Demand Estimation

Mixtape Session

Jeff Gortmaker and Ariel Pakes

Last Class

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- In each market $t \in \mathcal{T}$, individuals with types $i \in \mathcal{I}_t$ choose a $j \in \mathcal{J}_t \cup \{0\}$.

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- Let's go over your first coding exercise.

Unrealistic Substitution Patterns

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- Last week we derived the own-price elasticity. What about the cross-price one?

$$\eta_{jkt} = \frac{\partial \log q_{jt}}{\partial \log p_{kt}} = \frac{\partial q_{jt}}{\partial p_{kt}} \frac{p_{kt}}{q_{jt}} = \frac{\partial s_{jt}}{\partial p_{kt}} \frac{p_{kt}}{s_{jt}} = -\alpha \cdot p_{kt} \cdot s_{kt}$$

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- Doesn't depend on the characteristics of j!
  - $\rightarrow$ Independence of Irrelevant Alternatives (IIA) property.

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- There are two options: buying a car or a blue bus. Each has a 50% market share.
- Introduce a second bus, but it's red. Pure logit (IIA) predicts 33% market shares.
  - $\rightarrow$ In your exercise, consumers substituted *proportionally* from each cereal.
- In reality, we'd expect the car to still have 50% and each bus to have 25%.
  - $\rightarrow$ In your exercise, we'd hope for more substitution from more similar cereals.

Roadmap

Preference Heterogeneity

Mixed Logit Estimation

Numerical Best Practices

Differentiation Instruments

Coding Exercise 2

Red Bus/Blue Bus Solution

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- This will allow 50% of consumers to really like cars and 50% to really like buses.
  - $\rightarrow$ When a new bus is introduced, this doesn't really affect the car-lovers' choice.
- Want μ_{ijt} to dominate logit substitution from convenient but unrealistic ε_{ijt}.
  - $\rightarrow$ Want to add multiple dimensions of heterogeneity that really matter in our setting.

Random Coefficients

$$u_{ijt} = x'_{jt}\beta + \xi_{jt} + \varepsilon_{ijt}$$

- How to add preference heterogeneity to our pure logit model?
  - $\rightarrow$ For simplicity, I'll just let x_{jt} denote all characteristics, including prices p_{jt}.

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- Intuitively, we want to replace β with *random coefficients* β_{it}.
  - $\rightarrow$ *Random* in that they're drawn from a distribution of consumer types $i \in \mathcal{I}_t$.
  - $\rightarrow$ For $x_{jt} = \text{car}_{jt}$ and $\mathcal{I}_t = \{\text{car-lovers}, \text{bus-lovers}\}$, want $\beta_{it} \gg 0$ for car-lovers.

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$$u_{ijt} = x'_{jt} \underbrace{(\beta + \Pi y_{it} + \Sigma \nu_{it})}_{\beta_{it}} + \xi_{jt} + \varepsilon_{ijt}$$

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- Most common specification is $\beta_{it} \sim N(\beta + \Pi y_{it}, \Sigma\Sigma')$.
  - $\rightarrow$ Π shifts preferences according to "observed" demographics $y_{it} \sim$ census.
  - $\rightarrow$ Σ shifts preferences according to "unobserved" preferences $\nu_{it} \sim N(0, I)$.
  - $\rightarrow$ Σ is the *Cholesky root* of the variance matrix. Usually diagonal with standard deviations.

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Random Coefficients in Practice

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- In practice, we implement random coefficients by making a new dataset.
  - $\rightarrow$ In PyBLP lingo, “product data” rows are (j, t)’s, and new “agent data” rows are (i, t)’s.

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- In your coding exercise, you'll just draw $|\mathcal{I}_t| = 100$ types per market.
  - $\rightarrow$ Draw $\nu_{it} \sim N(0, I)$ from a random number generator.
  - $\rightarrow$ Draw y_{it} from census data on demographics: income, etc.
  - $\rightarrow$ Each type is equally-likely, so use equal sampling weights $w_{it} = 1/|\mathcal{I}_t|$.

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  - → Draw y_{it} from census data on demographics: income, etc.
  - → Each type is equally-likely, so use equal sampling weights $w_{it} = 1/|\mathcal{I}_t|$.
- The goal is to have a dataset that reflects the *distribution* of individuals.
  - → Realism aside, this allows us to address distributional concerns.
  - → E.g. will a tax or price change affect high- or low-income individuals differently?

Roadmap

- Preference Heterogeneity
- Mixed Logit Estimation
- Numerical Best Practices
- Differentiation Instruments
- Coding Exercise 2

From Linear Regression to GMM

$$\log \frac{s_{jt}}{s_{0t}} = \delta_{jt} = x'_{jt}\beta + \xi_{jt}$$

- In your exercise, you estimated β by running the above regression.
  - $\rightarrow$ Again, let x_{jt} include price, a constant, any other characteristics.
  - $\rightarrow$ Let z_{jt} include our price IV and exogenous characteristics in x_{jt}.

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- Our exclusion restriction implies the moment condition $\mathbb{E}[\xi_{jt} \cdot z_{jt}] = 0$.
- We'd get the exact same $\hat{\beta}$ by optimizing the following GMM objective:

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}}\, g(\beta) W g(\beta)' \quad \text{where} \quad g(\beta) = \frac{1}{N} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}_t} (\delta_{jt} - x_{jt}'\beta) \cdot z_{jt}$$

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- BLP's (1995) big advancement was how to incorporate flexible preference heterogeneity.
  - $\rightarrow$ Built on simulation estimator advancements (Pakes and Pollard, 1989; McFadden, 1989).

The BLP Estimator

$$\hat{\theta} = \operatorname*{argmin}_{\theta} g(\theta) W g(\theta)' \quad \text{where} \quad g(\theta) = \frac{1}{N} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}_t} (\delta_{jt}(\Sigma, \Pi) - x_{jt}'\beta) \cdot z_{jt}$$

- BLP estimation consists of two nested loops.
  1. In the “outer” loop, we optimize over $\theta = (\beta, \Sigma, \Pi)$.
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- Actually, since $g(\theta)$ is linear in x_{jt}, we can "concentrate out" β and optimize (Σ, Π).
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  - $\rightarrow$ Get $\hat{\beta}$ by running an IV regression of $\delta_{jt}(\Sigma, \Pi)$ on x_{jt}, like in the pure logit exercise.

- What about the GMM weighting matrix W?
  - $\rightarrow$ If you're just-identified ($\dim z_{jt} = \dim \theta$), it doesn't matter. You'll get a zero objective.
  - $\rightarrow$ Otherwise, you may want to repeat optimization with optimal the two-step GMM $\hat{W}$.

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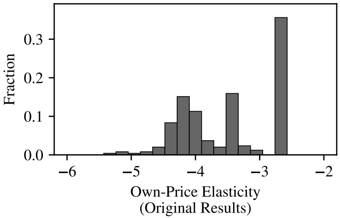
Motivation for Numerical Best Practices



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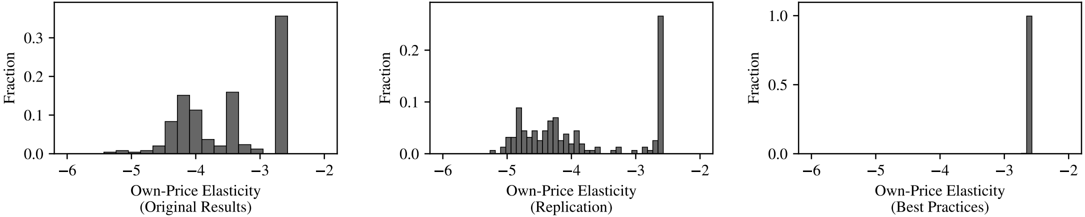

- Variation in BLP estimates across different optimization algorithms and starting values has disillusioned some researchers (Knittel and Metaxoglou, 2014).
- But there are some numerical best practices that you can follow to avoid these kinds of issues (Conlon and Gortmaker, 2020).
  - → They're likely to be useful for most computation-heavy structural estimation, not just BLP!

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- Set box constraints $\theta \in [\underline{\theta}, \overline{\theta}]$ to preclude unrealistic and unstable guesses of θ.
  - $\rightarrow$ E.g. huge Σ values can make the inner loop unstable.
  - $\rightarrow$ Economic intuition and initial estimates will give a sense for reasonable bounds.

Nonlinear Optimization

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- Set box constraints $\theta \in [\underline{\theta}, \overline{\theta}]$ to preclude unrealistic and unstable guesses of θ.
- Check that 3-5 different starting values $\theta \sim U(\underline{\theta}, \overline{\theta})$ give the same $\hat{\theta}$.
  - $\rightarrow$ For 2-step GMM, do this twice, once for each step (6-10 jobs total).
  - $\rightarrow$ If you have access to a cluster, each can be a separate job, run in parallel.

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- Check that 3-5 different starting values $\theta \sim U(\underline{\theta}, \overline{\theta})$ give the same $\hat{\theta}$.
- Prefer using gradient-based algorithms for "smooth" problems like BLP.
  - $\rightarrow$ Avoid derivative-free methods like Nelder-Mead/simplex, which tend to work worse.
  - $\rightarrow$ I prefer trust-region algorithms, e.g. SciPy's `trust-constr` or Knitro if you have it.

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- Prefer using gradient-based algorithms for "smooth" problems like BLP.
- Try to terminate on strict first-order conditions, e.g. $||\text{gradient}||_\infty <$ `1e-8`.
  - $\rightarrow$ Inner loop should be tighter to prevent error "bubbling up." PyBLP default is very tight.
  - $\rightarrow$ Can also check second-order conditions, i.e. Hessian eigenvalues are positive.

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- Prefer using gradient-based algorithms for “smooth” problems like BLP.
- Try to terminate on strict first-order conditions, e.g. $||\text{gradient}||_\infty <$ `1e-8`.
- Configure your optimizer! Defaults may not work for your setting.

Numerical Integration

$$s_{jt} = \sum_{i \in \mathcal{I}_t} w_{it} \cdot \frac{\exp(\delta_{jt} + \mu_{ijt})}{1 + \sum_{k \in \mathcal{J}_t} \exp(\delta_{kt} + \mu_{ikt})}$$

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- Sometimes there are only a few types that we can integrate exactly.
  - $\rightarrow$ E.g. high- and low-income types $i \in \{1, 2\}$ with known shares w_{1t} and $w_{2t} = 1 - w_{1t}$.

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- Sometimes there are only a few types that we can integrate exactly.
- But usually we approximate the distribution with Monte Carlo integration.
  - → Use a random number generator (RNG) to draw $|\mathcal{I}_t| \approx$ 1,000 of (ν_{it}, y_{it})'s per market.
  - → Even better than your default RNG are quasi-Monte Carlo sequences.
  - → I recommend scrambled Halton sequences. R: Owen (2017). Python: SciPy or PyBLP.

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- But usually we approximate the distribution with Monte Carlo integration.
- If you just need a few $\nu_{it} \sim N(0, I)$'s, try out Gauss-Hermite quadrature.
  - $\rightarrow$ 10-100$\times$ fewer carefully-chosen (w_{it}, ν_{it})'s that do just as well as Monte Carlo.
  - $\rightarrow$ Chosen to exactly integrate a polynomial expansion of the integrand.

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- But usually we approximate the distribution with Monte Carlo integration.
- If you just need a few $\nu_{it} \sim N(0, I)$'s, try out Gauss-Hermite quadrature.
- Keep increasing $|\mathcal{I}_t|$ until your estimates stabilize across draws/starting values.

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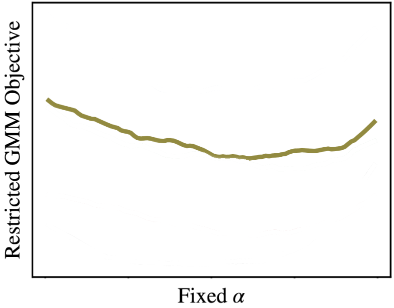

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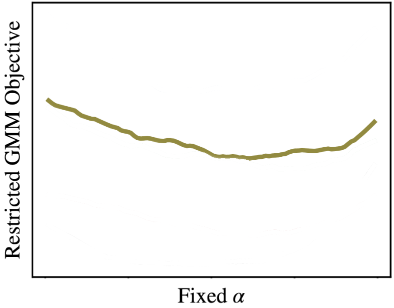
- Here, there's a minimum but also some challenges.



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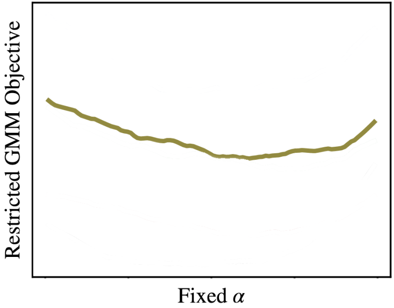

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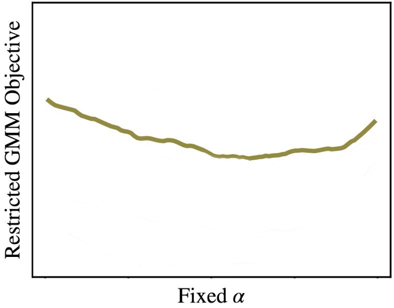

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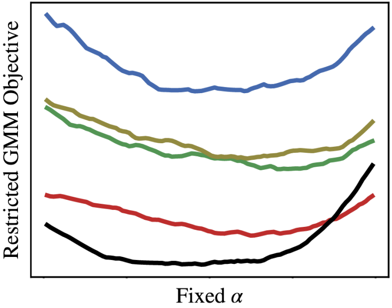

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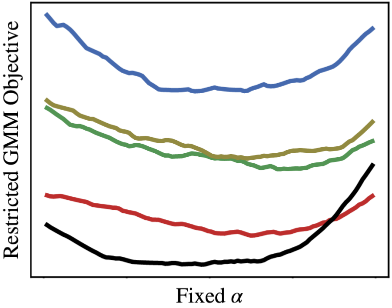
- Different instruments give different objectives.
  - $\rightarrow$ Even if they're all valid, some may be weaker.
  - $\rightarrow$ Weaker means flatter and harder to optimize.



Roadmap

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- Later, adding more can help with weakness and testing exclusion restrictions.

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- Let's use our stronger intuition about linear regression to think about instruments!

Linear Regression Intuition

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- If we set $\sigma = \pi = 0$ like on day 1, we get our familiar pure logit regression.
  - $\rightarrow$ Use the same IV as before to target β: if $x_{jt} = p_{jt}$, a price IV; if exogenous, x_{jt} itself.

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$$\log \frac{s_{jt}}{s_{0t}} \approx \beta x_{jt} + \sigma^2 d^x_{jt} + \xi_{jt} \quad \text{where} \quad d^x_{jt} = \Big(\frac{x_{jt}}{2} - \sum_{k \in \mathcal{J}_t} s_{kt} x_{kt}\Big) x_{jt}$$

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- To target σ, need a measure of how “differentiated” j is in terms of x_{jt} within t.
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  - → Can technically identify π from higher-order variation, e.g. in variance v_t^y.

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- In your exercise, you'll target (β, σ, π) with $z_{jt} = (x_{jt}, \sum_{k \neq j}(x_{jt} - x_{kt})^2, m^y_t x_{jt})$.
  - $\rightarrow$ If $x_{jt} = p_{jt}$, can replace x_{jt} with fitted values $\hat{p}_{jt}$ from the price IV's first stage.

Optimal Instruments

- There are many valid instruments that satisfy exclusion restrictions $\mathbb{E}[\xi_{jt} \mid z_{jt}] = 0$.
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- But adding a ton of instruments will bias your estimator.
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- Can be a bit tricky to compute, but with PyBLP it's just one line of code.
  - $\rightarrow$ In practice, can update your IVs along with your weighting matrix for a second GMM step.

Roadmap

Preference Heterogeneity

Mixed Logit Estimation

Numerical Best Practices

Differentiation Instruments

Coding Exercise 2

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- If you have time, try the supplemental exercises.
    $\rightarrow$ Numerical integration alternatives.
    $\rightarrow$ Optimal weights and instruments.
    $\rightarrow$ Supply-side restrictions.

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