Chapter 3. The Laplace Transform

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Introduction

- **Frequency domain analysis of continuous time signals and systems**
  - Both **Laplace and Fourier transforms** provide **complementary representations of a signal** to its own in time domain, and **algebraic characteristics of the systems**.

- **Damping and frequency characterization of continuous-time signals**
  - By the **location of the roots of the numerator and denominator**, or **zeros** and **poles**, of the Laplace transform of the signal.

- **Transfer function characterization of continuous-time LTI systems**
  - The **transfer function** is the **ratio of the Laplace transform of the output to that of the input**.

Introduction

- **Stability, and transient and steady-state responses**
  - Can only be **verified or understood** via the Laplace transform.

- **One- and two-sided Laplace transform**
  - Given the prevalence of the **causal signals and systems**, the Laplace transform is typically known as **one-sided** but the **two-sided** transform also exists.

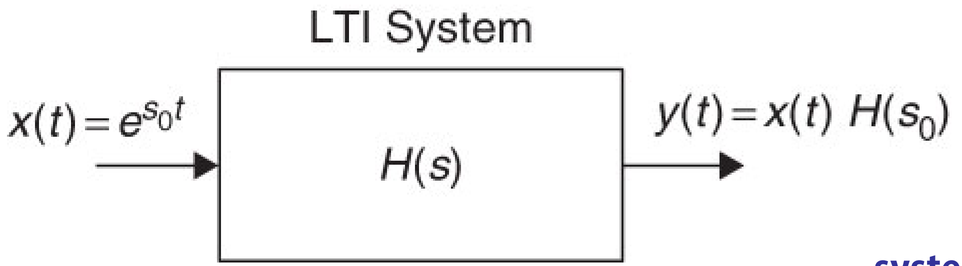
- **Region of convergence and the Fourier transform**
  - If and where the **integration of Laplace transform converges**.

- **Eigenfunctions of LTI systems**
  - The output of complex exponential is the **exponential with its magnitude and phase** changed by the response of the system.

Eigenfunctions of LTI Systems

- We focus on the **frequency domain representation** of **signals and their responses when applied to an LTI system**.
  - In **Laplace transform**, we use **complex exponentials** or **sinusoids** that depend on frequency as a basic function.

$$x(t) = e^{s_0 t} \quad s_0 = \sigma_0 + j\Omega_0 \text{ for } -\infty < t < \infty$$



system or transfer function

  - The **output of the LTI system** with impulse response $h(t)$ is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\ d\tau = e^{s_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-\tau s_0}\ d\tau = \boxed{x(t)}\boxed{H(s_0)}$$

eigenfunction

Two-Sided Laplace Transform

The **two-sided Laplace transform** of a **continuous-time function** $\boldsymbol{f(t)}$ is

$$\boldsymbol{F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-st}\, dt \qquad s \in ROC}$$

where the variable $s = \sigma + j\Omega$.

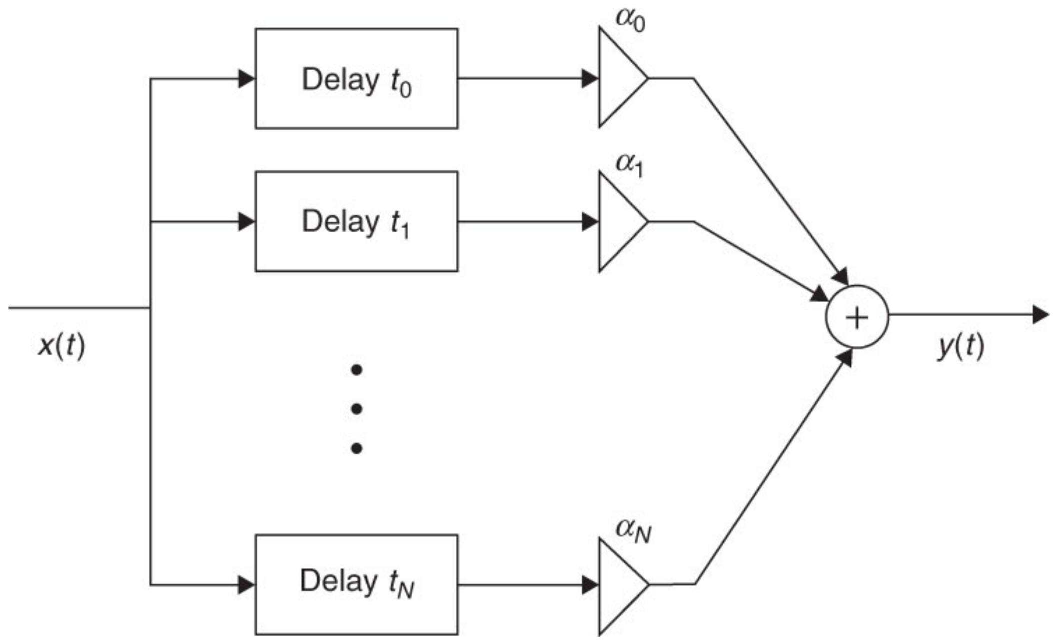
– $\boldsymbol{\Omega}$ is the **frequency** in rad/sec, $\boldsymbol{\sigma}$ is a **damping** factor, and ROC stands for the **region of convergence** – that is, where the **integral exists**.

The **inverse Laplace transform** is given by

$$\boldsymbol{f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st}\, ds \qquad \sigma \in ROC}$$

Example

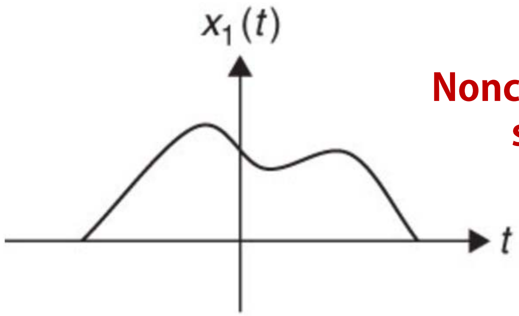
[Ex 3.1] Use the eigenfunction property to find the system function of the channel causing the multipath effect.



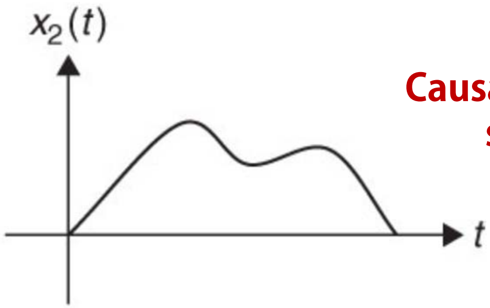
[Ex 3.2] Find the Laplace transform of $\delta(t), u(t)$, and a pulse $p(t) = u(t) - u(t-1)$.

Type of Functions for Laplace Transform

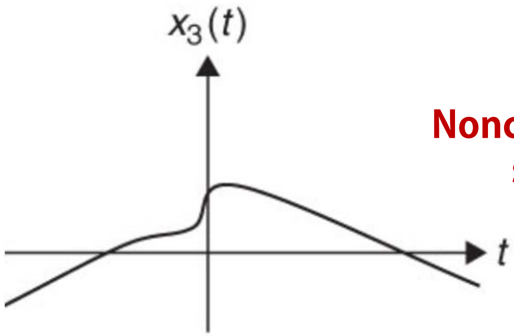
- Type of functions for calculating Laplace transform
  - **Finite-support** and **infinite-support** functions
  - **Causal, anti-causal**, and **non-causal** functions



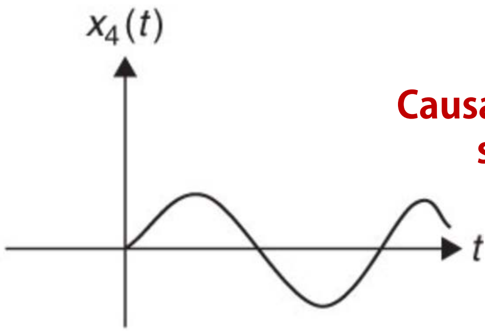
Noncausal finite support signal



Causal finite support signal



Noncausal infinite support signal



Causal infinite support signal

Existence of Laplace Transform

For the **Laplace transform of $f(t)$ to exist** we need that

$$\left|\int_{-\infty}^{\infty} f(t)e^{-st}\,dt\right| = \left|\int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\Omega t}dt\right| \leq \int_{-\infty}^{\infty} |f(t)e^{-\sigma t}|dt < \infty$$

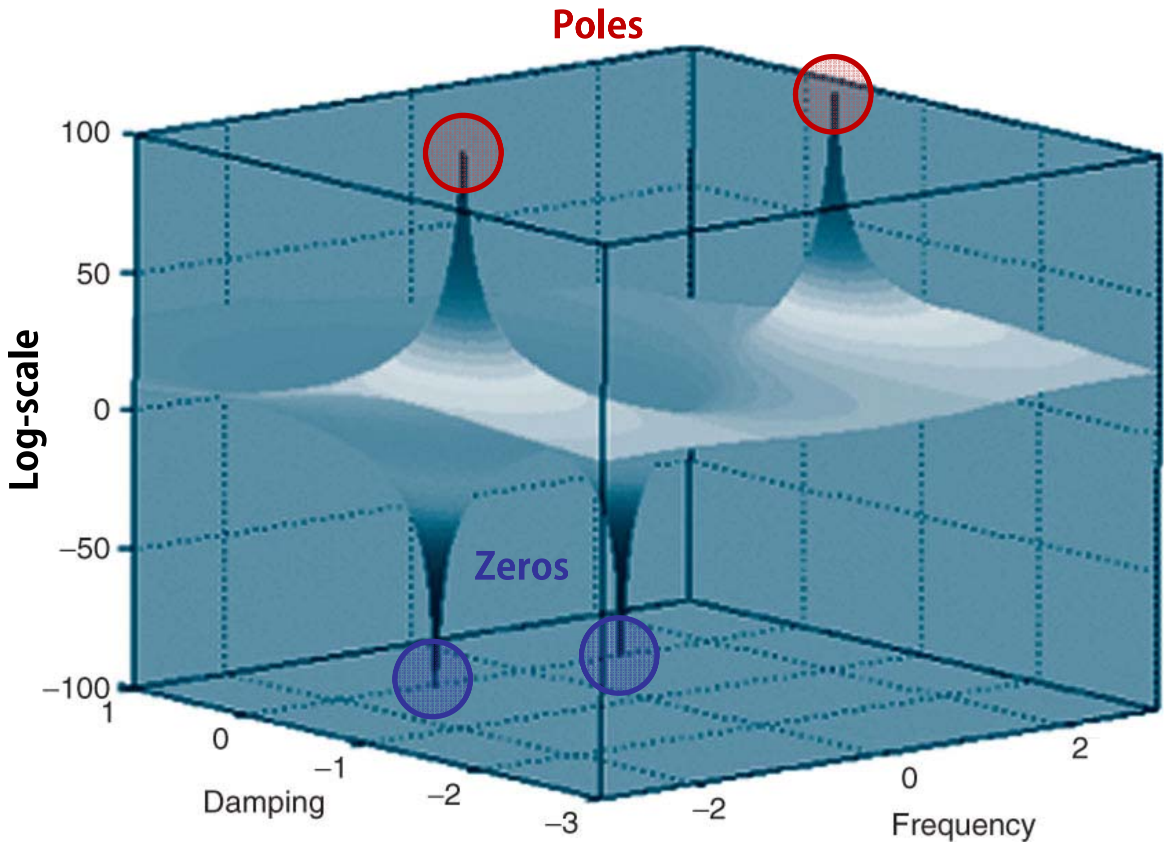
or that $f(t)e^{-\sigma t}$ be **absolutely integrable**.

- The value chosen for σ **determines the ROC of** $F(s)$; the **frequency** Ω **does not affect the ROC**.

For a rational function $F(s) = \mathcal{L}[f(t)] = N(s)/D(s)$, its **zeros** are the **values of s that makes the function $F(s) = 0$**, and its **poles** are the **values of s that makes the function $F(s) \to \infty$**.

Example of Poles and Zeros

$$F(s) = \frac{2(s^2+1)}{s^2+2s+5} = \frac{2(s+j)(s-j)}{(s+1+2j)(s+1-2j)}$$



Region of Convergence

The **Laplace transform** of a signal $f(t)$ with $|f(t)| < A$

- **Finite support** function ($f(t) = 0$ for $t < t_1$ and $t > t_2$) is

$$\mathcal{L}[f(t)] = \mathcal{L}[f(t)[u(t-t_1) - u(t-t_2)]] \qquad \textbf{whole } s\textbf{-plane}$$

- **Causal** function ($f(t) = 0$ for $t < 0$) is

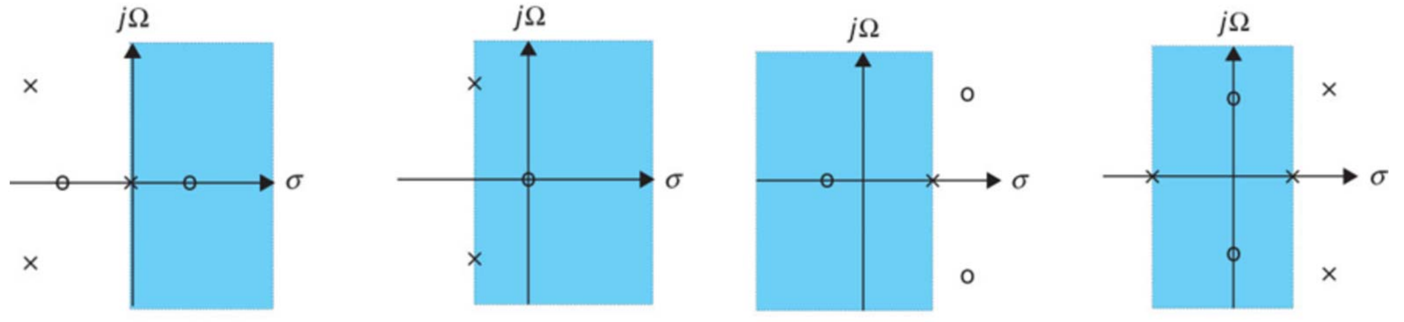
$$\mathcal{L}[f(t)u(t)] \qquad \mathcal{R}_c = \{(\sigma, \Omega)\colon \sigma > \max\{\sigma_i\}, -\infty < \Omega < \infty\}$$

- **Anti-causal** function ($f(t) = 0$ for $t > 0$) is

$$\mathcal{L}[f(t)u(-t)] \qquad \mathcal{R}_{ac} = \{(\sigma, \Omega)\colon \sigma < \min\{\sigma_i\}, -\infty < \Omega < \infty\}$$

- **Noncausal** function ($f(t) = f_{ac}(t) + f_c(t) = f(t)u(-t) + f(t)u(t)$) is

$$\mathcal{L}[f(t)] = \mathcal{L}[f_{ac}(-t)u(t)]_{(-s)} + \mathcal{L}[f_c(t)u(t)] \qquad \mathcal{R}_c \cap \mathcal{R}_{ac}$$



One-Sided Laplace Transform

The **one-sided Laplace transform** is defined as

$$\boldsymbol{F(s) = \mathcal{L}[f(t)u(t)] = \int_{0-}^{\infty} f(t)u(t)e^{-st}\, dt}$$

where $f(t)$ is either a **causal function** or made into a **casual function by the multiplication by $u(t)$**.

- The **advantage of the one-sided Laplace transform** is that it can be used in the **solution of differential equations with initial conditions**.

Examples

[Ex 3.3] Let us find the Laplace transform of $e^{j(\Omega_0 t+\theta)}u(t)$ to obtain the Laplace transform of $x(t) = \cos(\Omega_0 t + \theta)\, u(t)$. Consider the special case for $\theta = 0$ and $\theta = -\pi/2$. Determine the ROCs.

[Ex 3.4] Find the Laplace transform of a real exponential $x(t) = e^{-t}u(t)$, and of $x(t)$ modulated by a cosine or $y(t) = e^{-t}\cos(10t)\, u(t)$.

[Ex 3.5] Let the autocorrelation function $c(t) = e^{-a|t|}$, where $a > 0$. Find its Laplace transform.

[Ex 3.6] Consider a noncausal LTI system with impulse response

$$h(t) = e^{-t}u(t) + e^{2t}u(-t) = h_c(t) + h_{ac}(t)$$

Compute $H(s)$ and find out whether we compute $H(j\Omega)$ from $H(s)$.

Basic Properties of Laplace Transform

[Ex 3.7] Find the Laplace transform of the ramp function $r(t) = tu(t)$ and use it to find the Laplace of a triangular pulse $\Lambda(t) = r(t+1) - 2r(t) + r(t-1)$.

- We'll consider the **basic properties** of the **one-sided Laplace transform**
  - Many of these properties will be encountered in the **Fourier analysis**.
  - Pay attention to the **duality** that exists between the **time and the frequency domains**.

Laplace Transform - Linearity

For signals $f(t)$ and $g(t)$ with Laplace transform $F(s)$ and $G(s)$, and constants a and b, we have the Laplace transform is **linear**:

$$\mathcal{L}[\boldsymbol{af(t)u(t) + bg(t)u(t)}] = \boldsymbol{aF(s) + bG(s)}$$

- A signal is characterized by the **poles** of its Laplace transform:
  - The **σ axis** of Laplace plane corresponds to **damping**.
    - Single pole at the **left-hand s-plane** in $f(t) = e^{at}$: **decaying exponential**
    - Single pole at the **right-hand s-plane** in $f(t) = e^{at}$: **growing exponential**
  - The Laplace transform of a **sinusoid** has a pair of **poles on the $j\Omega$-axis**.
    - Poles move away from the origin: frequency increases
    - Poles move toward the origin: frequency decreases

Laplace Transform - Differentiation

For a signal $f(t)$ with Laplace transform $F(s)$ the **one-sided Laplace transform** of its **first-** and **Nth-order derivatives** are

$$\mathcal{L}\left[\frac{df(t)}{dt}u(t)\right] = sF(s) - f(0-)$$

$$\mathcal{L}\left[f^{(N)}(t)u(t)\right] = s^N F(s) - \sum_{k=0}^{N-1} f^{(k)}(0-)s^{N-1-k}$$

[Ex 3.8] Find the impulse response of RL circuit in series with voltage source $v_s(t)$, where the current $i(t)$ is the output.

[Ex 3.9] Obtain from the Laplace transform of $x(t) = \cos\Omega_0 t\ u(t)$ the Laplace transform of $\sin\Omega_0 t\ u(t)$ using the derivative property.

Laplace Transform - Integration

The Laplace transform of the **integral** of a **causal signal $\boldsymbol{y(t)}$** is given by

$$\mathcal{L}\left[\int_0^t y(\tau)d\tau \; u(t)\right] = \frac{Y(s)}{s}$$

[Ex 3.10] Suppose that

$$\int_0^t y(\tau)\, d\tau = 3u(t) - 2y(t)$$

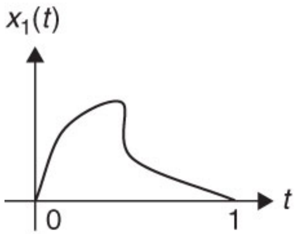
Find the Laplace transform of $y(t)$, a causal signal.

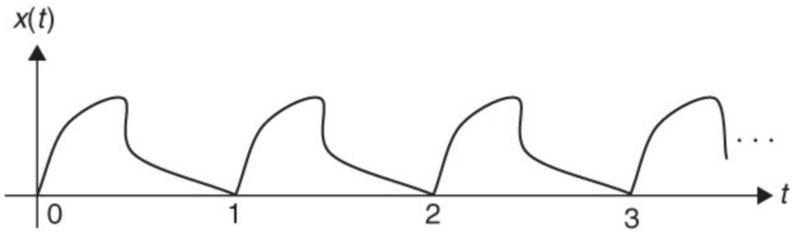
Laplace Transform - Time Shifting

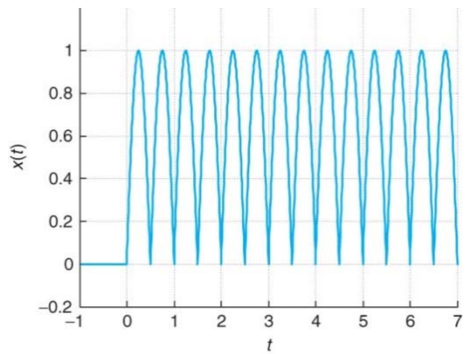
If the Laplace transform of $f(t)u(t)$ is $F(s)$, the Laplace transform of the **time-shifted signal $f(t-\tau)u(t-\tau)$** is

$$\mathcal{L}[f(t-\tau)u(t-\tau)] = e^{-\tau s}F(s)$$

[Ex 3.11] Suppose we wish to find the Laplace transform of the causal sequence of pulses $x(t)$ shown in below. Let $x_1(t)$ denote the first pulse.







[Ex 3.12] Consider the causal full-wave rectified signal shown above. Find its Laplace transform.

Laplace Transform - Duality

For a causal function $f(t)$, such that $f(0-) = 0$, we have **duality in derivatives and integrals**:

$$\mathcal{L}[f(t)] = F(s)$$

$$\mathcal{L}[df(t)/dt] = sF(s)$$

$$\mathcal{L}[tf(t)] = -dF(s)/ds$$

$$\mathcal{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = F(s)/s$$

$$\mathcal{L}[f(t)/tu(t)] = \int_{-\infty}^{-s} F(-\rho)d\rho$$

Laplace Transform - Duality

We also have **duality in time and frequency shifts**:

$$\mathcal{L}[f(t-\tau)u(t-\tau)] = e^{-\tau s}F(s)$$

$$\mathcal{L}[e^{-\alpha t}f(t)u(t)] = F(s+\alpha)$$

We finally have **duality in time expansion, contraction, and reflection** from the **two-sided Laplace transform**:

$$\mathcal{L}[f(\alpha t)u(\alpha t)] = 1/|\alpha|F(s/\alpha)$$

$$\mathcal{L}[(1/|\alpha|)f(t/\alpha)u(t/\alpha)] = F(\alpha s)$$

Laplace Transform - Convolution Integral

The Laplace transform of the **convolution integral** of a causal signal $x(t)$ with Laplace transform $X(s)$ and a causal impulse response $h(t)$ with Laplace transform $H(s)$ is given by
$\boldsymbol{\mathcal{L}[(x * h)(t)] = X(s)H(s)}$.

The **system/transfer function** $H(s) = \mathcal{L}[h(t)]$ can be expressed as the ratio

$$\boldsymbol{H(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[x(t)]} = \frac{\mathcal{L}[output]}{\mathcal{L}[input]}}$$

It transfers the Laplace transform of the input to the output.

Inverse Laplace Transform - Overview

- Inverting the Laplace transform consists in **finding function** that has the given transform with the given region of convergence.

- Inverse of one-sided Laplace transform
  - We **only consider causal signals**, the region of convergence is assumed and is not shown.
  - **Partial fraction expansion**
    - Expanding the given function in s into a sum of **components of which the inverse Laplace transforms** are known.

$$X(s) = \frac{N(s)}{D(s)} = g_0 + g_1 s + \cdots + g_m s^m + \frac{B(s)}{D(s)}$$

  - In the inverse, $\boldsymbol{u(t)}$ **should be included** since the **result of the inverse is causal**.

One-Sided Laplace Transforms

One-sided Laplace Transforms

	Function of time		Function of s, ROC
(1)	$\delta(t)$	$\Leftrightarrow$	1, whole s-plane
(2)	$u(t)$	$\Leftrightarrow$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
(3)	$r(t)$	$\Leftrightarrow$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
(4)	$e^{-at}u(t),\ a > 0$	$\Leftrightarrow$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
(5)	$\cos(\Omega_0 t)u(t)$	$\Leftrightarrow$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
(6)	$\sin(\Omega_0 t)u(t)$	$\Leftrightarrow$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
(7)	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\Leftrightarrow$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
(8)	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\Leftrightarrow$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
(9)	$2A\,e^{-at}\cos(\Omega_0 t+\theta)u(t),\ a > 0$	$\Leftrightarrow$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
(10)	$\frac{1}{(N-1)!}t^{N-1}u(t)$	$\Leftrightarrow$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
(11)	$\frac{1}{(N-1)!}t^{N-1}e^{-at}u(t)$	$\Leftrightarrow$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
(12)	$\frac{2A}{(N-1)!}t^{N-1}e^{-at}\cos(\Omega_0 t+\theta)u(t)$	$\Leftrightarrow$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

Single Real Poles

If $X(s)$ is a proper rational function

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_k (s - p_k)}$$

where $\{p_k\}$ are **simple real poles of $X(s)$**, its partial fraction expansion and its inverse are given by

$$\boldsymbol{X(s) = \sum_k \frac{A_k}{s - p_k} \Leftrightarrow x(t) = \sum_k A_k e^{p_k t}\, u(t)}$$

Where the expansion coefficients are computed as

$$\boldsymbol{A_k = X(s)(s - p_k)|_{s=p_k}}$$

[Ex 3.14] Find the causal inverse of the proper rational function , and use MATLAB to validate your answer.

$$X(s) = \frac{3s + 5}{s^2 + 3s + 2}$$

Simple Complex Conjugate Poles

The partial fraction expansion of a proper rational fn.

$$\boldsymbol{X(s) = \frac{N(s)}{(s+\alpha)^2 + \Omega_0^2}} = \frac{N(s)}{(s+\alpha-j\Omega_0)(s+\alpha+j\Omega_0)}$$

with complex conjugate poles is given by

$$\boldsymbol{X(s) = \frac{A}{s+\alpha-j\Omega_0} + \frac{A^*}{s+\alpha+j\Omega_0}}$$

where

$$\boldsymbol{A = X(s)(s+\alpha-j\Omega_0)|_{s=-\alpha+j\Omega_0} = |A|e^{j\theta}}$$

so that the inverse is the function

$$\boldsymbol{x(t) = 2|A|e^{-\alpha t}\cos(\Omega_0 t + \theta)\, u(t)}$$

[Ex 3.15] Find the causal inverse of the proper rational function, and use MATLAB to validate your answer.

$$X(s) = \frac{2s+3}{s^2+2s+4}$$

Double Real Poles

If a proper rational function has double real poles

$$\boldsymbol{X(s) = \frac{N(s)}{(s+\alpha)^2}} = \frac{a}{(s+\alpha)^2} + \frac{b}{s+\alpha}$$

then its inverse is

$$\boldsymbol{x(t) = [ate^{-\alpha t} + be^{-\alpha t}]u(t)}$$

where a can be computed as

$$\boldsymbol{a = X(s)(s+\alpha)^2|_{s=-\alpha}}$$

After replacing it, b is found by computing $X(s_0)$ for a value $s_0 \neq \alpha$.

[Ex 3.16] Find the causal inverse of the following function

$$X(s) = \frac{4}{s(s+2)^2}$$

$e^{-\rho s}$ Terms

When $X(s)$ has exponentials $e^{-\rho s}$ in the numerator or denominator, **ignore these terms and perform partial fraction expansion on the rest**

$$\boldsymbol{X(s) = \frac{N(s)}{D(s)(1 \pm e^{-\alpha s})} = \frac{N(s)}{D(s)} \mp \frac{N(s)e^{-\alpha s}}{D(s)} + \frac{N(s)e^{-2\alpha s}}{D(s)} \mp \cdots}$$

If $f(t)$ is the inverse of $N(s)/D(s)$, then

$$\boldsymbol{x(t) = f(t) \mp f(t-\alpha) + f(t-2\alpha) \mp \cdots}$$

[Ex 3.19] Find the causal inverse of

$$H(s) = \frac{1 - e^{-s}}{(s+1)(1 - e^{-2s})}$$

Inverse of Two-Sided Laplace Transforms

- Pay attention to the **region of convergence** and to the **location of poles with respect to the $j\Omega$ axis**.
  - **Causal signal**: ROC as a plane to the **right of all the poles**
  - **Anti-causal signal**: ROC as a plane to the **left of all poles**
  - **Two-sided signal**: ROC as a region between **poles on the right** and **the poles on the left**

[Ex 3.21] Consider the transfer function

$$H(s) = \frac{s}{(s+2)(s-1)}$$

Find out how many impulse responses can be connected with $H(s)$ by considering different ROC and by determining in which cases the system with $H(s)$ as its transfer function is BIBO stable.

LTI Systems Represented by ODEs

The response $y(t)$ of a system represented by an Nth-order differential equation with constant coefficients

$$\boldsymbol{y^{(N)}(t) + \sum_{k=0}^{N-1} a_k y^{(k)}(t) = \sum_{l=0}^{M} b_l x^{(l)}(t) \qquad N > M}$$

where $x(t)$ is the input and the initial conditions

$$\boldsymbol{\{y^{(k)}(t), \qquad 0 \le k \le N-1\}}$$

is obtained by inverting the Laplace transform

$$\boldsymbol{Y(s) = \frac{B(s)}{A(s)} X(s) + \frac{1}{A(s)} I(s)}$$

where $Y(s) = \mathcal{L}[y(t)]$, $X(s) = \mathcal{L}[x(t)]$ and

$$\boldsymbol{A(s) = \sum_{k=0}^{N} a_k s^k, B(s) = \sum_{l=0}^{M} b_l s^l, I(s) = \sum_{k=1}^{N} a_k \left(\sum_{m=0}^{k-1} s^{k-m-1} y^{(m)}(0) \right)}$$

LTI Systems Represented by ODEs

Let $H(s) = B(s)/A(s)$ and $H_1(s) = 1/A(s)$, the response $y(t) = \mathcal{L}^{-1}[Y(s)]$ of the system is obtained by

$$\boldsymbol{Y(s) = H(s)X(s) + H_1(s)I(s)}$$

which gives

$$\boldsymbol{y(t) = y_{zs}(t) + y_{zi}(t) = \mathcal{L}^{-1}[H(s)X(s)] + \mathcal{L}^{-1}[H_1(s)I(s)]}$$

In terms of convolution integrals,

$$\boldsymbol{y(t) = \int_0^t x(\tau)h(t-\tau)\,d\tau + \int_0^t i(\tau)h_1(t-\tau)\,d\tau}$$

where

$$\boldsymbol{i(t) = \mathcal{L}^{-1}[I(s)] = \sum_{k=1}^{N} a_k \left(\sum_{m=0}^{k-1} y^{(m)}(0)\delta^{k-m-1}(t) \right)}$$

Transient and Steady-State Responses

If the **poles of** the Laplace transform of **the output $\boldsymbol{Y(s)}$** of an LTI system are **open left-hand $\boldsymbol{s}$-plane**, the steady-state response is

$$\boldsymbol{y_{ss}(t) = \lim_{t\to\infty} y(t) = 0}$$

When solving ODEs using Laplace transform:

(i) The **steady-state component** is given by the inverse Laplace transforms of **terms that have simple poles in the $\boldsymbol{j\Omega}$ axis**.

(ii) The **transient response** is given by the inverse Laplace transforms of **terms with poles in the left-hand $\boldsymbol{s}$-plane**.

(iii) Multiple poles in the $\boldsymbol{j\Omega}$ axis and poles in the right-hand $\boldsymbol{s}$-plane give terms that will increase as t increases making the **complete response unbounded**.

Examples

[Ex 3.23] Consider a second-order differential equation,

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Find the impulse response $h(t)$ and the unit-step response $s(t)$ of the system.

[Ex 3.24] Consider again the second-order differential equation in Ex 3.22 but now with initial conditions $y(0) = 1$ and $dy(t)/dt|_{t=0} = 0$, and $x(t) = u(t)$. Find the complete response $y(t)$. Could we find the impulse response $h(t)$ from the response? How could we do it?

Computation of Convolution Integral

The Laplace transform of the convolution $y(t) = [x * h](t)$ is given by the product

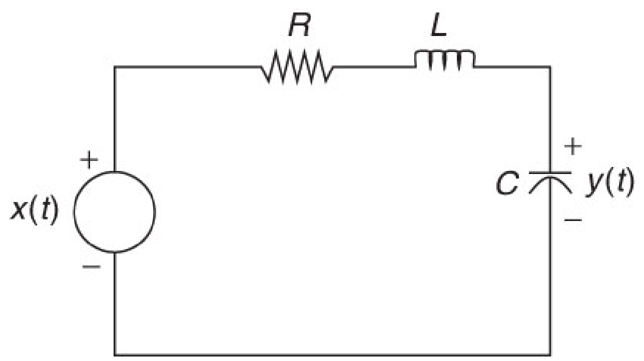
$$\boldsymbol{Y(s) = X(s)H(s)}$$

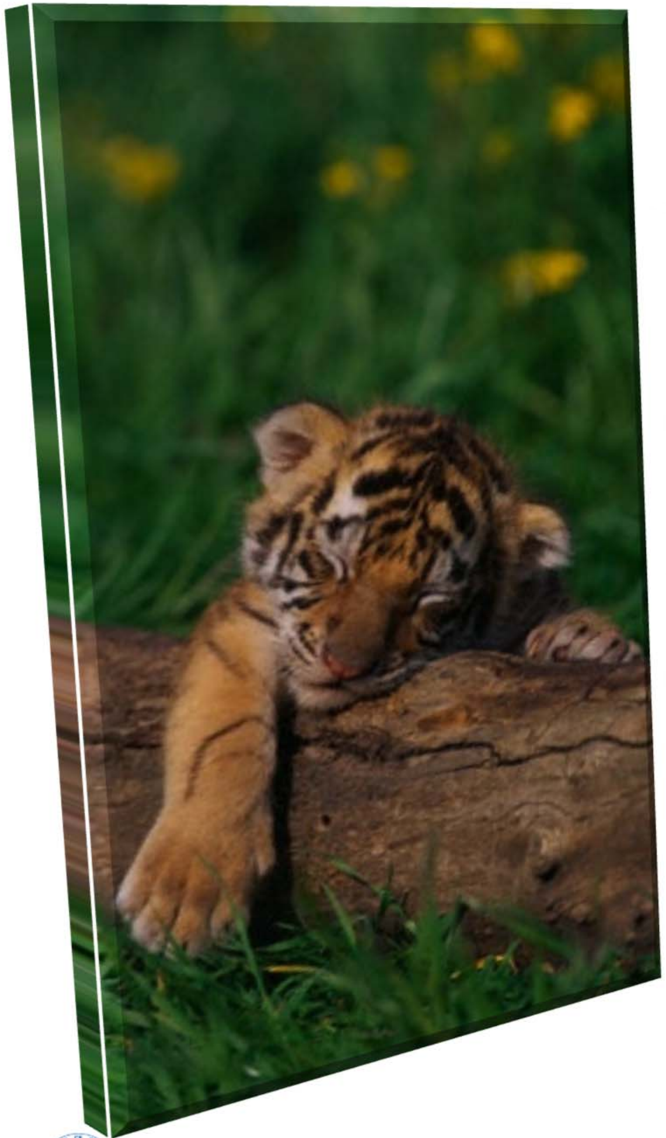
where $X(s) = \mathcal{L}[x(t)]$ and $H(s) = \mathcal{L}[h(t)]$.

The **transfer function** of the system $H(s)$ is defined as

$$\boldsymbol{H(s) = \mathcal{L}[h(t)] = \frac{Y(s)}{X(s)}}$$

[Ex 3.28] Consider an RLC circuit in series with input a voltage source $x(t)$ and as output the voltage $y(t)$ across the capacitor. Find its impulse response $h(t)$ and its unit-step response $s(t)$. Let $LC = 1$ and $R/L = 2$.





Thank You