Demand Estimation

Mixtape Session

Jeff Gortmaker and Ariel Pakes

Last Class

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- Let's go over your first coding exercise.

Unrealistic Substitution Patterns

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- Last week we derived the own-price elasticity. What about the cross-price one?

$$\eta_{jkt} = \frac{\partial \log q_{jt}}{\partial \log p_{kt}} = \frac{\partial q_{jt}}{\partial p_{kt}}\frac{p_{kt}}{q_{jt}} = \frac{\partial s_{jt}}{\partial p_{kt}}\frac{p_{kt}}{s_{jt}} = -\alpha \cdot p_{kt} \cdot s_{kt}$$

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- Doesn't depend on the characteristics of j!
  - $\rightarrow$ Independence of Irrelevant Alternatives (IIA) property.

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  - $\rightarrow$ In your exercise, consumers substituted *proportionally* from each cereal.

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- Introduce a second bus, but it's red. Pure logit (IIA) predicts 33% market shares.
  - → In your exercise, consumers substituted *proportionally* from each cereal.

- In reality, we'd expect the car to still have 50% and each bus to have 25%.
  - → In your exercise, we'd hope for more substitution from more similar cereals.

Roadmap

Preference Heterogeneity

Mixed Logit Estimation

Numerical Best Practices

Differentiation Instruments

Coding Exercise 2

Red Bus/Blue Bus Solution

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  - $\rightarrow$ When a new bus is introduced, this doesn't really affect the car-lovers' choice.

- Want μ_{ijt} to dominate logit substitution from convenient but unrealistic ε_{ijt}.
  - $\rightarrow$ Want to add multiple dimensions of heterogeneity that really matter in our setting.

Random Coefficients

$$u_{ijt} = x'_{jt}\beta + \xi_{jt} + \varepsilon_{ijt}$$

- How to add preference heterogeneity to our pure logit model?
  - $\rightarrow$ For simplicity, I'll just let x_{jt} denote all characteristics, including prices p_{jt}.

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- Intuitively, we want to replace β with *random coefficients* β_{it}.
    - $\rightarrow$ *Random* in that they're drawn from a distribution of consumer types $i \in \mathcal{I}_t$.
    - $\rightarrow$ For $x_{jt} = \text{car}_{jt}$ and $\mathcal{I}_t = \{\text{car-lovers}, \text{bus-lovers}\}$, want $\beta_{it} \gg 0$ for car-lovers.

Random Coefficients

$$u_{ijt} = x_{jt}' \underbrace{(\beta + \Pi y_{it} + \Sigma \nu_{it})}_{\beta_{it}} + \xi_{jt} + \varepsilon_{ijt}$$

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- Most common specification is $\beta_{it} \sim N(\beta + \Pi y_{it}, \Sigma\Sigma')$.
  - $\rightarrow$ Π shifts preferences according to "observed" demographics $y_{it} \sim$ census.
  - $\rightarrow$ Σ shifts preferences according to "unobserved" preferences $\nu_{it} \sim N(0, I)$.
  - $\rightarrow$ Σ is the *Cholesky root* of the variance matrix. Usually diagonal with standard deviations.

Random Coefficients

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Random Coefficients in Practice

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- In practice, we implement random coefficients by making a new dataset.
  - $\rightarrow$ In PyBLP lingo, "product data" rows are (j, t)'s, and new "agent data" rows are (i, t)'s.

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- In your coding exercise, you'll just draw $|\mathcal{I}_t| = 100$ types per market.
  - $\rightarrow$ Draw $\nu_{it} \sim N(0, I)$ from a random number generator.
  - $\rightarrow$ Draw y_{it} from census data on demographics: income, etc.
  - $\rightarrow$ Each type is equally-likely, so use equal sampling weights $w_{it} = 1/|\mathcal{I}_t|$.

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- The goal is to have a dataset that reflects the *distribution* of individuals.
  - $\rightarrow$ Realism aside, this allows us to address distributional concerns.
  - $\rightarrow$ E.g. will a tax or price change affect high- or low-income individuals differently?

Roadmap

From Linear Regression to GMM

$$\log \frac{s_{jt}}{s_{0t}} = \delta_{jt} = x'_{jt}\beta + \xi_{jt}$$

- In your exercise, you estimated β by running the above regression.
    - $\rightarrow$ Again, let x_{jt} include price, a constant, any other characteristics.
    - $\rightarrow$ Let z_{jt} include our price IV and exogenous characteristics in x_{jt}.

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- We'd get the exact same $\hat{\beta}$ by optimizing the following GMM objective:

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}}\, g(\beta)Wg(\beta)' \quad \text{where} \quad g(\beta) = \frac{1}{N}\sum_{t \in \mathcal{T}}\sum_{j \in \mathcal{J}_t}(\delta_{jt} - x'_{jt}\beta) \cdot z_{jt}$$

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$$s_{jt} = \sum_{i \in \mathcal{I}_t} w_{it} \cdot \frac{\exp[\delta_{jt} + \mu_{ijt}(\Sigma, \Pi)]}{1 + \sum_{k \in \mathcal{J}_t} \exp[\delta_{kt} + \mu_{ikt}(\Sigma, \Pi)]} \quad \text{for all} \quad j \in \mathcal{J}_t$$

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- Many ways to solve and speed up BLP's (1995) contraction.
    - $\rightarrow$ PyBLP will take care of this, but see Conlon and Gortmaker (2020) if interested.

The BLP Estimator

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}}\, g(\theta) W g(\theta)' \quad \text{where} \quad g(\theta) = \frac{1}{N} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}_t} (\delta_{jt}(\Sigma, \Pi) - x'_{jt}\beta) \cdot z_{jt}$$

- BLP estimation consists of two nested loops.
    1. In the "outer" loop, we optimize over $\theta = (\beta, \Sigma, \Pi)$.
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- Actually, since $g(\theta)$ is linear in x_{jt}, we can "concentrate out" β and optimize (Σ, Π).
    - $\rightarrow$ Get $\hat{\beta}$ by running an IV regression of $\delta_{jt}(\Sigma, \Pi)$ on x_{jt}, like in the pure logit exercise.

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- What about the GMM weighting matrix W?
    - $\rightarrow$ If you're just-identified ($\dim z_{jt} = \dim \theta$), it doesn't matter. You'll get a zero objective.
    - $\rightarrow$ Otherwise, you may want to repeat optimization with optimal the two-step GMM $\hat{W}$.
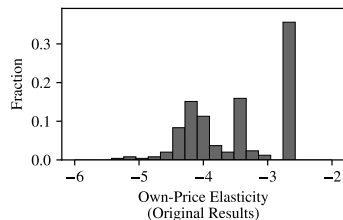
Roadmap

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Mixed Logit Estimation
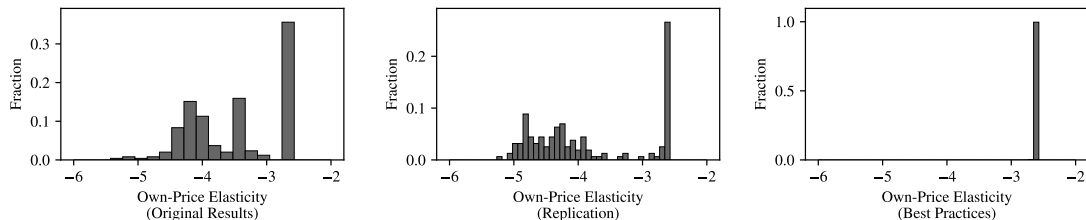
Numerical Best Practices

Differentiation Instruments

Coding Exercise 2

Motivation for Numerical Best Practices

Own-Price Elasticity
(Original Results)

- Variation in BLP estimates across different optimization algorithms and starting values has disillusioned some researchers (Knittel and Metaxoglou, 2014).

Motivation for Numerical Best Practices

- Variation in BLP estimates across different optimization algorithms and starting values has disillusioned some researchers (Knittel and Metaxoglou, 2014).

- But there are some numerical best practices that you can follow to avoid these kinds of issues (Conlon and Gortmaker, 2020).
    - → They're likely to be useful for most computation-heavy structural estimation, not just BLP!

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- Set box constraints $\theta \in [\underline{\theta}, \overline{\theta}]$ to preclude unrealistic and unstable guesses of θ.
    - $\rightarrow$ E.g. huge Σ values can make the inner loop unstable.
    - $\rightarrow$ Economic intuition and initial estimates will give a sense for reasonable bounds.

Nonlinear Optimization

$$\hat{\theta} = \operatorname*{argmin}_{\theta} Q(\theta)$$

- Set box constraints $\theta \in [\underline{\theta}, \overline{\theta}]$ to preclude unrealistic and unstable guesses of θ.

- Check that 3-5 different starting values $\theta \sim U(\underline{\theta}, \overline{\theta})$ give the same $\hat{\theta}$.
    - $\rightarrow$ For 2-step GMM, do this twice, once for each step (6-10 jobs total).
    - $\rightarrow$ If you have access to a cluster, each can be a separate job, run in parallel.

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- Prefer using gradient-based algorithms for "smooth" problems like BLP.
  - $\rightarrow$ Avoid derivative-free methods like Nelder-Mead/simplex, which tend to work worse.
  - $\rightarrow$ I prefer trust-region algorithms, e.g. SciPy's `trust-constr` or Knitro if you have it.

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- Try to terminate on strict first-order conditions, e.g. $\|\text{gradient}\|_\infty < \texttt{1e-8}$.
    - $\rightarrow$ Inner loop should be tighter to prevent error "bubbling up." PyBLP default is very tight.
    - $\rightarrow$ Can also check second-order conditions, i.e. Hessian eigenvalues are positive.

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- Try to terminate on strict first-order conditions, e.g. $\|\text{gradient}\|_\infty < \texttt{1e-8}$.

- Configure your optimizer! Defaults may not work for your setting.

Numerical Integration

$$s_{jt} = \sum_{i \in \mathcal{I}_t} w_{it} \cdot \frac{\exp(\delta_{jt} + \mu_{ijt})}{1 + \sum_{k \in \mathcal{J}_t} \exp(\delta_{kt} + \mu_{ikt})}$$

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- Sometimes there are only a few types that we can integrate exactly.
  - $\rightarrow$ E.g. high- and low-income types $i \in \{1, 2\}$ with known shares w_{1t} and $w_{2t} = 1 - w_{1t}$.

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- Sometimes there are only a few types that we can integrate exactly.

- But usually we approximate the distribution with Monte Carlo integration.
    - $\rightarrow$ Use a random number generator (RNG) to draw $|\mathcal{I}_t| \approx 1{,}000$ of (ν_{it}, y_{it})'s per market.
    - $\rightarrow$ Even better than your default RNG are quasi-Monte Carlo sequences.
    - $\rightarrow$ I recommend scrambled Halton sequences. R: Owen (2017). Python: SciPy or PyBLP.

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- If you just need a few $\nu_{it} \sim N(0, I)$'s, try out Gauss-Hermite quadrature.
    - $\rightarrow$ 10-100$\times$ fewer carefully-chosen (w_{it}, ν_{it})'s that do just as well as Monte Carlo.
    - $\rightarrow$ Chosen to exactly integrate a polynomial expansion of the integrand.
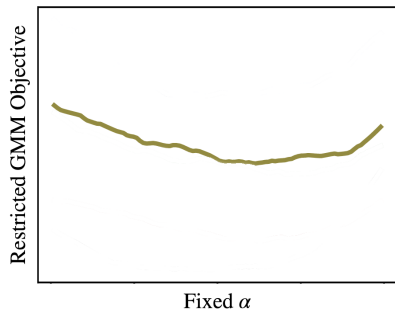
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- Keep increasing $|\mathcal{I}_t|$ until your estimates stabilize across draws/starting values.
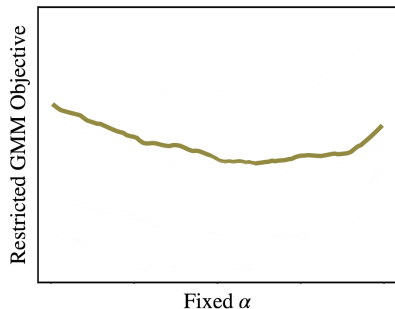
What Typically Goes Wrong

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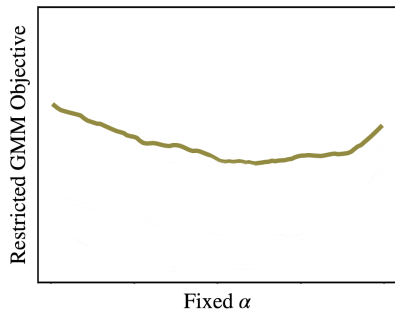- Can see how $Q(\theta) = g(\theta)Wg(\theta)'$ varies with θ.

Restricted GMM Objective

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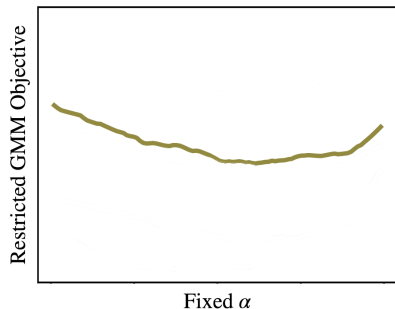

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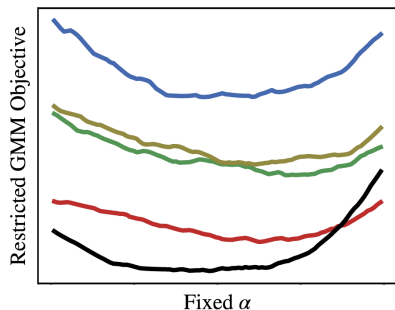
(y-axis: Restricted GMM Objective)

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  - $\rightarrow$ Even if they're all valid, some may be weaker.
  - $\rightarrow$ Weaker means flatter and harder to optimize.
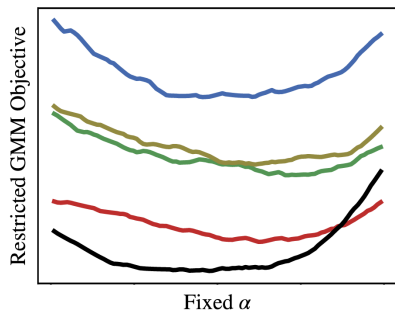
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Roadmap

Adding Instruments

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- Later, adding more can help with weakness and testing exclusion restrictions.

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- Let's use our stronger intuition about linear regression to think about instruments!

Linear Regression Intuition

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- If we set $\sigma = \pi = 0$ like on day 1, we get our familiar pure logit regression.
  - $\rightarrow$ Use the same IV as before to target β: if $x_{jt} = p_{jt}$, a price IV; if exogenous, x_{jt} itself.

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$$\log \frac{s_{jt}}{s_{0t}} \approx \beta x_{jt} + \sigma^2 d_{jt}^x + \xi_{jt} \quad \text{where} \quad d_{jt}^x = \Big(\frac{x_{jt}}{2} - \sum_{k \in \mathcal{J}_t} s_{kt} x_{kt}\Big) x_{jt}$$

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    - $\rightarrow$ Can technically identify π from higher-order variation, e.g. in variance v_t^y.

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- In your exercise, you'll target (β, σ, π) with $z_{jt} = (x_{jt}, \sum_{k \neq j}(x_{jt} - x_{kt})^2, m_t^y x_{jt})$.
  - $\rightarrow$ If $x_{jt} = p_{jt}$, can replace x_{jt} with fitted values $\hat{p}_{jt}$ from the price IV's first stage.

Optimal Instruments

- There are many valid instruments that satisfy exclusion restrictions $\mathbb{E}[\xi_{jt} \mid z_{jt}] = 0$.
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- But adding a ton of instruments will bias your estimator.
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- Can be a bit tricky to compute, but with PyBLP it's just one line of code.
  - $\rightarrow$ In practice, can update your IVs along with your weighting matrix for a second GMM step.

Roadmap

Coding Exercise 2

- Try to do the second exercise before day 3's class, when I'll do it live.
    1. Incorporating preference heterogeneity.
    2. Mixed logit estimation.
    3. Evaluating improvements to the price cut counterfactual.

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- If you have time, try the supplemental exercises.
    - $\rightarrow$ Numerical integration alternatives.
    - $\rightarrow$ Optimal weights and instruments.
    - $\rightarrow$ Supply-side restrictions.

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