

1、

解：

$$\begin{aligned}(1) f(x) &= \frac{1 - \cos x + \sin x}{1 + \cos x + \sin x} \\&= \frac{\sin^2 \frac{1}{2}x + \cos^2 \frac{1}{2}x - (\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x) + \frac{1}{2} \sin \frac{1}{2}x \cos \frac{1}{2}x}{\sin^2 \frac{1}{2}x + \cos^2 \frac{1}{2}x + (\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x) + \frac{1}{2} \sin \frac{1}{2}x \cos \frac{1}{2}x} = \tan \frac{1}{2}x\end{aligned}$$

$$\because 1 + \cos x + \sin x \neq 0$$

$$\therefore \sin(x + \frac{\pi}{4}) \neq -\frac{\sqrt{2}}{2}$$

$$\therefore x \neq -\frac{\pi}{2} + 2k\pi, x \neq \pi + 2k\pi,$$

$\therefore f(x)$  定义域不关于  $x$  轴对称

$\therefore f(x)$  为非奇非偶函数

$$(2) f(x) = \tan \frac{1}{2}x$$

$$\therefore T = 2\pi$$

2、

$$(1) S_{\triangle BCD} = \frac{\sqrt{3}}{4} \cdot 36^2 = 324\sqrt{3}, h = \sqrt{36^2 - \left(\frac{36}{\sqrt{3}}\right)^2} = \frac{36\sqrt{6}}{3} = 12\sqrt{6}$$

$$\therefore V = \frac{1}{3} S_{\triangle BCD} h = 3888\sqrt{2}$$

(2) 过C点作底面ABD的垂线交底面于H,

以H为坐标原点, HB为x轴, HC为z轴,

过H作y轴分别垂直于x, z轴建立坐标系

记内切球球心为O, 则O(0,0,3√6)

设M(x, y, z)

$$x^2 + y^2 + (z - 3\sqrt{6})^2 = 54$$

$$C(0,0,12\sqrt{6}), B(12\sqrt{3},0,0)$$

$$|MB| + \frac{1}{3}|MC| = \sqrt{(x-12\sqrt{3})^2 + y^2 + z^2} + \frac{1}{3}\sqrt{x^2 + y^2 + (z-12\sqrt{6})^2}$$

$$\therefore \frac{1}{3}\sqrt{x^2 + y^2 + (z-12\sqrt{6})^2} = \frac{1}{3}\sqrt{x^2 + y^2 + (z-3\sqrt{6}-9\sqrt{6})^2}$$

$$= \frac{1}{3}\sqrt{x^2 + y^2 + (z-3\sqrt{6})^2 - 18\sqrt{6}(z-3\sqrt{6}) + (9\sqrt{6})^2}$$

$$= \frac{1}{3}\sqrt{x^2 + y^2 + (z-3\sqrt{6})^2 - 18\sqrt{6}(z-3\sqrt{6}) + 8(x^2 + y^2 + (z-3\sqrt{6})^2) + 54}$$

$$= \frac{1}{3}\sqrt{9(x^2 + y^2 + (z-3\sqrt{6})^2) - 18\sqrt{6}(z-3\sqrt{6}) + 54} = \sqrt{x^2 + y^2 + (z-3\sqrt{6})^2 - 2\sqrt{6}(z-3\sqrt{6}) + 6}$$

$$= \sqrt{x^2 + y^2 + (z-4\sqrt{6})^2}$$

$$\therefore |MB| + \frac{1}{3}|MC| = \sqrt{(x-12\sqrt{3})^2 + y^2 + z^2} + \sqrt{x^2 + y^2 + (z-4\sqrt{6})^2} \geq \sqrt{(12\sqrt{3})^2 + (4\sqrt{6})^2} = 4\sqrt{33}$$

3、

$$(1) \text{证明: } a_{n+1} = a_n^2 - 2a_n + 2 = (a_n - 1)^2 + 1$$

$$\because a_1 = \frac{3}{2} > 1 \therefore \text{迭代可得 } a_n > 1, n \in N^*$$

$$\text{同理 } a_n < 2, n \in N^* \text{ 即 } 1 < a_n < 2, n \in N^*$$

$$\therefore a_{n+1} - a_n = (a_n - 1)(a_n - 2) < 0$$

$$\therefore 1 < a_{n+1} < a_n < 2$$

(2)

$$a_{k+1} = (a_k - 1)^2 + 1 > 1$$

$$\text{当 } k \geq 2 \text{ 时, } \frac{6}{2^k + 3} < 1 < a_{k+1}$$

$$\text{当 } k = 1 \text{ 时, } a_2 = \frac{5}{4} > \frac{6}{2^{2-1} + 3} = \frac{6}{5}$$

$$a_1 = \frac{3}{2} = \frac{6}{2^{1-1} + 3}$$

$$\therefore a_n \geq \frac{6}{2^{n-1} + 3}, n \in N^*$$

对右边不等式进行归纳证明如下:

$$(i) \text{当 } n = 1 \text{ 时, } a_1 = \frac{3}{2} = \frac{2^{1-1} + 2}{2^{1-1} + 1} \text{ 即命题对 } n = 1 \text{ 成立}$$

$$(ii) \text{假设当 } n = k \text{ 时命题成立即 } a_k \leq \frac{2^{k-1} + 2}{2^{k-1} + 1} = 1 + \frac{1}{2^{k-1} + 1}$$

$$a_{k+1} = (a_k - 1)^2 + 1 \leq \frac{1}{(2^{k-1} + 1)^2} + 1 = \frac{1}{2^{2k-2} + 2^k + 1} + 1 < \frac{1}{2^k + 1} + 1 = \frac{2^k + 2}{2^k + 1}$$

即命题对  $n = k + 1$  也成立

由(i)(ii)可知命题对  $n \in N^*$  成立

$$\therefore \frac{6}{2^{n-1} + 3} \leq a_n \leq \frac{2^{n-1} + 2}{2^{n-1} + 1}$$

(3)证明:

$$\because a_n > 1 \therefore S_n > n \because a_n \leq \frac{2^{n-1} + 2}{2^{n-1} + 1} < 1 + \frac{1}{2^{n-1}}$$

$$\therefore S_n < n + \sum_{i=1}^n \frac{1}{2^{i-1}} < n + 2 \therefore n < S_n < n + 2$$

$$(4) a_{n+1} - 1 = (a_n - 1)^2$$

$$\because a_n > 1$$

$$\therefore \ln(a_{n+1} - 1) = 2 \ln(a_n - 1)$$

$$\therefore \ln(a_n - 1) = 2^{n-1} \ln(a_1 - 1) = \ln\left(\frac{1}{2}\right)^{2^{n-1}}$$

$$\therefore a_n - 1 = \left(\frac{1}{2}\right)^{2^{n-1}} \text{ 即 } a_n = 1 + \left(\frac{1}{2}\right)^{2^{n-1}}$$

4、

$$(1) \because e = \frac{\sqrt{6}}{3}$$

$$\therefore \frac{c}{a} = \frac{\sqrt{6}}{3} \text{ 即 } \frac{c^2}{a^2} = \frac{2}{3}$$

$$\therefore c^2 = 2b^2, a^2 = 3b^2$$

$$\text{令 } l: y = kx + m$$

$$x^2 + 3(kx + m)^2 = 3b^2$$

$$\therefore (3k^2 + 1)x^2 + 6kmx + 3m^2 - 3b^2 = 0$$

$$\therefore x_A + x_B = -\frac{6km}{3k^2 + 1}$$

$$\therefore y_A + y_B = -\frac{6km}{3k^2 + 1} \cdot k + 2m = \frac{2m}{3k^2 + 1}$$

$$\therefore (x_A, y_A) + (x_B, y_B) = (x_A + x_B, y_A + y_B) = \left(-\frac{6km}{3k^2 + 1}, \frac{2m}{3k^2 + 1}\right) \text{ 与 } (3, -1) \text{ 共线}$$

$$\therefore k = 1$$

$$(2) l: y = x - c \text{ 即 } m = -c$$

设  $M(x_0, y_0)$ , 则

$$x_0 = x_A \sin \alpha + x_B \cos \alpha$$

$$y_0 = y_A \sin \alpha + y_B \cos \alpha$$

$$\text{由(1)得} \begin{cases} x_A + x_B = \frac{3}{2}c \\ x_A x_B = \frac{3}{4}b^2 \end{cases}$$

$$\text{而 } y_A y_B = (x_A - c)(x_B - c) = \frac{3}{4}b^2 - \frac{3}{2}c^2 + c^2 = -\frac{1}{4}b^2$$

$$\therefore x_A x_B + 3y_A y_B = 0$$

$$\therefore \begin{cases} \frac{x_A^2}{3b^2} + \frac{y_A^2}{b^2} = 1 \\ \frac{x_B^2}{3b^2} + \frac{y_B^2}{b^2} = 1 \end{cases}$$

$$\begin{aligned} \therefore \frac{x_0^2}{3b^2} + \frac{y_0^2}{b^2} &= \frac{(x_A \sin \alpha + x_B \cos \alpha)^2}{3b^2} + \frac{3(y_A \sin \alpha + y_B \cos \alpha)^2}{3b^2} \\ &= \frac{(x_A^2 + 3y_A^2) \sin^2 \alpha + (x_B^2 + 3y_B^2) \cos^2 \alpha + 2(x_A x_B + 3y_A y_B) \sin \alpha \cos \alpha}{3b^2} \\ &= \frac{3b^2(\sin^2 \alpha + \cos^2 \alpha)}{3b^2} = 1 \end{aligned}$$

$\therefore M$  在椭圆上

5、

$$(1) f(x) = \ln(x+1) - x, (x > 0)$$

$$f'(x) = -\frac{x}{x+1} < 0 \therefore f(x) \text{ 单调递减而 } f(0) = 0 \therefore -m \leq 0 \text{ 即 } m \geq 0$$

$$(2) \text{ 令 } F(x) = \ln(x+1) - x + \frac{1}{2}x^2, x > 0$$

$$F'(x) = \frac{1}{x+1} - 1 + x = \frac{x^2}{x+1} > 0$$

$$\therefore F(x) \text{ 单调递增而 } F(0) = 0 \therefore \ln(x+1) - x > -\frac{1}{2}x^2$$

$$\text{而 } -\frac{1}{2}(\sqrt{2m})^2 + m = 0 \therefore x_1 > \sqrt{2m}$$

$$(3) \text{ 证明: } g(x) = \ln(x+1) - \frac{1}{2}x, (x > -1)$$

$$\therefore g(x-1) + g(x) = \ln(x+1)x - x + \frac{1}{2}, (x > 0)$$

$$\text{令 } h(x) = \ln(x+1)x - x + \frac{1}{2}, x > 0$$

$$h'(x) = \frac{2x+1}{(x+1)x} - 1 = \frac{-x^2+x+1}{x^2+x}$$

$$h'(1) > 0, h'(2) < 0, \text{ 令 } h'(x_0) = 0 \text{ 则 } 1 < x_0 < 2$$

$$\therefore h(x) \text{ 在 } (1, x_0) \text{ 上递增, } (x_0, 2) \text{ 上递减}$$

$$\text{而 } h(1) = \ln 2 - \frac{1}{2}, h(2) = \ln 6 - \frac{3}{2} > h(1)$$

$$\therefore \text{ 当 } x \in (1, 2) \text{ 时, } h(x)_{\min} > h(1) = \ln 2 - \frac{1}{2}$$

$$\therefore g(x-1) + g(x) > \ln 2 - \frac{1}{2}, 1 < x < 2$$

$$g(x) = \ln(x+1) - \frac{1}{2}x, g'(x) = \frac{1}{x+1} - \frac{1}{2}, g'(1) = 0$$

$$\therefore x_1 < 1 < x_2, x_3 < 1 < x_4, g(x) \text{ 在 } (0, 1) \text{ 单调递增, } (1, +\infty) \text{ 单调递减}$$

$$(i) \text{ 若 } x_2 > 2, \text{ 则 } x_2 - x_3 > 2 - 1 = 1$$

$$(ii) \text{ 若 } x_2 \leq 2 \text{ 则 } 1 < x_2 \leq 2$$

$$g(x_2-1) + g(x_2) > \ln 2 - \frac{1}{2} = g(x_3) + g(x_2)$$

$$\therefore g(x_2-1) > g(x_3)$$

$$\because x_2-1 < 1, x_3 < 1 \therefore x_2-1 > x_3 \text{ 即 } x_2 - x_3 > 1$$

$$\because b < a \therefore b < \frac{\ln 2 - \frac{1}{2}}{2} \text{ 而 } g(2) = \ln 3 - 1 > \frac{\ln 2 - \frac{1}{2}}{2} > b = g(x_4) \therefore x_4 > 2 \text{ 而 } x_1 > 1$$

$$\therefore x_4 - x_2 > 1 \therefore |x_4 - x_3| + |x_2 - x_1| = x_4 - x_3 + x_2 - x_1 = (x_4 - x_2) + (x_3 - x_1) > 1 + 1 = 2$$