Mathematical Statistics (I) Assignment 2

Spring, 2021

1. Let X and Y have the joint pmf

$$p(x, y) = \frac{e^{-2}}{x!(y-x)!} I(x = 0, 1, \cdots, y, y = 0, 1, 2, cdots).$$

(a) Find the mgf $M(t_1, t_2)$ of $(X, Y)'$.
(b) Compute the means, the variances, and the correlation coefficient of X and Y.
(c) Determine the conditional expectation $E(X|Y = y)$.

2. Let X_1 and X_2 be independent variables and let $Y = X_1 + X_2$. Show that if $Y \sim \chi^2(r)$ and $X_1 \sim \chi^2(r_1)$, then $X_2 \sim \chi^2(r - r_1)$.

3. Consider a random variable X of continuous type with cdf $F(x)$ and pdf $f(x)$. The hazard rate is defined by

$$r(x) = \lim_{\Delta \to 0} \frac{P(x \leq X < x + \Delta | X \geq x)}{\Delta}.$$

In the case that X represent the failure time of an item, $r(x)$ is viewed as the rate of instantaneous failure at time $x > 0$.
(a) Show that $r(x) = f(x)/(1 - F(x))$.
(b) When $r(x) = cx^b$ (where $c > 0$ and $b \geq 0$ are constants), find the pdf of X.

4. Let $X_1 \sim N(6, 1)$ and $X_2 \sim N(7, 1)$ be two independent random variables.
Find $P(X_1 > X_2)$.

5. Let X and Y be two independent standard normal random variables. Find the mgf of the random variable $W = XY$.

6. Suppose that $X = (X_1, X_2)' \sim N_2(\mu, \Sigma)$. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.
(a) Find the distribution of $Y = (Y_1, Y_2)'$.
(b) Find a necessary and sufficient condition that Y_1 and Y_2 are independent.

7. Let X and Y be random variables with $\mu_X = 1, \mu_Y = 4, \sigma_X^2 = 4, \sigma_Y^2 = 6, \rho_{XY} = 1/2$. Find the mean and variance of $Z = 3X - 2Y$.

8. Let $X_1, \cdots, X_n$ be a random sample from a population with mean μ and variance $\sigma^2 > 0$ and let $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$ be the sample variance. Is S be an unbiased estimator of σ?