1. INTRODUCTION

Interest rates are both a barometer of the economy and an instrument for its control. The term structure of interest rates—market interest rates at various maturities—is a vital input into the valuation of many financial products. The goal of this chapter is to explain the term structure and interest rate dynamics—that is, the process by which the yields and prices of bonds evolve over time.

A spot interest rate (in this chapter, "spot rate") is a rate of interest on a security that makes a single payment at a future point in time. The forward rate is the rate of interest set today for a single-payment security to be issued at a future date. Section 2 explains the relationship between these two types of interest rates and why forward rates matter to active bond portfolio managers. Section 2 also briefly covers other important return concepts.

The swap rate curve is the name given to the swap market's equivalent of the yield curve. Section 3 describes in more detail the swap rate curve and a related concept, the swap spread, and describes their use in valuation.

Sections 4 and 5 describe traditional and modern theories of the term structure of interest rates, respectively. Traditional theories present various largely qualitative perspectives on economic forces that may affect the shape of the term structure. Modern theories model the term structure with greater rigor.

Section 6 describes yield curve factor models. The focus is a popular three-factor term structure model in which the yield curve changes are described in terms of three independent movements: level, steepness, and curvature. These factors can be extracted from the variance–covariance matrix of historical interest rate movements.

A summary of key points concludes the chapter.

2. SPOT RATES AND FORWARD RATES

In this section, we will first explain the relationships among spot rates, forward rates, yield to maturity, expected and realized returns on bonds, and the shape of the yield curve. We will then discuss the assumptions made about forward rates in active bond portfolio management.

At any point in time, the price of a risk-free single-unit payment (e.g., \$1, €1, or £1) at time T is called the **discount factor** with maturity T, denoted by $P(T)$. The yield to maturity of the payment is called a **spot rate**, denoted by $r(T)$. That is,

$$P(T) = \frac{1}{\left[1 + r(T)\right]^T} \tag{1}$$

The discount factor, $P(T)$, and the spot rate, $r(T)$, for a range of maturities in years $T > 0$ are called the **discount function** and the **spot yield curve** (or, more simply, **spot curve**), respectively. The spot curve represents the term structure of interest rates at any point in time. Note that the discount function completely identifies the spot curve and vice versa. The discount function and the spot curve contain the same set of information about the time value of money.

The spot curve shows, for various maturities, the annualized return on an option-free and default-risk-free **zero-coupon bond** (**zero** for short) with a single payment of principal at maturity. The spot rate as a yield concept avoids the complications associated with the need for a reinvestment rate assumption for coupon-paying securities. Because the spot curve depends on the market pricing of these option-free zero-coupon bonds at any point in time, the shape and level of the spot yield curve are dynamic—that is, continually changing over time.

As Equation 1 suggests, the default-risk-free spot curve is a benchmark for the time value of money received at any future point in time as determined by the market supply and demand for funds. It is viewed as the most basic term structure of interest rates because there is no reinvestment risk involved; the stated yield equals the actual realized return if the zero is held to maturity. Thus, the yield on a zero-coupon bond maturing in year T is regarded as the most accurate representation of the T-year interest rate.

A **forward rate** is an interest rate that is determined today for a loan that will be initiated in a future time period. The term structure of forward rates for a loan made on a specific initiation date is called the **forward curve**. Forward rates and forward curves can be mathematically derived from the current spot curve.

Denote the forward rate of a loan initiated T^* years from today with tenor (further maturity) of T years by $f(T^*,T)$. Consider a forward contract in which one party to the contract, the buyer, commits to pay the other party to the contract, the seller, a forward contract price, denoted by $F(T^*,T)$, at time T^* years from today for a zero-coupon bond with maturity T years and unit principal. This is only an agreement to do something in the future at the time the contract is entered into; thus, no money is exchanged between the two parties at contract initiation. At T^*, the buyer will pay the seller the contracted forward price value and will receive from the seller at time $T^* + T$ the principal payment of the bond, defined here as a single currency unit.

The **forward pricing model** describes the valuation of forward contracts. The no-arbitrage argument that is used to derive the model is frequently used in modern financial theory; the model can be adopted to value interest rate futures contracts and related instruments, such as options on interest rate futures.

The no-arbitrage principle is quite simple. It says that tradable securities with identical cash flow payments must have the same price. Otherwise, traders would be able to generate risk-free arbitrage profits. Applying this argument to value a forward contract, we consider the discount factors—in particular, the values $P(T^*)$ and $P(T^* + T)$ needed to price a forward contract, $F(T^*,T)$. This forward contract price has to follow Equation 2, which is known as the forward pricing model.

$$P(T^* + T) = P(T^*)F(T^*,T) \tag{2}$$

To understand the reasoning behind Equation 2, consider two alternative investments: (1) buying a zero-coupon bond that matures in $T^* + T$ years at a cost of $P(T^*+ T)$, and (2) entering into a forward contract valued at $F(T^*,T)$ to buy at T^* a zero-coupon bond with maturity T at a cost today of $P(T^*)F(T^*,T)$. The payoffs for the two investments at time $T^* + T$ are the same. For this reason, the initial costs of the investments have to be the same, and therefore, Equation 2 must hold. Otherwise, any trader could sell the overvalued investment and buy the undervalued investment with the proceeds to generate risk-free profits with zero net investment.

Working the problems in Example 1 should help confirm your understanding of discount factors and forward prices. Please note that the solutions in the examples that follow may be rounded to two or four decimal places.

EXAMPLE 1 Spot and Forward Prices and Rates (1)

Consider a two-year loan ($T = 2$) beginning in one year ($T^* = 1$). The one-year spot rate is $r(T^*) = r(1) = 7\% = 0.07$. The three-year spot rate is $r(T^* + T) = r(1 + 2) = r(3) = 9\% = 0.09$.

1. Calculate the one-year discount factor: $P(T^*) = P(1)$.
2. Calculate the three-year discount factor: $P(T^* + T) = P(1 + 2) = P(3)$.

3. Calculate the forward price of a two-year bond to be issued in one year: $F(T^*, T) = F(1,2)$.
4. Interpret your answer to Problem 3.

Solution to 1: Using Equation 1,

$$P(1) = \frac{1}{(1+0.07)^1} = 0.9346$$

Solution to 2:

$$P(3) = \frac{1}{(1+0.09)^3} = 0.7722$$

Solution to 3: Using Equation 2,

$$0.7722 = 0.9346 \times F(1,2).$$

$$F(1,2) = 0.7722 \div 0.9346 = 0.8262.$$

Solution to 4: The forward contract price of $F(1,2) = 0.8262$ is the price, agreed on today, that would be paid one year from today for a bond with a two-year maturity and a risk-free unit-principal payment (e.g., \$1, €1, or £1) at maturity. As shown in the solution to 3, it is calculated as the three-year spot rate, $P(3) = 0.7722$, divided by the one-year spot rate, $P(1) = 0.9346$.

2.1. The Forward Rate Model

This section uses the forward rate model to establish that when the spot curve is upward sloping, the forward curve will lie above the spot curve, and that when the spot curve is downward sloping, the forward curve will lie below the spot curve.

The forward rate $f(T^*, T)$ is the discount rate for a risk-free unit-principal payment $T^* + T$ years from today, valued at time T^*, such that the present value equals the forward contract price, $F(T^*,T)$. Then, by definition,

$$F(T^*,T) = \frac{1}{[1+f(T^*,T)]^T} \tag{3}$$

By substituting Equations 1 and 3 into Equation 2, the forward pricing model can be expressed in terms of rates as noted by Equation 4, which is the **forward rate model**:

$$[1+r(T^*+T)]^{(T^*+T)} = [1+r(T^*)]^{T^*}[1+f(T^*,T)]^T \tag{4}$$

Thus, the spot rate for $T^* + T$, which is $r(T^* + T)$, and the spot rate for T^*, which is $r(T^*)$, imply a value for the T-year forward rate at T^*, $f(T^*,T)$. Equation 4 is important because it shows how forward rates can be extrapolated from spot rates; that is, they are implicit in the spot rates at any given point in time.[1]

Equation 4 suggests two interpretations or ways to look at forward rates. For example, suppose $f(7,1)$, the rate agreed on today for a one-year loan to be made seven years from today, is 3%. Then 3% is the

- reinvestment rate that would make an investor indifferent between buying an eight-year zero-coupon bond or investing in a seven-year zero-coupon bond and at maturity reinvesting the proceeds for one year. In this sense, the forward rate can be viewed as a type of breakeven interest rate.
- one-year rate that can be locked in today by buying an eight-year zero-coupon bond rather than investing in a seven-year zero-coupon bond and, when it matures, reinvesting the proceeds in a zero-coupon instrument that matures in one year. In this sense, the forward rate can be viewed as a rate that can be locked in by extending maturity by one year.

Example 2 addresses forward rates and the relationship between spot and forward rates.

EXAMPLE 2 Spot and Forward Prices and Rates (2)

The spot rates for three hypothetical zero-coupon bonds (zeros) with maturities of one, two, and three years are given in the following table.

Maturity (T)	1	2	3
Spot rates	$r(1) = 9\%$	$r(2) = 10\%$	$r(3) = 11\%$

1. Calculate the forward rate for a one-year zero issued one year from today, $f(1,1)$.
2. Calculate the forward rate for a one-year zero issued two years from today, $f(2,1)$.
3. Calculate the forward rate for a two-year zero issued one year from today, $f(1,2)$.
4. Based on your answers to 1 and 2, describe the relationship between the spot rates and the implied one-year forward rates.

Solution to 1: $f(1,1)$ is calculated as follows (using Equation 4):

$$[1+r(2)]^2 = [1+r(1)]^1[1+f(1,1)]^1$$

$$(1+0.10)^2 = (1+0.09)^1[1+f(1,1)]^1$$

$$f(1,1) = \frac{(1.10)^2}{1.09} - 1 = 11.01\%$$

[1]An approximation formula that is based on taking logs of both sides of Equation 4 and using the approximation $\ln(1 + x) \approx x$ for small x is $f(T^*,T) \approx [(T^* + T)r(T^* + T) - T^*r(T^*)]/T$. For example, $f(1,2)$ in Example 2 could be approximated as $(3 \times 11\% - 1 \times 9\%)/2 = 12\%$, which is very close to 12.01%.

Solution to 2: $f(2,1)$ is calculated as follows:

$$[1+r(3)]^3 = [1+r(2)]^2 [1+f(2,1)]^1$$
$$(1+0.11)^3 = (1+0.10)^2 [1+f(2,1)]^1$$
$$f(2,1) = \frac{(1.11)^3}{(1.10)^2} - 1 = 13.03\%$$

Solution to 3: $f(1,2)$ is calculated as follows:

$$[1+r(3)]^3 = [1+r(1)]^1 [1+f(1,2)]^2$$
$$(1+0.11)^3 = (1+0.09)^1 [1+f(1,2)]^2$$
$$f(1,2) = \sqrt[2]{\frac{(1.11)^3}{1.09}} - 1 = 12.01\%$$

Solution to 4: The upward-sloping zero-coupon yield curve is associated with an upward-sloping forward curve (a series of increasing one-year forward rates because 13.03% is greater than 11.01%). This point is explained further in the following paragraphs.

The analysis of the relationship between spot rates and one-period forward rates can be established by using the forward rate model and successive substitution, resulting in Equations 5a and 5b:

$$\begin{aligned}[1+r(T)]^T = {} & [1+r(1)][1+f(1,1)][1+f(2,1)][1+f(3,1)]\ldots \\ & [1+f(T-1,1)]\end{aligned} \tag{5a}$$

$$r(T) = \{[1+r(1)][1+f(1,1)][1+f(2,1)][1+f(3,1)]\ldots[1+f(T-1,1)]\}^{(1/T)} - 1 \tag{5b}$$

Equation 5b shows that the spot rate for a security with a maturity of $T > 1$ can be expressed as a geometric mean of the spot rate for a security with a maturity of $T = 1$ and a series of $T - 1$ forward rates.

Whether the relationship in Equation 5b holds in practice is an important consideration for active portfolio management. If an active trader can identify a series of short-term bonds whose actual returns will exceed today's quoted forward rates, then the total return over his or her investment horizon would exceed the return on a maturity-matching, buy-and-hold strategy. Later, we will use this same concept to discuss dynamic hedging strategies and the local expectations theory.

Examples 3 and 4 explore the relationship between spot and forward rates.

EXAMPLE 3 Spot and Forward Prices and Rates (3)

Given the data and conclusions for $r(1)$, $f(1,1)$, and $f(2,1)$ from Example 2:

$$r(1) = 9\%$$

$$f(1,1) = 11.01\%$$

$$f(2,1) = 13.03\%$$

Show that the two-year spot rate of $r(2) = 10\%$ and the three-year spot rate of $r(3) = 11\%$ are geometric averages of the one-year spot rate and the forward rates.

Solution: Using Equation 5a,

$$\left[1+r(2)\right]^2 = \left[1+r(1)\right]\left[1+f(1,1)\right]$$
$$r(2) = \sqrt[2]{(1+0.09)(1+0.1101)} - 1 \approx 10\%$$

$$\left[1+r(3)\right]^3 = \left[1+r(1)\right]\left[1+f(1,1)\right]\left[1+f(2,1)\right]$$
$$r(3) = \sqrt[3]{(1+0.09)(1+0.1101)(1+0.1303)} - 1 \approx 11\%$$

We can now consolidate our knowledge of spot and forward rates to explain important relationships between the spot and forward rate curves. The forward rate model (Equation 4) can also be expressed as Equation 6.

$$\left\{\frac{\left[1+r(T^*+T)\right]}{\left[1+r(T^*)\right]}\right\}^{\frac{T^*}{T}}\left[1+r(T^*+T)\right] = \left[1+f(T^*,T)\right] \tag{6}$$

To illustrate, suppose $T^* = 1$, $T = 4$, $r(1) = 2\%$, and $r(5) = 3\%$; the left-hand side of Equation 6 is

$$\left(\frac{1.03}{1.02}\right)^{\frac{1}{4}}(1.03) = (1.0024)(1.03) = 1.0325$$

so $f(1,4) = 3.25\%$. Given that the yield curve is upward sloping—so, $r(T^* + T) > r(T^*)$—Equation 6 implies that the forward rate from T^* to T is greater than the long-term $(T^* + T)$ spot rate: $f(T^*,T) > r(T^* + T)$. In the example given, 3.25% > 3%. Conversely, when the yield curve is downward sloping, then $r(T^* + T) < r(T^*)$ and the forward rate from T^* to T is lower than the long-term spot rate: $f(T^*,T) < r(T^* + T)$. Equation 6 also shows that if the spot curve is flat, all one-period forward rates are equal to the spot rate. For an upward-sloping yield curve—$r(T^* + T) > r(T^*)$—the forward rate rises as T^* increases. For a downward-sloping yield curve—$r(T^* + T) < r(T^*)$—the forward rate declines as T^* increases.

EXAMPLE 4 Spot and Forward Prices and Rates (4)

Given the spot rates $r(1) = 9\%$, $r(2) = 10\%$, and $r(3) = 11\%$, as in Examples 2 and 3:

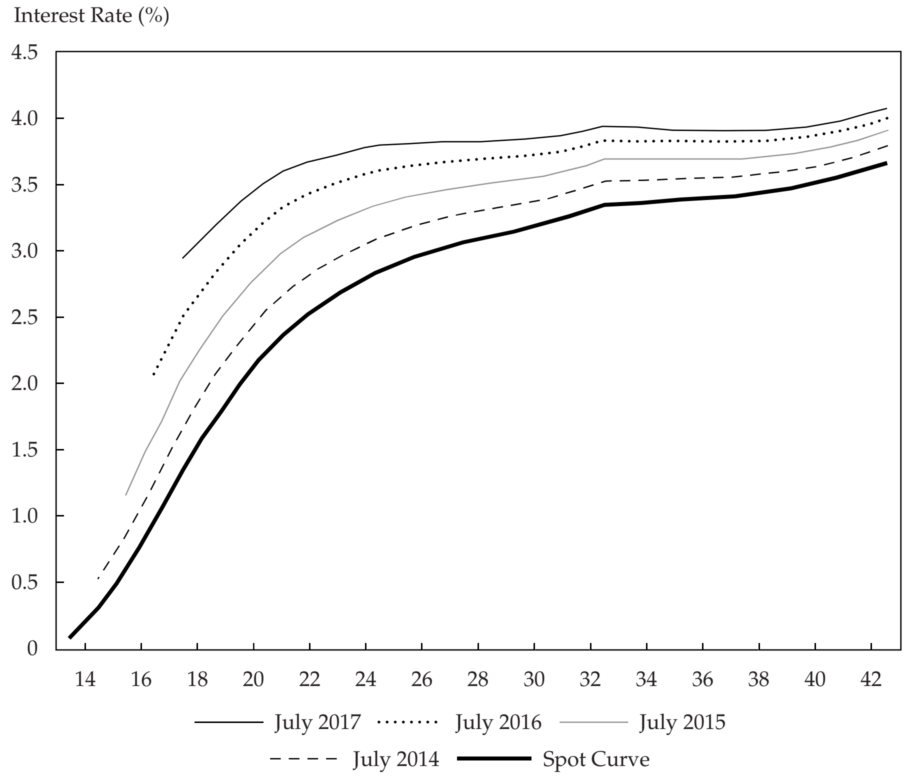
1. Determine whether the forward rate $f(1,2)$ is greater than or less than the long-term rate, $r(3)$.
2. Determine whether forward rates rise or fall as the initiation date, T^*, for the forward rate is increased.

Solution to 1: The spot rates imply an upward-sloping yield curve, $r(3) > r(2) > r(1)$, or in general, $r(T^* + T) > r(T^*)$. Thus, the forward rate will be greater than the long-term rate, or $f(T^*,T) > r(T^* + T)$. Note from Example 2 that $f(1,2) = 12.01\% > r(1 + 2) = r(3) = 11\%$.

Solution to 2: The spot rates imply an upward-sloping yield curve, $r(3) > r(2) > r(1)$. Thus, the forward rates will rise with increasing T^*. This relationship was shown in Example 2, in which $f(1,1) = 11.01\%$ and $f(2,1) = 13.03\%$.

These relationships are illustrated in Exhibit 1, using actual data. The spot rates for US Treasuries as of 31 July 2013 are represented by the lowest curve in the exhibit, which was constructed using interpolation between the data points, shown in the table following the exhibit. Note that the spot curve is upward sloping. The spot curve and the forward curves for the end of July 2014, July 2015, July 2016, and July 2017 are also presented in Exhibit 1. Because the yield curve is upward sloping, the forward curves lie above the spot curve and increasing the initiation date results in progressively higher forward curves. The highest forward curve is that for July 2017. Note that the forward curves in Exhibit 1 are progressively flatter at later start dates because the spot curve flattens at the longer maturities.

EXHIBIT 1 Spot Curve vs. Forward Curves, 31 July 2013



Maturity (years)	1	2	3	5	7	10	20	30
Spot rate (%)	0.11	0.33	0.61	1.37	2.00	2.61	3.35	3.66

When the spot yield curve is downward sloping, the forward yield curve will be below the spot yield curve. Spot rates for US Treasuries as of 31 December 2006 are presented in the table following Exhibit 2. We used linear interpolation to construct the spot curve based on these data points. The yield curve data were also somewhat modified to make the yield curve more downward sloping for illustrative purposes. The spot curve and the forward curves for the end of December 2007, 2008, 2009, and 2010 are presented in Exhibit 2.