**Mathematical Statistics (I)** **Assignment 1**

**Spring, 2021**

1. Define $\mathcal{F} = \cap\{\mathcal{G}; \mathcal{F}_0 \subset \mathcal{G}$, where $\mathcal{G}$ is a $\sigma$-field $\}$.
Show that $\mathcal{F}$ is a $\sigma$-field.
( $\mathcal{F}$ is the smallest $\sigma$-field which contains $\mathcal{F}_0$ and is referred to as the $\sigma$-field generated by $\mathcal{F}_0$.)

2. Prove Theorem 1.3 for decreasing sequence of events.

3. Consider the sample space is $\mathcal{C} = R = (-\infty, \infty)$ with the Borel $\sigma$-field $\mathcal{B}$. Define a set function $Q(B) = \int_B e^{-|x|}dx \quad \forall B \in \mathcal{B}$. Show that $Q$ is not a probability defined on $\mathcal{B}$. What constant do we multiply the integrand by to make it a Probability?

4. A median of a distribution of one random variable $X$ of the discrete or continuous type is a value of $x$ such that $P(X < x) \leq \frac{1}{2}$ and $P(X \leq x) \geq \frac{1}{2}$. Find a median of each of the following distributions:
(a) $p(x) = \frac{4!}{x!(4-x)!}(\frac{1}{4})^x(\frac{3}{4})^{4-x}, \ \ x = 0, 1, 2, 3, 4$, zero elsewhere.
(b) $f(x) = 3x^2 I(0 < x < 1)$.
(c) $f(x) = \frac{1}{\pi(1+x^2)} I(-\infty < x < \infty)$.

5. Let $X$ have the uniform pdf $f(x) = \frac{1}{\pi} I(-\frac{\pi}{2} < x < \frac{\pi}{2})$. Find the pdf of $Y = \tan X$.

6. Let $f(x) = \frac{1}{4} I(-1 < x < 3)$ be the pdf of random variable $X$. Find the cdf and pdf of $Y = X^2$.

7. A median of a distribution of a random variable $X$ of the discrete or continuous type is a value of $x$ such that $P(X < x) \leq \frac{1}{2}$ and $P(X \leq x) \geq \frac{1}{2}$. Let $X$ be a random variable of the continuous type that has pdf $f(x)$. If $m$ is the unique median of the distribution of $X$ and $b$ is a real constant, show that

$$E(|X - b|) = E(|X - m|) + 2\int_m^b (b - x) f(x) dx,$$

provided that the expectations exist. For what value of $b$ is $E(|X - b|)$ a minimum?

8. Let $X$ be a random variable of the continuous type with pdf $f(x)$ of which support is $(0, b)$, where $0 < b < \infty$. Show that

$$E(X) = \int_0^b (1 - F(x))dx,$$

where $F(x)$ is the cdf of $X$.

9. Let $X$ have the pdf $f(x) = \lambda e^{-\lambda x} I(0 < x < \infty)$. Find the mgf, the mean, and the variance of $X$.

10. Let $X$ and $Y$ have the joint pdf

$$f(x, y) = 6(1 - x - y)I(x + y < 1, 0 < x, 0 < y).$$

Compute $P(2X + 3Y < 1)$ and $E(XY + 2X^2)$.

11. Let $X_1$ and $X_2$ be continuous random variables with the joint pdf, $f_{X_1,X_2}(x_1, x_2)$. Let $Y = X_1 + X_2$. Show that the pdf $f_Y(y)$ can be obtained by

$$f_Y(y) = \int f_{X_1,X_2}(y - x, x)dx.$$

12. Let the joint pdf of random variables $X$ and $Y$ be

$$f(x, y) = 2I(0 < x < y, 0 < y < 1).$$

(a) Find the conditional means $E(X|Y), E(Y|X)$.
(b) Find the correlation coefficient $\rho$ of $X$ and $Y$.

13. Let the joint pdf of random variables $X$ and $Y$ be

$$f(x, y) = \frac{1}{\pi} 2I((x - 1)^2 + (y + 2)^2 < 1).$$

(a) Find the marginal pdf's of $X$ and $Y$.
(b) Are they independent?

14. Suppose that $X_1, X_2, X_3$, and $X_4$ are four independent random variables with the same pdf.

$$f(x) = 3(1 - x)^2 I(0 < x < 1).$$

Let $Y$ be the smallest of the four, that is, $Y = \min\{X_1, X_2, X_3, X_4\}$. Find the cdf and pdf of $Y$.

15. Let $X_1, X_2, X_3, X_4$ have the joint pdf $f(x_1, x_2, x_3, x_4) = 24I(0 < x_1 < x_2 < x_3 < x_4 < 1)$. Let

$$Y_1 = X_1/X_2, \quad Y_2 = X_2/X_3, \quad Y_3 = X_3/X_4, \quad Y_4 = X_4.$$

Show that $Y_1, Y_2, Y_3, Y_4$ are mutually independent.

16. Let $(X_1, \cdots, X_{k-1})'$ have a multinomial distribution, that is,

$$(X_1, \cdots, X_{k-1})' \sim m(n, p_1, \cdots, p_{k-1}).$$

(a) Find the mgf of $(X_2, \cdots, X_{k-1})'$.
(b) What is the pmf of $(X_2, \cdots, X_{k-1})'$?
(c) Determine the conditional pmf of $X_1$ given that $X_2 = x_2, \cdots, X_{k-1} = x_{k-1}$.
(d) Find the conditional expectation $E(X_1 | X_2 = x_2, \cdots, X_{k-1} = x_{k-1})$.

17. Let $(X_1, X_2)' \sim m(n, p_1, p_2)$ (trinomial distribution). Find $Var(X_1 - X_2)$.