PART II

ANALYSIS OF RISK

CHAPTER 4

UNDERSTANDING FIXED-INCOME RISK AND RETURN

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LEARNING OUTCOMES

After completing this chapter, you will be able to do the following:

- calculate and interpret the sources of return from investing in a fixed-rate bond;
- define, calculate, and interpret Macaulay, modified, and effective durations;
- explain why effective duration is the most appropriate measure of interest rate risk for bonds with embedded options;
- define key rate duration and describe the key use of key rate durations in measuring the sensitivity of bonds to changes in the shape of the benchmark yield curve;
- explain how a bond's maturity, coupon, embedded options, and yield level affect its interest rate risk;
- calculate the duration of a portfolio and explain the limitations of portfolio duration;
- calculate and interpret the money duration of a bond and price value of a basis point (PVBP);
- calculate and interpret approximate convexity and distinguish between approximate and effective convexity;
- estimate the percentage price change of a bond for a specified change in yield, given the bond's approximate duration and convexity;
- describe how the term structure of yield volatility affects the interest rate risk of a bond;
- describe the relationships among a bond's holding period return, its duration, and the investment horizon;
- explain how changes in credit spread and liquidity affect yield-to-maturity of a bond and how duration and convexity can be used to estimate the price effect of the changes.

1. INTRODUCTION

It is important for analysts to have a well-developed understanding of the risk and return characteristics of fixed-income investments. Beyond the vast worldwide market for publicly and privately issued fixed-rate bonds, many financial assets and liabilities with known future cash flows may be evaluated using the same principles. The starting point for this analysis is the yield-to-maturity, or internal rate of return on future cash flows, which was introduced in the fixed-income valuation chapter. The return on a fixed-rate bond is affected by many factors, the most important of which is the receipt of the interest and principal payments in the full amount and on the scheduled dates. Assuming no default, the return is also affected by changes in interest rates that affect coupon reinvestment and the price of the bond if it is sold before it matures. Measures of the price change can be derived from the mathematical relationship used to calculate the price of the bond. The first of these measures (duration) estimates the change in the price for a given change in interest rates. The second measure (convexity) improves on the duration estimate by taking into account the fact that the relationship between price and yield-to-maturity of a fixed-rate bond is not linear.

Section 2 uses numerical examples to demonstrate the sources of return on an investment in a fixed-rate bond, which includes the receipt and reinvestment of coupon interest payments and the redemption of principal if the bond is held to maturity. The other source of return is capital gains (and losses) on the sale of the bond prior to maturity. Section 2 also shows that fixed-income investors holding the same bond can have different exposures to interest rate risk if their investment horizons differ. Discussion of credit risk, although critical to investors, is postponed to Section 5 so that attention can be focused on interest rate risk.

Section 3 provides a thorough review of bond duration and convexity, and shows how the statistics are calculated and used as measures of interest rate risk. Although procedures and formulas exist to calculate duration and convexity, these statistics can be approximated using basic bond-pricing techniques and a financial calculator. Commonly used versions of the statistics are covered, including Macaulay, modified, effective, and key rate durations. The distinction is made between risk measures that are based on changes in the bond's yield-to-maturity (i.e., *yield* duration and convexity) and on benchmark yield curve changes (i.e., *curve* duration and convexity).

Section 4 returns to the issue of the investment horizon. When an investor has a short-term horizon, duration (and convexity) is used to estimate the change in the bond price. In this case, yield volatility matters. In particular, bonds with varying times-to-maturity have different degrees of yield volatility. When an investor has a long-term horizon, the interaction between coupon reinvestment risk and market price risk matters. The relationship among interest rate risk, bond duration, and the investment horizon is explored.

Section 5 discusses how the tools of duration and convexity can be extended to credit and liquidity risks and highlights how these different factors can affect a bond's return and risk.

A summary of key points and practice problems in the CFA Institute multiple-choice format conclude the chapter.

2. SOURCES OF RETURN

An investor in a fixed-rate bond has three sources of return: (1) receipt of the promised coupon and principal payments on the scheduled dates, (2) reinvestment of coupon payments, and (3) potential capital gains or losses on the sale of the bond prior to maturity. In this section, it is assumed that the issuer makes the coupon and principal payments as scheduled. This chapter focuses primarily on interest rate risk (the risk that interest rates will change), which affects the reinvestment of coupon payments and the market price if the bond is sold prior to maturity. Credit risk is considered in Section 5 of this chapter and is the primary subject of the chapter "Fundamentals of Credit Analysis."

When a bond is purchased at a premium or a discount, it adds another aspect to the rate of return. Recall from the chapter on fixed-income valuation that a discount bond offers the investor a "deficient" coupon rate, or one below the market discount rate. The amortization of the discount in each period brings the return in line with the market discount rate as the bond's carrying value is "pulled to par." For a premium bond, the coupon rate exceeds the market discount rate and the amortization of the premium adjusts the return to match the market discount rate. Through amortization, the bond's carrying value reaches par value at maturity.

A series of examples will demonstrate the effect of a change in interest rates on two investors' realized rate of returns. Interest rates are the rates at which coupon payments are reinvested and the market discount rates at the time of purchase and at the time of sale if the bond is not held to maturity. In Examples 1 and 2, interest rates are unchanged. The two investors, however, have different time horizons for holding the bond. Examples 3 and 4 show the impact of an increase in interest rates on the two investors' total return. Examples 5 and 6 show the impact of a decrease in interest rates. In each of the six examples, an investor initially buys a 10-year, 8% annual coupon payment bond at a price of 85.503075 per 100 of par value. The bond's yield-to-maturity is 10.40%.

$$85.503075 = \frac{8}{(1+r)^1} + \frac{8}{(1+r)^2} + \frac{8}{(1+r)^3} + \frac{8}{(1+r)^4} + \frac{8}{(1+r)^5} + \frac{8}{(1+r)^6} + \frac{8}{(1+r)^7} + \frac{8}{(1+r)^8} + \frac{8}{(1+r)^9} + \frac{108}{(1+r)^{10}}, \quad r = 0.1040$$

EXAMPLE 1

A "buy-and-hold" investor purchases a 10-year, 8% annual coupon payment bond at 85.503075 per 100 of par value and holds it until maturity. The investor receives the series of 10 coupon payments of 8 (per 100 of par value) for a total of 80, plus the redemption of principal (100) at maturity. In addition to collecting the coupon interest and the principal, the investor has the opportunity to reinvest the cash flows. If the coupon payments are reinvested at 10.40%, the future value of the coupons on the bond's maturity date is 129.970678 per 100 of par value.

$$\left[8\times(1.1040)^9\right]+\left[8\times(1.1040)^8\right]+\left[8\times(1.1040)^7\right]+\left[8\times(1.1040)^6\right]+\left[8\times(1.1040)^5\right]+\left[8\times(1.1040)^4\right]+\left[8\times(1.1040)^3\right]+\left[8\times(1.1040)^2\right]+\left[8\times(1.1040)^1\right]+8=129.970678$$

The first coupon payment of 8 is reinvested at 10.40% for nine years until maturity, the second is reinvested for eight years, and so forth. The future value of the annuity is obtained easily on a financial calculator, using 8 for the payment that is received at the end of each of the 10 periods. The amount in excess of the coupons, 49.970678 (= 129.970678 – 80), is the "interest-on-interest" gain from compounding.

The investor's total return is 229.970678, the sum of the reinvested coupons (129.970678) and the redemption of principal at maturity (100). The realized rate of return is 10.40%.

$$85.503075 = \frac{229.970678}{(1+r)^{10}}, \quad r = 0.1040$$

Example 1 demonstrates that the yield-to-maturity at the time of purchase measures the investor's rate of return under three assumptions: (1) The investor holds the bond to maturity, (2) there is no default by the issuer, and (3) the coupon interest payments are reinvested at that same rate of interest.

Example 2 considers another investor who buys the 10-year, 8% annual coupon payment bond and pays the same price. This investor, however, has a four-year investment horizon. Therefore, coupon interest is only reinvested for four years, and the bond is sold immediately after receiving the fourth coupon payment.

EXAMPLE 2

A second investor buys the 10-year, 8% annual coupon payment bond and sells the bond after four years. Assuming that the coupon payments are reinvested at 10.40% for four years, the future value of the reinvested coupons is 37.347111 per 100 of par value.

$$\left[8 \times (1.1040)^3\right] + \left[8 \times (1.1040)^2\right] + \left[8 \times (1.1040)^1\right] + 8 = 37.347111$$

The interest-on-interest gain from compounding is 5.347111 (= 37.347111 – 32). After four years, when the bond is sold, it has six years remaining until maturity. If the yield-to-maturity remains 10.40%, the sale price of the bond is 89.668770.

$$\frac{8}{(1.1040)^1} + \frac{8}{(1.1040)^2} + \frac{8}{(1.1040)^3} + \frac{8}{(1.1040)^4} + \frac{8}{(1.1040)^5} + \frac{108}{(1.1040)^6} = 89.668770$$

The total return is 127.015881 (= 37.347111 + 89.668770) and the realized rate of return is 10.40%.

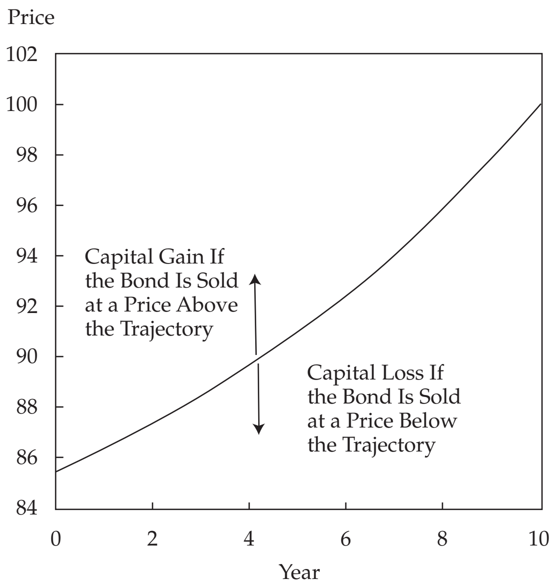
$$85.503075 = \frac{127.015881}{(1+r)^4}, \quad r = 0.1040$$

In Example 2, the investor's **horizon yield** is 10.40%. A horizon yield is the internal rate of return between the total return (the sum of reinvested coupon payments and the sale price or redemption amount) and the purchase price of the bond. The horizon yield on a bond investment is the annualized holding-period rate of return.

Example 2 demonstrates that the realized horizon yield matches the original yield-to-maturity if (1) coupon payments are reinvested at the same interest rate as the original yield-to-maturity, and (2) the bond is sold at a price on the constant-yield price trajectory, which implies that the investor does not have any capital gains or losses when the bond is sold.

Capital gains arise if a bond is sold at a price above its constant-yield price trajectory and capital losses occur if a bond is sold at a price above below its constant-yield price trajectory. This trajectory is based on the yield-to-maturity when the bond is purchased. The trajectory is shown in Exhibit 1 for a 10-year, 8% annual payment bond purchased at a price of 85.503075 per 100 of par value.

EXHIBIT 1 Constant-Yield Price Trajectory for a 10-Year, 8% Annual Payment Bond



Note: Price is price per 100 of par value.

A point on the trajectory represents the **carrying value** of the bond at that time. The carrying value is the purchase price plus the amortized amount of the discount if the bond is purchased at a price below par value. If the bond is purchased at a price above par value, the carrying value is the purchase price minus the amortized amount of the premium.

The amortized amount for each year is the change in the price between two points on the trajectory. The initial price of the bond is 85.503075 per 100 of par value. Its price (the carrying value) after one year is 86.393394, calculated using the original yield-to-maturity of 10.40%. Therefore, the amortized amount for the first year is 0.890319 (= 86.393394 – 85.503075). The bond price in Example 2 increases from 85.503075 to 89.668770, and that increase over the four years is movement *along* the constant-yield price trajectory. At the time the bond is sold, its carrying value is also 89.668770, so there is no capital gain or loss.

Examples 3 and 4 demonstrate the impact on investors' realized horizon yields if interest rates go up by 100 basis points (bps). The market discount rate on the bond increases from 10.40% to 11.40%. Coupon reinvestment rates go up by 100 bps as well.

EXAMPLE 3

The buy-and-hold investor purchases the 10-year, 8% annual payment bond at 85.503075. After the bond is purchased and before the first coupon is received, interest rates go up to 11.40%. The future value of the reinvested coupons at 11.40% for 10 years is 136.380195 per 100 of par value.

$$\left[8\times(1.1140)^9\right]+\left[8\times(1.1140)^8\right]+\left[8\times(1.1140)^7\right]+\left[8\times(1.1140)^6\right]+$$
$$\left[8\times(1.1140)^5\right]+\left[8\times(1.1140)^4\right]+\left[8\times(1.1140)^3\right]+\left[8\times(1.1140)^2\right]+$$
$$\left[8\times(1.1140)^1\right]+8=136.380195$$

The total return is 236.380195 (= 136.380195 + 100). The investor's realized rate of return is 10.70%.

$$85.503075=\frac{236.380195}{(1+r)^{10}},\quad r=0.1070$$

In Example 3, the buy-and-hold investor benefits from the higher coupon reinvestment rate. The realized horizon yield is 10.70%, 30 bps higher than the outcome in Example 1, when interest rates are unchanged. There is no capital gain or loss because the bond is held until maturity. The carrying value at the maturity date is par value, the same as the redemption amount.

EXAMPLE 4

The second investor buys the 10-year, 8% annual payment bond at 85.503075 and sells it in four years. After the bond is purchased, interest rates go up to 11.40%. The future value of the reinvested coupons at 11.40% after four years is 37.899724 per 100 of par value.

$$\left[8\times(1.1140)^3\right]+\left[8\times(1.1140)^2\right]+\left[8\times(1.1140)^1\right]+8=37.899724$$

The sale price of the bond after four years is 85.780408.

$$\frac{8}{(1.1140)^1}+\frac{8}{(1.1140)^2}+\frac{8}{(1.1140)^3}+\frac{8}{(1.1140)^4}+$$
$$\frac{8}{(1.1140)^5}+\frac{108}{(1.1140)^6}=85.780408$$

The total return is 123.680132 (= 37.899724 + 85.780408), resulting in a realized four-year horizon yield of 9.67%.

$$85.503075 = \frac{123.680132}{(1+r)^4}, \quad r = 0.0967$$

In Example 4, the second investor has a lower realized rate of return compared with the investor in Example 2, in which interest rates are unchanged. The future value of reinvested coupon payments goes up by 0.552613 (= 37.899724 – 37.347111) per 100 of par value because of the higher interest rates. But there is a *capital loss* of 3.888362 (= 89.668770 – 85.780408) per 100 of par value. Notice that the capital loss is measured from the bond's carrying value, the point on the constant-yield price trajectory, and not from the original purchase price. The bond is now sold at a price below the constant-yield price trajectory. The reduction in the realized four-year horizon yield from 10.40% to 9.67% is a result of the capital loss being greater than the gain from reinvesting coupons at a higher rate, which reduces the investor's total return.

Examples 5 and 6 complete the series of rate-of-return calculations for the two investors. Interest rates decline by 100 bps. The required yield on the bond falls from 10.40% to 9.40% after the purchase of the bond. The interest rates at which the coupon payments are reinvested fall as well.

EXAMPLE 5

The buy-and-hold investor purchases the 10-year bond at 85.503075 and holds the security until it matures. After the bond is purchased and before the first coupon is received, interest rates go down to 9.40%. The future value of reinvesting the coupon payments at 9.40% for 10 years is 123.888356 per 100 of par value.

$$\left[8\times(1.0940)^9\right]+\left[8\times(1.0940)^8\right]+\left[8\times(1.0940)^7\right]+\left[8\times(1.0940)^6\right]+$$
$$\left[8\times(1.0940)^5\right]+\left[8\times(1.0940)^4\right]+\left[8\times(1.0940)^3\right]+\left[8\times(1.0940)^2\right]+$$
$$\left[8\times(1.0940)^1\right]+8=123.888356$$

The total return is 223.888356, the sum of the future value of reinvested coupons and the redemption of par value. The investor's realized rate of return is 10.10%.

$$85.503075 = \frac{223.888356}{(1+r)^{10}}, \quad r = 0.1010$$

In Example 5, the buy-and-hold investor suffers from the lower coupon reinvestment rates. The realized horizon yield is 10.10%, 30 bps lower than the result in Example 1, when interest rates are unchanged. There is no capital gain or loss because the bond is held until maturity. Examples 1, 3, and 5 indicate that the interest rate risk for a buy-and-hold investor arises entirely from changes in coupon reinvestment rates.

EXAMPLE 6

The second investor buys the 10-year bond at 85.503075 and sells it in four years. After the bond is purchased, interest rates go down to 9.40%. The future value of the reinvested coupons at 9.40% is 36.801397 per 100 of par value.

$$\left[8\times(1.0940)^3\right]+\left[8\times(1.0940)^2\right]+\left[8\times(1.0940)^1\right]+8=36.801397$$

This reduction in future value is offset by the higher sale price of the bond, which is 93.793912 per 100 of par value.

$$\frac{8}{(1.0940)^1}+\frac{8}{(1.0940)^2}+\frac{8}{(1.0940)^3}+\frac{8}{(1.0940)^4}+\frac{8}{(1.0940)^5}+\frac{108}{(1.0940)^6}=93.793912$$

The total return is 130.595309 (= 36.801397 + 93.793912), and the realized yield is 11.17%.

$$85.503075=\frac{130.595309}{(1+r)^4},\quad r=0.1117$$

The investor in Example 6 has a capital gain of 4.125142 (= 93.793912 – 89.668770). The capital gain is measured from the carrying value, the point on the constant-yield price trajectory. That gain offsets the reduction in the future value of reinvested coupons of 0.545714 (= 37.347111 – 36.801397). The total return is higher than that in Example 2, in which the interest rate remains at 10.40%.

In these examples, interest income for the investor is the return associated with the *passage of time*. Therefore, interest income includes the receipt of coupon interest, the reinvestment of those cash flows, and the amortization of the discount from purchase at a price below par value (or the premium from purchase at a price above par value) to bring the return back in line with the market discount rate. A capital gain or loss is the return to the investor associated with the *change in the value* of the security. On the fixed-rate bond, a change in value arises from a change in the yield-to-maturity, which is the implied market discount rate. In practice, the manner in which interest income and capital gains and losses are calculated and reported on financial statements depends on financial and tax accounting rules.

This series of examples illustrates an important point about fixed-rate bonds: The *investment horizon* is at the heart of understanding interest rate risk and return. There are two offsetting types of interest rate risk that affect the bond investor: coupon reinvestment risk and market price risk. The future value of reinvested coupon payments (and in a portfolio, the principal on bonds that mature before the horizon date) *increases* when interest rates go up and *decreases* when rates go down. The sale price on a bond that matures after the horizon date (and thus needs to be sold) *decreases* when interest rates go up and *increases* when rates go down. Coupon reinvestment risk matters more when the investor has a long-term horizon relative to the time-to-maturity of the bond. For instance, a buy-and-hold investor only has coupon reinvestment risk. Market price risk matters more when the investor has a short-term horizon relative to the time-to-maturity. For example, an investor who sells the bond before the first coupon is received has only market price risk. Therefore, two investors holding the same bond (or bond portfolio) can have different exposures to interest rate risk if they have different investment horizons.

EXAMPLE 7

An investor buys a four-year, 10% annual coupon payment bond priced to yield 5.00%. The investor plans to sell the bond in two years once the second coupon payment is received. Calculate the purchase price for the bond and the horizon yield assuming that the coupon reinvestment rate after the bond purchase and the yield-to-maturity at the time of sale are (1) 3.00%, (2) 5.00%, and (3) 7.00%.

Solution: The purchase price is 117.729753.

$$\frac{10}{(1.0500)^1}+\frac{10}{(1.0500)^2}+\frac{10}{(1.0500)^3}+\frac{110}{(1.0500)^4}=117.729753$$

1. 3.00%: The future value of reinvested coupons is 20.300.

   $$(10 \times 1.0300) + 10 = 20.300$$

   The sale price of the bond is 113.394288.

   $$\frac{10}{(1.0300)^1}+\frac{110}{(1.0300)^2}=113.394288$$

   Total return: 20.300 + 113.394288 = 133.694288.
   If interest rates go down from 5.00% to 3.00%, the realized rate of return over the two-year investment horizon is 6.5647%, higher than the original yield-to-maturity of 5.00%.

   $$117.729753=\frac{133.694288}{(1+r)^2}, \qquad r=0.065647$$

2. 5.00%: The future value of reinvested coupons is 20.500.

   $$(10 \times 1.0500) + 10 = 20.500$$

   The sale price of the bond is 109.297052.

   $$\frac{10}{(1.0500)^1}+\frac{110}{(1.0500)^2}=109.297052$$

Total return: 20.500 + 109.297052 = 129.797052.
If interest rates remain 5.00% for reinvested coupons and for the required yield on the bond, the realized rate of return over the two-year investment horizon is equal to the yield-to-maturity of 5.00%.

$$117.729753=\frac{129.797052}{(1+r)^2}, \quad r=0.050000$$

3. 7.00%: The future value of reinvested coupons is 20.700.

$$(10 \times 1.0700) + 10 = 20.700$$

The bond is sold at 105.424055.

$$\frac{10}{(1.0700)^1}+\frac{110}{(1.0700)^2}=105.424055$$

Total return: 20.700 + 105.424055 = 126.124055.

$$117.729753=\frac{126.124055}{(1+r)^2}, \quad r=0.035037$$

If interest rates go up from 5.00% to 7.00%, the realized rate of return over the two-year investment horizon is 3.5037%, lower than the yield-to-maturity of 5.00%.

3. INTEREST RATE RISK ON FIXED-RATE BONDS

This section covers two commonly used measures of interest rate risk: duration and convexity. It distinguishes between risk measures based on changes in a bond's own yield to maturity (yield duration and convexity) and those that affect the bond based on changes in a benchmark yield curve (curve duration and convexity).

3.1. Macaulay, Modified, and Approximate Duration

The duration of a bond measures the sensitivity of the bond's full price (including accrued interest) to changes in the bond's yield-to-maturity or, more generally, to changes in benchmark interest rates. Duration estimates changes in the bond price assuming that variables other than the yield-to-maturity or benchmark rates are held constant. Most importantly, the time-to-maturity is unchanged. Therefore, duration measures the *instantaneous* (or, at least, same-day) change in the bond price. The accrued interest is the same, so it is the flat price that goes up or down when the full price changes. Duration is a useful measure because it represents the approximate amount of time a bond would have to be held for the market discount rate at purchase to be realized if there is a single change in interest rate. If the bond is held for the duration period, an increase from reinvesting coupons is offset by a decrease in price if interest rates increase and a decrease from reinvesting coupons is offset by an increase in price if interest rates decrease.

There are several types of bond duration. In general, these can be divided into **yield duration** and **curve duration**. Yield duration is the sensitivity of the bond price with respect to the bond's own yield-to-maturity. Curve duration is the sensitivity of the bond price (or more generally, the market value of a financial asset or liability) with respect to a benchmark yield curve. The benchmark yield curve could be the government yield curve on coupon bonds, the spot curve, or the forward curve, but in practice, the government par curve is often used. Yield duration statistics used in fixed-income analysis include Macaulay duration, modified duration, money duration, and the price value of a basis point (PVBP). A curve duration statistic often used is effective duration. Effective duration is covered in Section 3.2.

Macaulay duration is named after Frederick Macaulay, the Canadian economist who first wrote about the statistic in a book published in 1938.[1] Equation 1 is a general formula to calculate the Macaulay duration (MacDur) of a traditional fixed-rate bond.

$$\text{MacDur} = \left[\frac{\dfrac{(1-t/T)\times PMT}{(1-r)^{1-t/T}}+\dfrac{(2-t/T)\times PMT}{(1+r)^{2-t/T}}+\cdots+\dfrac{(N-t/T)\times(PMT+FV)}{(1+r)^{N-t/T}}}{\dfrac{PMT}{(1+r)^{1-t/T}}+\dfrac{PMT}{(1+r)^{2-t/T}}+\cdots+\dfrac{PMT+FV}{(1+r)^{N-t/T}}}\right] \tag{1}$$

where

t = the number of days from the last coupon payment to the settlement date
T = the number of days in the coupon period
t/T = the fraction of the coupon period that has gone by since the last payment
PMT = the coupon payment per period
FV = the future value paid at maturity, or the par value of the bond
r = the yield-to-maturity, or the market discount rate, per period
N = the number of evenly spaced periods to maturity as of the beginning of the current period

The denominator in Equation 1 is the full price (PV^{Full}) of the bond including accrued interest. It is the present value of the coupon interest and principal payments, with each cash flow discounted by the same market discount rate, r.

$$PV^{Full} = \frac{PMT}{(1+r)^{1-t/T}}+\frac{PMT}{(1+r)^{2-t/T}}+\cdots+\frac{PMT+FV}{(1+r)^{N-t/T}} \tag{2}$$

Equation 3 combines Equations 1 and 2 to reveal an important aspect of the Macaulay duration: Macaulay duration is a weighted average of the time to receipt of the bond's promised payments, where the weights are the shares of the full price that correspond to each of the bond's promised future payments.

[1] Frederick R. Macaulay, *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1856* (New York: National Bureau of Economic Research, 1938).

$$\text{MacDur} = \left\{ \begin{matrix} (1 - t/T)\left[\dfrac{\dfrac{PMT}{(1+r)^{1-t/T}}}{PV^{Full}}\right] + (2 - t/T)\left[\dfrac{\dfrac{PMT}{(1+r)^{2-t/T}}}{PV^{Full}}\right] + \cdots + \\ (N - t/T)\left[\dfrac{\dfrac{PMT + FV}{(1+r)^{N-t/T}}}{PV^{Full}}\right] \end{matrix} \right\} \quad (3)$$

The time to receipt of cash flow measured in terms of time periods are $1 - t/T$, $2 - t/T$, …, $N - t/T$. The weights are the present values of the cash flows divided by the full price. Therefore, Macaulay duration is measured in terms of time periods. A couple of examples will clarify this calculation.

Consider first the 10-year, 8% annual coupon payment bond used in Examples 1–6. The bond's yield-to-maturity is 10.40%, and its price is 85.503075 per 100 of par value. This bond has 10 evenly spaced periods to maturity. Settlement is on a coupon payment date so that $t/T = 0$. Exhibit 2 illustrates the calculation of the bond's Macaulay duration.

EXHIBIT 2 Macaulay Duration of a 10-Year, 8% Annual Payment Bond

Period	Cash Flow	Present Value	Weight	Period × Weight
1	8	7.246377	0.08475	0.0847
2	8	6.563747	0.07677	0.1535
3	8	5.945423	0.06953	0.2086
4	8	5.385347	0.06298	0.2519
5	8	4.878032	0.05705	0.2853
6	8	4.418507	0.05168	0.3101
7	8	4.002271	0.04681	0.3277
8	8	3.625245	0.04240	0.3392
9	8	3.283737	0.03840	0.3456
10	108	40.154389	0.46963	4.6963
		85.503075	1.00000	7.0029

The first two columns of Exhibit 2 show the number of periods to the receipt of the cash flow and the amount of the payment per 100 of par value. The third column is the present value of the cash flow. For example, the final payment is 108 (the last coupon payment plus the redemption of principal) and its present value is 40.154389.

$$\frac{108}{(1.1040)^{10}} = 40.154389$$

The sum of the present values is the full price of the bond. The fourth column is the weight, the share of total market value corresponding to each cash flow. The final payment of 108 per 100 of par value is 46.963% of the bond's market value.

$$\frac{40.154389}{85.503075} = 0.46963$$

The sum of the weights is 1.00000. The fifth column is the number of periods to the receipt of the cash flow (the first column) multiplied by the weight (the fourth column). The sum of that column is 7.0029, which is the Macaulay duration of this 10-year, 8% annual coupon payment bond. This statistic is sometimes reported as 7.0029 *years,* although the time frame is not needed in most applications.

Now consider an example *between* coupon payment dates. A 6% semiannual payment corporate bond that matures on 14 February 2022 is purchased for settlement on 11 April 2014. The coupon payments are 3 per 100 of par value, paid on 14 February and 14 August of each year. The yield-to-maturity is 6.00% quoted on a street-convention semiannual bond basis. The full price of this bond comprises the flat price plus accrued interest. The flat price for the bond is 99.990423 per 100 of par value. The accrued interest is calculated using the 30/360 method to count days. This settlement date is 57 days into the 180-day semiannual period, so $t/T = 57/180$. The accrued interest is 0.950000 (= 57/180 × 3) per 100 of par value. The full price for the bond is 100.940423 (= 99.990423 + 0.950000). Exhibit 3 shows the calculation of the bond's Macaulay duration.

EXHIBIT 3 Macaulay Duration of an Eight-Year, 6% Semiannual Payment Bond Priced to Yield 6.00%

Period	Time to Receipt	Cash Flow	Present Value	Weight	Time × Weight
1	0.6833	3	2.940012	0.02913	0.019903
2	1.6833	3	2.854381	0.02828	0.047601
3	2.6833	3	2.771244	0.02745	0.073669
4	3.6833	3	2.690528	0.02665	0.098178
5	4.6833	3	2.612163	0.02588	0.121197
6	5.6833	3	2.536080	0.02512	0.142791
7	6.6833	3	2.462214	0.02439	0.163025
8	7.6833	3	2.390499	0.02368	0.181959
9	8.6833	3	2.320873	0.02299	0.199652
10	9.6833	3	2.253275	0.02232	0.216159
11	10.6833	3	2.187645	0.02167	0.231536
12	11.6833	3	2.123927	0.02104	0.245834
13	12.6833	3	2.062065	0.02043	0.259102
14	13.6833	3	2.002005	0.01983	0.271389
15	14.6833	3	1.943694	0.01926	0.282740
16	15.6833	103	64.789817	0.64186	10.066535
			100.940423	1.00000	12.621268

There are 16 semiannual periods to maturity between the last coupon payment date of 14 February 2014 and maturity on 14 February 2022. The time to receipt of cash flow in semiannual periods is in the second column: 0.6833 = 1 – 57/180, 1.6833 = 2 – 57/180, etc.

The cash flow for each period is in the third column. The annual yield-to-maturity is 6.00%, so the yield per semiannual period is 3.00%. When that yield is used to get the present value of each cash flow, the full price of the bond is 100.940423, the sum of the fourth column. The weights, which are the shares of the full price corresponding to each cash flow, are in the fifth column. The Macaulay duration is the sum of the items in the sixth column, which is the weight multiplied by the time to receipt of each cash flow. The result, 12.621268, is the Macaulay duration on an eight-year, 6% semiannual payment bond for settlement on 11 April 2014 measured in *semiannual periods*. Similar to coupon rates and yields-to-maturity, duration statistics invariably are annualized in practice. Therefore, the Macaulay duration typically is reported as 6.310634 *years* (= 12.621268/2).[2] (Such precision for the duration statistic is not needed in practice. Typically, "6.31 years" is enough. The full precision is shown here to illustrate calculations.)

Another approach to calculating the Macaulay duration is to use a closed-form equation derived using calculus and algebra. Equation 4 is a general closed-form formula for determining the Macaulay duration of a fixed-rate bond, where c is the coupon rate per period (PMT/FV).[3]

$$\text{MacDur} = \left\{ \frac{1+r}{r} - \frac{1+r+\left[N \times (c-r)\right]}{c \times \left[(1+r)^N - 1\right] + r} \right\} - (t/T) \tag{4}$$

The Macaulay duration of the 10-year, 8% annual payment bond is calculated by entering $r = 0.1040$, $c = 0.0800$, $N = 10$, and $t/T = 0$ into Equation 4.

$$\text{MacDur} = \frac{1+0.1040}{0.1040} - \frac{1+0.1040+\left[10 \times (0.0800-0.1040)\right]}{0.0800 \times \left[(1+0.1040)^{10} - 1\right] + 0.1040} = 7.0029$$

Therefore, the weighted average time to receipt of the interest and principal payments that will result in realization of the initial market discount rate on this 10-year bond is 7.00 years.

The Macaulay duration of the 6% semiannual payment bond maturing on 14 February 2022 is obtained by entering $r = 0.0300$, $c = 0.0300$, $N = 16$, and $t/T = 57/180$ into Equation 4.

$$\begin{aligned}\text{MacDur} &= \left[\frac{1+0.0300}{0.0300} - \frac{1+0.0300+\left[16 \times (0.0300-0.0300)\right]}{0.0300 \times \left[(1+0.0300)^{16} - 1\right] + 0.0300} \right] - (57/180) \\ &= 12.621268\end{aligned}$$

Equation 4 uses the yield-to-maturity *per period*, the coupon rate *per period*, the number of *periods* to maturity, and the fraction of the current *period* that has gone by. Its output is the Macaulay duration in terms of *periods*. It is converted to annual duration by dividing by the number of periods in the year.

[2] Microsoft Excel users can obtain the Macaulay duration using the DURATION financial function: DURATION ("4/11/2014," "2/14/2022," 0.06, 0.06, 2, 0). The inputs are the settlement date, maturity date, annual coupon rate as a decimal, annual yield-to-maturity as a decimal, periodicity, and the code for the day count (0 for 30/360, 1 for actual/actual).

[3] The step-by-step derivation of this formula is in Donald J. Smith, *Bond Math: The Theory behind the Formulas* (Hoboken, NJ: John Wiley & Sons, 2011).

The calculation of the **modified duration** (ModDur) statistic of a bond requires a simple adjustment to Macaulay duration. It is the Macaulay duration statistic divided by one plus the yield per period.

$$\text{ModDur} = \frac{\text{MacDur}}{1+r} \tag{5}$$

For example, the modified duration of the 10-year, 8% annual payment bond is 6.3432.

$$\text{ModDur} = \frac{7.0029}{1.1040} = 6.3432$$

The modified duration of the 6% semiannual payment bond maturing on 14 February 2022 is 12.253658 semiannual periods.

$$\text{ModDur} = \frac{12.621268}{1.0300} = 12.253658$$

The annualized modified duration of the bond is 6.126829 (= 12.253658/2).[4]

Although modified duration might seem to be just a Macaulay duration with minor adjustments, it has an important application in risk measurement: Modified duration provides an estimate of the percentage price change for a bond given a change in its yield-to-maturity.

$$\%\Delta PV^{Full} \approx -\text{AnnModDur} \times \Delta\text{Yield} \tag{6}$$

The percentage price change refers to the full price, including accrued interest. The AnnModDur term in Equation 6 is the *annual* modified duration, and the ΔYield term is the change in the *annual* yield-to-maturity. The ≈ sign indicates that this calculation is an estimation. The minus sign indicates that bond prices and yields-to-maturity move inversely.

If the annual yield on the 6% semiannual payment bond that matures on 14 February 2022 jumps by 100 bps, from 6.00% to 7.00%, the estimated loss in value for the bond is 6.1268%.

$$\%\Delta PV^{Full} \approx -6.126829 \times 0.0100 = -0.061268$$

If the yield-to-maturity were to drop by 100 bps to 5.00%, the estimated gain in value is also 6.1268%.

$$\%\Delta PV^{Full} \approx -6.126829 \times -0.0100 = 0.061268$$

Modified duration provides a *linear* estimate of the percentage price change. In terms of absolute value, the change is the same for either an increase or decrease in the yield-to-maturity. Recall from "Introduction to Fixed-Income Valuation" that for a given coupon rate and time-to-maturity, the percentage price change is greater (in absolute value) when the market

[4] Microsoft Excel users can obtain the modified duration using the MDURATION financial function: MDURATION ("4/11/2014," "2/14/2022," 0.06, 0.06, 2, 0). The inputs are the same as for the Macaulay duration in Footnote 2.

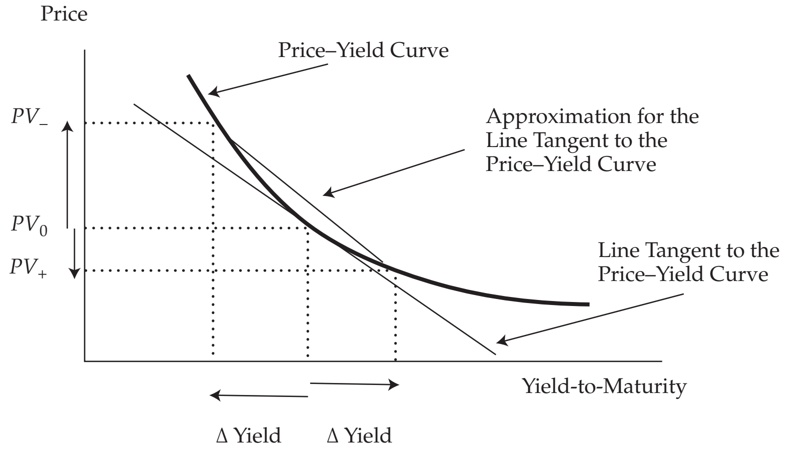
discount rate goes down than when it goes up. Later in this chapter, a "convexity adjustment" to duration is introduced. It improves the accuracy of this estimate, especially when a large change in yield-to-maturity (such as 100 bps) is considered.

The modified duration statistic for a fixed-rate bond is easily obtained if the Macaulay duration is already known. An alternative approach is to *approximate* modified duration directly. Equation 7 is the approximation formula for annual modified duration.

$$\text{ApproxModDur} = \frac{(PV_{-}) - (PV_{+})}{2 \times (\Delta\text{Yield}) \times (PV_0)} \tag{7}$$

The objective of the approximation is to estimate the slope of the line tangent to the price–yield curve. The slope of the tangent and the approximated slope are shown in Exhibit 4.

EXHIBIT 4 Approximate Modified Duration



To estimate the slope, the yield-to-maturity is changed up and down by the same amount—the ΔYield. Then the bond prices given the new yields-to-maturity are calculated. The price when the yield is increased is denoted PV_{+}. The price when the yield-to-maturity is reduced is denoted PV_{-}. The original price is PV_0. These prices are the full prices, including accrued interest. The slope of the line based on PV_{+} and PV_{-} is the approximation for the slope of the line tangent to the price–yield curve. The following example illustrates the remarkable accuracy of this approximation. In fact, as ΔYield approaches zero, the approximation approaches AnnModDur.

Consider the 6% semiannual coupon payment corporate bond maturing on 14 February 2022. For settlement on 11 April 2014, the full price (PV_0) is 100.940423 given that the yield-to-maturity is 6.00%.

$$PV_0 = \left[\frac{3}{(1.03)^1} + \frac{3}{(1.03)^2} + \cdots + \frac{103}{(1.03)^{16}}\right] \times (1.03)^{57/180} = 100.940423$$

Raise the annual yield-to-maturity by 5 bps, from 6.00% to 6.05%. This increase corresponds to an increase in the yield-to-maturity per semiannual period of 2.5 bps, from 3.00% to 3.025% per period. The new full price (PV_{+}) is 100.631781.

$$PV_+ = \left[\frac{3}{(1.03025)^1} + \frac{3}{(1.03025)^2} + \cdots + \frac{103}{(1.03025)^{16}}\right] \times (1.03025)^{57/180} = 100.631781$$

Lower the annual yield-to-maturity by 5 bps, from 6.00% to 5.95%. This decrease corresponds to a decrease in the yield-to-maturity per semiannual period of 2.5 bps, from 3.00% to 2.975% per period. The new full price (PV_-) is 101.250227.

$$PV_- = \left[\frac{3}{(1.02975)^1} + \frac{3}{(1.02975)^2} + \cdots + \frac{103}{(1.02975)^{16}}\right] \times (1.02975)^{57/180} = 101.250227$$

Enter these results into Equation 7 for the 5 bp change in the annual yield-to-maturity, or $\Delta\text{Yield} = 0.0005$:

$$\text{ApproxModDur} = \frac{101.250227 - 100.631781}{2 \times 0.0005 \times 100.940423} = 6.126842$$

The "exact" annual modified duration for this bond is 6.126829 and the "approximation" is 6.126842—virtually identical results. Therefore, although duration can be calculated using the approach in Exhibits 2 and 3—basing the calculation on the weighted average time to receipt of each cash flow—or using the closed-form formula as in Equation 4, it can also be estimated quite accurately using the basic bond-pricing equation and a financial calculator. The Macaulay duration can be approximated as well—the approximate modified duration multiplied by one plus the yield per period.

$$\text{ApproxMacDur} = \text{ApproxModDur} \times (1 + r) \tag{8}$$

The approximation formulas produce results for *annualized* modified and Macaulay durations. The frequency of coupon payments and the periodicity of the yield-to-maturity are included in the bond price calculations.

EXAMPLE 8

Assume that the 3.75% US Treasury bond that matures on 15 August 2041 is priced to yield 5.14% for settlement on 15 October 2014. Coupons are paid semiannually on 15 February and 15 August. The yield-to-maturity is stated on a street-convention semiannual bond basis. This settlement date is 61 days into a 184-day coupon period, using the actual/actual day-count convention. Compute the approximate modified duration and the approximate Macaulay duration for this Treasury bond assuming a 5 bp change in the yield-to-maturity.

Solution: The yield-to-maturity per semiannual period is 0.0257 (= 0.0514/2). The coupon payment per period is 1.875 (= 3.75/2). At the beginning of the period, there are 27 years (54 semiannual periods) to maturity. The fraction of the period that has passed is 61/184. The full price at that yield-to-maturity is 80.501507 per 100 of par value.

$$PV_0 = \left[\frac{1.875}{(1.0257)^1} + \frac{1.875}{(1.0257)^2} + \cdots + \frac{101.875}{(1.0257)^{54}}\right] \times (1.0257)^{61/184} = 80.501507$$

Raise the yield-to-maturity from 5.14% to 5.19%—therefore, from 2.57% to 2.595% per semiannual period—and the price becomes 79.886293 per 100 of par value.

$$PV_{+} = \left[\frac{1.875}{(1.02595)^1} + \frac{1.875}{(1.02595)^2} + \cdots + \frac{101.875}{(1.02595)^{54}}\right] \times (1.02595)^{61/184}$$
$$= 79.886293$$

Lower the yield-to-maturity from 5.14% to 5.09%—therefore, from 2.57% to 2.545% per semiannual period—and the price becomes 81.123441 per 100 of par value.

$$PV_{-} = \left[\frac{1.875}{(1.02545)^1} + \frac{1.875}{(1.02545)^2} + \cdots + \frac{101.875}{(1.02545)^{54}}\right] \times (1.02545)^{61/184}$$
$$= 81.123441$$

The approximate annualized modified duration for the Treasury bond is 15.368.

$$\text{ApproxModDur} = \frac{81.123441 - 79.886293}{2 \times 0.0005 \times 80.501507} = 15.368$$

The approximate annualized Macaulay duration is 15.763.

$$\text{ApproxMacDur} = 15.368 \times 1.0257 = 15.763$$

Therefore, from these statistics, the investor knows that the weighted average time to receipt of interest and principal payments is 15.763 years (the Macaulay duration) and that the estimated loss in the bond's market value is 15.368% (the modified duration) if the market discount rate were to suddenly go up by 1% from 5.14% to 6.14%.

3.2. Effective Duration

Another approach to assess the interest rate risk of a bond is to estimate the percentage change in price given a change in a benchmark yield curve—for example, the government par curve. This estimate, which is very similar to the formula for approximate modified duration, is called the **effective duration**. The effective duration of a bond is the sensitivity of the bond's price to a change in a benchmark yield curve. The formula to calculate effective duration (EffDur) is Equation 9.

$$\text{EffDur} = \frac{(PV_{-}) - (PV_{+})}{2 \times (\Delta\text{Curve}) \times (PV_0)} \tag{9}$$

The difference between approximate modified duration and effective duration is in the denominator. Modified duration is a *yield duration* statistic in that it measures interest rate risk in terms of a change in the bond's own yield-to-maturity (ΔYield). Effective duration is a *curve duration* statistic in that it measures interest rate risk in terms of a parallel shift in the benchmark yield curve (ΔCurve).

Effective duration is essential to the measurement of the interest rate risk of a complex bond, such as a bond that contains an embedded call option. The duration of a callable bond is *not* the sensitivity of the bond price to a change in the yield-to-worst (i.e., the lowest of the yield-to-maturity, yield-to-first-call, yield-to-second-call, and so forth). The problem is that future cash flows are uncertain because they are contingent on future interest rates. The issuer's decision to call the bond depends on the ability to refinance the debt at a lower cost of funds. In brief, a callable bond does not have a well-defined internal rate of return (yield-to-maturity). Therefore, yield duration statistics, such as modified and Macaulay durations, do not apply; effective duration is the appropriate duration measure.

The specific option-pricing models that are used to produce the inputs to effective duration for a callable bond are covered in later chapters. However, as an example, suppose that the full price of a callable bond is 101.060489 per 100 of par value. The option-pricing model inputs include (1) the length of the call protection period, (2) the schedule of call prices and call dates, (3) an assumption about credit spreads over benchmark yields (which includes any liquidity spread as well), (4) an assumption about future interest rate volatility, and (5) the level of market interest rates (e.g., the government par curve). The analyst then holds the first four inputs constant and raises and lowers the fifth input. Suppose that when the government par curve is raised and lowered by 25 bps, the new full prices for the callable bond from the model are 99.050120 and 102.890738, respectively. Therefore, PV_0 = 101.060489, PV_+ = 99.050120, PV_- = 102.890738, and ΔCurve = 0.0025. The effective duration for the callable bond is 7.6006.

$$\text{EffDur} = \frac{102.890738 - 99.050120}{2 \times 0.0025 \times 101.060489} = 7.6006$$

This curve duration measure indicates the bond's sensitivity to the benchmark yield curve—in particular, the government par curve—assuming no change in the credit spread. In practice, a callable bond issuer might be able to exercise the call option and obtain a lower cost of funds if (1) benchmark yields fall and the credit spread over the benchmark is unchanged or (2) benchmark yields are unchanged and the credit spread is reduced (e.g., because of an upgrade in the issuer's rating). A pricing model can be used to determine a "credit duration" statistic—that is, the sensitivity of the bond price to a change in the credit spread. On a traditional fixed-rate bond, modified duration estimates the percentage price change for a change in the benchmark yield and/or the credit spread. For bonds that do not have a well-defined internal rate of return because the future cash flows are not fixed—for instance, callable bonds and floating-rate notes—pricing models are used to produce different statistics for changes in benchmark interest rates and for changes in credit risk.

Another fixed-income security for which yield duration statistics, such as modified and Macaulay durations, are not relevant is a mortgage-backed bond. These securities arise from a residential (or commercial) loan portfolio securitization. The key point for measuring interest rate risk on a mortgage-backed bond is that the cash flows are contingent on homeowners' ability to refinance their debt at a lower rate. In effect, the homeowners have call options on their mortgage loans.

A practical consideration in using effective duration is in setting the change in the benchmark yield curve. With approximate modified duration, accuracy is improved by choosing a smaller yield-to-maturity change. But the pricing models for more-complex securities, such as callable and mortgage-backed bonds, include assumptions about the behavior of the corporate issuers, businesses, or homeowners. Rates typically need to change by a minimum amount to affect the decision to call a bond or refinance a mortgage loan because issuing new debt involves transaction costs. Therefore, estimates of interest rate risk using effective duration are not necessarily improved by choosing a smaller change in benchmark rates. Effective duration has become an important tool in the financial analysis of not only traditional bonds but also financial liabilities. Example 9 demonstrates such an application of effective duration.

EXAMPLE 9

Defined-benefit pension schemes typically pay retirees a monthly amount based on their wage level at the time of retirement. The amount could be fixed in nominal terms or indexed to inflation. These programs are referred to as "defined-benefit pension plans" when US GAAP or IFRS accounting standards are used. In Australia, they are called "superannuation funds."

A British defined-benefit pension scheme seeks to measure the sensitivity of its retirement obligations to market interest rate changes. The pension scheme manager hires an actuarial consultancy to model the present value of its liabilities under three interest rate scenarios: (1) a base rate of 5%; (2) a 100 bp increase in rates, up to 6%; and (3) a 100 bp drop in rates, down to 4%.

The actuarial consultancy uses a complex valuation model that includes assumptions about employee retention, early retirement, wage growth, mortality, and longevity. The following chart shows the results of the analysis.

Interest Rate Assumption	Present Value of Liabilities
4%	GBP973.5 million
5%	GBP926.1 million
6%	GBP871.8 million

Compute the effective duration of the pension scheme's liabilities.

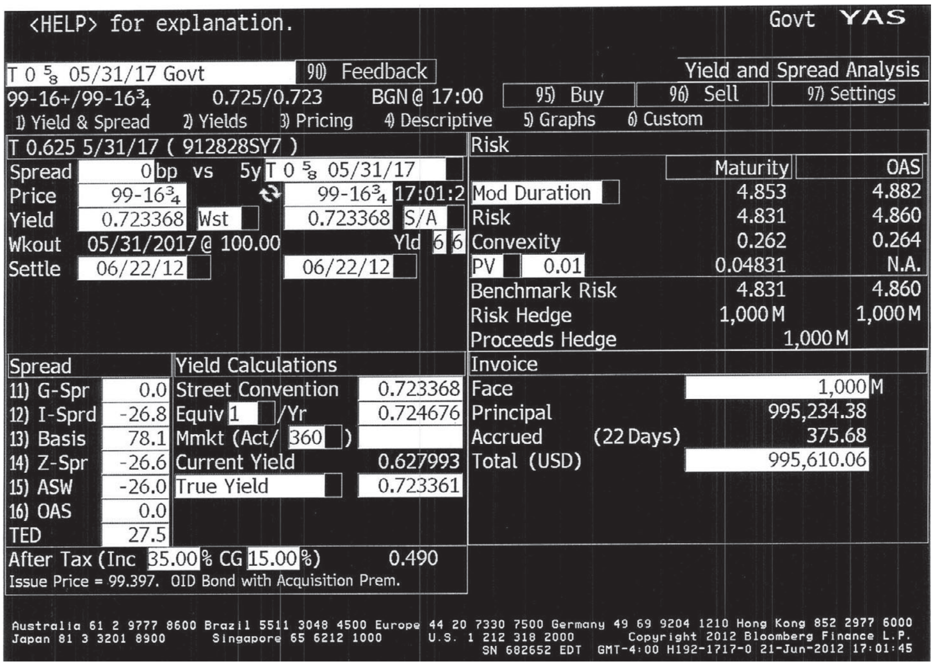
Solution: $PV_0 = 926.1$, $PV_+ = 871.8$, $PV_- = 973.5$, and $\Delta\text{Curve} = 0.0100$. The effective duration of the pension scheme's liabilities is 5.49.

$$\text{EffDur} = \frac{973.5 - 871.8}{2 \times 0.0100 \times 926.1} = 5.49$$

This effective duration statistic for the pension scheme's liabilities might be used in asset allocation decisions to decide the mix of equity, fixed income, and alternative assets.

Although effective duration is the most appropriate interest rate risk measure for bonds with embedded options, it also is useful with traditional bonds to supplement the information provided by the Macaulay and modified yield durations. Exhibit 5 displays the Bloomberg Yield and Spread (YAS) Analysis page for the 0.625% US Treasury note that matures on 31 May 2017.

EXHIBIT 5 Bloomberg YAS Page for the 0.625% US Treasury Note



In Exhibit 5, the quoted (flat) price for the bond is 99 – 16 ¾, which is equal to 99 and 16 ¾ 32nds per 100 of par value for settlement on 22 June 2012. Most bond prices are stated in decimals, but US Treasuries are usually quoted in fractions. As a decimal, the flat price is 99.523438. The accrued interest uses the actual/actual day-count method. That settlement date is 22 days into a 183-day semiannual coupon payment period. The accrued interest is 0.037568 per 100 of par value (= 22/183 × 0.00625/2 × 100). The full price of the bond is 99.561006. The yield-to-maturity of the bond is 0.723368%, stated on a street-convention semiannual bond basis.

The modified duration for the bond is shown in Exhibit 5 to be 4.853, which is the conventional *yield* duration statistic. Its *curve* duration, however, is 4.882, which is the price sensitivity with respect to changes in the US Treasury par curve. On Bloomberg, the effective duration is called the "OAS duration" because it is based on the option-pricing model that is also used to calculate the option-adjusted spread. The small difference arises because the government yield curve is not flat. When the par curve is shifted in the model, the government spot curve is also shifted, although not in the same "parallel" manner. Therefore, the change in the bond price is not exactly the same as it would be if its own yield-to-maturity changed by the same amount as the change in the par curve. In general, the modified duration and effective duration on a traditional option-free bond are not identical. The difference narrows when the yield curve is flatter, the time-to-maturity is shorter, and the bond is priced closer to par value (so that the difference between the coupon rate and the yield-to-maturity is smaller).

The modified duration and effective duration on an option-free bond are identical only in the rare circumstance of an absolutely flat yield curve.

Above, the effective duration for a sample callable bond was calculated as:

$$\text{EffDur} = \frac{102.890738 - 99.050120}{2 \times 0.0025 \times 101.060489} = 7.6006$$

This duration measure indicates the bond's sensitivity to the benchmark yield curve assuming that all yields change by the same amount.

3.3. Key Rate Duration

Key rate duration provides further insight into a bond's sensitivity to changes in the benchmark yield curve. A **key rate duration** (or **partial duration**) is a measure of a bond's sensitivity to a change in the benchmark yield curve at a specific maturity segment. In contrast to effective duration, key rate durations help identify "shaping risk" for a bond—that is, a bond's sensitivity to changes in the shape of the benchmark yield curve (e.g., the yield curve becoming steeper or flatter).

The previous illustration of effective duration assumed a parallel shift of 25 bps at all maturities. However, the analyst may want to know how the price of the callable bond is expected to change if benchmark rates at short maturities (say up to 2 years) shifted up by 25 bps but longer maturity benchmark rates remained unchanged. This scenario would represent a flattening of the yield curve, given that the yield curve is upward sloping. Using key rate durations, the expected price change would be approximately equal to minus the key rate duration for the short maturity segment times the 0.0025 interest rate shift at that segment. Of course, for parallel shifts in the benchmark yield curve, key rate durations will indicate the same interest rate sensitivity as effective duration.

3.4. Properties of Bond Duration

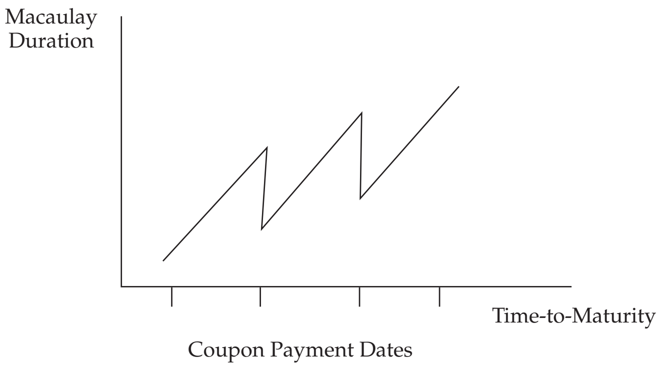
The Macaulay and modified yield duration statistics for a traditional fixed-rate bond are functions of the input variables: the coupon rate or payment per period, the yield-to-maturity per period, the number of periods to maturity (as of the beginning of the period), and the fraction of the period that has gone by. The properties of bond duration are obtained by changing one of these variables while holding the others constant. Because duration is the basic measure of interest rate risk on a fixed-rate bond, these properties are important to understand.

The closed-form formula for Macaulay duration, presented as Equation 4 and again here, is useful in demonstrating the characteristics of the bond duration statistic.

$$\text{MacDur} = \left\{ \frac{1+r}{r} - \frac{1+r+\left[N \times (c-r)\right]}{c \times \left[(1+r)^N - 1\right] + r} \right\} - \left(t/T\right)$$

The same characteristics hold for modified duration. Consider first the fraction of the period that has gone by (t/T). Macaulay and modified durations depend on the day-count basis used to obtain the yield-to-maturity. The duration of a bond that uses the actual/actual method to count days is slightly different from that of an otherwise comparable bond that uses the 30/360 method. The key point is that for a constant yield-to-maturity (r), the expression in braces is unchanged as time passes during the period. Therefore, the Macaulay duration decreases smoothly as t goes from $t = 0$ to $t = T$, which creates a "saw-tooth" pattern. This pattern for a typical fixed-rate bond is illustrated in Exhibit 6.

EXHIBIT 6 Macaulay Duration between Coupon Payments with a Constant Yield-to-Maturity



As times passes during the coupon period (moving from right to left in the diagram), the Macaulay duration declines smoothly and then jumps upward after the coupon is paid.

The characteristics of bond duration related to changes in the coupon rate, the yield-to-maturity, and the time-to-maturity are illustrated in Exhibit 7.

EXHIBIT 7 Properties of the Macaulay Yield Duration

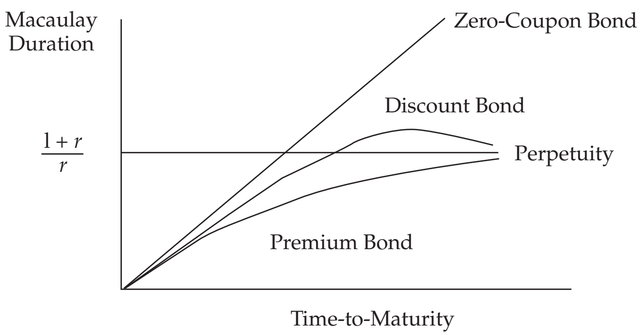


Exhibit 7 shows the graph for coupon payment dates when $t/T = 0$, thus not displaying the saw-tooth pattern between coupon payments. The relationship between the Macaulay duration and the time-to-maturity for a zero-coupon bond is the 45-degree line: MacDur = N when $c = 0$ (and $t/T = 0$). Therefore, the Macaulay duration of a zero-coupon bond is its time-to-maturity.

A **perpetuity** or perpetual bond, which also is called a consol, is a bond that does not mature. There is no principal to redeem. The investor receives a fixed coupon payment forever, unless the bond is callable. Non-callable perpetuities are rare, but they have an interesting Macaulay duration: MacDur = $(1 + r)/r$ as N approaches infinity. In effect, the second expression within the braces approaches zero as the number of periods to maturity increases because N in the numerator is a coefficient but N in the denominator is an exponent and the denominator increases faster than the numerator as N grows larger.

Typical fixed-rate coupon bonds with a stated maturity date are portrayed in Exhibit 7 as the premium and discount bonds. The usual pattern is that longer times-to-maturity correspond to higher Macaulay duration statistics. This pattern always holds for bonds trading at par value or at a premium above par. In Equation 4, the second expression within the braces is a positive number for premium and par bonds. The numerator is positive because the coupon rate (c) is greater than or equal to the yield-to-maturity (r), whereas the denominator is always positive. Therefore, the Macaulay duration is always less than $(1 + r)/r$, and it approaches that threshold from below as the time-to-maturity increases.

The curious result displayed in Exhibit 7 is in the pattern for discount bonds. Generally, the Macaulay duration increases for a longer time-to-maturity. But at some point when the time-to-maturity is high enough, the Macaulay duration exceeds $(1 + r)/r$, reaches a maximum, and then approaches the threshold from above. In Equation 4, such a pattern develops when the number of periods (N) is large and the coupon rate (c) is below the yield-to-maturity (r). Then the numerator of the second expression within the braces can become negative. The implication is that on long-term discount bonds, the interest rate risk can actually be less than on a shorter-term bond, which explains why the word "generally" is needed in describing the maturity effect for the relationship between bond prices and yields-to-maturity. Generally, for the same coupon rate, a longer-term bond has a greater percentage price change than a shorter-term bond when their yields-to-maturity change by the same amount. The exception is when the longer-term bond actually has a lower duration statistic.

Coupon rates and yields-to-maturity are both inversely related to the Macaulay duration. In Exhibit 7, for the same time-to-maturity and yield-to-maturity, the Macaulay duration is higher for a zero-coupon bond than for a low-coupon bond trading at a discount. Also, the low-coupon bond trading at a discount has a higher duration than a high-coupon bond trading at a premium. Therefore, all else being equal, a lower-coupon bond has a higher duration and more interest rate risk than a higher-coupon bond. The same pattern holds for the yield-to-maturity. A higher yield-to-maturity reduces the weighted average of the time to receipt of cash flow. More weight is on the cash flows received in the near term, and less weight is on the cash flows received in the more-distant future periods if those cash flows are discounted at a higher rate.

In summary, the Macaulay and modified duration statistics for a fixed-rate bond depend primarily on the coupon rate, yield-to-maturity, and time-to-maturity. A higher coupon rate or a higher yield-to-maturity reduces the duration measures. A longer time-to-maturity *usually* leads to a higher duration. It *always* does so for a bond priced at a premium or at par value. But if the bond is priced at a discount, a longer time-to-maturity *might* lead to a lower duration. This situation only occurs if the coupon rate is low (but not zero) relative to the yield and the time-to-maturity is long.

EXAMPLE 10

A hedge fund specializes in investments in emerging market sovereign debt. The fund manager believes that the implied default probabilities are too high, which means that the bonds are viewed as "cheap" and the credit spreads are too high. The hedge fund plans to take a position on one of these available bonds.

Bond	Time-to-Maturity	Coupon Rate	Price	Yield-to-Maturity
(A)	10 years	10%	58.075279	20%
(B)	20 years	10%	51.304203	20%
(C)	30 years	10%	50.210636	20%

The coupon payments are annual. The yields-to-maturity are effective annual rates. The prices are per 100 of par value.

1. Compute the approximate modified duration of each of the three bonds using a 1 bp change in the yield-to-maturity and keeping precision to six decimals (because approximate duration statistics are very sensitive to rounding).
2. Which of the three bonds is expected to have the highest percentage price increase if the yield-to-maturity on each decreases by the same amount—for instance, by 10 bps from 20% to 19.90%?

Solution to 1:

Bond A:

$$PV_0 = 58.075279$$

$$PV_+ = 58.047598$$

$$\frac{10}{(1.2001)^1} + \frac{10}{(1.2001)^2} + \cdots + \frac{110}{(1.2001)^{10}} = 58.047598$$

$$PV_- = 58.102981$$

$$\frac{10}{(1.1999)^1} + \frac{10}{(1.1999)^2} + \cdots + \frac{110}{(1.1999)^{10}} = 58.102981$$

The approximate modified duration of Bond A is 4.768.

$$\text{ApproxModDur} = \frac{58.102981 - 58.047598}{2 \times 0.0001 \times 58.075279} = 4.768$$

Bond B:

$$PV_0 = 51.304203$$

$$PV_+ = 51.277694$$

$$\frac{10}{(1.2001)^1}+\frac{10}{(1.2001)^2}+\cdots+\frac{110}{(1.2001)^{20}}=51.277694$$

$$PV_- = 51.330737$$

$$\frac{10}{(1.1999)^1}+\frac{10}{(1.1999)^2}+\cdots+\frac{110}{(1.1999)^{20}}=51.330737$$

The approximate modified duration of Bond B is 5.169.

$$\text{ApproxModDur}=\frac{51.330737-51.277694}{2\times 0.0001\times 51.304203}=5.169$$

Bond C:

$$PV_0 = 50.210636$$

$$PV_+ = 50.185228$$

$$\frac{10}{(1.2001)^1}+\frac{10}{(1.2001)^2}+\cdots+\frac{110}{(1.2001)^{30}}=50.185228$$

$$PV_- = 50.236070$$

$$\frac{10}{(1.1999)^1}+\frac{10}{(1.1999)^2}+\cdots+\frac{110}{(1.1999)^{30}}=50.236070$$

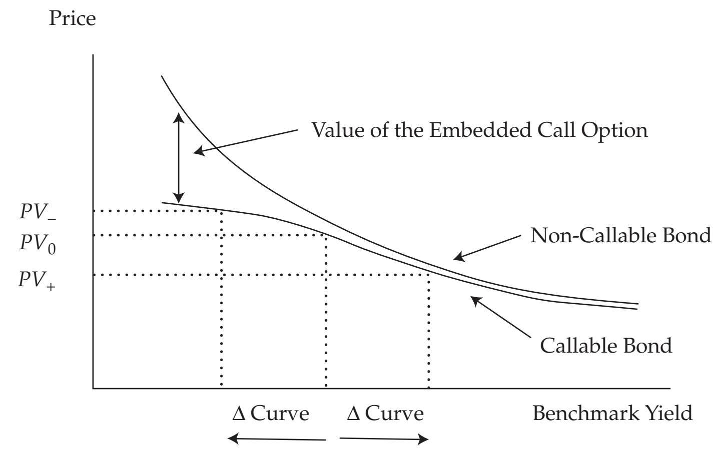
The approximate modified duration of Bond C is 5.063.

$$\text{ApproxModDur}=\frac{50.236070-50.185228}{2\times 0.0001\times 50.210636}=5.063$$

Solution to 2: Despite the significant differences in times-to-maturity (10, 20, and 30 years), the approximate modified durations on the three bonds are fairly similar (4.768, 5.169, and 5.063). Because the yields-to-maturity are so high, the additional time to receipt of interest and principal payments on the 20- and 30-year bonds have low weight. Nevertheless, Bond B, with 20 years to maturity, has the highest modified duration. If the yield-to-maturity on each is decreased by the same amount—for instance, by 10 bps, from 20% to 19.90%—Bond B would be expected to have the highest percentage price increase because it has the highest modified duration. This example illustrates the relationship between the Macaulay duration and the time-to-maturity on discount bonds in Exhibit 7. The 20-year bond has a higher duration than the 30-year bond.

Callable bonds require the use of effective duration because Macaulay and modified yield duration statistics are not relevant. The yield-to-maturity for callable bonds is not well-defined because future cash flows are uncertain. Exhibit 8 illustrates the impact of the change in the benchmark yield curve (ΔCurve) on the price of a callable bond price compared with that on a comparable non-callable bond. The two bonds have the same credit risk, coupon rate, payment frequency, and time-to-maturity. The vertical axis is the bond price. The horizontal axis is a particular benchmark yield—for instance, a point on the par curve for government bonds.

EXHIBIT 8 Interest Rate Risk Characteristics of a Callable Bond

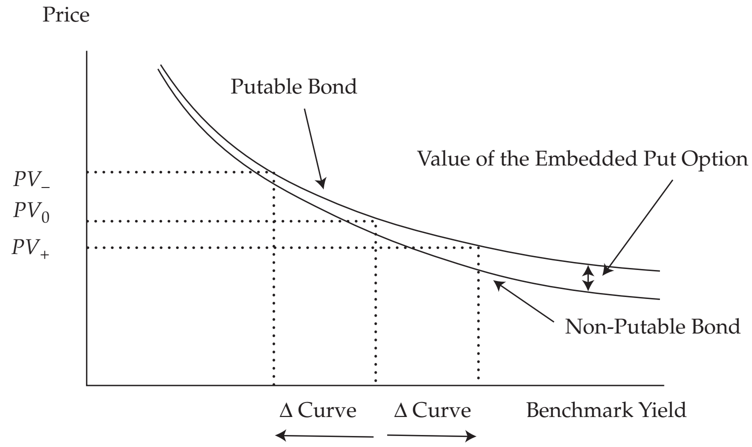


As shown in Exhibit 8, the price of the non-callable bond is always greater than that of the callable bond with otherwise identical features. The difference is the value of the embedded call option. Recall that the call option is an option to the issuer and not the holder of the bond. When interest rates are high compared with the coupon rate, the value of the call option is low. When rates are low, the value of the call option is much greater because the issuer is more likely to exercise the option to refinance the debt at a lower cost of funds. The investor bears the "call risk" because if the bond is called, the investor must reinvest the proceeds at a lower interest rate.

Exhibit 8 shows the inputs for calculating the effective duration of the callable bond. The entire benchmark curve is raised and lowered by the same amount, ΔCurve. The key point is that when benchmark yields are high, the effective durations of the callable and non-callable bonds are very similar. Although the exhibit does not illustrate it, the slopes of the lines tangent to the price–yield curve are about the same in such a situation. But when interest rates are low, the effective duration of the callable bond is lower than that of the otherwise comparable non-callable bond. That is because the callable bond price does not increase as much when benchmark yields fall. The slope of the line tangent to the price–yield curve would be flatter. The presence of the call option limits price appreciation. Therefore, an embedded call option reduces the effective duration of the bond, especially when interest rates are falling and the bond is more likely to be called. The lower effective duration can also be interpreted as a shorter expected life—the weighted average of time to receipt of cash flow is reduced.

Exhibit 9 considers another embedded option—a put option.

EXHIBIT 9 Interest Rate Risk Characteristics of a Putable Bond



A putable bond allows the investor to sell the bond back to the issuer prior to maturity, usually at par value, which protects the investor from higher benchmark yields or credit spreads that otherwise would drive the bond to a discounted price. Therefore, the price of a putable bond is always higher than that of an otherwise comparable non-putable bond. The price difference is the value of the embedded put option.

An embedded put option reduces the effective duration of the bond, especially when rates are rising. If interest rates are low compared with the coupon rate, the value of the put option is low and the impact of a change in the benchmark yield on the bond's price is very similar to the impact on the price of a non-putable bond. But when benchmark interest rates rise, the put option becomes more valuable to the investor. The ability to sell the bond at par value limits the price depreciation as rates rise. In summary, the presence of an embedded option reduces the sensitivity of the bond price to changes in the benchmark yield curve, assuming no change in credit risk.

3.5. Duration of a Bond Portfolio

Similar to equities, bonds are typically held in a portfolio. There are two ways to calculate the duration of a bond portfolio: (1) the weighted average of time to receipt of the *aggregate* cash flows and (2) the weighted average of the individual bond durations that comprise the portfolio. The first method is the theoretically correct approach, but it is difficult to use in practice. The second method is commonly used by fixed-income portfolio managers, but it has its own limitations. The differences in these two methods to compute portfolio duration can be examined with a numerical example.

Suppose an investor holds the following portfolio of two *zero-coupon* bonds:

Bond	Maturity	Price	Yield	Macaulay Duration	Modified Duration	Par Value	Market Value	Weight
(X)	1 year	98.00	2.0408%	1	0.980	10,000,000	9,800,000	0.50
(Y)	30 years	9.80	8.0503%	30	27.765	100,000,000	9,800,000	0.50

The prices are per 100 of par value. The yields-to-maturity are effective annual rates. The total market value for the portfolio is 19,600,000. The portfolio is evenly weighted in terms of market value between the two bonds.

The first approach views the portfolio as a series of aggregated cash flows. Its **cash flow yield** is 7.8611%. A cash flow yield is the internal rate of return on a series of cash flows, usually used on a complex security such as a mortgage-backed bond (using projected cash flows based on a model of prepayments as a result of refinancing) or a portfolio of fixed-rate bonds. It is the solution for *r* in the following equation.

$$19{,}600{,}000 = \frac{10{,}000{,}000}{(1+r)^1} + \frac{0}{(1+r)^2} + \cdots + \frac{0}{(1+r)^{29}} + \frac{100{,}000{,}000}{(1+r)^{30}}, \quad r = 0.078611$$

The Macaulay duration of the portfolio in this approach is the weighted average of time to receipt of aggregated cash flow. The cash flow yield is used to obtain the weights. This calculation is similar to Equation 1, and the portfolio duration is 16.2825.

$$\text{MacDur} = \left[\frac{\dfrac{1\times 10{,}000{,}000}{(1.078611)^1} + \dfrac{30\times 100{,}000{,}000}{(1.078611)^{30}}}{\dfrac{10{,}000{,}000}{(1.078611)^1} + \dfrac{100{,}000{,}000}{(1.078611)^{30}}}\right] = 16.2825$$

There are just two future cash flows in the portfolio—the redemption of principal on the two zero-coupon bonds. In more complex portfolios, a series of coupon and principal payments may occur on some dates, with an aggregated cash flow composed of coupon interest on some bonds and principal on those that mature.

The modified duration of the portfolio is the Macaulay duration divided by one plus the cash flow yield per period (here, the periodicity is 1).

$$\text{ModDur} = \frac{16.2825}{1.078611} = 15.0958$$

The modified duration for the portfolio is 15.0958. That statistic indicates the percentage change in the market value given a change in the cash flow yield. If the cash flow yield increases or decreases by 100 bps, the market value of the portfolio is expected to increase or decrease by about 15.0958%.

Although this approach is "theoretically correct," it is difficult to use in practice. First, the cash flow yield is not commonly calculated for bond portfolios. Second, the amount and timing of future coupon and principal payments are uncertain if the portfolio contains callable or putable bonds or floating-rate notes. Third, interest rate risk is usually expressed as a change in benchmark interest rates, not as a change in the cash flow yield. Fourth, the change in the cash flow yield is not necessarily the same amount as the change in the yields-to-maturity on the individual bonds. For instance, if the yields-to-maturity on the two zero-coupon bonds in this portfolio both increase or decrease by 10 bps, the cash flow yield increases or decreases by only 9.52 bps.

In practice, the second approach to portfolio duration is commonly used. The Macaulay and modified durations for the portfolio are calculated as the weighted average of the statistics for the individual bonds. The shares of overall portfolio market value are the weights. This weighted average is an approximation of the "theoretically correct" portfolio duration, which is obtained using the first approach. This approximation becomes more accurate when the differences in the yields-to-maturity on the bonds in the portfolio are smaller. When the yield curve is flat, the two approaches produce the same portfolio duration.

Given the equal "50/50" weights in this simple numerical example, this version of portfolio duration is easily computed.

$$\text{Average Macaulay duration} = (1 \times 0.50) + (30 \times 0.50) = 15.50$$

$$\text{Average modified duration} = (0.980 \times 0.50) + (27.765 \times 0.50) = 14.3725$$

Note that 0.980 = 1/1.020404 and 27.765 = 30/1.080503. An advantage of the second approach is that callable bonds, putable bonds, and floating-rate notes can be included in the weighted average using the effective durations for these securities.

The main advantage to the second approach is that it is easily used as a measure of interest rate risk. For instance, if the yields-to-maturity on the bonds in the portfolio increase by 100 bps, the estimated drop in the portfolio value is 14.3725%. However, this advantage also indicates a limitation: This measure of portfolio duration implicitly assumes a **parallel shift** in the yield curve. A parallel yield curve shift implies that all rates change by the same amount in the same direction. In reality, interest rate changes frequently result in a steeper or flatter yield curve. Yield volatility is discussed later in this chapter.

EXAMPLE 11

An investment fund owns the following portfolio of three fixed-rate government bonds:

	Bond A	Bond B	Bond C
Par value	EUR25,000,000	EUR25,000,000	EUR50,000,000
Coupon rate	9%	11%	8%
Time-to-maturity	6 years	8 years	12 years
Yield-to-maturity	9.10%	9.38%	9.62%
Market value	EUR24,886,343	EUR27,243,887	EUR44,306,787
Macaulay duration	4.761	5.633	7.652

The total market value of the portfolio is EUR96,437,017. Each bond is on a coupon date so that there is no accrued interest. The market values are the full prices given the par value. Coupons are paid semiannually. The yields-to-maturity are stated on a semiannual bond basis, meaning an annual rate for a periodicity of 2. The Macaulay durations are annualized.

1. Calculate the average (annual) modified duration for the portfolio using the shares of market value as the weights.

2. Estimate the percentage loss in the portfolio's market value if the (annual) yield-to-maturity on each bond goes up by 20 bps.

Solution to 1: The average (annual) modified duration for the portfolio is 6.0495.

$$\left(\frac{4.761}{1+\frac{0.0910}{2}}\times\frac{24,886,343}{96,437,017}\right)+\left(\frac{5.633}{1+\frac{0.0938}{2}}\times\frac{27,243,887}{96,437,017}\right)+$$
$$\left(\frac{7.652}{1+\frac{0.0962}{2}}\times\frac{44,306,787}{96,437,017}\right)=6.0495$$

Note that the annual modified duration for each bond is the annual Macaulay duration, which is given, divided by one plus the yield-to-maturity per semiannual period.

Solution to 2: The estimated decline in market value if each yield rises by 20 bps is 1.21%: $-6.0495 \times 0.0020 = -0.0121$.

3.6. Money Duration of a Bond and the Price Value of a Basis Point

Modified duration is a measure of the *percentage price change* of a bond given a change in its yield-to-maturity. A related statistic is **money duration**. The money duration of a bond is a measure of the *price change* in units of the currency in which the bond is denominated. The money duration can be stated per 100 of par value or in terms of the actual position size of the bond in the portfolio. In the United States, money duration is commonly called "dollar duration."

Money duration (MoneyDur) is calculated as the annual modified duration times the full price (PV^{Full}) of the bond, including accrued interest.

$$\text{MoneyDur} = \text{AnnModDur} \times PV^{Full} \tag{10}$$

The estimated change in the bond price in currency units is calculated using Equation 11, which is very similar to Equation 6. The difference is that for a given change in the annual yield-to-maturity (ΔYield), modified duration estimates the percentage price change and money duration estimates the change in currency units.

$$\Delta PV^{Full} \approx -\text{MoneyDur x } \Delta\text{Yield} \tag{11}$$

For an example of money duration, consider the 6% semiannual coupon payment bond that matures on 14 February 2022 and is priced to yield 6.00% for settlement on 11 April 2014. The full price of the bond is 100.940423 per 100 of par value, and the annual modified duration is 6.1268. Suppose that a Hong Kong–based life insurance company has a position in the bond for a par value of HKD100,000,000. The market value of the investment is HKD100,940,423. The money duration of this bond is HKD618,441,784 (= 6.1268 × HKD100,940,423). Therefore, if the yield-to-maturity rises by 100 bps—from 6.00% to 7.00%—the expected loss is approximately HKD6,184,418 (= HKD618,441,784 × 0.0100). On a percentage basis, that expected loss is approximately 6.1268%. The "convexity adjustment" introduced in the next section makes these estimates more accurate.

Another version of money duration is the **price value of a basis point** (PVBP) for the bond. The PVBP is an estimate of the change in the full price given a 1 bp change in the yield-to-maturity. The PVBP can be calculated using a formula similar to that for the approximate modified duration. Equation 12 is the formula for the PVBP.

$$\text{PVBP} = \frac{(PV_-) - (PV_+)}{2} \tag{12}$$

PV_- and PV_+ are the full prices calculated by decreasing and increasing the yield-to-maturity by 1 bp. The PVBP is also called the "PV01," standing for the "price value of an 01"or "present value of an 01," where "01" means 1 bp. In the United States, it is commonly called the "DV01," or the "dollar value of a 01." A related statistic, sometimes called a "basis point value" (or BPV), is the money duration times 0.0001 (1 bp).

For a numerical example of the PVBP calculation, consider the 0.625% semiannual coupon payment US Treasury note that matures on 31 May 2017. In Exhibit 5, the PVBP for the Treasury note is shown to be 0.04831. Its yield-to-maturity is 0.723368%, and the settlement date is 22 days into a 183-day period. To confirm this result, calculate the new prices by increasing and decreasing the yield-to-maturity. First, increase the yield by 1 bp (0.01%), from 0.723368% to 0.733368%, to solve for a PV_+ of 99.512707.

$$PV_+ = \left[\frac{0.3125}{\left(1+\frac{0.00733368}{2}\right)^1} + \cdots + \frac{100.3125}{\left(1+\frac{0.00733368}{2}\right)^{10}}\right] \times \left(1+\frac{0.00733368}{2}\right)^{22/183}$$
$$= 99.512707$$

Then, decrease the yield-to-maturity by 1 bp, from 0.723368% to 0.713368%, to solve for a PV_- of 99.609333.

$$PV_- = \left[\frac{0.3125}{\left(1+\frac{0.00713368}{2}\right)^1} + \cdots + \frac{100.3125}{\left(1+\frac{0.00713368}{2}\right)^{10}}\right] \times \left(1+\frac{0.00713368}{2}\right)^{22/183}$$
$$= 99.609333$$

The PVBP is obtained by substituting these results into Equation 12.

$$\text{PVBP} = \frac{99.609333 - 99.512707}{2} = 0.04831$$

Another money duration statistic reported on the Bloomberg YAS page is "risk." It is shown to be 4.831. Bloomberg's risk statistic is simply the PVBP (or PV01) times 100.

EXAMPLE 12

A life insurance company holds a USD10 million (par value) position in a 4.50% ArcelorMittal bond that matures on 25 February 2017. The bond is priced (flat) at 98.125 per 100 of par value to yield 5.2617% on a street-convention semiannual bond basis for settlement on 27 June 2014. The total market value of the position, including accrued interest, is USD9,965,000, or 99.650 per 100 of par value. The bond's (annual) Macaulay duration is 2.4988.

1. Calculate the money duration per 100 in par value for the ArcelorMittal bond.
2. Using the money duration, estimate the loss on the position for each 1 bp increase in the yield-to-maturity for that settlement date.

Solution to 1: The money duration is the annual modified duration times the full price of the bond per 100 of par value.

$$\left(\frac{2.4988}{1 + \frac{0.052617}{2}}\right) \times \text{USD}99.650 = \text{USD}242.62$$

Solution to 2: For each 1 bp increase in the yield-to-maturity, the loss is estimated to be USD0.024262 per 100 of par value: USD242.62 × 0.0001 = USD0.024262.

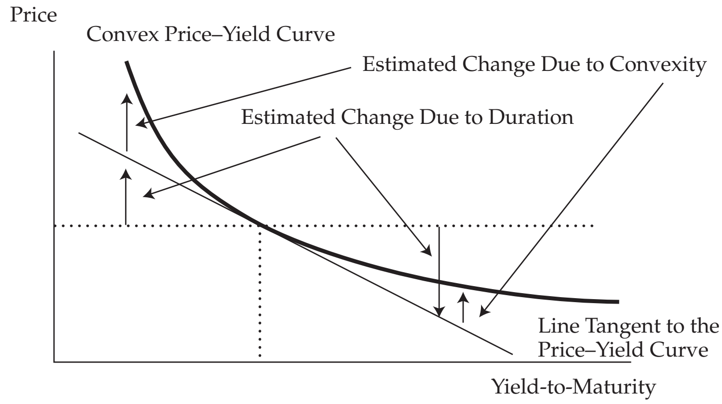
Given a position size of USD10 million in par value, the estimated loss per basis-point increase in the yield is USD2,426.20. The money duration is per 100 of par value, so the position size of USD10 million is divided by 100.

$$\text{USD}0.024262 \times \frac{\text{USD}10,000,000}{100} = \text{USD}2,426.20$$

3.7. Bond Convexity

Modified duration measures the primary effect on a bond's percentage price change given a change in the yield-to-maturity. A secondary effect is measured by the convexity statistic, which is illustrated in Exhibit 10 for a traditional (option-free) fixed-rate bond.

EXHIBIT 10 Convexity of a Traditional (Option-Free) Fixed-Rate Bond



The true relationship between the bond price and the yield-to-maturity is the curved (convex) line shown in Exhibit 10. This curved line shows the actual bond price given its market discount rate. Duration (in particular, money duration) estimates the change in the bond price along the straight line that is tangent to the curved line. For small yield-to-maturity changes, there is little difference between the lines. But for larger changes, the difference becomes significant.

The convexity statistic for the bond is used to improve the estimate of the percentage price change provided by modified duration alone. Equation 13 is the convexity-adjusted estimate of the percentage change in the bond's full price.[5]

$$\%\Delta PV^{Full} \approx (-\text{AnnModDur} \times \Delta\text{Yield}) + \left[\frac{1}{2} \times \text{AnnConvexity} \times (\Delta\text{Yield})^2\right] \tag{13}$$

The first bracketed expression, the "first-order" effect, is the same as Equation 6. The (annual) modified duration, AnnModDur, is multiplied by the change in the (annual) yield-to-maturity, ΔYield. The second bracketed expression, the "second-order" effect, is the **convexity adjustment**. The convexity adjustment is the annual convexity statistic, AnnConvexity, times one-half, multiplied by the change in the yield-to-maturity *squared*. This additional term is a positive amount on a traditional (option-free) fixed-rate bond for either an increase or decrease in the yield. In Exhibit 10, this amount adds to the linear estimate provided by the duration alone, which brings the adjusted estimate very close to the actual price on the curved line. But it still is an estimate, so the ≈ sign is used.

Similar to the Macaulay and modified durations, the annual convexity statistic can be calculated in several ways. It can be calculated using tables, such as Exhibits 2 and 3. It also is possible to derive a closed-form equation for the convexity of a fixed-rate bond on and between coupon payment dates using calculus and algebra.[6] But like modified duration, convexity can

[5] Readers who have studied calculus will recognize this equation as the first two terms of a Taylor series expansion. The first term, the modified duration, includes the first derivative of the bond price with respect to a change in the yield. The second term, the convexity, includes the second derivative.

[6] The step-by-step derivation for a closed-form equation for convexity on and between coupon payment dates is in Donald J. Smith, *Bond Math: The Theory behind the Formulas* (Hoboken, NJ: John Wiley & Sons, 2011).

be approximated with accuracy. Equation 14 is the formula for the approximate convexity statistic, ApproxCon.

$$\text{ApproxCon} = \frac{(PV_-) + (PV_+) - [2 \times (PV_0)]}{(\Delta\text{Yield})^2 \times (PV_0)} \tag{14}$$

This equation uses the same inputs as Equation 7 for ApproxModDur. The new price when the yield-to-maturity is increased is PV_+. The new price when the yield is decreased by the same amount is PV_-. The original price is PV_0. These are the full prices, including accrued interest, for the bond.

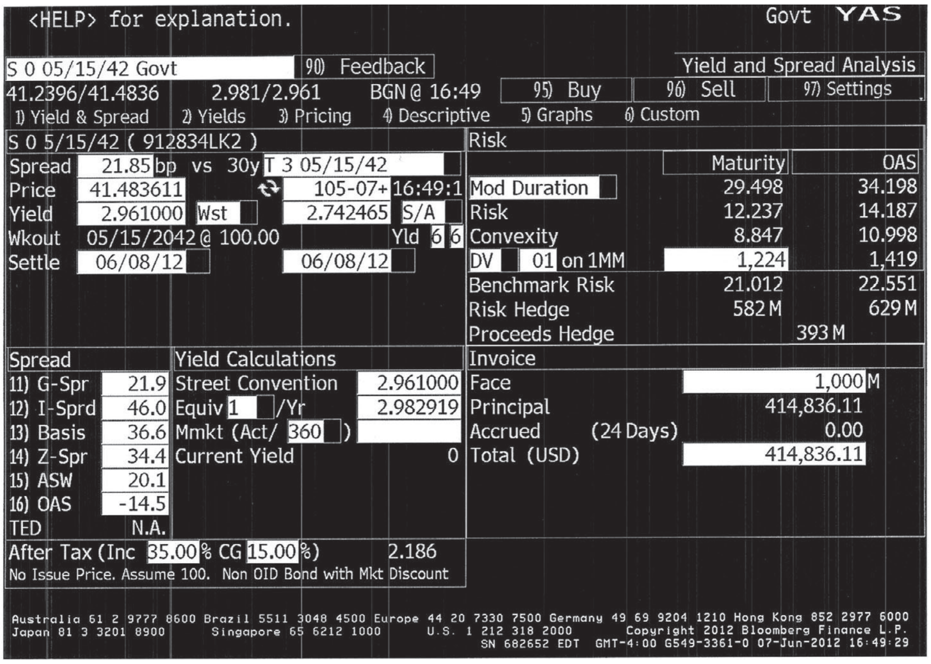
The accuracy of this approximation can be demonstrated with the special case of a zero-coupon bond. The absence of coupon payments simplifies the interest rate risk measures. The Macaulay duration of a zero-coupon bond is $N - t/T$ in terms of periods to maturity. The exact convexity statistic of a zero-coupon bond, also in terms of periods, is calculated with Equation 15.

$$\text{Convexity (of a zero-coupon bond)} = \frac{[N - (t/T)] \times [N + 1 - (t/T)]}{(1+r)^2} \tag{15}$$

N is the number of periods to maturity as of the beginning of the current period, t/T is the fraction of the period that has gone by, and r is the yield-to-maturity per period.

For an example of this calculation, consider a long-term, zero-coupon US Treasury bond. The bond's Bloomberg YAS page is shown in Exhibit 11.

EXHIBIT 11 Bloomberg YAS Page for the Zero-Coupon US Treasury Bond



The bond matures on 15 May 2042 and was priced at 41.483611 per 100 of par value for settlement on 8 June 2012. Its yield-to-maturity was 2.961% stated on a street-convention semiannual bond basis. Even though it is a zero-coupon bond, its yield-to-maturity is based on the actual/actual day-count convention. That settlement date was 24 days into a 184-day period. The annual modified duration was 29.498.

For this bond, $N = 60$, $t/T = 24/184$, and $r = 0.02961/2$. Entering these variables into Equation 15 produces a convexity of 3,538.68 in terms of semiannual periods.

$$\frac{\left[60-\left(24/184\right)\right]\times\left[60+1-\left(24/184\right)\right]}{\left(1+\dfrac{0.02961}{2}\right)^{2}} = 3{,}538.68$$

As with the other statistics, convexity is annualized in practice and for use in the convexity adjustment in Equation 13. It is divided by the periodicity *squared*. The yield-to-maturity on this zero-coupon bond is stated on a semiannual bond basis, meaning a periodicity of 2. Therefore, the annualized convexity statistic is 884.7.

$$\frac{3{,}538.68}{4} = 884.7$$

For example, suppose that the yield-to-maturity is expected to fall by 10 bps, from 2.961% to 2.861%. Given the (annual) modified duration of 29.498 and (annual) convexity of 884.7, the expected percentage price gain is 2.9940%.

$$\begin{aligned}\%\Delta PV^{Full} &\approx \left[-29.498\times -0.0010\right]+\left[\frac{1}{2}\times 884.7\times\left(-0.0010\right)^{2}\right]\\ &= 0.029498+0.000442\\ &= 0.029940\end{aligned}$$

Modified duration alone (under)estimates the gain to be 2.9498%. The convexity adjustment adds 4.42 bps.

The long-term, zero-coupon bond of Exhibit 11 demonstrates the significant difference between *yield* duration and convexity and *curve* duration and convexity, even on an option-free bond. Its modified duration is 29.498, whereas its effective duration is 34.198. Its yield convexity is reported on the Bloomberg page to be 8.847, and its effective convexity is 10.998. (Note that Bloomberg scales the convexity statistics by dividing by 100.) In general, the differences are heightened when the benchmark yield curve is not flat, when the bond has a long time-to-maturity, and the bond is priced at a significant discount or premium.

To obtain the ApproxCon for this long-term, zero-coupon bond, calculate PV_0, PV_+, and PV_- for yields-to-maturity of 2.961%, 2.971%, and 2.951%, respectively. For this exercise, $\Delta\text{Yield} = 0.0001$.

$$PV_0 = \frac{100}{\left(1+\dfrac{0.02961}{2}\right)^{60}}\times\left(1+\frac{0.02961}{2}\right)^{24/184} = 41.483617$$

$$PV_+ = \frac{100}{\left(1+\dfrac{0.02971}{2}\right)^{60}}\times\left(1+\frac{0.02971}{2}\right)^{24/184} = 41.361431$$

$$PV_- = \frac{100}{\left(1+\frac{0.02951}{2}\right)^{60}} \times \left(1+\frac{0.02951}{2}\right)^{24/184} = 41.606169$$

The price of the zero-coupon bond is actually 41.483611, not 41.483617. In this calculation, PV_0 is slightly different because the quoted yield-to-maturity is rounded.[7] It is appropriate to use the calculated PV_0 to be consistent with the change in the yield-to-maturity.

Using these results, first calculate ApproxModDur using Equation 7 to confirm that these inputs are correct. In Exhibit 11, modified duration is stated to be 29.498.

$$\text{ApproxModDur} = \frac{41.606169 - 41.361431}{2 \times 0.0001 \times 41.483617} = 29.498$$

Using Equation 14, ApproxCon is 882.3.

$$\text{ApproxCon} = \frac{41.606169 + 41.361431 - (2 \times 41.483617)}{(0.0001)^2 \times 41.483617} = 882.3$$

This result, 882.3, is an approximation for *annualized* convexity. The number of periods in the year is included in the price calculations. This approximation is quite close to the "exact" result using the closed-form equation for the special case of the zero-coupon bond, 884.7. The difference is not likely to be meaningful for practical applications.

Because this is an individual zero-coupon bond, it is easy to calculate the new price if the yield-to-maturity does go down by 10 bps, to 2.861%.

$$\frac{100}{\left(1+\frac{0.02861}{2}\right)^{60}} \times \left(1+\frac{0.02861}{2}\right)^{24/184} = 42.725841$$

Therefore, the actual percentage price increase is 2.9945%.

$$\frac{42.725841 - 41.483611}{41.483611} = 0.029945$$

The convexity-adjusted estimation of 2.9940% is very close to the actual change. Using the approximate convexity of 882.3 instead of the exact convexity of 884.7 would not have had a meaningful impact.

$$\begin{aligned} \%\Delta PV^{Full} &\approx (-29.458 \times -0.0010) + \left[\frac{1}{2} \times 882.3 \times (-0.0010)^2\right] \\ &= 0.029458 + 0.000441 \\ &= 0.029899 \end{aligned}$$

The "exact" convexity adjustment is 4.42 bps. The "approximate" convexity adjustment is 4.41 bps.

[7] Given the price of 41.483611, the yield-to-maturity is 2.96100046%.

EXAMPLE 13

An Italian bank holds a large position in a 7.25% annual coupon payment corporate bond that matures on 4 April 2029. The bond's yield-to-maturity is 7.44% for settlement on 27 June 2014, stated as an effective annual rate. That settlement date is 83 days into the 360-day year using the 30/360 method of counting days.

1. Calculate the full price of the bond per 100 of par value.
2. Calculate the approximate modified duration and approximate convexity using a 1 bp increase and decrease in the yield-to-maturity.
3. Calculate the estimated convexity-adjusted percentage price change resulting from a 100 bp increase in the yield-to-maturity.
4. Compare the estimated percentage price change with the actual change, assuming the yield-to-maturity jumps to 8.44% on that settlement date.

Solutions: There are 15 years from the beginning of the current period on 4 April 2014 to maturity on 4 April 2029.

1. The full price of the bond is 99.956780 per 100 of par value.

$$PV_0 = \left[\frac{7.25}{(1.0744)^1} + \cdots + \frac{107.25}{(1.0744)^{15}}\right] \times (1.0744)^{83/360} = 99.956780$$

2. $PV_+ = 99.869964$, and $PV_- = 100.043703$.

$$PV_+ = \left[\frac{7.25}{(1.0745)^1} + \cdots + \frac{107.25}{(1.0745)^{15}}\right] \times (1.0745)^{83/360} = 99.869964$$

$$PV_- = \left[\frac{7.25}{(1.0743)^1} + \cdots + \frac{107.25}{(1.0743)^{15}}\right] \times (1.0743)^{83/360} = 100.043703$$

The approximate modified duration is 8.6907.

$$\text{ApproxModDur} = \frac{100.043703 - 99.869964}{2 \times 0.0001 \times 99.956780} = 8.6907$$

The approximate convexity is 107.046.

$$\text{ApproxCon} = \frac{100.043703 + 99.869964 - (2 \times 99.956780)}{(0.0001)^2 \times 99.956780} = 107.046$$

3. The convexity-adjusted percentage price drop resulting from a 100 bp increase in the yield-to-maturity is estimated to be 8.1555%. Modified duration alone

estimates the percentage drop to be 8.6907%. The convexity adjustment adds 53.52 bps.

$$\begin{aligned}\%\Delta PV^{Full} &\approx (-8.6907 \times 0.0100) + \left[\frac{1}{2} \times 107.046 \times (0.0100)^2\right] \\ &= -0.086907 + 0.005352 \\ &= -0.081555\end{aligned}$$

4. The new full price if the yield-to-maturity goes from 7.44% to 8.44% on that settlement date is 91.780921.

$$PV^{Full} = \left[\frac{7.25}{(1.0844)^1} + \cdots + \frac{107.25}{(1.0844)^{15}}\right] \times (1.0844)^{83/360} = 91.780921$$

$$\%\Delta PV^{Full} = \frac{91.780921 - 99.956780}{99.956780} = -0.081794$$

The actual percentage change in the bond price is –8.1794%. The convexity-adjusted estimate is –8.1555%, whereas the estimated change using modified duration alone is –8.6907%.

The money duration of a bond indicates the first-order effect on the full price of a bond in units of currency given a change in the yield-to-maturity. The **money convexity** statistic (MoneyCon) is the second-order effect. The money convexity of the bond is the annual convexity multiplied by the full price, such that

$$\Delta PV^{Full} \approx -(\text{MoneyDur} \times \Delta\text{Yield}) + \left[\frac{1}{2} \times \text{MoneyCon} \times (\Delta\text{Yield})^2\right] \qquad (16)$$

For a money convexity example, consider again the Hong Kong–based life insurance company that has a HKD100,000,000 position in the 6.00% bond that matures on 14 February 2022. In Section 3.5, using the money duration alone, the estimated loss is HKD6,184,418 if the yield-to-maturity increases by 100 bps. The money duration for the position is HKD618,441,784. That estimation is improved by including the convexity adjustment. In Section 3.1, these inputs are calculated to obtain the approximate modified duration of 6.1268 for a 5 bp change in the yield-to-maturity (ΔYield = 0.0005): PV_0 = 100.940423, PV_+ = 100.631781, and PV_- = 101.250227. Enter these into Equation 14 to calculate the approximate convexity.

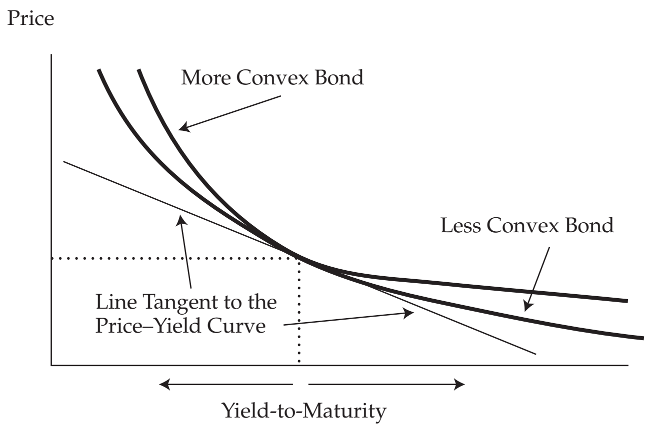
$$\text{ApproxCon} = \frac{101.250227 + 100.631781 - (2 \times 100.940423)}{(0.0005)^2 \times 100.940423} = 46.047$$

The money convexity is 46.047 times the market value of the position, HKD100,940,423. The convexity-adjusted loss given a 100 bp jump in the yield-to-maturity is HKD5,952,018.

$$-\left[(6.1268 \times \text{HKD}100{,}940{,}423) \times 0.0100\right] +$$
$$\left[\frac{1}{2} \times (46.047 \times \text{HKD}100{,}940{,}423) \times (0.0100)^2\right]$$
$$= -\text{HKD}6{,}184{,}418 + \text{HKD}232{,}400$$
$$= -\text{HKD}5{,}952{,}018$$

The factors that lead to greater convexity are the same as for duration. A fixed-rate bond with a longer time-to-maturity, a lower coupon rate, and a lower yield-to-maturity has greater convexity than a bond with a shorter time-to-maturity, a higher coupon rate, and a higher yield-to-maturity. Another factor is the dispersion of cash flows, meaning the degree to which payments are spread out over time. If two bonds have the same duration, the one that has the greater dispersion of cash flows has the greater convexity. The positive attributes of greater convexity for an investor are shown in Exhibit 12.

EXHIBIT 12 The Positive Attributes of Greater Bond Convexity on a Traditional (Option-Free) Bond



The two bonds in Exhibit 12 are assumed to have the same price, yield-to-maturity, and modified duration. Therefore, they share the same line tangent to their price–yield curves. The benefit of greater convexity occurs when their yields-to-maturity change. For the same decrease in yield-to-maturity, the more convex bond *appreciates more* in price. And for the same increase in yield-to-maturity, the more convex bond *depreciates less* in price. The conclusion is that the more convex bond outperforms the less convex bond in both bull (rising price) and bear (falling price) markets. This conclusion assumes, however, that this positive attribute is not "priced into" the bond. To the extent that it is included, the more convex bond would have a higher price (and lower yield-to-maturity). That does not diminish the value of convexity. It only suggests that the investor has to pay for it. As economists say, "There is no such thing as a free lunch."

EXAMPLE 14

The investment manager for a UK defined-benefit pension scheme is considering two bonds about to be issued by a large life insurance company. The first is a 30-year, 4% semiannual coupon payment bond. The second is a 100-year, 4% semiannual coupon payment "century" bond. Both bonds are expected to trade at par value at issuance.

Calculate the approximate modified duration and approximate convexity for each bond using a 5 bp increase and decrease in the annual yield-to-maturity. Retain accuracy to six decimals per 100 of par value.

Solution: In the calculations, the yield per semiannual period goes up by 2.5 bps to 2.025% and down by 2.5 bps to 1.975%. The 30-year bond has an approximate modified duration of 17.381 and an approximate convexity of 420.80.

$$PV_+ = \frac{2}{(1.02025)^1} + \cdots + \frac{102}{(1.02025)^{60}} = 99.136214$$

$$PV_- = \frac{2}{(1.01975)^1} + \cdots + \frac{102}{(1.01975)^{60}} = 100.874306$$

$$\text{ApproxModDur} = \frac{100.874306 - 99.136214}{2 \times 0.0005 \times 100} = 17.381$$

$$\text{ApproxCon} = \frac{100.874306 + 99.136214 - (2 \times 100)}{(0.0005)^2 \times 100} = 420.80$$

The 100-year century bond has an approximate modified duration of 24.527 and an approximate convexity of 1,132.88.

$$PV_+ = \frac{2}{(1.02025)^1} + \cdots + \frac{102}{(1.02025)^{200}} = 98.787829$$

$$PV_- = \frac{2}{(1.01975)^1} + \cdots + \frac{102}{(1.01975)^{200}} = 101.240493$$

$$\text{ApproxModDur} = \frac{101.240493 - 98.787829}{2 \times 0.0005 \times 100} = 24.527$$

$$\text{ApproxCon} = \frac{101.240493 + 98.787829 - (2 \times 100)}{(0.0005)^2 \times 100} = 1{,}132.88$$

The century bond offers a higher modified duration—24.527 compared with 17.381—and a much greater degree of convexity—1,132.88 compared with 420.80.

In the same manner that the primary, or first-order, effect of a shift in the benchmark yield curve is measured by effective duration, the secondary, or second-order, effect is measured by **effective convexity**. The effective convexity of a bond is a *curve convexity* statistic that measures the secondary effect of a change in a benchmark yield curve. A pricing model is used to determine the new prices when the benchmark curve is shifted upward (PV_+) and downward (PV_-) by the same amount (ΔCurve). These changes are made holding other factors constant—for example, the credit spread. Then, Equation 17 is used to calculate the effective convexity (EffCon) given the initial price (PV_0).

$$\text{EffCon} = \frac{\left[(PV_-)+(PV_+)\right]-\left[2\times(PV_0)\right]}{(\Delta\text{Curve})^2\times(PV_0)} \tag{17}$$

This equation is very similar to Equation 14, for approximate *yield* convexity. The difference is that in Equation 14, the denominator includes the change in the yield-to-maturity squared, $(\Delta\text{Yield})^2$. Here, the denominator includes the change in the benchmark yield curve squared, $(\Delta\text{Curve})^2$.

Consider again the callable bond example in Section 3.2. It is assumed that an option-pricing model is used to generate these callable bond prices: PV_0 = 101.060489, PV_+ = 99.050120, PV_- = 102.890738, and ΔCurve = 0.0025. The effective duration for the callable bond is 7.6006.

$$\text{EffDur} = \frac{102.890738-99.050120}{2\times0.0025\times101.060489} = 7.6006$$

Using these inputs in Equation 17, the effective convexity is –285.17.

$$\text{EffCon} = \frac{102.890738+99.050120-(2\times101.060489)}{(0.0025)^2\times101.060489} = -285.17$$

Negative convexity, which could be called "concavity," is an important feature of callable bonds. Putable bonds, on the other hand, always have positive convexity. As a second-order effect, effective convexity indicates the change in the first-order effect (i.e., effective duration) as the benchmark yield curve is changed. In Exhibit 8, as the benchmark yield goes down, the slope of the line tangent to the curve for the non-callable bond steepens, which indicates positive convexity. But the slope of the line tangent to the callable bond flattens as the benchmark yield goes down. Technically, it reaches an inflection point, which is when the effective convexity shifts from positive to negative.

In summary, when the benchmark yield is high and the value of the embedded call option is low, the callable and the non-callable bonds experience very similar effects from interest rate changes. They both have positive convexity. But as the benchmark yield is reduced, the curves diverge. At some point, the callable bond moves into the range of negative convexity, which indicates that the embedded call option has more value to the issuer and is more likely to be exercised. This situation limits the potential price appreciation of the bond arising from lower interest rates, whether because of a lower benchmark yield or a lower credit spread.

Another way to understand why a callable bond can have negative convexity is to rearrange Equation 17.

$$\text{EffCon} = \frac{\left[(PV_-)-(PV_0)\right]-\left[(PV_0)-(PV_+)\right]}{(\Delta\text{Curve})^2\times(PV_0)}$$

In the numerator, the first bracketed expression is the increase in price when the benchmark yield curve is lowered. The second expression is the decrease in price when the benchmark yield curve is raised. On a non-callable bond, the increase is always larger than the decrease (in absolute value). This result is the "convexity effect" for the relationship between bond prices and yields-to-maturity. On a callable bond, the increase can be smaller than the decrease (in absolute value). That creates negative convexity, as illustrated in Exhibit 8.

4. INTEREST RATE RISK AND THE INVESTMENT HORIZON

This section explores the effect of yield volatility on the investment horizon and on the interaction between the investment horizon, market price risk, and coupon reinvestment risk.

4.1. Yield Volatility

An important aspect in understanding the interest rate risk and return characteristics of an investment in a fixed-rate bond is the time horizon. This section considers a short-term horizon. A primary concern for the investor is the change in the price of the bond given a sudden (i.e., same-day) change in its yield-to-maturity. The accrued interest does not change, so the impact of the change in the yield is on the flat price of the bond. Section 4.2 considers a long-term horizon. The reinvestment of coupon interest then becomes a key factor in the investor's horizon yield.

Bond duration is the primary measure of risk arising from a change in the yield-to-maturity. Convexity is the secondary risk measure. In the discussion of the impact on the bond price, the phrase "for a *given* change in the yield-to-maturity" is used repeatedly. For instance, the given change in the yield-to-maturity could be 1 bp, 25 bps, or 100 bps. In comparing two bonds, it is assumed that the "given change" is the same for both securities. When the government bond par curve is shifted up or down by the same amount to calculate effective duration and effective convexity, the events are described as "parallel" yield curve shifts. Because yield curves are rarely (if ever) straight lines, this shift may also be described as a "shape-preserving" shift to the yield curve. The key assumption is that all yields-to-maturity under consideration rise or fall by the same amount across the curve.

Although the assumption of a parallel shift in the yield curve is common in fixed-income analysis, it is not always realistic. In reality, the shape of the yield curve changes based on factors affecting the supply and demand of shorter-term versus longer-term securities. In fact, the term structure of bond yields (also called the "term structure of interest rates") is typically upward sloping. However, the **term structure of yield volatility** may have a different shape depending on a number of factors. The term structure of yield volatility is the relationship between the volatility of bond yields-to-maturity and times-to-maturity.

For example, a central bank engaging in expansionary monetary policy might cause the yield curve to steepen by reducing short-term interest rates. But this policy might cause greater *volatility* in short-term bond yields-to-maturity than in longer-term bonds, resulting in a downward-sloping term structure of yield volatility. Longer-term bond yields are mostly determined by future inflation and economic growth expectations. Those expectations often tend to be less volatile.

The importance of yield volatility in measuring interest rate risk is that bond price changes are products of two factors: (1) the impact *per* basis-point change in the yield-to-maturity and (2) the *number* of basis points in the yield-to-maturity change. The first factor is duration or the combination of duration and convexity, and the second factor is the yield volatility. For example, consider a 5-year bond with a modified duration of 4.5 and a 30-year bond with a modified duration of 18.0. Clearly, for a *given* change in yield-to-maturity, the 30-year bond represents more much more interest rate risk to an investor who has a short-term horizon. In fact, the 30-year bond appears to have *four times* the risk given the ratio of the modified durations. But that assumption neglects the possibility that the 30-year bond might have half the yield volatility of the 5-year bond.

Equation 13, restated here, summarizes the two factors.

$$\%\Delta PV^{Full} \approx (-\text{AnnModDur} \times \Delta\text{Yield}) + \left[\frac{1}{2} \times \text{AnnConvexity} \times (\Delta\text{Yield})^2\right]$$

The estimated percentage change in the bond price depends on the modified duration and convexity as well as on the yield-to-maturity change. Parallel shifts between two bond yields and along a benchmark yield curve are common assumptions in fixed-income analysis. However, an analyst must be aware that non-parallel shifts frequently occur in practice.

EXAMPLE 15

A fixed-income analyst is asked to rank three bonds in terms of interest rate risk. Interest rate risk here means the potential price decrease on a percentage basis given a sudden change in financial market conditions. The increases in the yields-to-maturity represent the "worst case" for the scenario being considered.

Bond	Modified Duration	Convexity	ΔYield
A	3.72	12.1	25 bps
B	5.81	40.7	15 bps
C	12.39	158.0	10 bps

The modified duration and convexity statistics are annualized. ΔYield is the increase in the annual yield-to-maturity. Rank the bonds in terms of interest rate risk.

Solution: Calculate the estimated percentage price change for each bond:

Bond A:

$$(-3.72 \times 0.0025) + \left[\frac{1}{2} \times 12.1 \times (0.0025)^2\right] = -0.009262$$

Bond B:

$$(-5.81 \times 0.0015) + \left[\frac{1}{2} \times 40.7 \times (0.0015)^2\right] = -0.008669$$

Bond C:

$$(-12.39 \times 0.0010) + \left[\frac{1}{2} \times 158.0 \times (0.0010)^2\right] = -0.012311$$

Based on these assumed changes in the yield-to-maturity and the modified duration and convexity risk measures, Bond C has the highest degree of interest rate risk (a potential loss of 1.2311%), followed by Bond A (a potential loss of 0.9262%) and Bond B (a potential loss of 0.8669%).

4.2. Investment Horizon, Macaulay Duration, and Interest Rate Risk

Although short-term interest rate risk is a concern to some investors, other investors have a long-term horizon. Day-to-day changes in bond prices cause *unrealized* capital gains and losses. Those unrealized gains and losses might need to be accounted for in financial statements. This section considers a long-term investor concerned only with the total return over the investment horizon. Therefore, interest rate risk is important to this investor. The investor faces coupon reinvestment risk as well as market price risk if the bond needs to be sold prior to maturity.

Section 2 included examples of interest rate risk using a 10-year, 8% annual coupon payment bond that is priced at 85.503075 per 100 of par value. The bond's yield-to-maturity is 10.40%. A key result in Example 3 is that an investor with a 10-year time horizon is concerned only with coupon reinvestment risk. This situation assumes, of course, that the issuer makes all of the coupon and principal payments as scheduled. The buy-and-hold investor has a higher total return if interest rates rise (see Example 3) and a lower total return if rates fall (see Example 5). The investor in Examples 4 and 6 has a four-year horizon. This investor faces market price risk in addition to coupon reinvestment risk. In fact, the market price risk dominates because this investor has a higher total return if interest rates fall (see Example 6) and a lower return if rates rise (see Example 4).

Now, consider a third investor who has a seven-year time horizon. If interest rates remain at 10.40%, the future value of reinvested coupon interest is 76.835787 per 100 of par value.

$$\left[8\times(1.1040)^6\right]+\left[8\times(1.1040)^5\right]+\left[8\times(1.1040)^4\right]+\left[8\times(1.1040)^3\right]+$$
$$\left[8\times(1.1040)^2\right]+\left[8\times(1.1040)^1\right]+8=76.835787$$

The bond is sold for a price of 94.073336, assuming that the bond stays on the constant-yield price trajectory and continues to be "pulled to par."

$$\frac{8}{(1.1040)^1}+\frac{8}{(1.1040)^2}+\frac{108}{(1.1040)^3}=94.073336$$

The total return is 170.909123 (= 76.835787 + 94.073336) per 100 of par value, and the horizon yield, as expected, is 10.40%.

$$85.503075=\frac{170.909123}{(1+r)^7},\qquad r=0.1040$$

Following Examples 3 and 4, assume that the yield-to-maturity on the bond rises to 11.40%. Also, coupon interest is now reinvested each year at 11.40%. The future value of reinvested coupons becomes 79.235183 per 100 of par value.

$$\left[8\times(1.1140)^6\right]+\left[8\times(1.1140)^5\right]+\left[8\times(1.1140)^4\right]+\left[8\times(1.1140)^3\right]+$$
$$\left[8\times(1.1140)^2\right]+\left[8\times(1.1140)^1\right]+8=79.235183$$

After receiving the seventh coupon payment, the bond is sold. There is a capital loss because the price, although much higher than at purchase, is below the constant-yield price trajectory.

$$\frac{8}{(1.1140)^1}+\frac{8}{(1.1140)^2}+\frac{108}{(1.1140)^3}=91.748833$$

The total return is 170.984016 (= 79.235183 + 91.748833) per 100 of par value and the holding-period rate of return is 10.407%.

$$85.503075=\frac{170.984016}{(1+r)^7}, \qquad r=0.10407$$

Following Examples 5 and 6, assume that the coupon reinvestment rates and the bond yield-to-maturity fall to 9.40%. The future value of reinvested coupons is 74.512177.

$$\left[8+(1.0940)^6\right]+\left[8+(1.0940)^5\right]+\left[8+(1.0940)^4\right]+\left[8+(1.0940)^3\right]+$$
$$\left[8+(1.0940)^2\right]+\left[8+(1.0940)^1\right]+8=74.52177$$

The bond is sold at a capital gain because the price is above the constant-yield price trajectory.

$$\frac{8}{(1.0940)^1}+\frac{8}{(1.0940)^2}+\frac{108}{(1.0940)^3}=96.481299$$

The total return is 170.993476 (= 74.512177 + 96.481299) per 100 of par value, and the horizon yield is 10.408%.

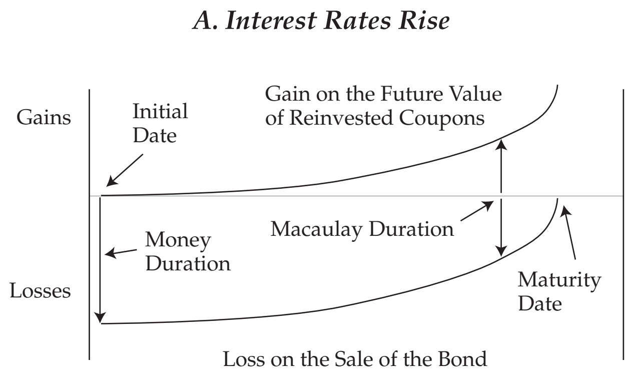
$$85.503075=\frac{170.993476}{(1+r)^7}, \qquad r=0.10408$$

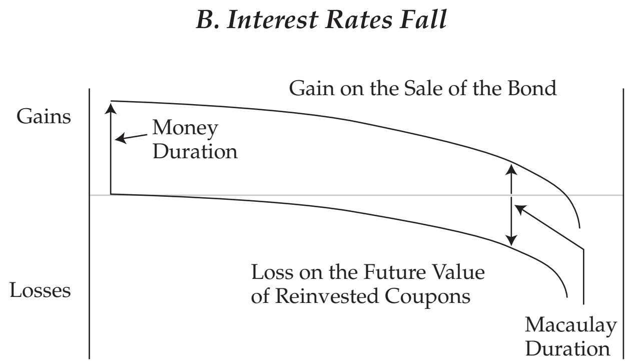
These results are summarized in the following table to reveal the remarkable outcome: The total returns and horizon yields are virtually the same. The investor with the 7-year horizon, unlike those having a 4- or 10-year horizon, achieves the same holding-period rate of return whether interest rates rise, fall, or remain the same. Note that the terms "horizon yield" and "holding-period rate of return" are used interchangeably in this chapter. Sometimes "horizon yield" refers to yields on bonds that need to be sold at the end of the investor's holding period.

Interest Rate	Future Value of Reinvested Coupon	Sale Price	Total Return	Horizon Yield
9.40%	74.512177	96.481299	170.993476	10.408%
10.40%	76.835787	94.073336	170.909123	10.400%
11.40%	79.235183	91.748833	170.984016	10.407%

This particular bond was chosen as an example to demonstrate an important property of Macaulay duration: For a particular assumption about yield volatility, Macaulay duration indicates the investment horizon for which coupon reinvestment risk and market price risk offset each other. In Section 3.1, the Macaulay duration of this 10-year, 8% annual payment bond is calculated to be 7.0029 years. This is one of the applications for duration in which "years" is meaningful and in which Macaulay duration is used rather than modified duration. The particular assumption about yield volatility is that there is a one-time "parallel" shift in the yield curve that occurs before the next coupon payment date. Exhibit 13 illustrates this property of bond duration, assuming that the bond is initially priced at par value.

EXHIBIT 13 Interest Rate Risk, Macaulay Duration, and the Investment Horizon





As demonstrated in Panel A of Exhibit 13, when interest rates rise, duration measures the immediate drop in value. In particular, the money duration indicates the change in price. Then as time passes, the bond price is "pulled to par." The gain in the future value of reinvested coupons starts small but builds over time as more coupons are received. The curve indicates the additional future value of reinvested coupons because of the higher interest rate. At some point in the lifetime of the bond, those two effects offset each other and the gain on reinvested coupons is equal to the loss on the sale of the bond. That point in time is the Macaulay duration statistic.

The same pattern is displayed in the Panel B when interest rates fall, which leads to a reduction in the bond yield and the coupon reinvestment rate. There is an immediate jump in the bond price, as measured by the money duration, but then the "pulled to par" effect brings the price down as time passes. The impact from reinvesting at a lower rate starts small but then becomes more significant over time. The loss on reinvested coupons is with respect to the future value if interest rates had not fallen. Once again, the bond's Macaulay duration indicates the point in time when the two effects offset each other and the gain on the sale of the bond matches the loss on coupon reinvestment.

The earlier numerical example and Exhibit 13 allow for a statement of the general relationships among interest rate risk, the Macaulay duration, and the investment horizon.

1. When the investment horizon is greater than the Macaulay duration of a bond, coupon reinvestment risk dominates market price risk. The investor's risk is to lower interest rates.
2. When the investment horizon is equal to the Macaulay duration of a bond, coupon reinvestment risk offsets market price risk.
3. When the investment horizon is less than the Macaulay duration of the bond, market price risk dominates coupon reinvestment risk. The investor's risk is to higher interest rates.

In the numerical example, the Macaulay duration of the bond is 7.0 years. Statement 1 reflects the investor with the 10-year horizon; Statement 2, the investor with the 7-year horizon; and Statement 3, the investor with the 4-year horizon.

The difference between the Macaulay duration of a bond and the investment horizon is called the **duration gap**. The duration gap is a bond's Macaulay duration minus the investment horizon. The investor with the 10-year horizon has a negative duration gap and currently is at risk of lower rates. The investor with the 7-year horizon has a duration gap of zero and currently is hedged against interest rate risk. The investor with the 4-year horizon has a positive duration gap and currently is at risk of higher rates. The word "currently" is important because interest rate risk is connected to an *immediate* change in the bond's yield-to-maturity and the coupon reinvestment rates. As time passes, the investment horizon is reduced and the Macaulay duration of the bond also changes. Therefore, the duration gap changes as well.

EXAMPLE 16

An investor plans to retire in 10 years. As part of the retirement portfolio, the investor buys a newly issued, 12-year, 8% annual coupon payment bond. The bond is purchased at par value, so its yield-to-maturity is 8.00% stated as an effective annual rate.

1. Calculate the approximate Macaulay duration for the bond, using a 1 bp increase and decrease in the yield-to-maturity and calculating the new prices per 100 of par value to six decimal places.
2. Calculate the duration gap at the time of purchase.
3. Does this bond at purchase entail the risk of higher or lower interest rates? Interest rate risk here means an immediate, one-time, parallel yield curve shift.

Solution to 1: The approximate modified duration of the bond is 7.5361. $PV_0 = 100$, $PV_+ = 99.924678$, and $PV_- = 100.075400$.

$$PV_+ = \frac{8}{(1.0801)^1} + \cdots + \frac{108}{(1.0801)^{12}} = 99.924678$$

$$PV_{-} = \frac{8}{(1.0799)^1} + \cdots + \frac{108}{(1.0799)^{12}} = 100.075400$$

$$\text{ApproxModDur} = \frac{100.075400 - 99.924678}{2 \times 0.0001 \times 100} = 7.5361$$

The approximate Macaulay duration is 8.1390 (= 7.5361 × 1.08).

Solution to 2: Given an investment horizon of 10 years, the duration gap for this bond at purchase is negative: 8.1390 – 10 = –1.8610.

Solution to 3: A negative duration gap entails the risk of lower interest rates. To be precise, the risk is an immediate, one-time, parallel, downward yield curve shift because coupon reinvestment risk dominates market price risk. The loss from reinvesting coupons at a rate lower than 8% is larger than the gain from selling the bond at a price above the constant-yield price trajectory.

5. CREDIT AND LIQUIDITY RISK

The focus of this chapter is to demonstrate how bond duration and convexity estimate the bond price change, either in percentage terms or in units of currency, given an assumed yield-to-maturity change. This section addresses the *source* of the change in the yield-to-maturity. In general, the yield-to-maturity on a corporate bond is composed of a government *benchmark* yield and a *spread* over that benchmark. A change in the bond's yield-to-maturity can originate in either component or a combination of the two.

The key point is that for a traditional (option-free) fixed-rate bond, the same duration and convexity statistics apply for a change in the benchmark yield as for a change in the spread. The "building blocks" approach from "Introduction to Fixed-Income Valuation" shows that these yield-to-maturity changes can be broken down further. A change in the benchmark yield can arise from a change in either the expected inflation rate or the expected real rate of interest. A change in the spread can arise from a change in the credit risk of the issuer or in the liquidity of the bond. Therefore, for a fixed-rate bond, the "inflation duration," the "real rate duration," the "credit duration," and the "liquidity duration" are all the same number. The inflation duration would indicate the change in the bond price if expected inflation were to change by a certain amount. In the same manner, the real rate duration would indicate the bond price change if the real rate were to go up or down. The credit duration and liquidity duration would indicate the price sensitivity that would arise from changes in those building blocks in the yield-to-maturity. A bond with a modified duration of 5.00 and a convexity of 32.00 will appreciate in value by about 1.26% if its yield-to-maturity goes down by 25 bps: $(-5.00 \times -0.0025) + [1/2 \times 32.00 \times (-0.0025)^2] = +\ 0.0126$, regardless of the *source* of the yield-to-maturity change.

Suppose that the yield-to-maturity on a corporate bond is 6.00%. If the benchmark yield is 4.25%, the spread is 1.75%. An analyst believes that credit risk makes up 1.25% of the spread and liquidity risk, the remaining 0.50%. Credit risk includes the probability of default as well as the recovery of assets if default does occur. A credit rating downgrade or an adverse change in the ratings outlook for a borrower reflects a higher risk of default. Liquidity risk refers to the transaction costs associated with selling a bond. In general, a bond with greater frequency of trading and a higher volume of trading provides fixed-income investors with more opportunity to purchase or sell the security and thus has less liquidity risk. In practice, there is a difference between the *bid* (or purchase) and the *offer* (or sale) price. This difference depends on the type of bond, the size of the transaction, and the time of execution, among other factors. For instance, government bonds often trade at just a few basis points between the purchase and sale prices. More thinly traded corporate bonds can have a much wider difference between the bid and offer prices.

The problem for a fixed-income analyst is that it is rare for the changes in the components of the overall yield-to-maturity to occur in isolation. In practice, the analyst is concerned with the *interaction* between changes in benchmark yields and spreads, between changes in expected inflation and the expected real rate, and between changes in credit and liquidity risk. For example, during a financial crisis, a "flight to quality" can cause government benchmark yields to fall as credit spreads widen. An unexpected credit downgrade on a corporate bond can result in greater credit as well as liquidity risk.

EXAMPLE 17

The (flat) price on a fixed-rate corporate bond falls one day from 92.25 to 91.25 per 100 of par value because of poor earnings and an unexpected ratings downgrade of the issuer. The (annual) modified duration for the bond is 7.24. Which of the following is *closest* to the estimated change in the credit spread on the corporate bond, assuming benchmark yields are unchanged?

A. 15 bps
B. 100 bps
C. 108 bps

Solution: Given that the price falls from 92.25 to 91.25, the percentage price decrease is 1.084%.

$$\frac{91.25 - 92.25}{92.25} = -0.01084$$

Given an annual modified duration of 7.24, the change in the yield-to-maturity is 14.97 bps.

$$-0.01084 \approx -7.24 \times \Delta\text{Yield},\ \Delta\text{Yield} = 0.001497$$

Therefore, the answer is A. The change in price reflects a credit spread increase on the bond of about 15 bps.

6. SUMMARY

This chapter covers the risk and return characteristics of fixed-rate bonds. The focus is on the widely used measures of interest rate risk—duration and convexity. These statistics are used extensively in fixed-income analysis. The following are the main points made in the chapter:

- The three sources of return on a fixed-rate bond purchased at par value are (1) receipt of the promised coupon and principal payments on the scheduled dates, (2) reinvestment of coupon payments, and (3) potential capital gains, as well as losses, on the sale of the bond prior to maturity.
- For a bond purchased at a discount or premium, the rate of return also includes the effect of the price being "pulled to par" as maturity nears, assuming no default.
- The total return is the future value of reinvested coupon interest payments and the sale price (or redemption of principal if the bond is held to maturity).
- The horizon yield (or holding period rate of return) is the internal rate of return between the total return and purchase price of the bond.
- Coupon reinvestment risk increases with a higher coupon rate and a longer reinvestment time period.
- Capital gains and losses are measured from the carrying value of the bond and not from the purchase price. The carrying value includes the amortization of the discount or premium if the bond is purchased at a price below or above par value. The carrying value is any point on the constant-yield price trajectory.
- Interest income on a bond is the return associated with the passage of time. Capital gains and losses are the returns associated with a change in the value of a bond as indicated by a change in the yield-to-maturity.
- The two types of interest rate risk on a fixed-rate bond are coupon reinvestment risk and market price risk. These risks offset each other to a certain extent. An investor gains from higher rates on reinvested coupons but loses if the bond is sold at a capital loss because the price is below the constant-yield price trajectory. An investor loses from lower rates on reinvested coupon but gains if the bond is sold at a capital gain because the price is above the constant-yield price trajectory.
- Market price risk dominates coupon reinvestment risk when the investor has a short-term horizon (relative to the time-to-maturity on the bond).
- Coupon reinvestment risk dominates market price risk when the investor has a long-term horizon (relative to the time-to-maturity)—for instance, a buy-and-hold investor.
- Bond duration, in general, measures the sensitivity of the full price (including accrued interest) to a change in interest rates.
- Yield duration statistics measuring the sensitivity of a bond's full price to the bond's own yield-to-maturity include the Macaulay duration, modified duration, money duration, and price value of a basis point.
- Curve duration statistics measuring the sensitivity of a bond's full price to the benchmark yield curve are usually called "effective durations."
- Macaulay duration is the weighted average of the time to receipt of coupon interest and principal payments, in which the weights are the shares of the full price corresponding to each payment. This statistic is annualized by dividing by the periodicity (number of coupon payments or compounding periods in a year).
- Modified duration provides a linear estimate of the percentage price change for a bond given a change in its yield-to-maturity.
- Approximate modified duration approaches modified duration as the change in the yield-to-maturity approaches zero.
- Effective duration is very similar to approximate modified duration. The difference is that approximate modified duration is a yield duration statistic that measures interest rate risk in

terms of a change in the bond's own yield-to-maturity, whereas effective duration is a curve duration statistic that measures interest rate risk assuming a parallel shift in the benchmark yield curve.

- Key rate duration is a measure of a bond's sensitivity to a change in the benchmark yield curve at specific maturity segments. Key rate durations can be used to measure a bond's sensitivity to changes in the shape of the yield curve.
- Bonds with an embedded option do not have a meaningful internal rate of return because future cash flows are contingent on interest rates. Therefore, effective duration is the appropriate interest rate risk measure, not modified duration.
- The effective duration of a traditional (option-free) fixed-rate bond is its sensitivity to the benchmark yield curve, which can differ from its sensitivity to its own yield-to-maturity. Therefore, modified duration and effective duration on a traditional (option-free) fixed-rate bond are not necessarily equal.
- During a coupon period, Macaulay and modified durations decline smoothly in a "sawtooth" pattern, assuming the yield-to-maturity is constant. When the coupon payment is made, the durations jump upward.
- Macaulay and modified durations are inversely related to the coupon rate and the yield-to-maturity.
- Time-to-maturity and Macaulay and modified durations are *usually* positively related. They are *always* positively related on bonds priced at par or at a premium above par value. They are *usually* positively related on bonds priced at a discount below par value. The exception is on long-term, low-coupon bonds, on which it is possible to have a lower duration than on an otherwise comparable shorter-term bond.
- The presence of an embedded call option reduces a bond's effective duration compared with that of an otherwise comparable non-callable bond. The reduction in the effective duration is greater when interest rates are low and the issuer is more likely to exercise the call option.
- The presence of an embedded put option reduces a bond's effective duration compared with that of an otherwise comparable non-putable bond. The reduction in the effective duration is greater when interest rates are high and the investor is more likely to exercise the put option.
- The duration of a bond portfolio can be calculated in two ways: (1) the weighted average of the time to receipt of *aggregate* cash flows and (2) the weighted average of the durations of individual bonds that compose the portfolio.
- The first method to calculate portfolio duration is based on the cash flow yield, which is the internal rate of return on the aggregate cash flows. It cannot be used for bonds with embedded options or for floating-rate notes.
- The second method is simpler to use and quite accurate when the yield curve is relatively flat. Its main limitation is that it assumes a parallel shift in the yield curve in that the yields on all bonds in the portfolio change by the same amount.
- Money duration is a measure of the price change in terms of units of the currency in which the bond is denominated.
- The price value of a basis point (PVBP) is an estimate of the change in the full price of a bond given a 1 bp change in the yield-to-maturity.
- Modified duration is the primary, or first-order, effect on a bond's percentage price change given a change in the yield-to-maturity. Convexity is the secondary, or second-order, effect. It indicates the change in the modified duration as the yield-to-maturity changes.
- Money convexity is convexity times the full price of the bond. Combined with money duration, money convexity estimates the change in the full price of a bond in units of currency given a change in the yield-to-maturity.

- Convexity is a positive attribute for a bond. Other things being equal, a more convex bond appreciates in price more than a less convex bond when yields fall and depreciates less when yields rise.
- Effective convexity is the second-order effect on a bond price given a change in the benchmark yield curve. It is similar to approximate convexity. The difference is that approximate convexity is based on a yield-to-maturity change and effective convexity is based on a benchmark yield curve change.
- Callable bonds have negative effective convexity when interest rates are low. The increase in price when the benchmark yield is reduced is less in absolute value than the decrease in price when the benchmark yield is raised.
- The change in a bond price is the product of (1) the impact per basis-point change in the yield-to-maturity and (2) the number of basis points in the yield change. The first factor is estimated by duration and convexity. The second factor depends on yield volatility.
- The investment horizon is essential in measuring the interest rate risk on a fixed-rate bond.
- For a particular assumption about yield volatility, the Macaulay duration indicates the investment horizon for which coupon reinvestment risk and market price risk offset each other. The assumption is a one-time parallel shift to the yield curve in which the yield-to-maturity and coupon reinvestment rates change by the same amount in the same direction.
- When the investment horizon is greater than the Macaulay duration of the bond, coupon reinvestment risk dominates price risk. The investor's risk is to lower interest rates. The duration gap is negative.
- When the investment horizon is equal to the Macaulay duration of the bond, coupon reinvestment risk offsets price risk. The duration gap is zero.
- When the investment horizon is less than the Macaulay duration of the bond, price risk dominates coupon reinvestment risk. The investor's risk is to higher interest rates. The duration gap is positive.
- Credit risk involves the probability of default and degree of recovery if default occurs, whereas liquidity risk refers to the transaction costs associated with selling a bond.
- For a traditional (option-free) fixed-rate bond, the same duration and convexity statistics apply if a change occurs in the benchmark yield or a change occurs in the spread. The change in the spread can result from a change in credit risk or liquidity risk.
- In practice, there often is interaction between changes in benchmark yields and in the spread over the benchmark.

PROBLEMS

These questions were developed by Danny Hassett, CFA (Cedar Hill, TX, USA), Lou Lemos, CFA (Louisville, KY, USA), and Bin Wang, CFA (Austin, TX, USA).

1. A "buy-and-hold" investor purchases a fixed-rate bond at a discount and holds the security until it matures. Which of the following sources of return is *least likely* to contribute to the investor's total return over the investment horizon, assuming all payments are made as scheduled?
   A. Capital gain
   B. Principal payment
   C. Reinvestment of coupon payments

2. Which of the following sources of return is *most likely* exposed to interest rate risk for an investor of a fixed-rate bond who holds the bond until maturity?
   A. Capital gain or loss
   B. Redemption of principal
   C. Reinvestment of coupon payments
3. An investor purchases a bond at a price above par value. Two years later, the investor sells the bond. The resulting capital gain or loss is measured by comparing the price at which the bond is sold to the:
   A. carrying value.
   B. original purchase price.
   C. original purchase price value plus the amortized amount of the premium.

The following information relates to Problems 4–6

An investor purchases a nine-year, 7% annual coupon payment bond at a price equal to par value. After the bond is purchased and before the first coupon is received, interest rates increase to 8%. The investor sells the bond after five years. Assume that interest rates remain unchanged at 8% over the five-year holding period.

4. Per 100 of par value, the future value of the reinvested coupon payments at the end of the holding period is *closest* to:
   A. 35.00.
   B. 40.26.
   C. 41.07.
5. The capital gain/loss per 100 of par value resulting from the sale of the bond at the end of the five-year holding period is *closest* to a:
   A. loss of 8.45.
   B. loss of 3.31.
   C. gain of 2.75.
6. Assuming that all coupons are reinvested over the holding period, the investor's five-year horizon yield is *closest* to:
   A. 5.66%.
   B. 6.62%.
   C. 7.12%.

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7. An investor buys a three-year bond with a 5% coupon rate paid annually. The bond, with a yield-to-maturity of 3%, is purchased at a price of 105.657223 per 100 of par value. Assuming a 5 bp change in yield-to-maturity, the bond's approximate modified duration is *closest* to:
   A. 2.78.
   B. 2.86.
   C. 5.56.
8. Which of the following statements about duration is correct? A bond's:
   A. effective duration is a measure of yield duration.
   B. modified duration is a measure of curve duration.
   C. modified duration cannot be larger than its Macaulay duration.
9. An investor buys a 6% annual payment bond with three years to maturity. The bond has a yield-to-maturity of 8% and is currently priced at 94.845806 per 100 of par. The bond's Macaulay duration is *closest* to:

A. 2.62.
B. 2.78.
C. 2.83.

10. The interest rate risk of a fixed-rate bond with an embedded call option is *best* measured by:
    A. effective duration.
    B. modified duration.
    C. Macaulay duration.
11. Which of the following is *most* appropriate for measuring a bond's sensitivity to shaping risk?
    A. Key rate duration
    B. Effective duration
    C. Modified duration
12. A Canadian pension fund manager seeks to measure the sensitivity of her pension liabilities to market interest rate changes. The manager determines the present value of the liabilities under three interest rate scenarios: a base rate of 7%, a 100 bp increase in rates up to 8%, and a 100 bp drop in rates down to 6%. The results of the manager's analysis are presented below:

Interest Rate Assumption	Present Value of Liabilities
6%	CAD510.1 million
7%	CAD455.4 million
8%	CAD373.6 million

    The effective duration of the pension fund's liabilities is *closest* to:
    A. 1.49.
    B. 14.99.
    C. 29.97.
13. Which of the following statements about Macaulay duration is correct?
    A. A bond's coupon rate and Macaulay duration are positively related.
    B. A bond's Macaulay duration is inversely related to its yield-to-maturity.
    C. The Macaulay duration of a zero-coupon bond is less than its time-to-maturity.
14. Assuming no change in the credit risk of a bond, the presence of an embedded put option:
    A. reduces the effective duration of the bond.
    B. increases the effective duration of the bond.
    C. does not change the effective duration of the bond.
15. A bond portfolio consists of the following three fixed-rate bonds. Assume annual coupon payments and no accrued interest on the bonds. Prices are per 100 of par value.

Bond	Maturity	Market Value	Price	Coupon	Yield-to-Maturity	Modified Duration
A	6 years	170,000	85.0000	2.00%	4.95%	5.42
B	10 years	120,000	80.0000	2.40%	4.99%	8.44
C	15 years	100,000	100.0000	5.00%	5.00%	10.38

    The bond portfolio's modified duration is *closest* to:
    A. 7.62.
    B. 8.08.
    C. 8.20.

16. A limitation of calculating a bond portfolio's duration as the weighted average of the yield durations of the individual bonds that compose the portfolio is that it:
    A. assumes a parallel shift to the yield curve.
    B. is less accurate when the yield curve is less steeply sloped.
    C. is not applicable to portfolios that have bonds with embedded options.
17. Using the information below, which bond has the *greatest* money duration per 100 of par value assuming annual coupon payments and no accrued interest?

Bond	Time-to-Maturity	Price Per 100 of Par Value	Coupon Rate	Yield-to-Maturity	Modified Duration
A	6 years	85.00	2.00%	4.95%	5.42
B	10 years	80.00	2.40%	4.99%	8.44
C	9 years	85.78	3.00%	5.00%	7.54

    A. Bond A
    B. Bond B
    C. Bond C
18. A bond with exactly nine years remaining until maturity offers a 3% coupon rate with annual coupons. The bond, with a yield-to-maturity of 5%, is priced at 85.784357 per 100 of par value. The estimated price value of a basis point for the bond is *closest* to:
    A. 0.0086.
    B. 0.0648.
    C. 0.1295.
19. The "second-order" effect on a bond's percentage price change given a change in yield-to-maturity can be *best* described as:
    A. duration.
    B. convexity.
    C. yield volatility.
20. A bond is currently trading for 98.722 per 100 of par value. If the bond's yield-to-maturity (YTM) rises by 10 bps, the bond's full price is expected to fall to 98.669. If the bond's YTM decreases by 10 bps, the bond's full price is expected to increase to 98.782. The bond's approximate convexity is *closest* to:
    A. 0.071.
    B. 70.906.
    C. 1,144.628.
21. A bond has an annual modified duration of 7.020 and annual convexity of 65.180. If the bond's yield-to-maturity decreases by 25 bps, the expected percentage price change is *closest* to:
    A. 1.73%.
    B. 1.76%.
    C. 1.78%.
22. A bond has an annual modified duration of 7.140 and annual convexity of 66.200. The bond's yield-to-maturity is expected to increase by 50 bps. The expected percentage price change is *closest* to:
    A. –3.40%.
    B. –3.49%.
    C. –3.57%.

23. Which of the following statements relating to yield volatility is *most* accurate? If the term structure of yield volatility is downward sloping, then:
    A. short-term rates are higher than long-term rates.
    B. long-term yields are more stable than short-term yields.
    C. short-term bonds will always experience greater price fluctuation than long-term bonds.
24. The holding period for a bond at which the coupon reinvestment risk offsets the market price risk is *best* approximated by:
    A. duration gap.
    B. modified duration.
    C. Macaulay duration.
25. When the investor's investment horizon is less than the Macaulay duration of the bond she owns:
    A. the investor is hedged against interest rate risk.
    B. reinvestment risk dominates, and the investor is at risk of lower rates.
    C. market price risk dominates, and the investor is at risk of higher rates.
26. An investor purchases an annual coupon bond with a 6% coupon rate and exactly 20 years remaining until maturity at a price equal to par value. The investor's investment horizon is eight years. The approximate modified duration of the bond is 11.470 years. The duration gap at the time of purchase is *closest* to:
    A. –7.842.
    B. 3.470.
    C. 4.158.
27. A manufacturing company receives a ratings upgrade and the price increases on its fixed-rate bond. The price increase was *most likely* caused by a(n):
    A. decrease in the bond's credit spread.
    B. increase in the bond's liquidity spread.
    C. increase of the bond's underlying benchmark rate.