Chapter 14　Boundary-Value Problems in Other Coordinates Systems

14.1 Problems Involving Laplace's Equation in Polar Coordinates

■ Introduction

- 원판, 원통, 공 모양의 영역에서의 해석 → 극좌표, 원통좌표, 구면좌표를 사용하는 것이 유리

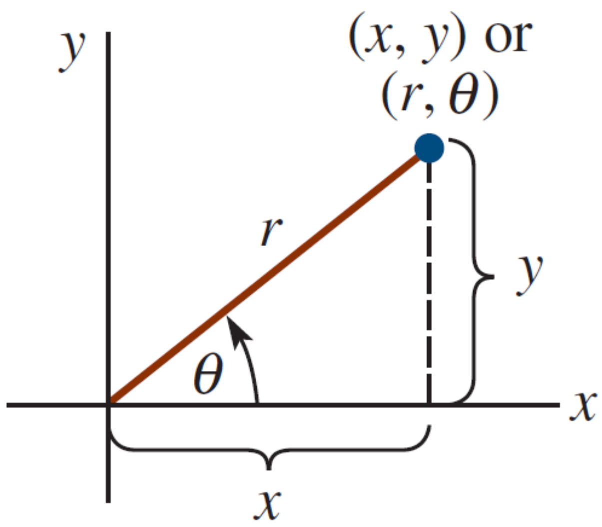
- 정상상태의 해석을 위하여 Laplacian 을 표현하는 방법을 고려함

■ Laplacian in Polar Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x},$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$



※ $r^2 = x^2 + y^2 \rightarrow 2r\frac{\partial r}{\partial x} = 2x \rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$

※ $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta$

※ $\tan\theta = \frac{y}{x} \rightarrow \sec^2\theta\,\frac{\partial\theta}{\partial x} = y\left(-x^{-2}\right) \rightarrow \frac{\partial\theta}{\partial x} = \frac{-y\cos^2\theta}{x^2} = -\frac{\sin\theta}{r^2}$

※ $\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{-y\frac{1}{x^2}}{\frac{x^2+y^2}{x^2}} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}\left(= \frac{y}{r^2}\right)$

※ $r^2 = x^2 + y^2 \rightarrow 2r\frac{\partial r}{\partial y} = 2y \rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$

※ $\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta$

※ $\tan\theta = \frac{y}{x} \rightarrow \sec^2\theta\,\frac{\partial\theta}{\partial y} = \frac{1}{x} \rightarrow \frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{x} = \frac{\cos\theta}{r}$

※ $\frac{\partial\theta}{\partial y} = \frac{\partial}{\partial y}tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{\frac{1}{x}}{\frac{x^2+y^2}{x^2}} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}\left(= \frac{y}{r^2}\right)$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) \\
&= \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x} \\
&= (\cos\theta\frac{\partial}{\partial r}(\frac{\partial u}{\partial r}) + \frac{\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial}{\partial r}(\frac{\partial u}{\partial \theta}))\cos\theta \\
&\quad + (-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial}{\partial \theta}(\frac{\partial u}{\partial r}) - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}(\frac{\partial u}{\partial \theta}))(-\frac{\sin\theta}{r})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right) \\
&= \frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial y} \\
&= \left(\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) - \frac{\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right)\right)\sin\theta \\
&\quad + \left(\cos\theta\frac{\partial u}{\partial r} + \sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right) - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\right)\left(\frac{\cos\theta}{r}\right)
\end{aligned}
$$

$$
\frac{\partial^2 u}{\partial x^2} = \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \tag{1}
$$

$$
\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \tag{2}
$$

식 (1) + 식 (2); 극좌표에서 u 에 대한 Laplacian;

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

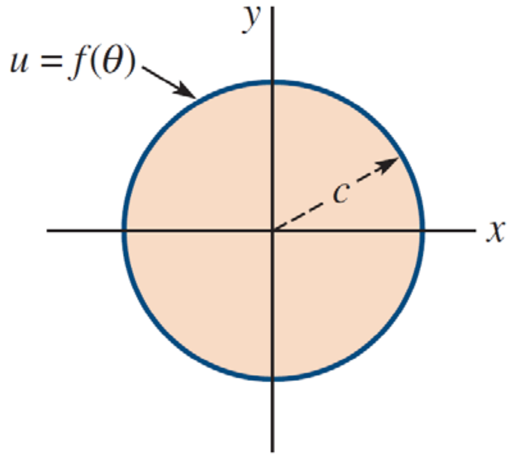
극좌표계에서 Laplace 방정식은

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \qquad (3)$$

Example 1   Steady Temperatures in a Circular Plate

반경 c 인 둥근 원판에서 둘레의 온도가 $u(c,\theta)=f(\theta),\ 0<\theta<2\pi$ 일 때

정상상태 온도 $u(r,\theta)$ 구하기.



Solution

- 경계조건은 Nonhomogeneous 임
- 물리학적으로, $u(r,\theta)$ 은 연속(continuous), 유계(Bounded)임.
- $u(r,\theta)$ 은 유일한 값을 가져야함.
- 기하학적 조건에서 $u(r,\theta)=u(r,\theta+2\pi)$
- 변수분리형 함수 $u(r,\theta)=R(r)\Theta(\theta)$ 를 시도함, 단 $\Theta(\theta)=\Theta(\theta+2\pi)$

주어진 식 (3)에 대입하면

$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda.$$

$$\rightarrow\ r^2R''+rR'-\lambda R=0 \qquad (4)$$

$$\Theta''+\lambda\Theta=0 \qquad (5)$$

식 (5)는 다음의 형태이다

$$\Theta'' + \lambda\Theta = 0, \ \Theta(\theta) = \Theta(\theta + 2\pi). \tag{6}$$

(5)를 분리상수의 부호에 따라 각각 풀면,

$$\Theta(\theta) = c_1 + c_2\theta, \qquad \lambda = 0 \tag{7}$$

$$\Theta(\theta) = c_1 \cosh\alpha\theta + c_2 \sinh\alpha\theta, \qquad \lambda = -\alpha^2 < 0 \tag{8}$$

$$\Theta(\theta) = c_1 \cos\alpha\theta + c_2 \sin\alpha\theta, \qquad \lambda = \alpha^2 > 0 \tag{9}$$

(7)은 $c_2 = 0, \ c_1 \neq 0$ 인 경우만 2π-peroidic 조건을 만족시킴 → $\lambda_0 = 0$

(8)은 2π-peroidic 조건을 만족시킬 수 없으므로 제외함

(9)는 2π-peroidic 조건을 만족시킴. → $\lambda_n = n^2, \ n = 1, 2, \ldots$

따라서 (6)의 고유함수는

$\Theta(\theta) = c_1, \ n = 0,$ and $\Theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta, \ n = 1, 2, \ldots.$

$\lambda_n = n^2,\ n = 0,1,2,\dots$ 일 때 식 (4) $r^2R'' + rR' - \lambda R = 0$ (Cauchy-Euler equi-dimensional equation) 의 해는

$$R(r) = c_3 + c_4 \ln r,\ n = 0, \tag{10}$$

$$R(r) = c_3 r^n + c_4 r^{-n},\ n = 1,2,\dots. \tag{11}$$

여기서 $r = 0$ 일 때 유한한 값을 갖기 위해 $c_4 = 0$ 이어야 함.

※ $\lambda_n = n^2,\ n = 0$, $r^2R'' + rR' - \lambda R = 0$

r^m, $m(m-1) + m = 0 \rightarrow m = 0$(중근) $\therefore R(r) = c_3 + c_4 \ln r,\ n = 0$

※ $\lambda_n = n^2,\ n = 1,2,\dots$, $r^2R'' + rR' - \lambda R = 0$

r^m, $m(m-1) + m - n^2 = 0 \rightarrow m = \pm n$ $\therefore R(r) = c_3 r^n + c_4 r^{-n},\ n = 1,2,\dots.$

따라서

$u_0 = A_0,\ n = 0,$ and $u_n = r^n \left(A_n \cos n\theta + B_n \sin n\theta\right),\ n = 1,2,\dots$

해를 중첩하면

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta). \tag{12}$$

(12)에 경계조건을 대입하면

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} c^n \left(A_n \cos n\theta + B_n \sin n\theta\right)$$

여기서 계수는

$$A_0 = \frac{a_0}{2},\ c^n A_n = a_n, \quad \text{and} \quad c^n B_n = b_n.$$

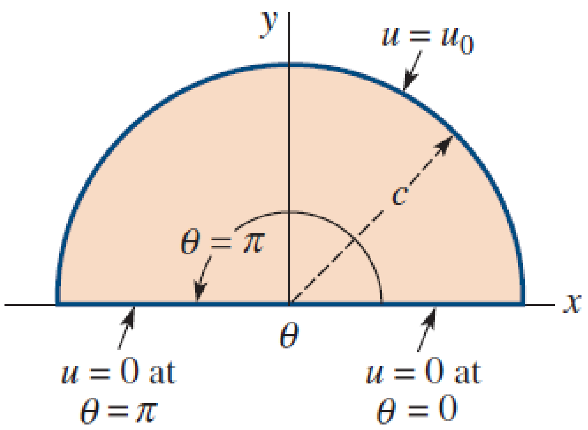
$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) d\theta$$

$$A_n = \frac{1}{c^n \pi}\int_0^{2\pi} f(\theta) \cos n\theta d\theta$$

$$B_n = \frac{1}{c^n \pi}\int_0^{2\pi} f(\theta) \sin \theta d\theta.$$

Example 2 Steady Temperatures in a Semicircular Plate

그림의 반원판에서 정상상태의 온도분포 $u(r,\theta)$ 구하기



Solution

시스템 방정식은

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0,\ 0<\theta<\pi,\ 0<r<c$$

$$u(c,\theta)=u_0, \qquad 0<\theta<\pi$$

$$u(r,0)=0, \qquad u(r,\pi)=0, \qquad 0<r<c.$$

$u(r,\theta)=R(r)\Theta(\theta)$ 로 정의하고 변수분리를 하면,

$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda$$

$$r^2R''+rR'-\lambda R=0 \tag{16}$$

$$\Theta''+\lambda\Theta=0. \tag{17}$$

$u(r,0)=R(r)\Theta(0)=0\rightarrow\Theta(0)=0, \qquad u(r,\pi)=R(r)\Theta(\pi)=0\rightarrow\Theta(\pi)=0 \quad [0<r<c]$

$$\Theta'' + \lambda\Theta = 0,\ \Theta(0) = 0,\ \Theta(\pi) = 0. \tag{18}$$

$$\begin{cases} r^2R'' + rR' - \lambda R = 0,\ 0 \le r \le c,\ 0 \le \theta \le \pi \\ \Theta'' + \lambda\Theta = 0,\ \Theta(0) = \Theta(\pi) = 0 \end{cases}$$

① $\lambda = 0$

$$\Rightarrow \begin{cases} \Theta''(\theta) = 0,\ \Theta(0) = \Theta(\pi) = 0 \\ r^2R''(r) + rR'(r) = 0 \end{cases}$$

$\Theta(\theta) = c_1 + c_2\theta \rightarrow 0 = \Theta(0) = c_1,\ 0 = \Theta(\pi) = c_1 + c_2\pi \therefore \Theta(\theta) = 0$

$r^2R''(r) + rR'(r) = 0$의 특성(보조)방정식은 $m(m-1) + m = 0$, 특성근은 $m = 0$(중근)이므로

$r^2R''(r) + rR'(r) = 0$의 일반해는

$$R(r) = c_3r^0 + c_4r^0\ln r = c_3 + c_4\ln r$$

정상상태의 해(steady-state) $u(r,\theta) = R(r)\Theta(\theta)$ 는 $r = 0$에서도 유계이므로 $R(r) = c_3 + c_4\ln r$ 에서

$c_4 = 0$이어야 한다. 따라서 $\lambda = 0$일 때 해는

$$u_0(r,\theta) = R(r)\Theta(\theta) = c_2c_3 = A_0 \quad \text{(임의의 상수)}$$

② $\lambda = -\alpha^2 < 0 \, (\alpha > 0)$

$$\Rightarrow \begin{cases} \Theta''(\theta) - \alpha^2 \Theta(\theta) = 0, \Theta(0) = \Theta(\pi) = 0 \\ r^2 R''(r) + rR'(r) + \alpha^2 R(r) = 0 \end{cases}$$

$\Theta(\theta) = c_1 \cosh(\alpha\theta) + c_2 \sinh(\alpha\theta)$
$\rightarrow 0 = \Theta(0) = c_1 \rightarrow c_1 = 0$
$\Rightarrow 0 = \Theta(\pi) = c_1 \cosh(\alpha\pi) + c_2 \sinh(\alpha\pi) = c_2 \sinh(\alpha\pi) \; [\alpha > 0] \rightarrow c_2 = 0$
$\therefore \Theta(\theta) = 0$

$r^2 R''(r) + rR'(r) + \alpha^2 R(r) = 0$의 특성방정식은 $m(m-1) + m + \alpha^2 = 0$, 특성근은 $m = \pm \alpha i$이므로

$r^2 R''(r) + rR'(r) + \alpha^2 R(r) = 0$의 일반해는

$$R(r) = c_3 \cos(\alpha \ln r) + c_4 \sin(\alpha \ln r)$$

정상상태의 해(steady-state) $u(r,\theta) = R(r)\Theta(\theta)$는 $r = 0$에서도 유계이어야 하므로

$R(r) = c_3 \cos(\alpha \ln r) + c_4 \sin(\alpha \ln r)$에서 $c_3 = 0, c_4 = 0$이어야 한다. 즉, $R(r) = 0$

따라서 $\lambda = -\alpha^2 < 0$일 때 해는

$$u(r,\theta) = R(r)\Theta(\theta) = 0$$

③ $\lambda = \alpha^2 > 0\ (\alpha > 0)$

$$\Rightarrow \begin{cases} \Theta''(\theta) + \alpha^2\Theta(\theta) = 0,\ \Theta(0) = \Theta(\pi) = 0 \\ r^2R''(r) + rR'(r) - \alpha^2R(r) = 0 \end{cases}$$

$\Theta(\theta) = c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta)$
$\rightarrow 0 = \Theta(0) = c_1 \rightarrow c_1 = 0$
$\Rightarrow 0 = \Theta(\pi) = c_2\sin(\alpha\pi)\ [\alpha > 0] \rightarrow \alpha n\ (n = 1, 2, \cdots)$
$\therefore\ \Theta(\theta) = c_1\sin(n\theta)\ \ (n = 1, 2, \cdots)$

$r^2R''(r) + rR'(r) - \alpha^2R(r) = 0$의 특성방정식은 $m(m-1) + m - \alpha^2 = 0$, 특성근은 $m = \pm\alpha = \pm n$이므로

$r^2R''(r) + rR'(r) + \alpha^2R(r) = 0$의 일반해는

$$R(r) = c_3r^{\alpha} + c_4r^{-\alpha} = c_3r^n + c_4r^{-n}\ (n = 1, 2, \cdots)$$

정상상태의 해(steady-state) $u(r,\theta) = R(r)\Theta(\theta)$는 $r = 0$에서도 유계이어야 하므로

$R(r) = c_3r^n + c_4r^{-n}\ (n = 1, 2, \cdots)$에서 $c_4 = 0$이어야 한다.

따라서 $\lambda = -\alpha^2 < 0$일 때 해는

$u_n(r,\theta) = R_n(r)\Theta_n(\theta) = c_3c_5r^n\sin(n\theta) = A_nr^n\sin(n\theta)\ (n = 1, 2, \cdots)$

이 해를 중첩하면

$$u(r,\theta)=\sum_{n=1}^{\infty}A_n r^n \sin n\theta$$

$r=c$ 에서의 경계조건 $u(c,\theta)=u_0$ 를 적용하면, Fourier sine series

$$u_0=\sum_{n=1}^{\infty}A_n c^n \sin n\theta$$

→ Fourier sine series 의 계수(coefficient) $A_n c^n=\frac{2}{\pi}\int_0^{\pi}u_0 \sin n\theta d\theta$ → $A_n=\frac{2u_0}{\pi c^n}\frac{1-(-1)^n}{n}$

따라서 최종해는: $u(r,\theta)=\frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\left(\frac{r}{c}\right)^n \sin n\theta.$