1、

$$(1)f(x)=\sin^2\omega x+\sqrt{3}\sin\omega x\sin(\omega x+\frac{\pi}{2})$$

$$=\sin^2\omega x+\sqrt{3}\sin\omega x\cos\omega x=\frac{1-\cos 2\omega x}{2}+\frac{\sqrt{3}}{2}\sin 2\omega x$$

$$=\frac{1}{2}+\frac{\sqrt{3}}{2}\sin 2\omega x-\frac{1}{2}\cos 2\omega x=\frac{1}{2}\sin(2\omega x-\frac{\pi}{6})$$

$$\frac{2\pi}{2\omega}=\pi\therefore\omega=1$$

$$(2)f(x)=\frac{1}{2}\sin(2x-\frac{\pi}{6})$$

$$x\in[0,\frac{\pi}{3}]\therefore 2x-\frac{\pi}{6}\in[-\frac{\pi}{6},\frac{\pi}{2}]$$

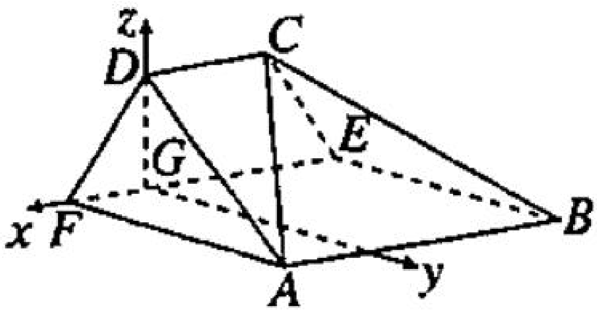
$$\therefore f(x)=\frac{1}{2}\sin(2\omega x-\frac{\pi}{6})\in[-\frac{1}{4},\frac{1}{2}]$$

2、

(1)证明: $AF\perp DF,AF\perp FE\Rightarrow AF\perp$ 平面 $EFDC$

$\because AF\subset$ 平面 $ABEF\therefore$ 平面 $ABEF\perp$ 平面 $EFDC$

(2)如图建立坐标系



由(1)得$\angle DFE$为二面角$D-AF-E$得平面角

即$\angle DFE=60^{\circ}$则$DF=2, DG=\sqrt{3}$

$\therefore A(1,4,0), B(-3,4,0), E(-3,0,0), D(0,0,\sqrt{3})$

$\because AB // EF \therefore AB //$平面$EFDC$

$\therefore \angle CEF$为二面角$C-BE-F$的平面角

$\therefore \angle CEF=60^{\circ} \therefore C(-2,0,\sqrt{3})$

连接AC，则$\overrightarrow{EC}=(1,0,\sqrt{3}), \overrightarrow{EB}=(0,4,0)$,

$\overrightarrow{AC}=(-3,-4,\sqrt{3}), \overrightarrow{AB}=(-4,0,0)$

设$\vec{n}=(x,y,z)$是平面BCE的法向量

则$\begin{cases}\vec{n}\cdot\overrightarrow{EC}=0\\ \vec{n}\cdot\overrightarrow{EB}=0\end{cases}$即$\begin{cases}x+\sqrt{3}z=0\\ 4y=0\end{cases}$取$\vec{n}=(3,0,-\sqrt{3})$

设$\vec{m}$是平面$ABCD$的法向量

则$\begin{cases}\vec{m}\cdot\overrightarrow{AC}=0\\ \vec{m}\cdot\overrightarrow{AB}=0\end{cases}$取$\vec{m}=(0,\sqrt{3},4)$

$\therefore \cos<\vec{n},\vec{m}>=\dfrac{\vec{n}\cdot\vec{m}}{|\vec{n}||\vec{m}|}=-\dfrac{2\sqrt{19}}{19}$

3、

(1)$f(x)=x^2-2x+2+\ln x$即$f'(x)=2x-2+\dfrac{1}{x}$

$\therefore f'(1)=1 \therefore y=x$

(2)$f'(x)=\dfrac{2x^2-2x+a}{x} \therefore 2{x_2}^2-2x_2+a=0$即$a=2x_2-2{x_2}^2$

$\because x_1, x_2>0$且$x_2>x_1$而$\begin{cases}x_1+x_2=1\\ x_1x_2=\dfrac{a}{2}\end{cases} \therefore \dfrac{1}{2}<x_2<1$

$\therefore f(x_2)={x_2}^2-2x_2+2+(2x_2-2{x_2}^2)\ln x_2$

$={x_2}^2-2x_2+2+2x_2\ln x_2-2{x_2}^2\ln x_2$

令$g(x)=x^2-2x+2+2x\ln x-2x^2\ln x, \dfrac{1}{2}<x<1$

$\therefore g'(x)=2x-2+2(1+\ln x)-2(x+2x\ln x)=(2-4x)\ln x$

$\because x\in(\dfrac{1}{2},1) \therefore g'(x)>0$

$\therefore g(x)>g(\dfrac{1}{2})=\dfrac{5-\ln 2}{4}$

4、

(1)略,结果与(2)一致

(2)$x_I=\dfrac{PF_1+F_1F_2-PF_2}{2}-c$

设$PF_1=2a-m,PF_2=m$

则$x_I=\dfrac{2a-m+2c-m}{2}=a+c-m-c=a-m$

$y_I=r$且$r(F_1F_2+PF_1+PF_2)=2S_{\Delta}PF_1F_2$

而$\cos\angle PF_1F_2=\dfrac{(2a-m)^2+4c^2-m^2}{2(2a-m)\cdot 2c}=\dfrac{a^2-am+c^2}{c(2a-m)}$

$F_1(-c,0),F_2(c,0),I(a+c-m,r)$

$k_{IF_1}\cdot k_{IF_2}=\dfrac{r}{a-m+c}\cdot\dfrac{r}{a-m-c}=\dfrac{r^2}{(a-m+c)(a-m-c)}$

$\cos^2\angle PF_1F_2=\dfrac{(a^2-am+c^2)^2}{c^2(2a-m)^2}$

$\therefore\sin^2\angle PF_1F_2=\dfrac{c^2(2a-m)^2-(a^2-am+c^2)^2}{c^2(2a-m)^2}$

$\because r(F_1F_2+PF_1+PF_2)=2S_{\Delta}PF_1F_2=PF_1\cdot F_1F_2\cdot\sin\angle PF_1F_2$

$\therefore(2a-m)^2(2c)^2\cdot\dfrac{c^2(2a-m)^2-(a^2-am+c^2)^2}{c^2(2a-m)^2}=r^2(2c+2a)^2$

$\therefore c^2(2a-m)^2-(a^2-am+c^2)^2=r^2(c+a)^2$

$\therefore(2ac-cm)^2-(a^2-am+c^2)^2=r^2(c+a)^2$

$\therefore r^2(c+a)^2=(2ac-cm+a^2-am+c^2)(2ac-cm-a^2+am-c^2)$

$\therefore-(a+c)(a+c-m)(a-c)(a-c-m)=r^2(c+a)^2$即$r^2=\dfrac{(a-c)(a+c-m)(a-c-m)}{a+c}$

$\therefore k_{IF_1}\cdot k_{IF_2}=\dfrac{r^2}{(a-m+c)(a-m-c)}=\dfrac{a-c}{a+c}$

5、

$(1)a_{n+1}=\begin{cases}a_n+n,若a_n\le n\\ a_n-n,若a_n>n\end{cases},n\in N^*$

$\therefore a_{n+1}\le a_n+n,n\in N^*$

$\therefore \sum_{i=2}^{n+1}a_i\le\sum_{i=1}^{n}a_i+\frac{n(n+1)}{2}$

$\therefore a_{n+1}\le 1+\frac{n(n+1)}{2}$

$\therefore a_n\le 1+\frac{n(n-1)}{2},n\ge 2且n\in N^*$

当$n=1$时,$a_1=1$符合上式$\therefore a_n\le 1+\frac{n(n-1)}{2},n\in N^*$

(2)

$a_1=1,a_2=2$

假设对某个整数$r\ge 2$有$a_r=r$,我们证明对$t=1,\ldots,r-1$有

$a_{r+2t-1}=2r+t-1>r+2t-1,a_{r+2t}=r-t<r+2t,(*)$

对t归纳证明:

(i)当$t=1$时，由于$a_r=r\le r$,由定义$a_{r+1}=2r>r+1$

$a_{r+2}=a_{r+1}-(r+1)=r-1<r+2$即当$t=1$时$(*)$成立

(ii)假设某个$1\le t<r-1$使得$(*)$成立,则由定义

$a_{r+2t+1}=a_{r+2t}+(r+2t)=r-t+r+2t=2r+t>r+2t+1$

$a_{r+2t+2}=a_{r+2t+1}-(r+2t+1)=2r+t-(r+2t+1)=r-t-1<r+2t+2$

由此$(*)$对$t+1$也成立

由$(i)(ii)$可知$(*)$对所有$t=1,2,\ldots,r-1$成立

特别地,当$t=r-1$时,$a_{3r-2}=1$，从而$a_{3r-1}=1+3r-2=3r-1$

若将所有满足$a_r=r$的正整数r从小到大记为$r_1,r_2,\ldots,r_m$则$r_1=1,r_2=2,r_{k+1}=3r_k-1,k\ge 2且k\in N^*$

由此可知$r_{k+1}-\frac{1}{2}=3(r_k-\frac{1}{2}),k=1,2,\ldots m-1$

从而$r_m=3^{m-1}\cdot(r_1-\frac{1}{2})+-\frac{1}{2}=\frac{3^{m-1}+1}{2}$

由于$r_{2022}=\frac{3^{2021}+1}{2}<3^{2021}<\frac{3^{2022}+1}{2}=r_{2023}$

$\therefore$在$1,2,\ldots,3^{2021}$中满足$a_r=r$的数r共有2022个

由$(*)$值每个$k=1,2,\ldots 2021$,有$r_{k+1},r_{k+2},\ldots,3r_k-2$中恰有一半满足$a_r<r$

对于任意$k\in N,(4k+3)^{2m-1}\equiv 3(\bmod 4),(4k+3)^{2m}\equiv 1(\bmod 4),m\in N^*$

$\therefore r_{2022}+1=\frac{3^{2021}+1}{2}+1$为奇数,而$3^{2021}$为奇数

在$r_{2022}+1,\ldots,3^{2021}$中，奇数均满足$a_r>r$,偶数均满足$a_r<r$

其中的偶数比奇数少一个，因此

满足$a_r<r\le 3^{2021}$的正整数个数为$\frac{1}{2}(3^{2021}-2022-1)=\frac{3^{2021}-2023}{2}$