower bounds as a function of the normalized minimal nodal degree, d_{min}/N, for a fixed graph size of $N = 70$. It can be seen that both the mutual coherence and the proposed bounds decrease as d_{min}/N increases, as expected. Additionally, the proposed upper and lower bounds effectively create a narrow range around the mutual coherence, where the upper bound is tighter for smaller values of d_{min}/N and the lower bound is tighter for higher values of d_{min}/N. The inset in this figure shows a nearly linear relationship between the d_{min}/N and the edge presence probability p. The lower plot of Fig. 1 presents the averaged mutual coherence and the proposed bounds as a function of the number of nodes in the graph, N, where $p = 0.3, 0.8$. For both cases, the mutual coherence and the proposed bounds decrease as N increases. For the bounds, this trend is due to an increase in minimal nodal degree with graph size, which is expected in the Erdős-Réyni $\mathcal{G}(N,p)$ graph model. In contrast to the upper plot, here the relationship between the bounds and the mutual coherence remains consistent across all observed graph sizes.

V. Conclusions

In this letter, we derived theoretical guarantees for sparse recovery problems in which the dictionary is a Laplacian matrix. By computing and analyzing the inner products between two Laplacian matrix columns, we established upper and lower bounds on the mutual coherence of the Laplacian matrix. It is shown that the proposed bounds depend on the graph's minimal nodal degree. These bounds can be utilized to evaluate various CS measures and to determine the conditions for reliable sparse recovery using standard methods. Simulations demonstrate that the bounds closely approximate the behavior of the practical mutual coherence of the Laplacian of an Erdős-Rényi graph, where the upper bound is tighter for sparser graphs, while the lower bound is tighter for dense graphs. Future work should extend this framework to scenarios with partial observability, where measurements are available from a subset of nodes, higher-order GSP filter models, and other GSOs to broaden the method's applicability.

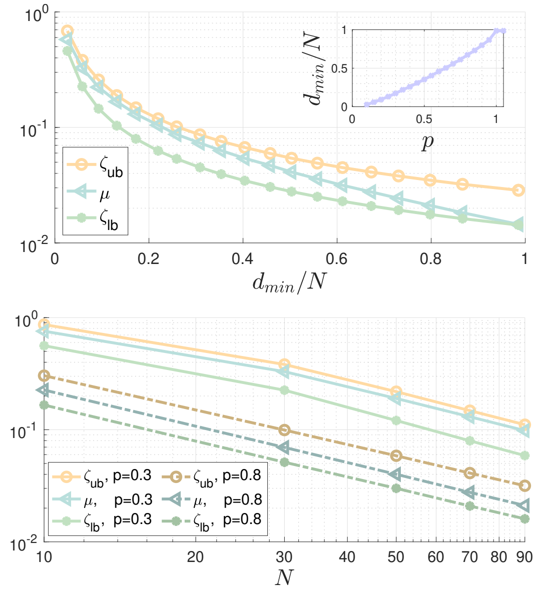

Figure 1: Averaged mutual coherence with upper and lower bounds versus the normalized minimal nodal degree, d_{min}/N (top), and the graph size, N, for $p = 0.3, 0.8$ (bottom). The inset shows d_{min}/N versus the edge-presence probability p.

Appendix: Analysis of (14)

Given the variables $a, b \in \mathbb{R}$, such that $a \geq 1$ and $b \geq 1$, we consider the function

$$f(a,b) \triangleq \frac{a+b}{\sqrt{a^2+a}\sqrt{b^2+b}}. \tag{24}$$

The partial derivative of $f(a,b)$ with respect to (w.r.t.) a is

$$\begin{aligned}
&\frac{\partial f(a,b)}{\partial a} \\
&= \frac{1}{\sqrt{b^2+b}\sqrt{a^2+a}} \frac{a^2+a-\frac{1}{2}(2a^2+a+2ab+b)}{a^2+a} \\
&= \frac{1}{\sqrt{b^2+b}\sqrt{a^2+a}} \frac{\frac{1}{2}a-ab-\frac{1}{2}b}{a^2+a}.
\end{aligned} \tag{25}$$

Now, for $a \geq 1$ and $b \geq 1$, it can be verified from (25) that $\frac{\partial f(a,b)}{\partial a}$ is always negative. Therefore, for $a \geq 1$ and $b \geq 1$, it is a monotonic decreasing function of a. In addition, it can be seen from (24) that $f(a,b)$ is symmetric in the sense that $f(a,b) = f(b,a)$. Thus, for $a \geq 1$ and $b \geq 1$, the function $f(a,b)$ is also a monotonic decreasing function of b.

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