

Demand Estimation

MIXTAPE SESSION

Jeff Gortmaker and Ariel Pakes



Last Class

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- In each market $t \in \mathcal{T}$, individuals with types $i \in \mathcal{I}_t$ choose a $j \in \mathcal{J}_t \cup \{0\}$.

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- Let's go over your first coding exercise.

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$$\eta_{jkt} = \frac{\partial \log q_{jt}}{\partial \log p_{kt}} = \frac{\partial q_{jt}}{\partial p_{kt}} \frac{p_{kt}}{q_{jt}} = \frac{\partial s_{jt}}{\partial p_{kt}} \frac{p_{kt}}{s_{jt}} = -\alpha \cdot p_{kt} \cdot s_{kt}$$

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- Doesn't depend on the characteristics of j !
→ Independence of Irrelevant Alternatives (IIA) property.

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 - In your exercise, consumers substituted *proportionally* from each cereal.

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- There are two options: buying a car or a blue bus. Each has a 50% market share.
- Introduce a second bus, but it's red. Pure logit (IIA) predicts 33% market shares.
 - In your exercise, consumers substituted *proportionally* from each cereal.
- In reality, we'd expect the car to still have 50% and each bus to have 25%.
 - In your exercise, we'd hope for more substitution from more similar cereals.

Roadmap

Preference Heterogeneity

Mixed Logit Estimation

Numerical Best Practices

Differentiation Instruments

Coding Exercise 2

Red Bus/Blue Bus Solution

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- This will allow 50% of consumers to really like cars and 50% to really like buses.
 - When a new bus is introduced, this doesn't really affect the car-lovers' choice.
- Want μ_{ijt} to dominate logit substitution from convenient but unrealistic ε_{ijt} .
 - Want to add multiple dimensions of heterogeneity that really matter in our setting.

Random Coefficients

$$u_{ijt} = x'_{jt}\beta + \xi_{jt} + \varepsilon_{ijt}$$

- How to add preference heterogeneity to our pure logit model?
 - For simplicity, I'll just let x_{jt} denote all characteristics, including prices p_{jt} .

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- How to add preference heterogeneity to our pure logit model?
 - For simplicity, I'll just let x_{jt} denote all characteristics, including prices p_{jt} .
- Intuitively, we want to replace β with *random coefficients* β_{it} .
 - *Random* in that they're drawn from a distribution of consumer types $i \in \mathcal{I}_t$.
 - For $x_{jt} = \text{car}_{jt}$ and $\mathcal{I}_t = \{\text{car-lovers}, \text{bus-lovers}\}$, want $\beta_{it} \gg 0$ for car-lovers.

Random Coefficients

$$u_{ijt} = x'_{jt} \underbrace{(\beta + \Pi y_{it} + \Sigma \nu_{it})}_{\beta_{it}} + \xi_{jt} + \varepsilon_{ijt}$$

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- Most common specification is $\beta_{it} \sim N(\beta + \Pi y_{it}, \Sigma \Sigma')$.
 - Π shifts preferences according to “observed” demographics $y_{it} \sim \text{census}$.
 - Σ shifts preferences according to “unobserved” preferences $\nu_{it} \sim N(0, I)$.
 - Σ is the *Cholesky root* of the variance matrix. Usually diagonal with standard deviations.

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- In practice, we implement random coefficients by making a new dataset.
 - In PyBLP lingo, “product data” rows are (j, t) ’s, and new “agent data” rows are (i, t) ’s.

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- In your coding exercise, you’ll just draw $|\mathcal{I}_t| = 100$ types per market.
 - Draw $\nu_{it} \sim N(0, I)$ from a random number generator.
 - Draw y_{it} from census data on demographics: income, etc.
 - Each type is equally-likely, so use equal sampling weights $w_{it} = 1/|\mathcal{I}_t|$.

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 - Each type is equally-likely, so use equal sampling weights $w_{it} = 1/|\mathcal{I}_t|$.
- The goal is to have a dataset that reflects the *distribution* of individuals.
 - Realism aside, this allows us to address distributional concerns.
 - E.g. will a tax or price change affect high- or low-income individuals differently?

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Coding Exercise 2

From Linear Regression to GMM

$$\log \frac{s_{jt}}{s_{0t}} = \delta_{jt} = x'_{jt} \beta + \xi_{jt}$$

- In your exercise, you estimated β by running the above regression.
 - Again, let x_{jt} include price, a constant, any other characteristics.
 - Let z_{jt} include our price IV and exogenous characteristics in x_{jt} .

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- Our exclusion restriction implies the moment condition $\mathbb{E}[\xi_{jt} \cdot z_{jt}] = 0$.
- We'd get the exact same $\hat{\beta}$ by optimizing the following GMM objective:

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} g(\beta)' W g(\beta) \quad \text{where} \quad g(\beta) = \frac{1}{N} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}_t} (\delta_{jt} - x'_{jt}\beta) \cdot z_{jt}$$

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$$s_{jt} = \sum_{i \in \mathcal{I}_t} w_{it} \cdot \frac{\exp[\delta_{jt} + \mu_{ijt}(\Sigma, \Pi)]}{1 + \sum_{k \in \mathcal{J}_t} \exp[\delta_{kt} + \mu_{ikt}(\Sigma, \Pi)]} \quad \text{for all } j \in \mathcal{J}_t$$

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- Many ways to solve and speed up [BLP's \(1995\)](#) contraction.
 - PyBLP will take care of this, but see [Conlon and Gortmaker \(2020\)](#) if interested.

The BLP Estimator

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} g(\theta)Wg(\theta)' \quad \text{where} \quad g(\theta) = \frac{1}{N} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}_t} (\delta_{jt}(\Sigma, \Pi) - x'_{jt}\beta) \cdot z_{jt}$$

- BLP estimation consists of two nested loops.
 1. In the “outer” loop, we optimize over $\theta = (\beta, \Sigma, \Pi)$.
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 - Get $\hat{\beta}$ by running an IV regression of $\delta_{jt}(\Sigma, \Pi)$ on x_{jt} , like in the pure logit exercise.
- What about the GMM weighting matrix W ?
 - If you’re just-identified ($\dim z_{jt} = \dim \theta$), it doesn’t matter. You’ll get a zero objective.
 - Otherwise, you may want to repeat optimization with optimal the two-step GMM \hat{W} .

Roadmap

Preference Heterogeneity

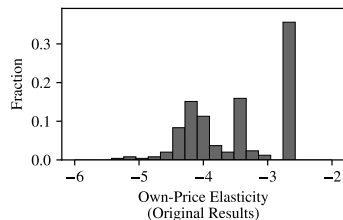
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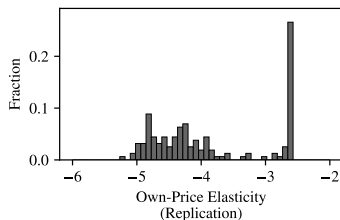
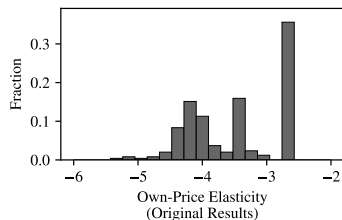
Coding Exercise 2

Motivation for Numerical Best Practices



- Variation in BLP estimates across different optimization algorithms and starting values has disillusioned some researchers ([Knittel and Metaxoglou, 2014](#)).

Motivation for Numerical Best Practices



- Variation in BLP estimates across different optimization algorithms and starting values has disillusioned some researchers ([Knittel and Metaxoglou, 2014](#)).
- But there are some numerical best practices that you can follow to avoid these kinds of issues ([Conlon and Gortmaker, 2020](#)).
 - They're likely to be useful for most computation-heavy structural estimation, not just BLP!

Nonlinear Optimization

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- Set **box constraints** $\theta \in [\underline{\theta}, \bar{\theta}]$ to preclude unrealistic and unstable guesses of θ .
 - E.g. huge Σ values can make the inner loop unstable.
 - Economic intuition and initial estimates will give a sense for reasonable bounds.

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- Set **box constraints** $\theta \in [\underline{\theta}, \bar{\theta}]$ to preclude unrealistic and unstable guesses of θ .
- Check that 3-5 **different starting values** $\theta \sim U(\underline{\theta}, \bar{\theta})$ give the same $\hat{\theta}$.
 - For 2-step GMM, do this twice, once for each step (6-10 jobs total).
 - If you have access to a cluster, each can be a separate job, run in parallel.

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- Check that 3-5 **different starting values** $\theta \sim U(\underline{\theta}, \bar{\theta})$ give the same $\hat{\theta}$.
- Prefer using **gradient-based algorithms** for “smooth” problems like BLP.
 - Avoid derivative-free methods like Nelder-Mead/simplex, which tend to work worse.
 - I prefer trust-region algorithms, e.g. SciPy’s `trust-constr` or Knitro if you have it.

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- Prefer using **gradient-based algorithms** for “smooth” problems like BLP.
- Try to terminate on **strict first-order conditions**, e.g. $\|\text{gradient}\|_{\infty} < 1\text{e-}8$.
 - Inner loop should be tighter to prevent error “bubbling up.” PyBLP default is very tight.
 - Can also check second-order conditions, i.e. Hessian eigenvalues are positive.

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- Prefer using **gradient-based algorithms** for “smooth” problems like BLP.
- Try to terminate on **strict first-order conditions**, e.g. $\|\text{gradient}\|_{\infty} < 1\text{e-}8$.
- **Configure your optimizer!** Defaults may not work for your setting.

Numerical Integration

$$s_{jt} = \sum_{i \in \mathcal{I}_t} w_{it} \cdot \frac{\exp(\delta_{jt} + \mu_{ijt})}{1 + \sum_{k \in \mathcal{J}_t} \exp(\delta_{kt} + \mu_{ikt})}$$

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- Sometimes there are only a few types that we can integrate exactly.
 - E.g. high- and low-income types $i \in \{1, 2\}$ with known shares w_{1t} and $w_{2t} = 1 - w_{1t}$.

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- Sometimes there are only a few types that we can integrate exactly.
- But usually we approximate the distribution with **Monte Carlo** integration.
 - Use a random number generator (RNG) to draw $|\mathcal{I}_t| \approx 1,000$ of (ν_{it}, y_{it}) 's per market.
 - Even better than your default RNG are **quasi-Monte Carlo** sequences.
 - I recommend scrambled Halton sequences. R: **Owen (2017)**. Python: SciPy or PyBLP.

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- But usually we approximate the distribution with **Monte Carlo** integration.
- If you just need a few $\nu_{it} \sim N(0, I)$'s, try out **Gauss-Hermite quadrature**.
 - 10-100× fewer carefully-chosen (w_{it}, ν_{it}) 's that do just as well as Monte Carlo.
 - Chosen to exactly integrate a polynomial expansion of the integrand.

Numerical Integration

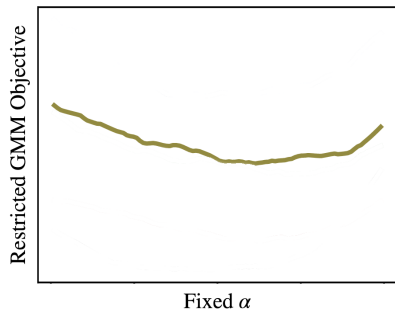
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- **Keep increasing $|\mathcal{I}_t|$** until your estimates stabilize across draws/starting values.

What Typically Goes Wrong

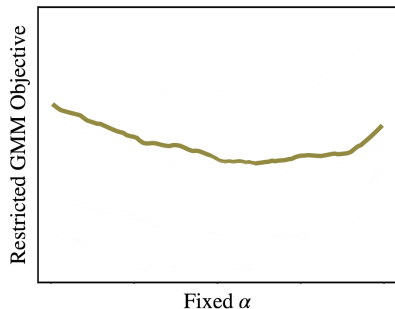
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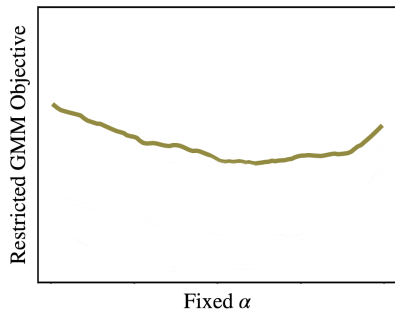
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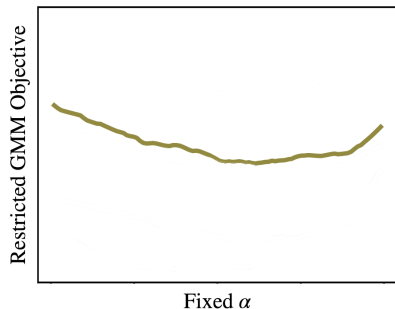
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 - Too few draws $|\mathcal{I}_t|$ makes the objective “choppy.”



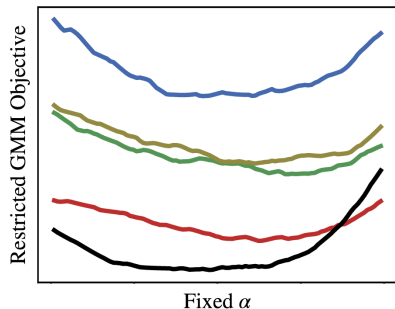
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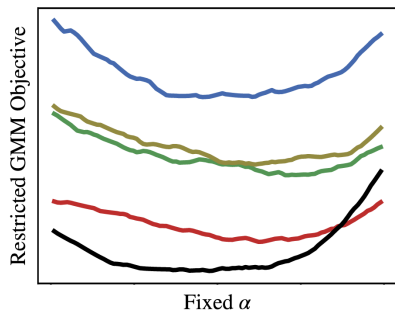
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 - Poorly-configured optimizers can stop too early.
- Different instruments give different objectives.
 - Even if they're all valid, some may be weaker.
 - Weaker means flatter and harder to optimize.



Roadmap

Preference Heterogeneity

Mixed Logit Estimation

Numerical Best Practices

Differentiation Instruments

Coding Exercise 2

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- Later, adding more can help with weakness and testing exclusion restrictions.

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- Let's use our stronger intuition about linear regression to think about instruments!

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 - Use the same IV as before to target β : if $x_{jt} = p_{jt}$, a price IV; if exogenous, x_{jt} itself.

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$$\log \frac{s_{jt}}{s_{0t}} \approx \beta x_{jt} + \sigma^2 d_{jt}^x + \xi_{jt} \quad \text{where} \quad d_{jt}^x = \left(\frac{x_{jt}}{2} - \sum_{k \in \mathcal{J}_t} s_{kt} x_{kt} \right) x_{jt}$$

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 - Can technically identify π from higher-order variation, e.g. in variance v_t^y .

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- To target π , we can interact x_{jt} with mean within-market income m_t^y .
- In your exercise, you’ll target (β, σ, π) with $z_{jt} = (x_{jt}, \sum_{k \neq j} (x_{jt} - x_{kt})^2, m_t^y x_{jt})$.
→ If $x_{jt} = p_{jt}$, can replace x_{jt} with fitted values \hat{p}_{jt} from the price IV’s first stage.

Optimal Instruments

- There are many valid instruments that satisfy exclusion restrictions $\mathbb{E}[\xi_{jt} \mid z_{jt}] = 0$.
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- But adding a ton of instruments will bias your estimator.
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- Can be a bit tricky to compute, but with PyBLP it’s just one line of code.
 - In practice, can update your IVs along with your weighting matrix for a second GMM step.

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Coding Exercise 2

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 - What dimensions of preference heterogeneity are missing?
- If you have time, try the supplemental exercises.
 - Numerical integration alternatives.
 - Supply-side restrictions.
 - Optimal weights and IVs.

References I

- Angrist, Joshua D, Guido W Imbens, and Alan B Krueger**, “Jackknife instrumental variables estimation,” *Journal of Applied Econometrics*, 1999, 14 (1), 57–67.
- Berry, Steven, James Levinsohn, and Ariel Pakes**, “Automobile prices in market equilibrium,” *Econometrica*, 1995, 63 (4), 841–890.
- Berry, Steven T and Philip A Haile**, “Identification in differentiated products markets using market level data,” *Econometrica*, 2014, 82 (5), 1749–1797.
- **and** — , “Nonparametric identification of differentiated products demand using micro data,” 2023.
- Chamberlain, Gary**, “Asymptotic efficiency in estimation with conditional moment restrictions,” *Journal of Econometrics*, 1987, 34 (3), 305–334.

References II

- Conlon, Christopher and Jeff Gortmaker**, “Best practices for differentiated products demand estimation with PyBLP,” *The RAND Journal of Economics*, 2020, 51 (4), 1108–1161.
- Gandhi, Amit and Jean-François Houde**, “Measuring substitution patterns in differentiated-products industries,” 2020.
- Han, Chirok and Peter CB Phillips**, “GMM with many moment conditions,” *Econometrica*, 2006, 74 (1), 147–192.
- Knittel, Christopher R and Konstantinos Metaxoglou**, “Estimation of random-coefficient demand models: Two empiricists’ perspective,” *Review of Economics and Statistics*, 2014, 96 (1), 34–59.

References III

Newey, Whitney K and Frank Windmeijer, “Generalized method of moments with many weak moment conditions,” *Econometrica*, 2009, 77 (3), 687–719.

Owen, Art B, “A randomized Halton algorithm in R,” 2017.

Salanié, Bernard and Frank A Wolak, “Fast, detail-free, and approximately correct: Estimating mixed demand systems,” 2022.