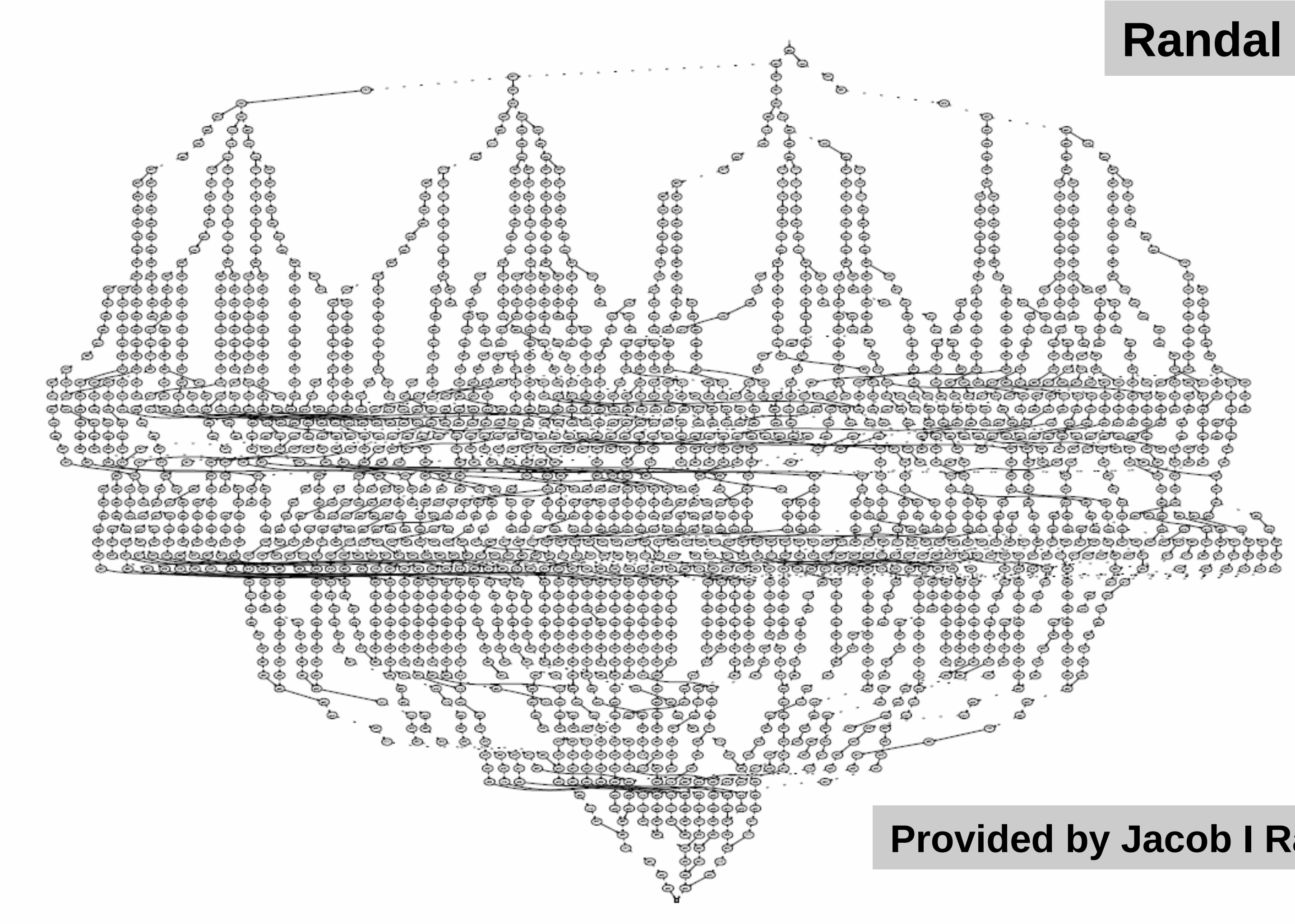
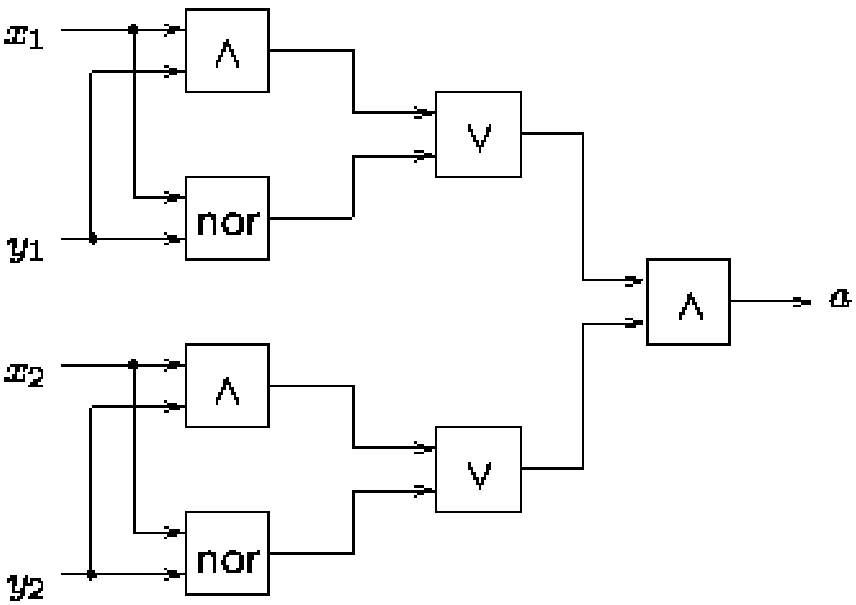
Binary Decision Diagrams

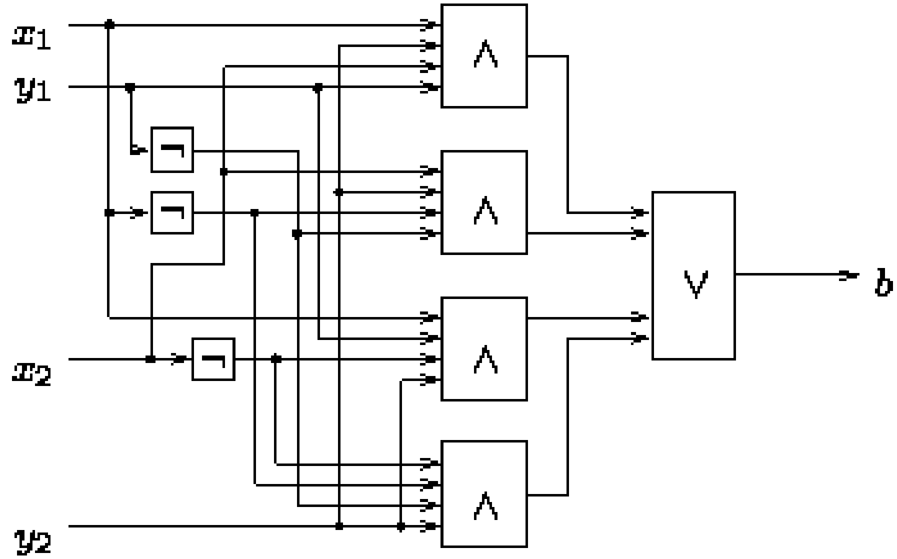
Randal Bryant ’86



Provided by Jacob I Rasmussen

Combinatorial Circuits

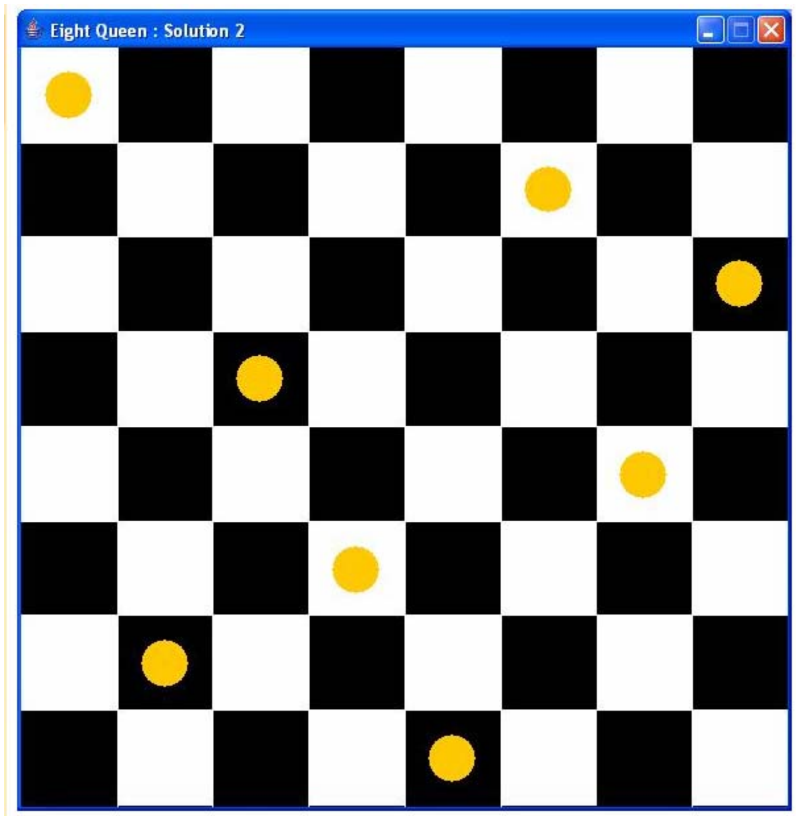




Combinatorial Problems

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

Sudoku

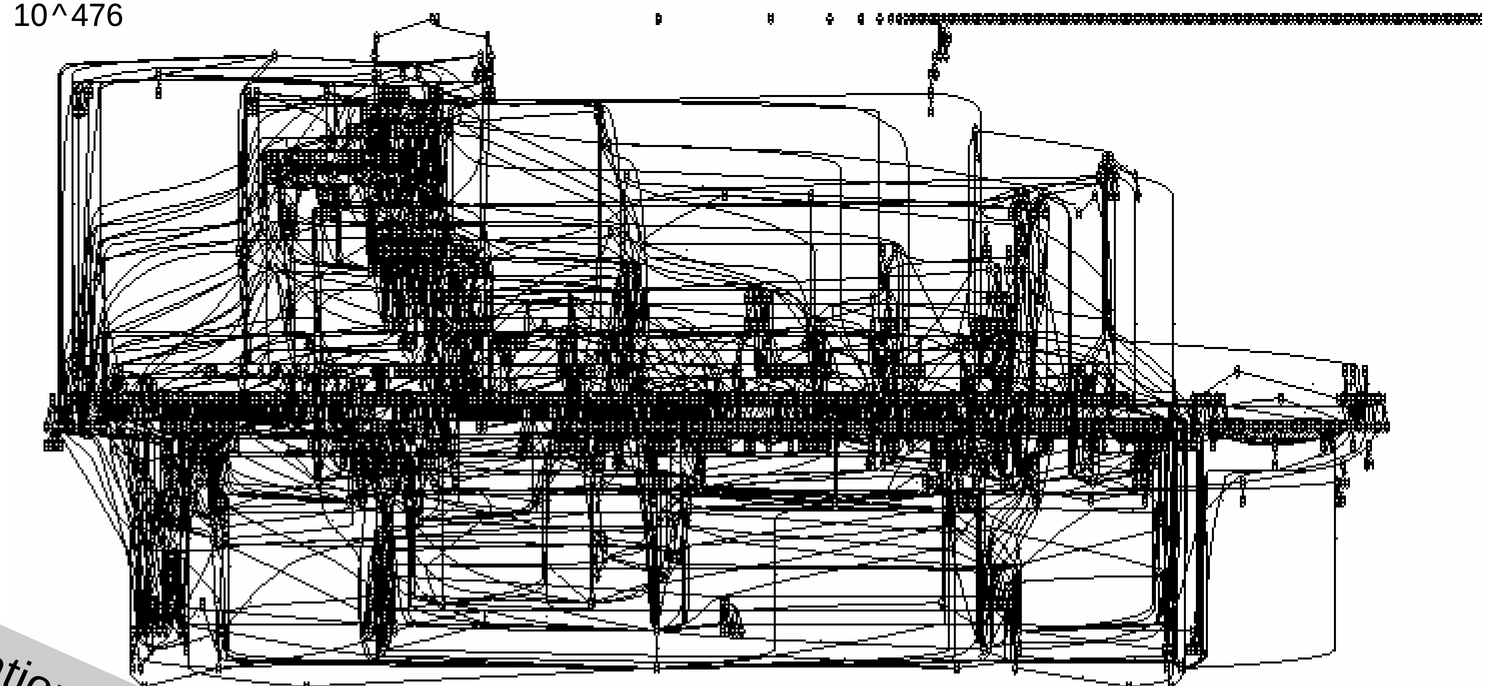


Eight Queen

Control Programs
A Train Simulator, visualSTATE (VVS)

1421 machines
11102 transitions
2981 inputs
2667 outputs
3204 local states
Declare state sp.: 10^476

BUGS ?



“Ideal” presentation: 1 bit/state will clearly NOT work!

Reduced Ordered Binary Decision Diagrams [Bryant'86]

- Compact represetation of *boolean functions* allowing effective manipulation (satifiability, validity,….)

or

- Compact representation of *sets* over finite universe allowing effective manipulations.

Boolean Logic

Boolean Functions

Boolean functions: $\mathbb{B} = \{0, 1\}$,

$$f : \mathbb{B} \times \cdots \times \mathbb{B} \to \mathbb{B}$$

Boolean expressions:

$$t \ ::= \ x \mid 0 \mid 1 \mid \neg t \mid t \wedge t \mid t \vee t \mid t \Rightarrow t \mid t \Leftrightarrow t$$

Truth assignments: ρ,

$$[v_1/x_1, v_2/x_2, \ldots, v_n/x_n]$$

Satisfiable: Exists ρ such that $t[\rho] = 1$

Tautology: Forall ρ, $t[\rho] = 1$

Truth Tables

	$\neg$
0	1
1	0

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\Rightarrow$	0	1
0	1	1
1	0	1

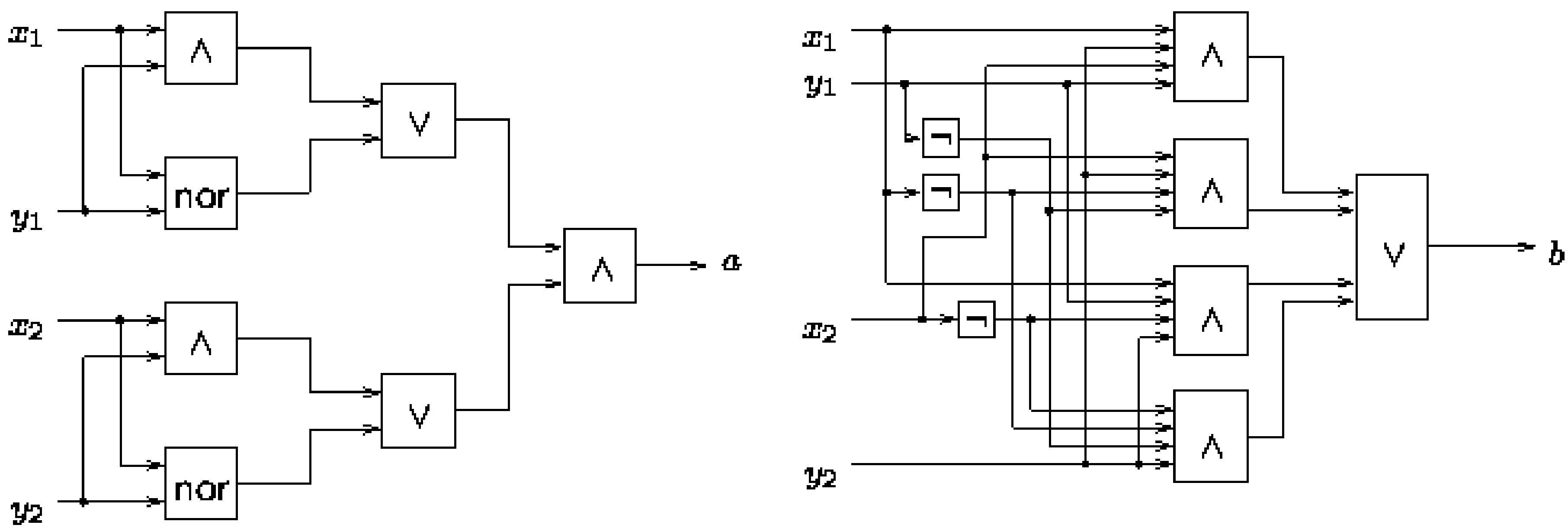
$\Leftrightarrow$	0	1
0	1	0
1	0	1

xyz	$x \to y, z$
000	0
001	1
010	0
011	1
100	0
101	0
110	1
111	1

$x_1 \cdots x_n$	$f(x_1, \ldots, x_n)$
$0 \cdots 0$	1
$0 \cdots 1$	0
$\vdots$	$\vdots$
$1 \cdots 1$	0

2^n entries

Combinatorial Circuits



Are they two circuits equivalent?

“Good” Representations of Boolean Functions

Always perfect representations are hopeless

Normalforms

- Disjunctive NF
- Conjunctive NF
- If-then-else NF
- …….

THEOREM (Cook’s theorem)

Satisfiability of Boolean expressions is NP-complete

Compact representations are

- **compact** and
- **efficient**

on **real-life** examples

If-Then-Else Operator

Let t, t_1 and t_2 be boolean expressions.

Syntax

$$t \longrightarrow t_1, t_2$$

Semantics

If-Then-Else operator $t \rightarrow t_1, t_2$ is equivalent to $(t \wedge t_1) \vee (\neg t \wedge t_2)$.

t	t_1	t_2	$t \rightarrow t_l, t_2$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

If-Then-Else Normal Form

Definition

A boolean expression is in If-Then-Else normal form (INF) iff it is given by the following abstract syntax

$$t, t_1, t_2 ::= 0 \mid 1 \mid x \rightarrow t_1, t_2$$

where x ranges over boolean variables.

Example: $x_1 \rightarrow (x_2 \rightarrow 1, 0), 0$ (equivalent to $x_1 \wedge x_2$)

Boolean expressions in INF can be drawn as decision trees.

Binary Decision Structures

Shannon Expansion

Let t be a boolean expression and x a variable. We define boolean expressions

- $t[0/x]$ where every occurrence of x in t is replaced with 0, and
- $t[1/x]$ where every occurrence of x in t is replaced with 1.

Shannon's Expansion Law

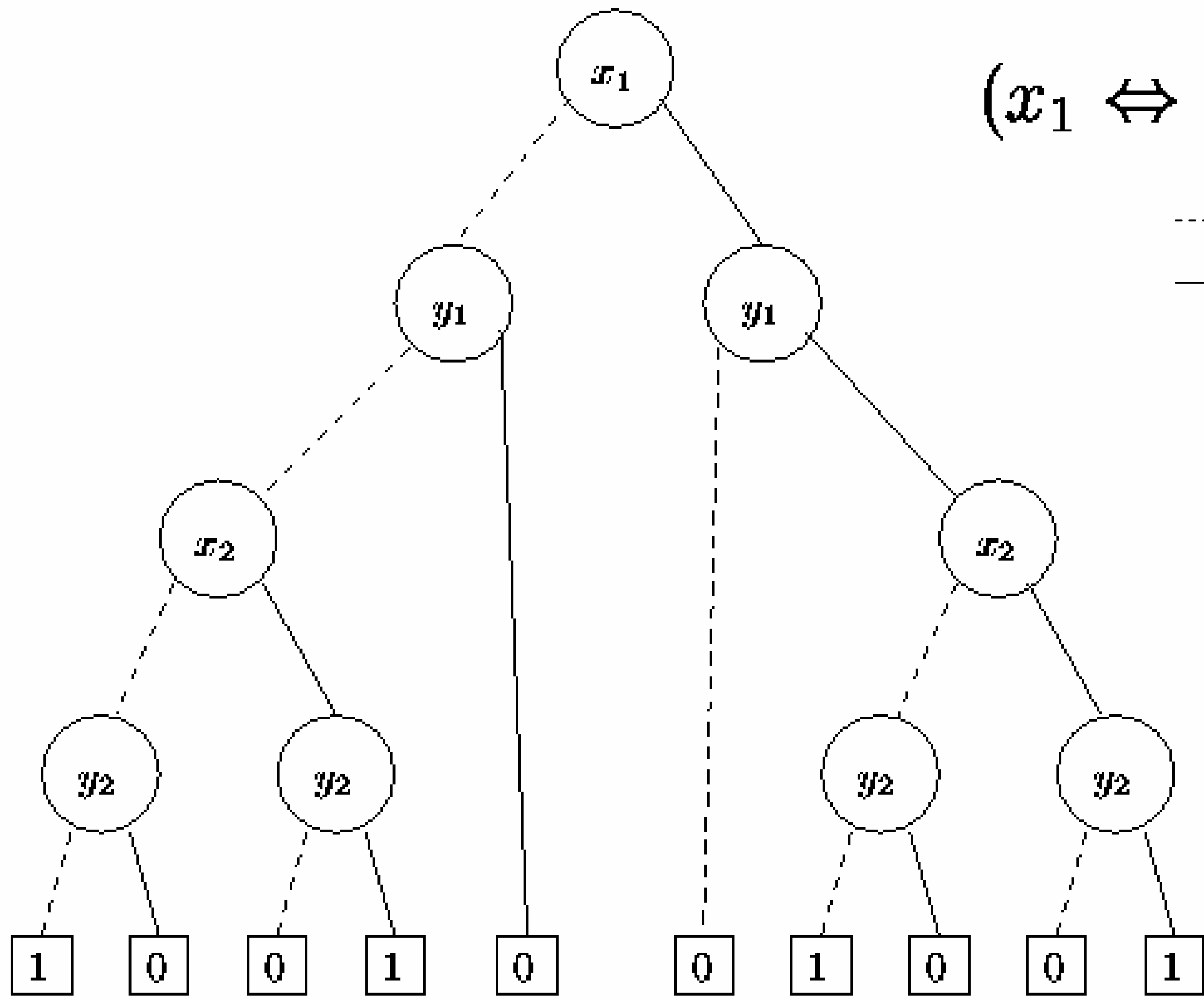
Let x be an arbitrary boolean variable. Any boolean expressions t is equivalent to

$$x \rightarrow t[1/x], t[0/x].$$

Corollary

For any boolean expression there is an equivalent one in INF.

Binary Decision *Trees*



$$(x_1 \Leftrightarrow y_1) \wedge (x_2 \Leftrightarrow y_2)$$

- - - - - Variable is set to 0

——— Variable is set to 1

Each path determines a partial (set of) truth assignments.

Result of the boolean expression under the given assigment found in value of terminal.

Binary Decision *Diagrams*
allows NODES to be shared

Equivalence ~ on nodes:

n ~ *m* **iff**

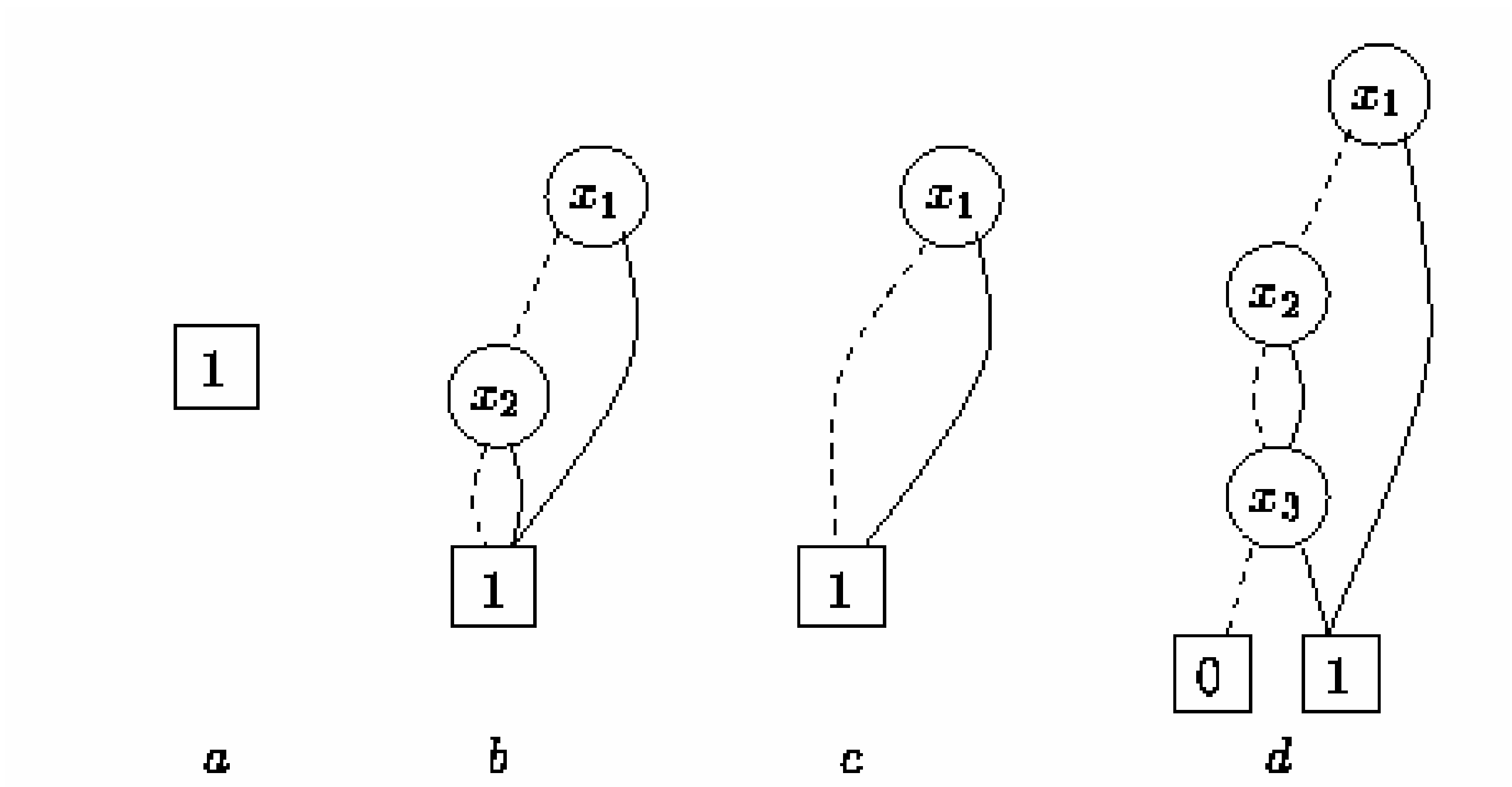
either both *n* and *m* are terminals
and have the same value

or both are non-terminals with var(*n*) = var(*m*) **and**

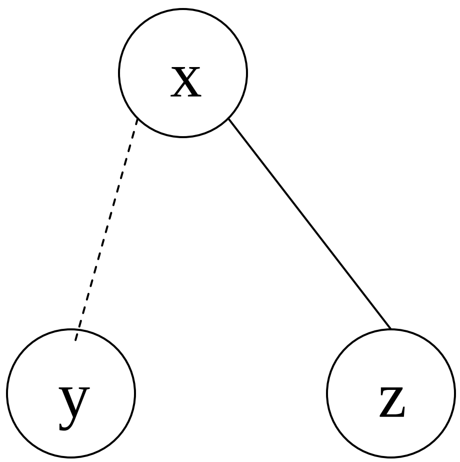
1. *n'* ~ *m'* **when** *n* -0-> *n'* , *m*-0->*m'* , **and**
2. *n'* ~ *m'* **when** *n* -1-> *n'* , *m*-1->*m'*

Have you seen this somewhere before ?

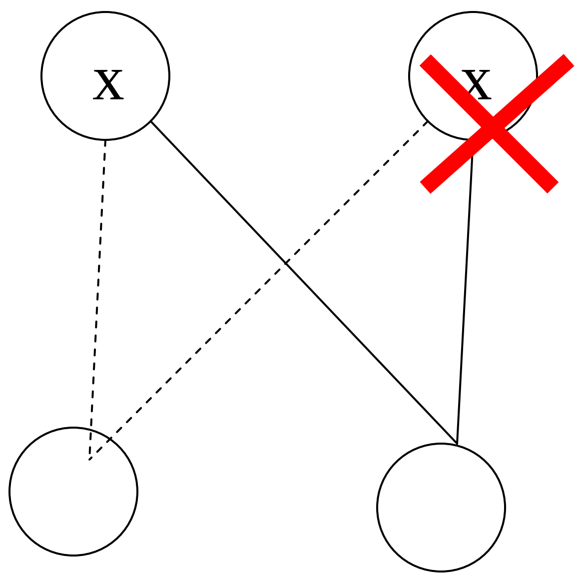
Orderedness & Redundant *TESTS*

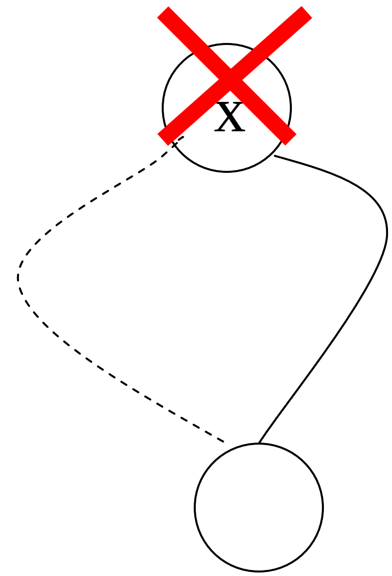


Orderedness & Reducedness

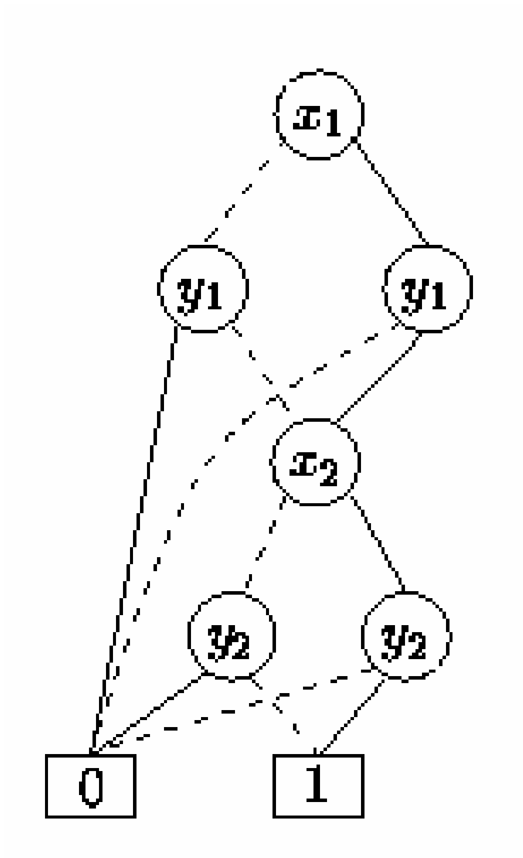


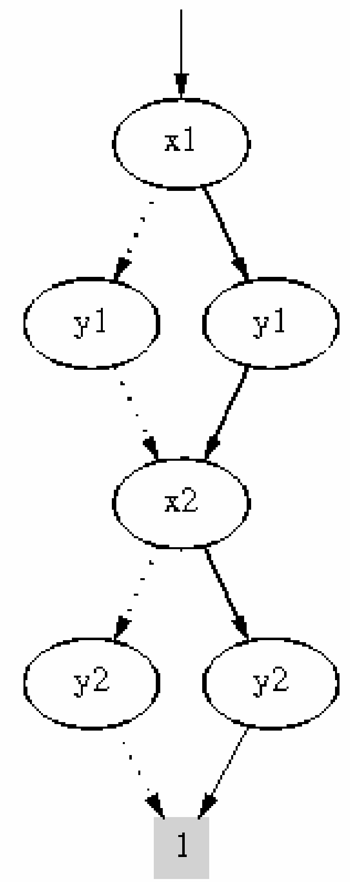
x<y   x<z





Reduced Ordered Binary Decision Diagrams





Iben

Edges to 0 implicit

ROBDDs formally

A *Binary Decision Diagram* is a rooted, directed, acyclic graph (V, E). V contains (up to) two *terminal* vertices, $0, 1 \in V$. $v \in V \setminus \{0, 1\}$ are *non-terminal* and has attributes *var*(v), and *low*(v), *high*$(v) \in V$.

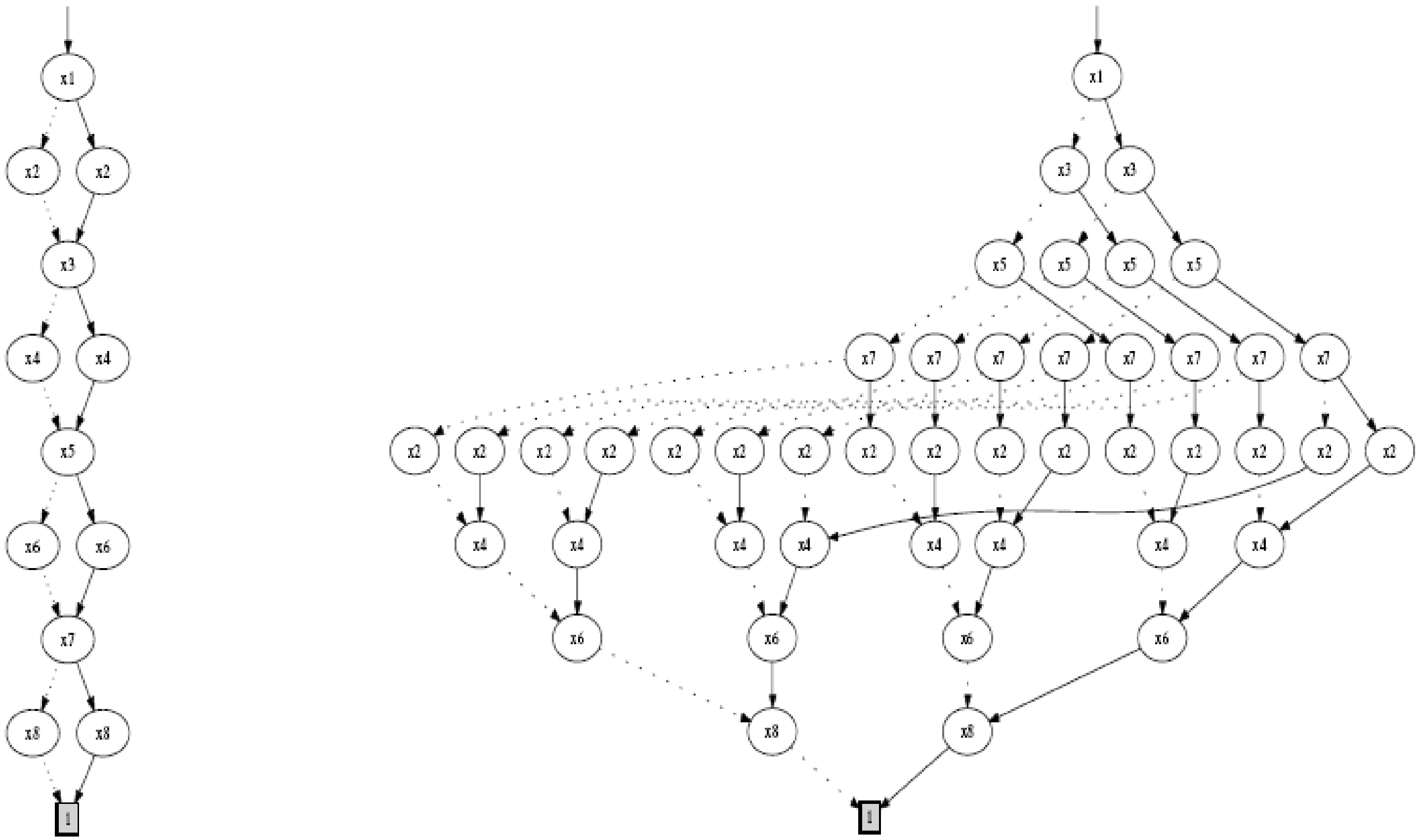
A BDD is *ordered* if on all paths from the root the variables respect a given total order.

A BDD is *reduced* if for all non-terminal vertices u, v,

1) *low*$(u) \neq$ *high*(u)
2) *low*$(u) =$ *low*(v), *high*$(u) =$ *high*(v), *var*$(u) =$ *var*(v) implies $u = v$

Ordering DOES matter

$$(x_1 \Leftrightarrow x_2) \wedge (x_3 \Leftrightarrow x_4) \wedge (x_5 \Leftrightarrow x_6) \wedge (x_7 \Leftrightarrow x_8)$$



$x_1 < x_2 < \cdots < x_8$ $\qquad$ $x_1 < x_3 < x_5 < x_7 < x_2 < x_4 < x_6 < x_8$

Canonicity of ROBDDs

$$
\begin{aligned}
t_0 &= 0 \\
t_1 &= 1 \\
t_u &= x \to t_h, t_l, \text{ if } u \text{ is a node } (x, l, h)
\end{aligned}
$$

Lemma 1 (Canonicity lemma) *For any function* $f : \mathbb{B}^n \to \mathbb{B}$ *there is exactly one ROBDD* b *with variables* $x_1 < x_2 < \ldots < x_n$ *such that*

$$t_b[v_1/x_1, \ldots, v_n/x_n] = f(v_1, \ldots, v_n)$$

for all $(v_1, \ldots, v_n) \in \mathbb{B}^n$.

Consequences: b is a tautology, if and only if, $b = \boxed{1}$
b is satisfiable, if and only if, $b \neq \boxed{0}$

Algorithms on ROBDDs

Array Implementation

Assume $x_1 < x_2 < x_3$.

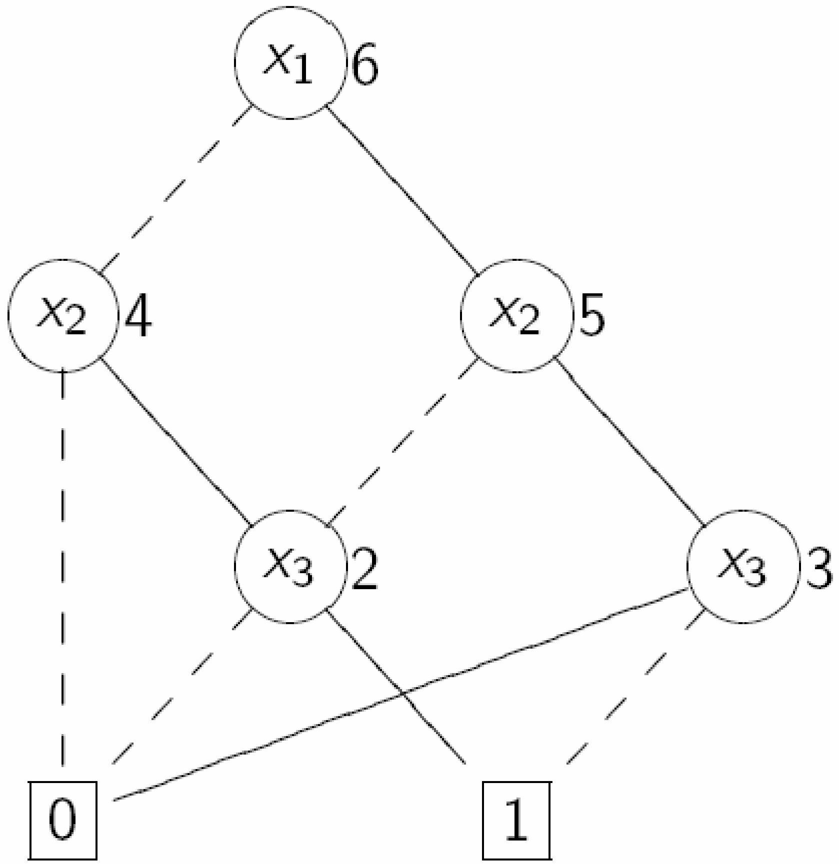


Table T :

$u \mapsto (var(u), low(u), high(u))$

u	*var*	*low*	*high*
0	4	-	-
1	4	-	-
2	3	0	1
3	3	1	0
4	2	0	2
5	2	2	3
6	1	4	5

Inverse table H :

$(var, low, high) \mapsto u.$

Example: $T(4) = (2, 0, 2)$, $H(1, 4, 5) = 6$, and $H(3, 0, 2) = undef$.

MakeNode

$T : u \mapsto (var(u), low(u), high(u)) \qquad H : (var, low, high) \mapsto u$

Makenode $(var, low, high)$: Node $=$

if $low = high$ **then**
  return low
else
  $u := H(var, low, high)$
  if $u \neq undef$ **then**
    return u
  else
    add a new node (row) to T with attributes $(var, low, high)$
    return $H(var, low, high)$
  end if
end if

Build

Let t be a boolean expression and $x_1 < x_2 < \cdots < x_n$.

Build$(t, 1)$ builds a corresponding ROBDD and returns its root.

Build(t, i): Node $=$
if $i > n$ **then**
  if t is true **then return** 0 **else return** 1
else
  low := Build$(t[0/x_i], i + 1)$
  $high$:= Build$(t[1/x_i], i + 1)$
  var := i
  return Makenode$(var, low, high)$
end if

Complexity ??

Boolean Operations on BDDs

Let us assume that ROBDDs for boolean expressions t_1 and t_2 are already constructed.

How to construct ROBDD for

- $\neg t_1$
- $t_1 \wedge t_2$
- $t_1 \vee t_2$
- $t_1 \Rightarrow t_2$
- $t_1 \Leftrightarrow t_2$

with an emphasis on efficiency?

Idea ($x_1 < x_2 < x_3 < \cdots x_n$)

- $x_i = x_i$

$$(x_i \rightarrow t_1, t_2) \wedge (x_i \rightarrow t'_1, t'_2)$$

$$\equiv$$

$$x_i \rightarrow (t_1 \wedge t'_1), (t_2 \wedge t'_2)$$

- $x_i < x_j$

$$(x_i \rightarrow t_1, t_2) \wedge (x_j \rightarrow t'_1, t'_2)$$

$$\equiv$$

$$x_i \rightarrow (t_1 \wedge (x_j \rightarrow t'_1, t'_2)), (t_2 \wedge (x_j \rightarrow t'_1, t'_2))$$

The same equivalences hold also for $\vee$, $\Rightarrow$ and $\Leftrightarrow$.

APPLY operation

Apply(op, b_1, b_2)

function $app(u_1, u_2) =$

  if $u_1 \in \{0,1\}$ **and** $u_2 \in \{0,1\}$ **then** $res \leftarrow op(u_1, u_2)$

  else if $u_1 \in \{0,1\}$ **and** $u_2 \geq 2$ **then**

    $res \leftarrow \mathrm{makenode}(var(u_2), app(u_1, low(u_2)), app(u_1, high(u_2)))$

  else if $u_1 \geq 2$ **and** $u_2 \in \{0,1\}$ **then**

    $res \leftarrow \mathrm{makenode}(var(u_1), app(low(u_1), u_2), app(high(u_1), u_2))$

  else if $var(u_1) = var(u_2)$ **then**

    $res \leftarrow \mathrm{makenode}(var(u_1), app(low(u_1), low(u_2)),$
      $app(high(u_1), high(u_2)))$

  else if $var(u_1) < var(u_2)$ **then**

    $res \leftarrow \mathrm{makenode}(var(u_1), app(low(u_1), u_2), app(high(u_1), u_2))$

  else (* $var(u_1) > var(u_2)$ *)

    $res \leftarrow \mathrm{makenode}(var(u_2), app(u_1, low(u_2)), app(u_1, high(u_2)))$

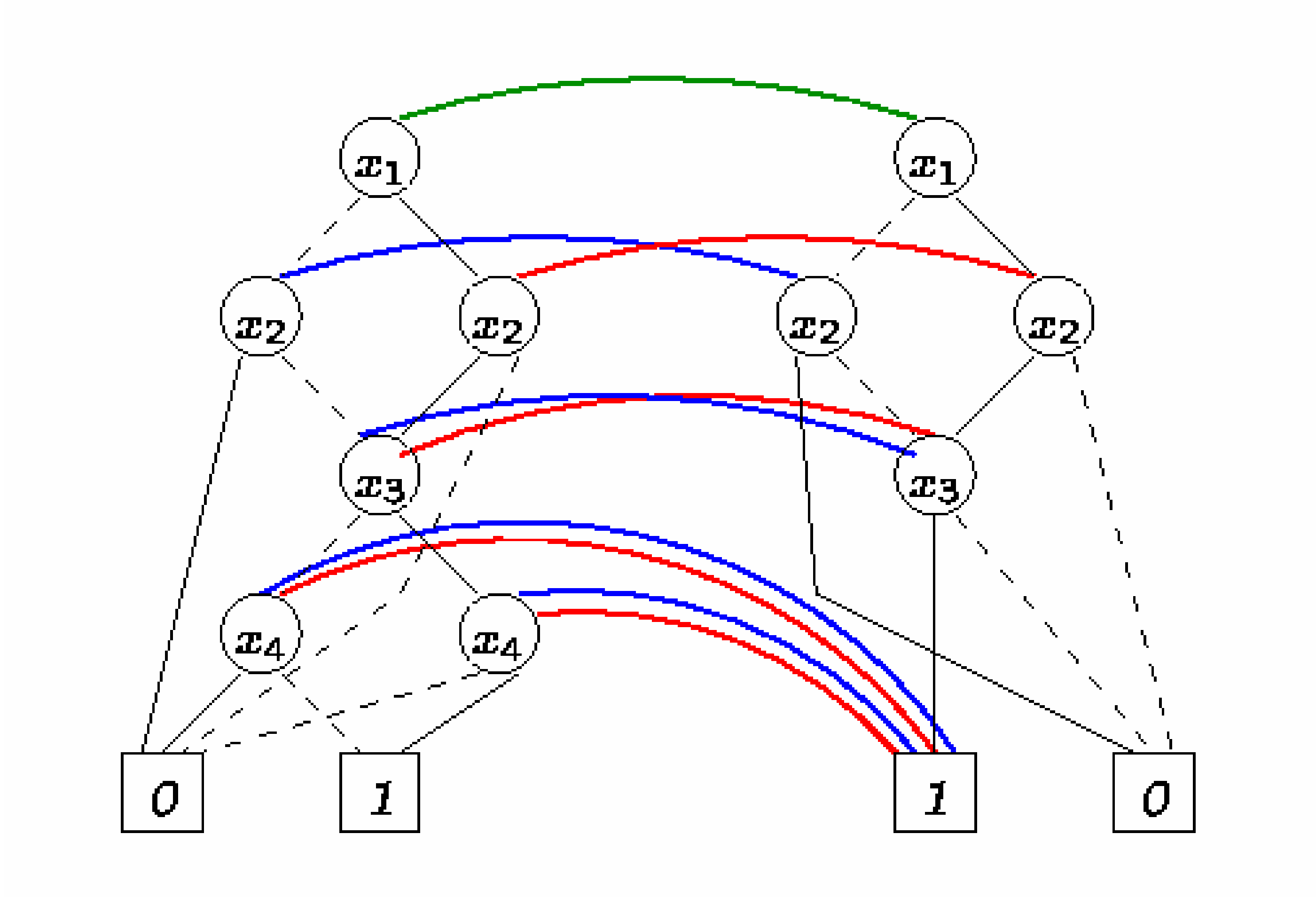
  return res

$b.root \leftarrow app(b_1.root, b_2.root)$

return b

Complexity ??

APPLY example



APPLY operation

with dynamic programming

```
Apply(op, b1, b2)
function app(u1, u2) =
    if G(u1, u2) ≠ empty then return G(u1, u2)
    else if u1 ∈ {0,1} and u2 ∈ {0,1} then res ← op(u1, u2)
        else if u1 ∈ {0,1} and u2 ≥ 2 then
            res ← makenode(var(u2), app(u1, low(u2)), app(u1, high(u2)))
        else if u1 ≥ 2 and u2 ∈ {0,1} then
            res ← makenode(var(u1), app(low(u1), u2), app(high(u1), u2))
        else if var(u1) = var(u2) then
            res ← makenode(var(u1), app(low(u1), low(u2)),
                                    app(high(u1), high(u2)))
        else if var(u1) < var(u2) then
            res ← makenode(var(u1), app(low(u1), u2), app(high(u1), u2))
        else (* var(u1) > var(u2) *)
            res ← makenode(var(u2), app(u1, low(u2)), app(u1, high(u2)))
        G(u1, u2) ← res
        return res

forall i ≤ max(b1), j ≤ max(b2) :  G(i, j) ← empty
b.root ← app(b1.root, b2.root)
return b
```

Complexity $O(|b_1||b_2|)$

Other operations

Let t be a boolean expression with its ROBDD representation.

The following operations can be done efficiently:

- Restriction $t[0/x_i]$ ($t[1/x_i]$): restricts the variable x_i to 0 (1)
- SatCount(t): returns the number of satisfying assignments
- AnySat(t): returns some satisfying assignment
- AllSat(t): returns all satisfying assignments
- Existential quantification $\exists x_i.t$: equivalent to $t[0/x_i] \vee t[1/x_i]$
- Composition $t[t'/x_i]$: equivalent to $t' \rightarrow t[1/x_i], t[0/x_i]$

Application of ROBDDs

Constraint Solving & Analysis
& IBEN

Mia’s Schedule 4th Grade

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
8-9	mat	eng	dan	tys	eng		
9-10	mat	tys	dan	geo	tys		
10-11	eng	dan	tys	dan	tys		
11-12	dan	dan	bio	mat	gym		
12-13	gym	fys	fys	fys	gym	gym	
13-14			bio	geo			
14-15			bio				

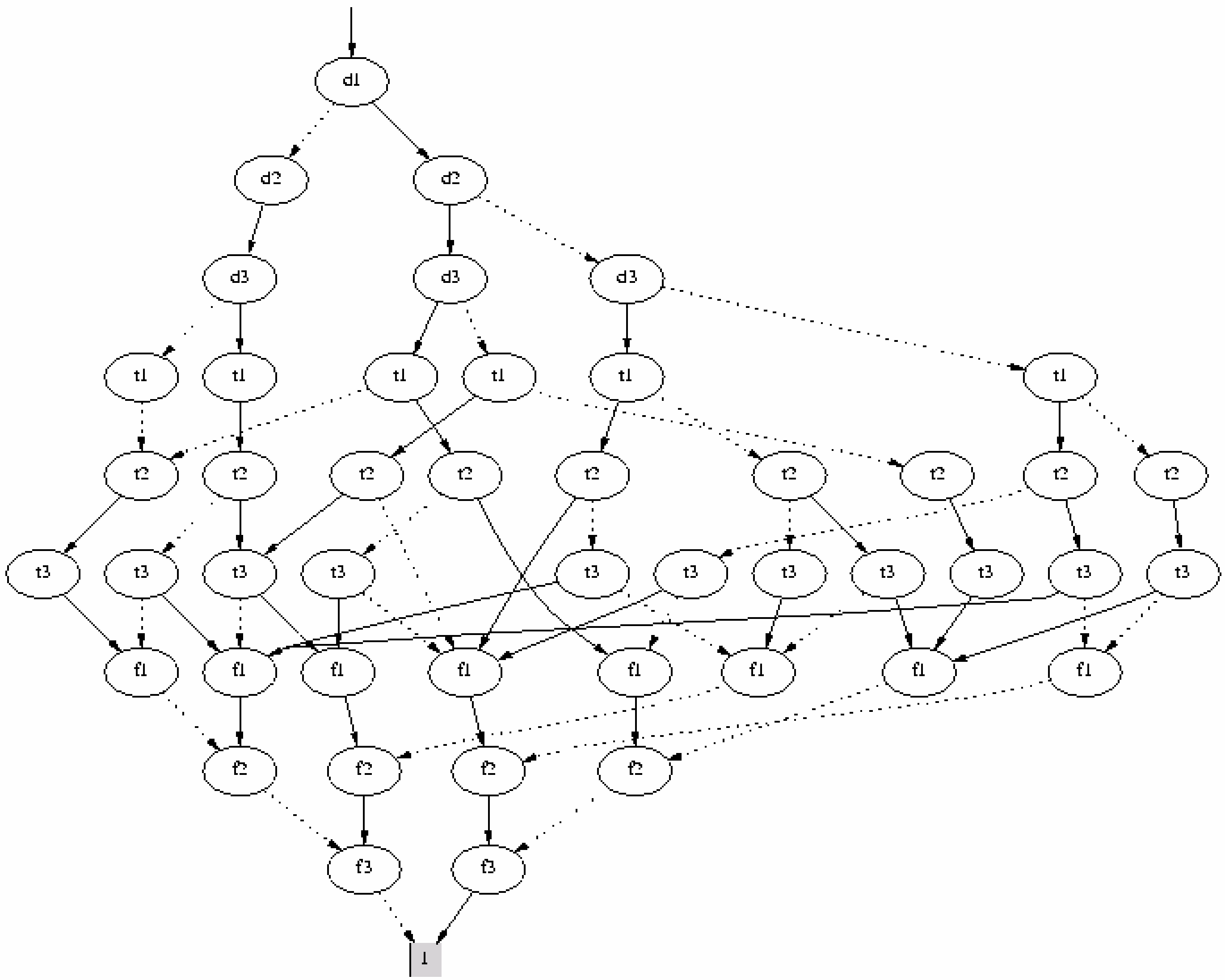
```
Boolean variables
=================
vars d1 d2 d3;
vars t1 t2 t3;
vars f1 f2 f3;
vars e1 e2 e3;
```

```
--Encodning of days –
=================
man := d1  & d2  & d3;
tir := d1  & d2  & !d3;
ons := d1  & !d2 & d3;
tor := d1  & !d2 & !d3;
fre := !d1 & d2  & d3;
lor := !d1 & d2  & !d3;
xxx := !d1 & !d2  & d3;
son := !d1 & !d2 & !d3;

uge := man + tir + ons + tor + fre;
weekend := lor + xxx  + son;
```

```
--Encodning of hours—
==================
h1 := t1 & t2 & t3;
h2 := t1 & t2 & !t3;
h3 := t1 & !t2 & t3;
h4 := t1 & !t2 & !t3;
h5 := !t1 & t2 & t3;
h6 := !t1 & t2 & !t3;
h7 := !t1 & !t2 & t3;
h8 := !t1 & !t2 & !t3;

formiddag := h1 + h2 + h3 + h4;
eftermiddag := ! formiddag;
```

```
--Encodning of topic
--==================
dan := f1 & f2 & f3;
eng := f1 & f2 & !f3;
mat := f1 & !f2 & f3;
tys := f1 & !f2 & !f3;
geo := !f1 & f2 & f3;
bio := !f1 & f2 & !f3;
fys := !f1 & !f2 & f3;
gym := !f1 & !f2 & !f3;
```

```
--Mia's Schedule —
==================
skema := man & h1 & mat +
         man & h2 & mat +
         man & h3 & eng  +
         man & h4 & dan +
         man & h5 & gym +
         tir & h1 & eng +
         tir & h2 & tys +
         tir & h3 & dan +
         tir & h4 & dan +
         tir & h5 & fys +
         ons & h1 & dan +
         ons & h2 & dan +
         ons & h3 & tys +
         ons & h4 & bio  +
         ons & h5 & fys +
         ons & h6 & bio +
         ons & h7 & bio +
         tor & h1 & tys +
         tor & h2 & geo +
         tor & h3 & dan +
         tor & h4 & mat +
         tor & h5 & fys +
         tor & h6 & geo +
         fre & h1 & eng +
         fre & h2 & tys +
         fre & h3 & tys +
         fre & h4 & gym +
         lor & h5 & gym;
```


d1
d2
d2
d3
d3
d3
t1
t1
t1
t1
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t1
t2
t2
t2
t2
t2
t2
t2
t2
t2
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t3
t3
t3
t3
t3
t3
f1
f1
f1
f1
f1
f1
f1
f1
f2
f2
f2
f2
f3
f3
1

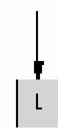
```
--Various questions –
======================
q1 := (skema & mat) => formiddag;

q2 := (skema & fys) => eftermiddag;

q3 := (skema & dan) => (man + tir + ons);

q4 := (skema & gym) => uge;

konfliktfri :=
  (( skema & (subst [e1/f1 e2/f2 e3/f3] (skema))) =>
        ((e1=f1) & (e2=f2) & (e3=f3)));
```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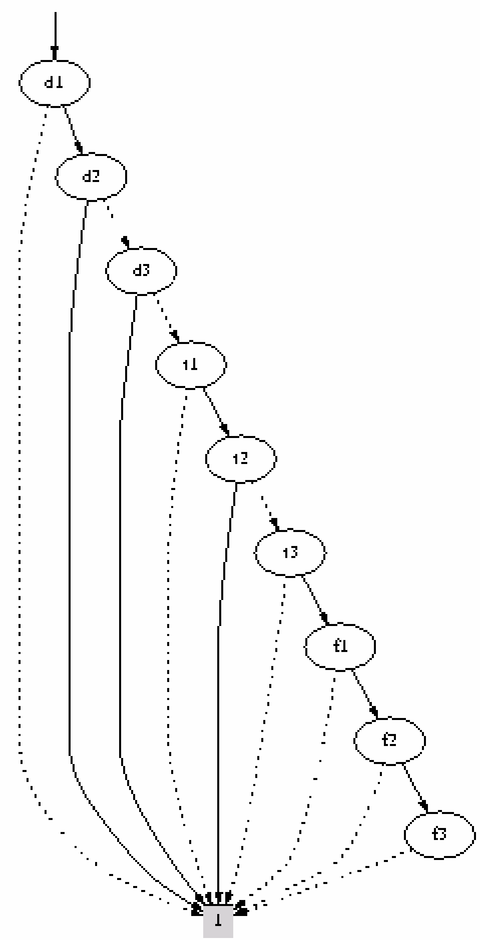
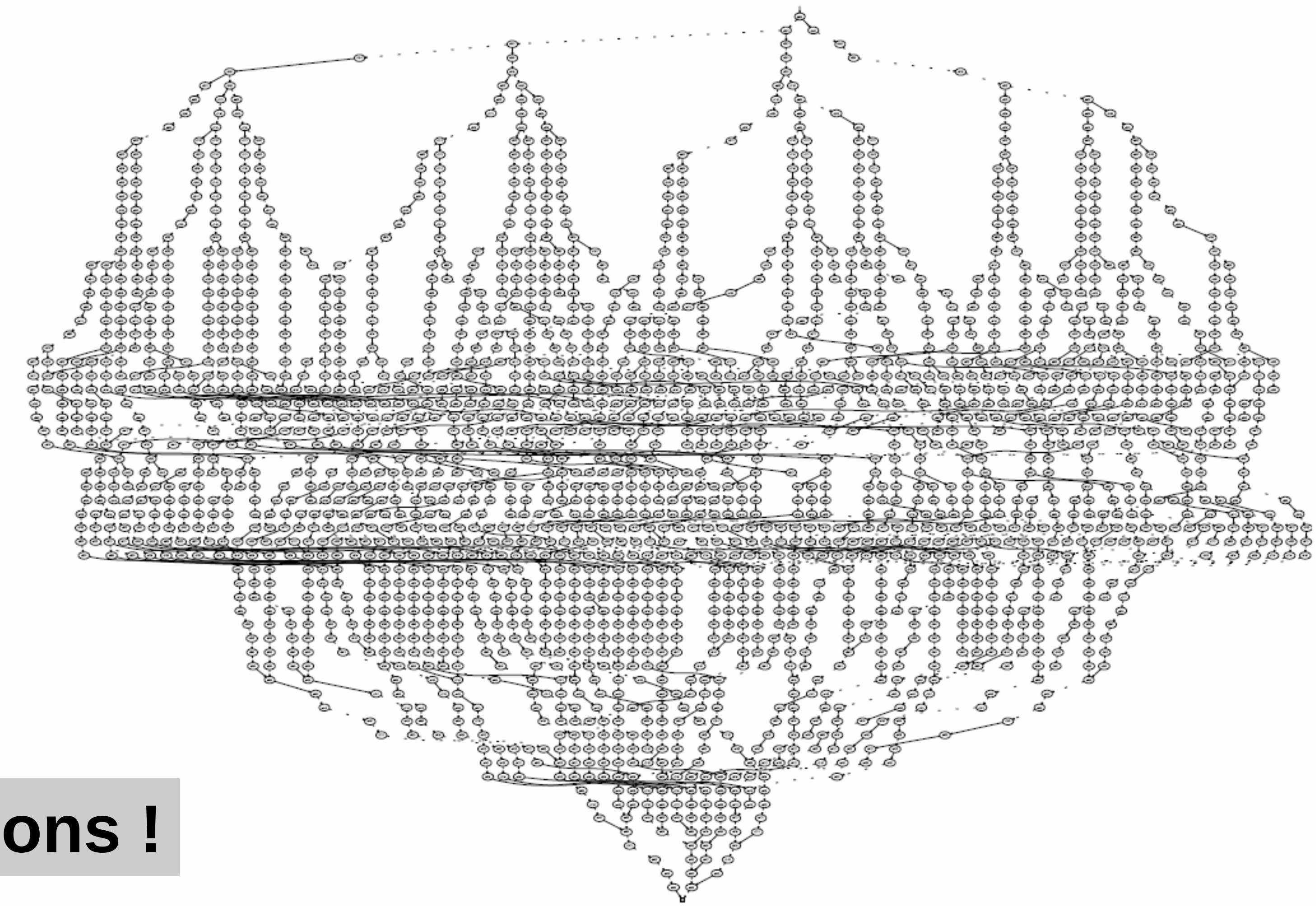

d1
d2
d3
t1
t2
t3
f1
f2
f3
1

Constraint Solving & Analysis & IBEN

1			
	2		
		3	
			4

4 x 4 Sudoku



288 solutions !

Encoding

	1	2	3	4
1	**1**			
2		**2**		
3			**3**	
4				**4**

$x_{2,2,2} = 1$
$x_{4,4,4} = 1$
$x_{2,2,1} = 0$

Boolean variables $x_{i,j,k}$ for all
$i, j, k \in \{1,2,3,4\}$.

Idea:

$x_{i,j,k} = 1$; if the number k is in position (i,j) in the solution
0 ; otherwise

	1	2	3	4
1	**1**			
2		**2**		
3			**3**	
4				**4**

Constraints

Precisely one value in **each position** i, j:

$x_{1,j,1} + x_{i,j,2} + x_{i,j,3} + x_{i,j,4} = 1$ for each i, j

Each value k appears in **each row** i exactly ones:

$x_{i,1,k} + x_{i,2,k} + x_{i,3,k} + x_{i,4,k} = 1$ for each i, k

Each value k appears in **each colomn** j exactly ones:

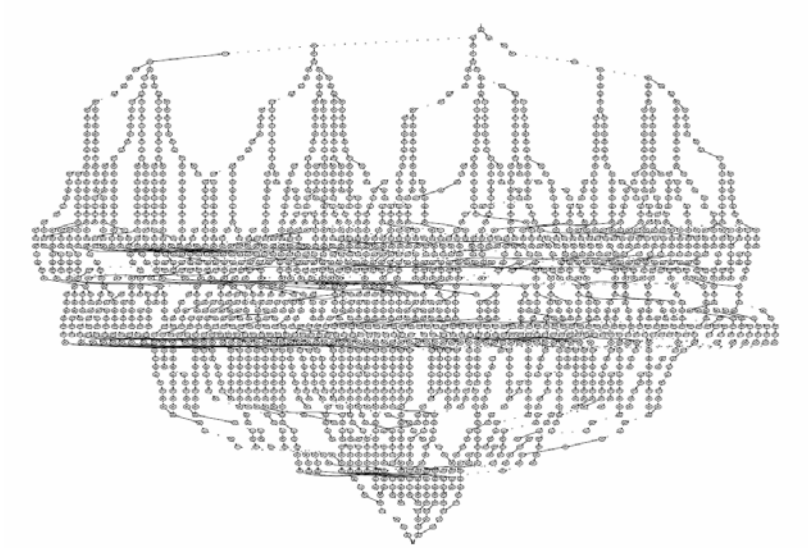
$x_{1,j,k} + x_{2,j,k} + x_{3,j,k} + x_{4,j,k} = 1$ for each j, k

Each value k appears in **each 2x2 box** exactly ones:

$x_{1,1,k} + x_{1,2,k} + x_{2,1,k} + x_{2,2,k} = 1$ (e.g.)

Solving Sudoku

	1	2	3	4
1				
2				
3				
4				



	1	2	3	4
1	**1**			
2		**2**		
3			**3**	
4				**4**