Quiz 2: Control Systems Eng.
2019/05/14

Student Number: [ ] Name: **Solution**

1. (20 points $= 2\times 10$ pts)

(1)

- Forward path gain: $T_1 = G_1G_2, \quad T_2 = G_1G_3$

- Loop gain: (1) G_1H_1 (2) $-G_1G_2H_2$ (3) $-G_1G_3H_2$

- Nontouching loops taken two at a time: 0

$\Delta_1 = 1$

$\Delta_2 = 1$

$$\frac{Y}{R} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta} = \frac{G_1G_2 + G_1G_3}{1 - G_1H_1 + G_1G_2H_2 + G_1G_3H_2}$$

(2)

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1 + 2u\end{aligned} \quad \rightarrow \quad \begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}-1 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} + \begin{pmatrix}0 \\ 2\end{pmatrix}u$$

$$y = 0.5x_1 + 0.5\dot{x}_1 = 0.5x_1 + 0.5(-x_1 + x_2) = 0x_1 + 0.5x_2 \quad \rightarrow \quad y = \begin{pmatrix}0 & 0.5\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix}$$

$$\Rightarrow \quad A = \begin{pmatrix}-1 & 1 \\ -1 & 0\end{pmatrix}, \quad B = \begin{pmatrix}0 \\ 2\end{pmatrix}, \quad C = \begin{pmatrix}0 & 0.5\end{pmatrix}, \quad D = 0$$

2. (20 points $= 2\times 10$ pts)

(1)

$$R = \frac{1}{s}, \quad E = R - Y$$

$$Y = \frac{2}{s+2}\left[R + (R - Y)\frac{3}{s}\right] = \frac{2}{s+2}\left[\left(1 + \frac{3}{s}\right)R - Y\left(\frac{3}{s}\right)\right]$$

$$Y + \frac{6}{s(s+2)}Y = \frac{2}{s+2}\left(1 + \frac{3}{s}\right)R$$

$$\frac{s(s+2)+6}{s(s+2)}Y = \frac{2(s+3)}{s(s+2)}R$$

$$\frac{Y(s)}{R(s)} = \frac{2(s+3)}{s(s+2)} \cdot \frac{s(s+2)}{s^2 + 2s + 6} = \frac{2(s+3)}{s^2 + 2s + 6}$$

(2)

$$Y(s)=\frac{2(s+3)}{s^2+2s+6}R(s)$$

$$E(s)=R(s)-Y(s)=R(s)-\frac{2(s+3)}{s^2+2s+6}R(s)=R(s)\left[1-\frac{2(s+3)}{s^2+2s+6}\right]$$

$$=R(s)\left(\frac{s^2+2s+6-2s-6}{s^2+2s+6}\right)=R(s)\left(\frac{s^2}{s^2+2s+6}\right)$$

$$e_{ss}=\lim_{s\to0}sE(s)=\lim_{s\to0}s\frac{1}{s}\left(\frac{s^2}{s^2+2s+6}\right)=\lim_{s\to0}\left(\frac{s^2}{s^2+2s+6}\right)=0$$

3. (20 points)

$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\dfrac{K}{s(s^2+s+1)(s+4)}}{1+\dfrac{K}{s(s^2+s+1)(s+4)}}=\frac{K}{s(s^2+s+1)(s+4)+K}$$

The characteristic equation is:

$$Q(s)=s(s^2+s+1)(s+4)+K=s^4+5s^3+5s^2+4s+K$$

$$\begin{array}{c|ccc}
s^4 & 1 & 5 & K \\
s^3 & 5 & 4 & 0 \\
s^2 & \frac{21}{5} & K & \\
s^1 & \frac{5}{21}\left(\frac{84}{5}-5K\right) & & \\
s^0 & K & &
\end{array}$$

$$b_1=-\frac{1}{5}\begin{vmatrix}1 & 5\\ 5 & 4\end{vmatrix}=\frac{21}{5},\quad b_2=-\frac{1}{5}\begin{vmatrix}1 & K\\ 5 & 0\end{vmatrix}=K$$

$$c_1=-\frac{1}{\frac{21}{5}}\begin{vmatrix}5 & 4\\ \frac{21}{5} & K\end{vmatrix}=\frac{5}{21}\left(\frac{84}{5}-5K\right)\qquad c_2=0$$

Stable condition: $\frac{5}{21}\left(\frac{84}{5}-5K\right)>0$ and $K>0$

$5K<\frac{84}{5}\ \rightarrow\ K<\frac{84}{25}$ and $K>0$

$\therefore\quad 0<K<\frac{84}{25}$

4. (20 points)

$$1+G(s)=1+\frac{K(1+s)}{s(1+Ts)(1+2s)}=0,\quad s(1+Ts)(1+2s)+Ks+K=0,\quad 2Ts^3+(2+T)s^2+(K+1)s+K=0$$

$$\begin{array}{lcc} s^3 & 2T & K+1 \\ s^2 & (2+T) & K \\ s^1 & \dfrac{(K+1)(2+T)-2KT}{(2+T)} & 0 \\ s^0 & K & \end{array}$$

$$b_1=-\frac{1}{(2+T)}\begin{vmatrix} 2T & K+1 \\ (2+T) & K \end{vmatrix}=\frac{(K+1)(2+T)-2KT}{(2+T)}$$

$$b_2=-\frac{1}{(2+T)}\begin{vmatrix} 2T & 0 \\ (2+T) & 0 \end{vmatrix}=0$$

$$c_1=-\frac{1}{b_1}\begin{vmatrix} (2+T) & K \\ b_1 & 0 \end{vmatrix}=\frac{1}{b_1}(Kb_1)=K,\quad c_2=0$$

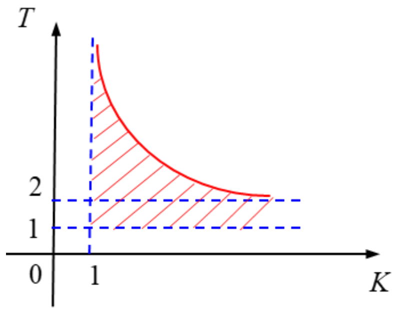
$2T>0,\ \ (2+T)>0\ \ \rightarrow$ From conditions: $T>1,\ K>1$

$$\frac{(K+1)(2+T)-2KT}{(2+T)}>0$$

$$2K+2+T-KT>0,\quad 2(K+1)-T(K-1)>0$$

$$T(K-1)<2(K+1)$$

$$T<\frac{2(K+1)}{(K-1)}=2+\frac{4}{(K-1)}\quad\rightarrow\quad T>1,\ K>1$$



5. (20 points)

(i) Find $E(s)$

$$Y=G(R-HY)=GR-GHY\ \rightarrow\ Y(1+GH)=GR\ \rightarrow\ Y=\frac{G}{1+GH}R$$

$$E=R-\frac{HG}{1+GH}R=\frac{R(1+GH)-HGR}{1+GH}=\frac{R}{1+GH}\quad\text{with}\quad R(s)=\frac{1}{s}$$

$$E(s)=\left(\frac{1}{s}\right)\cdot\frac{1}{1+GH}=\left(\frac{1}{s}\right)\cdot\frac{1}{1+\left(\frac{s+1}{s^2+5s+\alpha}\right)\left(\frac{1}{s+4}\right)}=\left(\frac{1}{s}\right)\cdot\frac{(s^2+5s+\alpha)(s+4)}{(s^2+5s+\alpha)(s+4)+(s+1)}$$

(ii) The steady-state error:

$$\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)$$

$$=\lim_{s\to 0}\frac{(s^2+5s+\alpha)(s+4)}{(s^2+5s+\alpha)(s+4)+(s+1)}$$

$$=\frac{(\alpha)(4)}{(\alpha)(4)+(1)}=\frac{4\alpha}{4\alpha+1}=0$$

$$\therefore\quad \alpha=0$$

OR

$$\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)$$

$$=\lim_{s\to 0}\frac{1}{1+\left(\frac{s+1}{s^2+5s+\alpha}\right)\left(\frac{1}{s+4}\right)}=\frac{1}{1+\left(\frac{1}{\alpha}\right)\left(\frac{1}{4}\right)}$$

$$=\frac{4\alpha}{4\alpha+1}=0$$

$$\therefore\quad \alpha=0$$