Chapter 5. Reduction of Multiple Subsystems

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Objectives

- How to reduce a block diagram of multiple subsystems to a single block representing the transfer function from input to output
- How to analyze and design transient response for a system consisting of multiple subsystems
- How to representing in state space a system consisting of multiple subsystems
- How to convert between alternate representations of a system in state space

5.1 Introduction

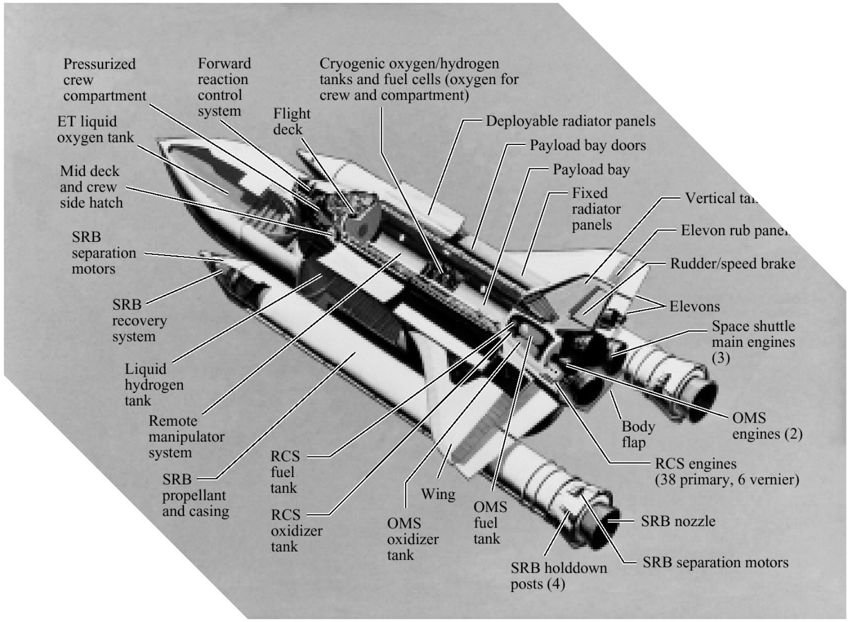
- Interconnection of many subsystems

  ⇒ frequency domain or state-space analysis and design

  ⇒ block diagrams / signal-flow graphs

- Block diagrams ⇐ frequency domain analysis

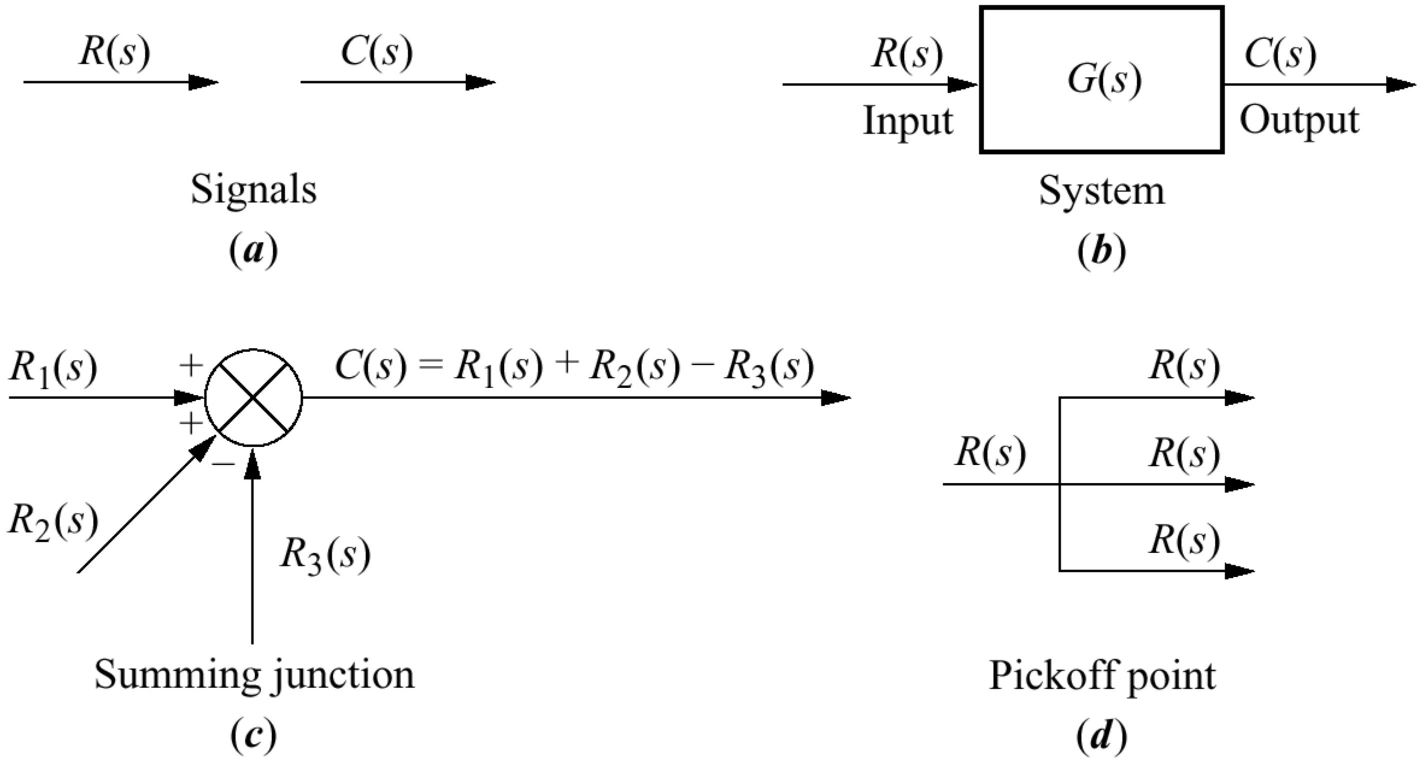
- Signal-flow graphs ⇐ state-space analysis



- The space shuttle consists of multiple subsystems.
- Can you identify those that are control systems, or parts of control systems?

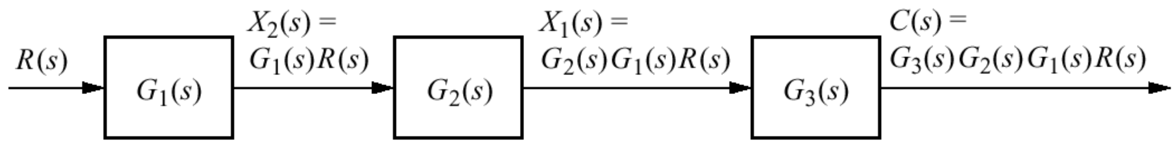
5.2 Block Diagrams

Figure 5.2
Components of a block diagram for a linear, time-invariant system

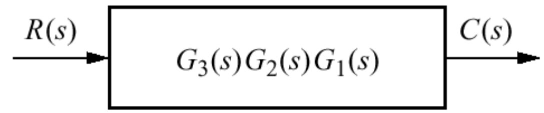


• *Cascade form*

a. Cascaded subsystems;
b. equivalent transfer function



(*a*)

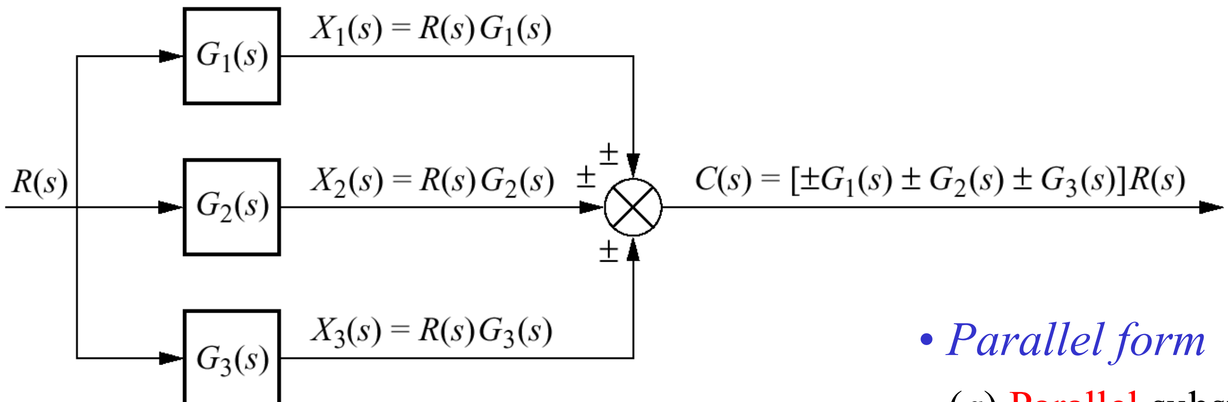


(*b*)

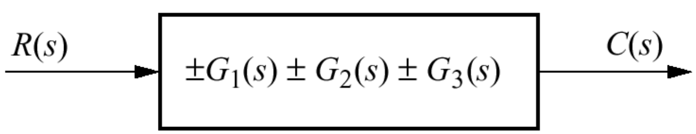
• *Parallel form*

(*a*) Parallel subsystems;

(*b*) equivalent transfer function



(*a*)



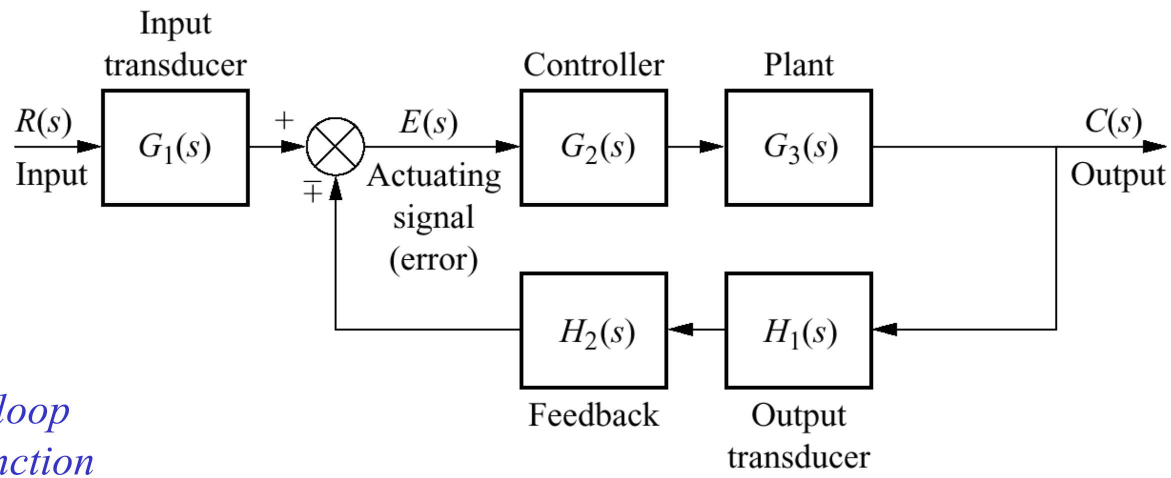
(*b*)

- *Feedback form*

  a. Feedback control system;
  b. simplified model;
  c. equivalent transfer function

$$G_e(s) = \frac{C(s)}{R(s)} = ?$$

$\Rightarrow$ *Closed-loop transfer function*

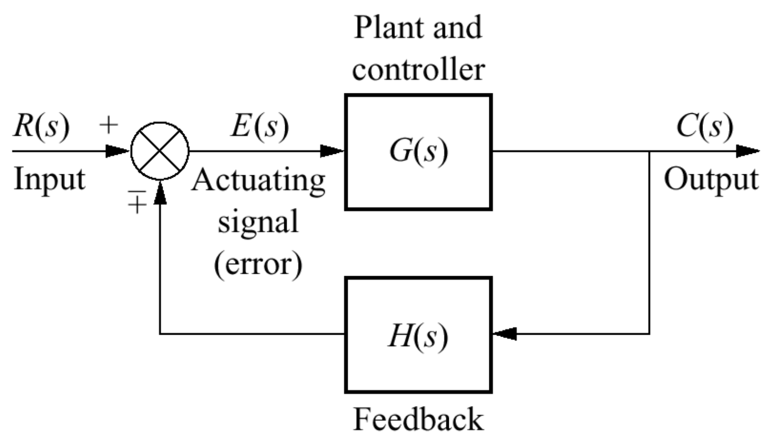


(*a*)

$$E(s) = R(s) \mp C(s)H(s)$$

$$C(s) = E(s)G(s) \Rightarrow E(s) = \frac{C(s)}{G(s)}$$

$$\frac{C(s)}{G(s)} = R(s) \mp C(s)H(s)$$

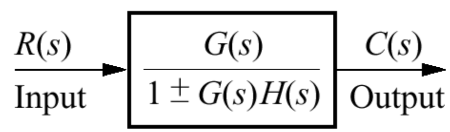


(*b*)

$$C(s) = G(s)R(s) \mp G(s)C(s)H(s),$$

$$C(s)[1 \pm G(s)H(s)] = G(s)R(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm \underline{G(s)H(s)}}$$



(*c*)

$\Rightarrow$ *open-loop transfer function* or *loop gain*

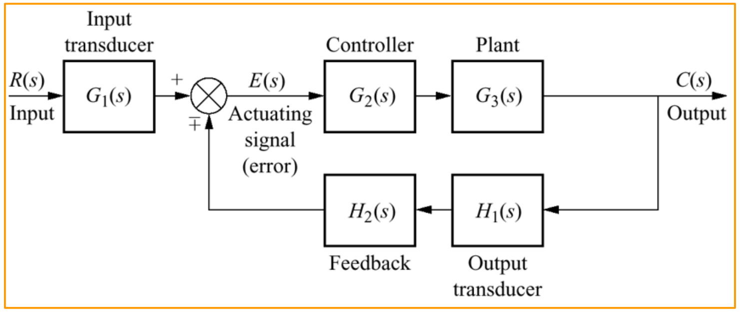
- *Moving blocks to create familiar forms*

  Block diagram algebra for summing junctions

  - equivalent forms for moving a block

  (a) to the <u>left past</u> a summing junction;

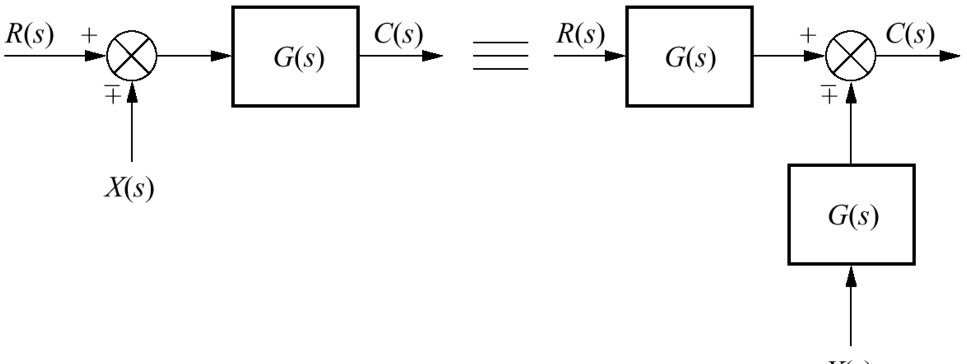
  (b) to the <u>right past</u> a summing junction



$$\left[R(s) \mp X(s)\right] G(s) = C(s)$$

$$R(s)G(s) \mp X(s)G(s) = C(s)$$

$\Rightarrow$

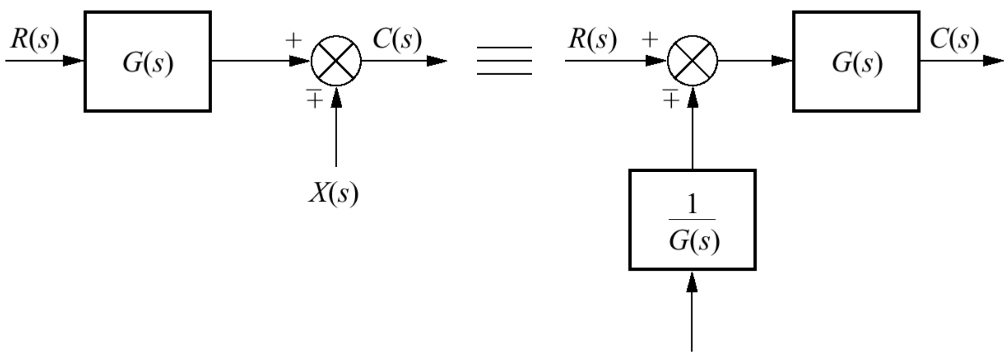


(*a*)

$$R(s)G(s) \mp X(s) = C(s)$$

$$\left[R(s) \mp \frac{1}{G(s)} X(s)\right] G(s) = C(s)$$

$\Rightarrow$



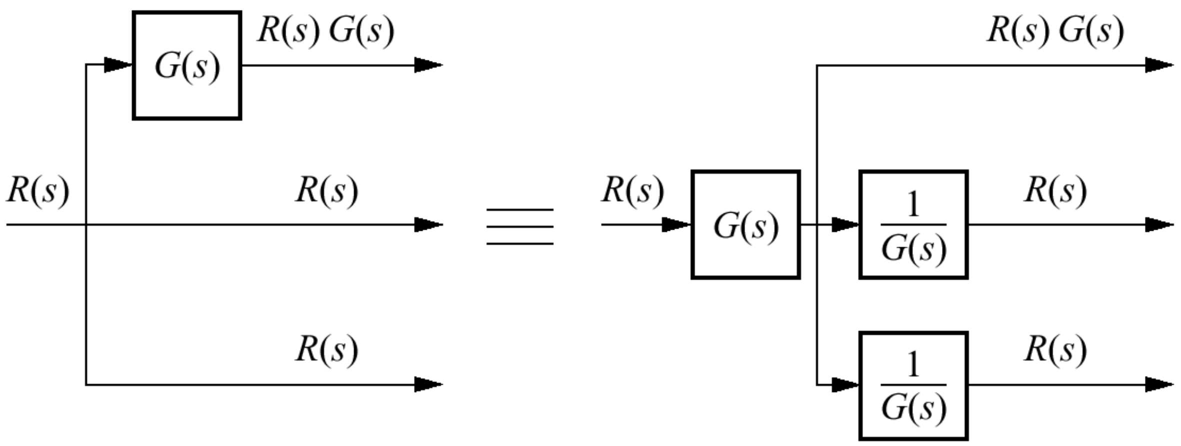
(*b*)

Figure 5.8

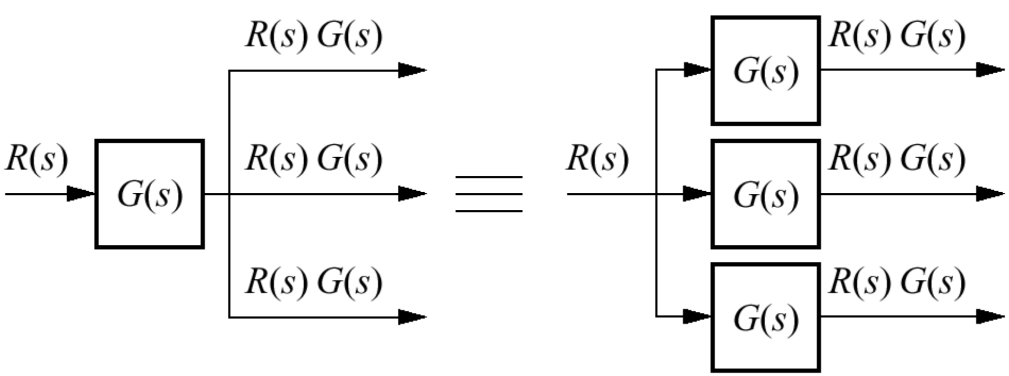
Block diagram algebra for *pickoff points* - equivalent forms for moving a block

(*a*) to the left past a pickoff point;

(*b*) to the right past a pickoff point



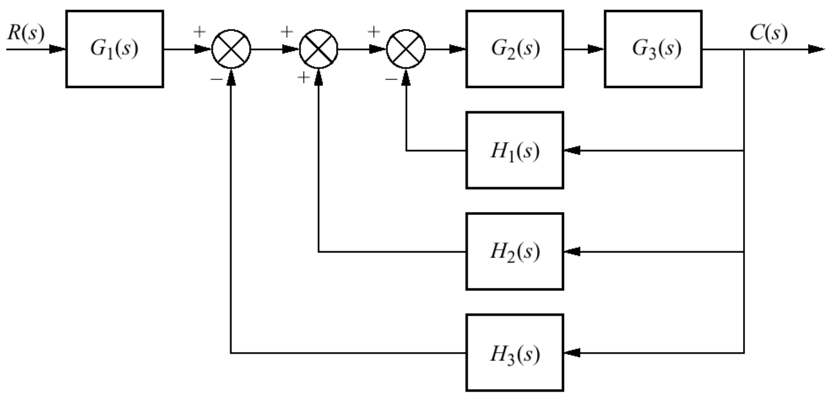
(*a*)



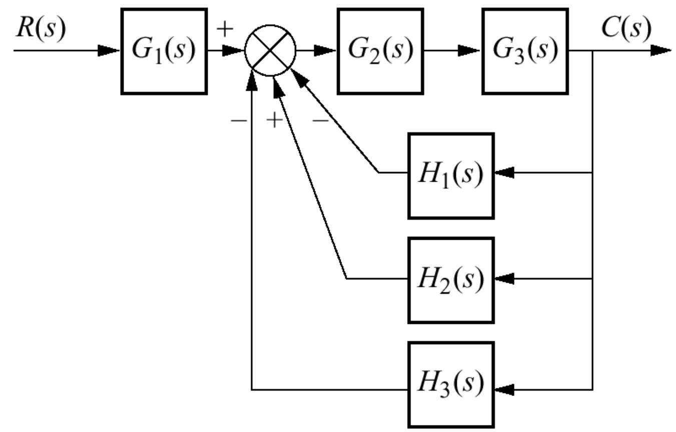
(*b*)

Example 5.1: Block diagram reduction via familiar forms (page 242)

Reduce the block diagram to a single transfer function



$\Rightarrow$

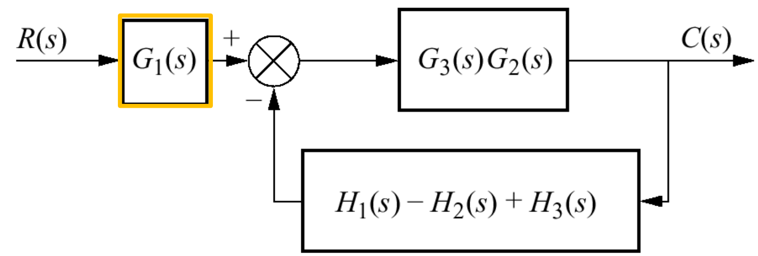


(*a*)

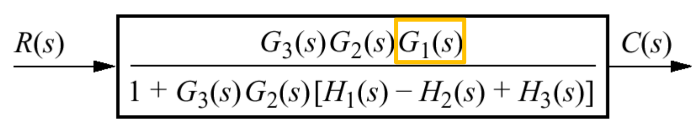
(*a*) collapse summing junctions;

(*b*) form equivalent cascaded system in the forward path and equivalent parallel system in the feedback path;

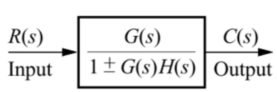
(*c*) form equivalent feedback system and multiply by cascaded $G_1(s)$



(*b*)



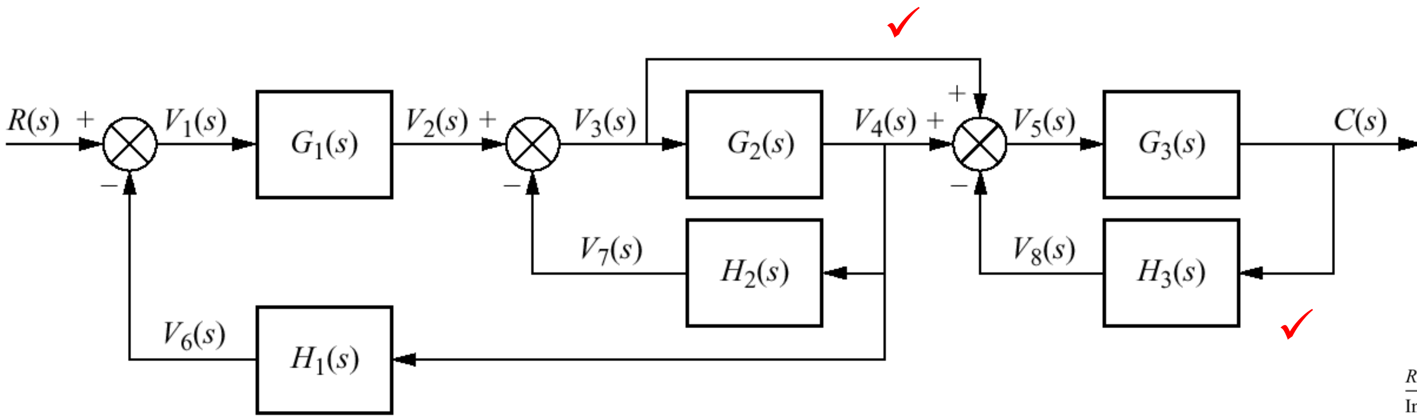
(*c*)

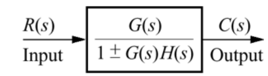


$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}}$$

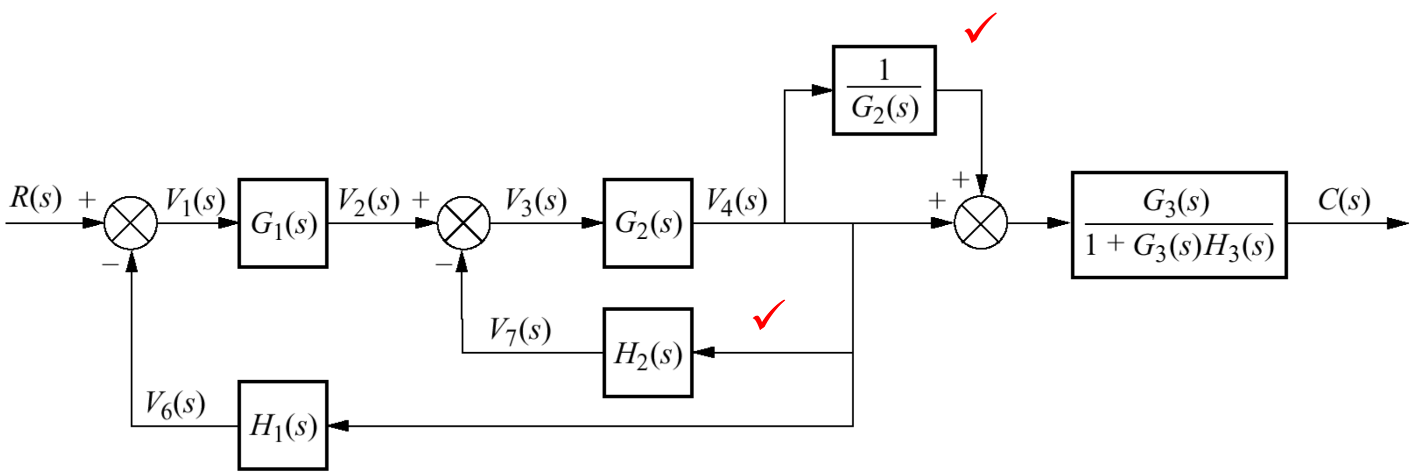
Example 5.2: Block diagram reduction via moving blocks (page 243)

Reduce the system to a single transfer function

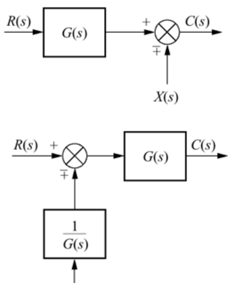




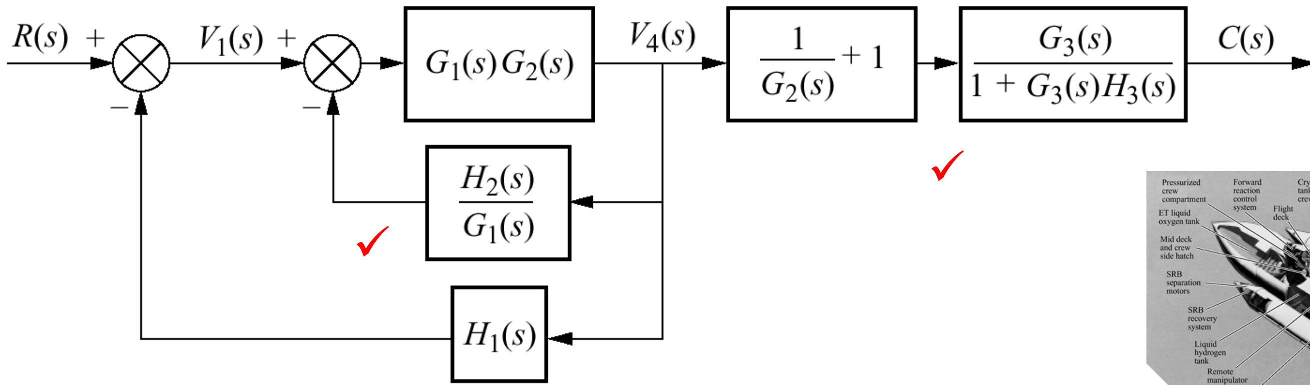
$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}}$$



(*a*)

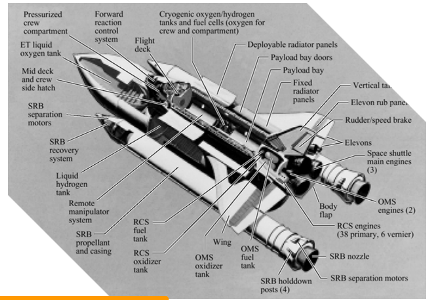


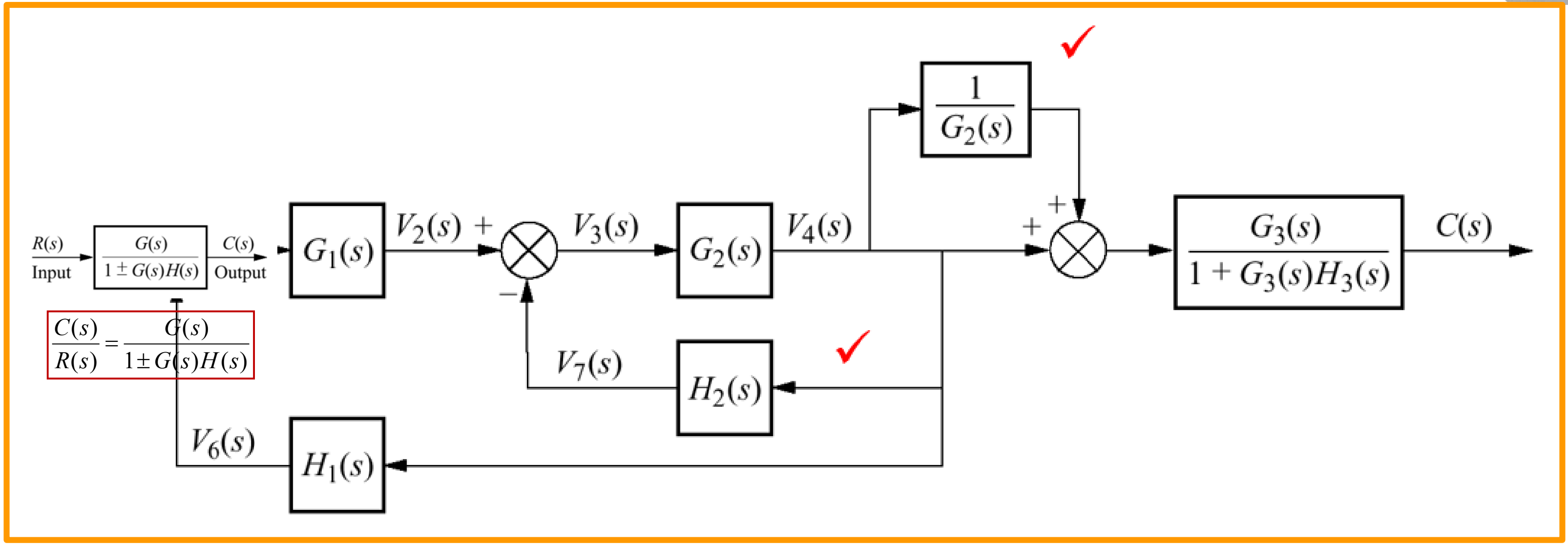
R(s)
G(s)
C(s)
X(s)
R(s)
G(s)
C(s)
1
G(s)



$R(s)$
$V_1(s)$
$G_1(s)G_2(s)$
$V_4(s)$
$\frac{1}{G_2(s)} + 1$
$\frac{G_3(s)}{1 + G_3(s)H_3(s)}$
$C(s)$
$\frac{H_2(s)}{G_1(s)}$
$H_1(s)$

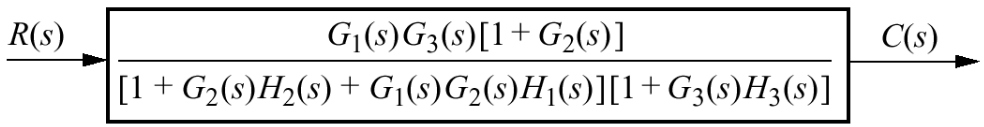
(*b*)





$R(s)$
Input
$\frac{G(s)}{1 \pm G(s)H(s)}$
$C(s)$
Output
$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$
$G_1(s)$
$V_2(s)$
$V_3(s)$
$G_2(s)$
$V_4(s)$
$\frac{1}{G_2(s)}$
$\frac{G_3(s)}{1 + G_3(s)H_3(s)}$
$C(s)$
$V_7(s)$
$H_2(s)$
$V_6(s)$
$H_1(s)$

(*d*)

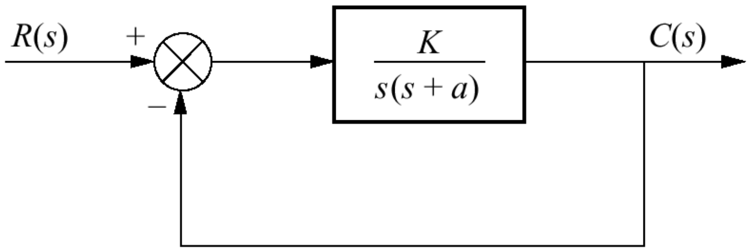


$R(s)$
$\frac{G_1(s)G_3(s)[1 + G_2(s)]}{[1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)][1 + G_3(s)H_3(s)]}$
$C(s)$

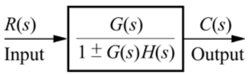
(*e*)

5.3 Analysis and Design of Feedback Systems

- Second-order feedback control system:



$$E(s)=R(s)-C(s)$$
$$C(s)=E(s)G(s)=R(s)G(s)-C(s)G(s)\ \Rightarrow$$
$$C(s)\left[1+G(s)\right]=R(s)G(s)$$



$$\frac{C(s)}{R(s)}=\frac{G(s)}{1\pm G(s)H(s)}$$

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}\quad\Rightarrow\quad T(s)=\frac{\dfrac{K}{s(s+a)}}{1+\dfrac{K}{s(s+a)}}=\frac{K}{s^2+as+K}$$

Root Locus

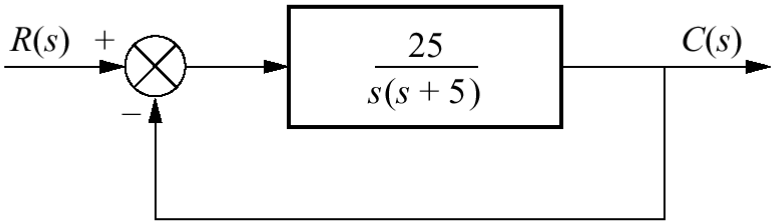
For $0<K<\dfrac{a^2}{4},\quad s_{1,2}=-\dfrac{a}{2}\pm\dfrac{\sqrt{a^2-4K}}{2}\ \Rightarrow$ *overdamped*

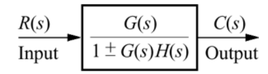
For $K=\dfrac{a^2}{4},\quad s_{1,2}=-\dfrac{a}{2}\ \Rightarrow$ *critically damped*

For $K>\dfrac{a^2}{4},\quad s_{1,2}=-\dfrac{a}{2}\pm j\dfrac{\sqrt{4K-a^2}}{2}\ \Rightarrow$ *underdamped*

Example 5.3: Finding transient response (page 246)

Find the peak time, % overshoot, and settling time.





$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

$$T(s) = \frac{25}{s^2 + 5s + 25}$$

$$\omega_n = \sqrt{25} = 5, \quad 2\zeta\omega_n = 5, \Rightarrow \zeta = 0.5$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$= \frac{\dfrac{25}{s(s+5)}}{1 + \dfrac{25}{s(s+5)}} = \frac{\dfrac{25}{s(s+5)}}{\dfrac{s^2 + 5s + 25}{s(s+5)}}$$

$$= \frac{25}{s^2 + 5s + 25}$$

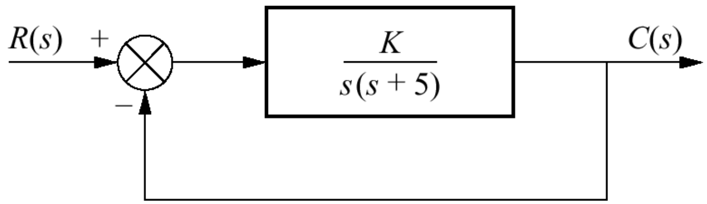
$$\Rightarrow \quad T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = 0.726 \text{ sec}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100\% = 16.30\%$$

$$T_s = \boxed{\frac{4}{\zeta\omega_n} = 1.6 \text{ sec}}$$

Example 5.4: Gain design for transient response (page 247)

Design the gain K so that the system will respond with 10% overshoot.



We want: $\%OS = 10\%$

$$T(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}$$

$$=\frac{\dfrac{K}{s(s+5)}}{1+\dfrac{K}{s(s+5)}}=\frac{\dfrac{K}{s(s+5)}}{\dfrac{s^2+5s+K}{s(s+5)}}$$

$$=\frac{K}{s^2+5s+25}$$

$$\zeta=\frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}=\frac{-\ln(0.1)}{\sqrt{\pi^2+\ln^2(0.1)}}=0.591$$

$$T(s)=\frac{K}{s^2+5s+K}$$

$$2\zeta\omega_n=5,\quad \omega_n=\sqrt{K}\quad\Rightarrow\quad \zeta=\frac{5}{2\sqrt{K}}$$

$$K=17.89$$

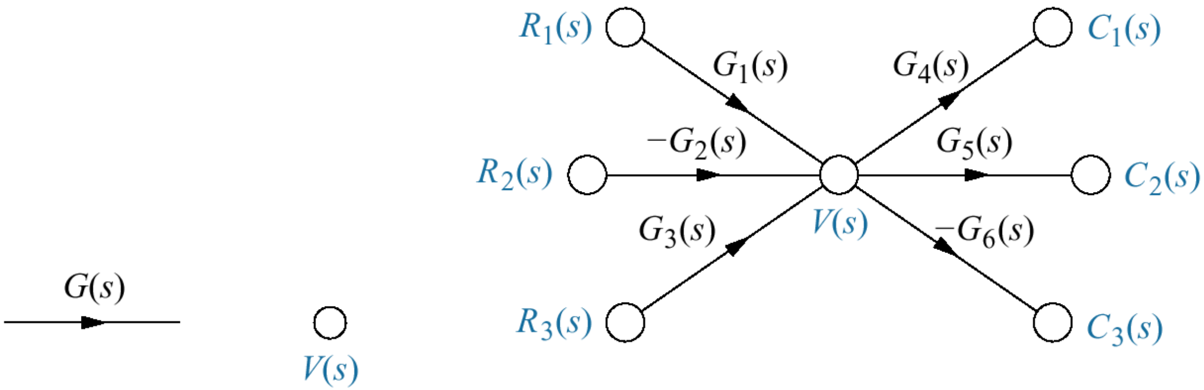
$$2\zeta\sqrt{K}=5$$

$$K=\left(\frac{5}{2\zeta}\right)^2$$

5.4 Signal-Flow Graphs

- Block diagrams: blocks, signals, summing junctions, and pickoff points
- Signal-flow graph: *branches* ⇒ systems

  nodes ⇒ signals



(*a*) *branches* ⇒ systems   **(*b*)** *nodes* ⇒ signals   **(*c*)**

Signal-flow graph components:
(*a*) system
(*b*) signal
(*c*) interconnection of systems and signals

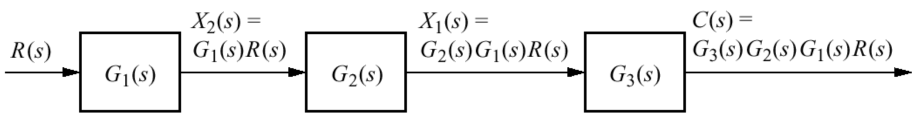
Example 5.5: Converting the block diagrams to signal-flow graphs (page 249)

Converting the block diagrams into signal-flow graphs.

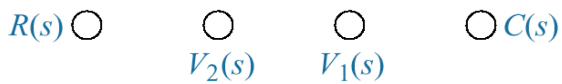
Building signal-flow graphs:

(*a*) cascaded system nodes (Figure 5.3(a));

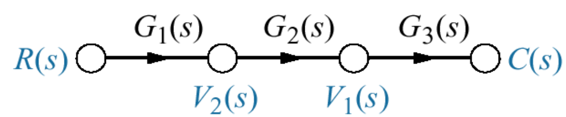
(*b*) cascaded system signal-flow graph;



(*a*)



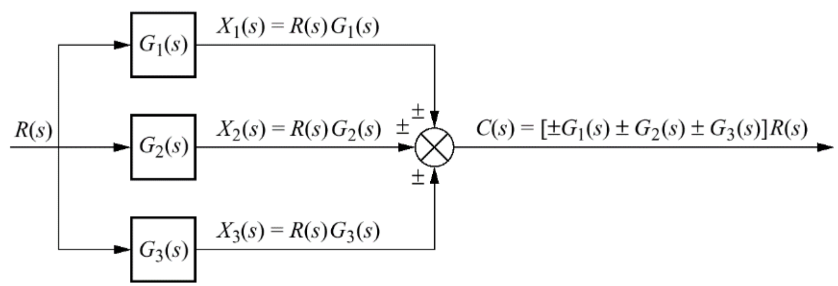
(*a*)



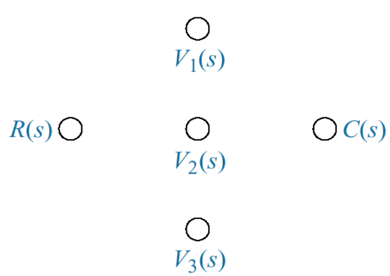
(*b*)

(*c*) parallel system nodes (Figure 5.5(*a*));

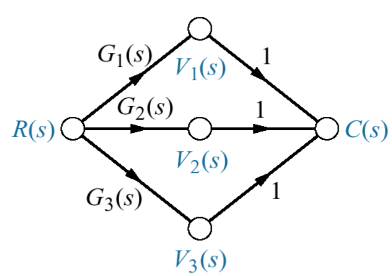
(*d*) parallel system signal-flow graph;



(*a*)

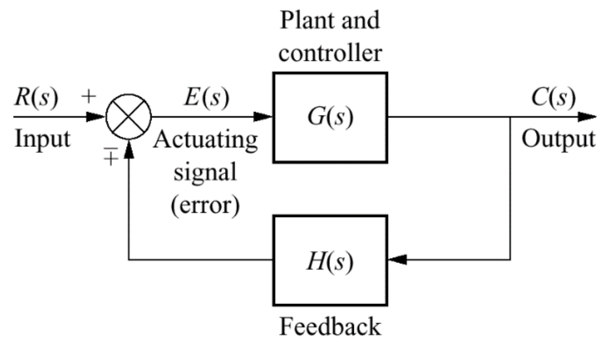


(*c*)

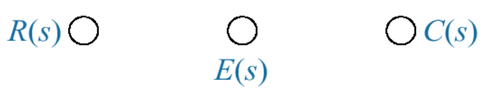


(*d*)

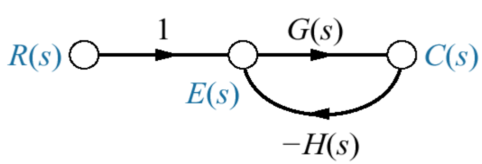
(*e*) feedback system nodes (from Figure 5.6(*b*));
(*f*) feedback system signal-flow graph



(*b*)



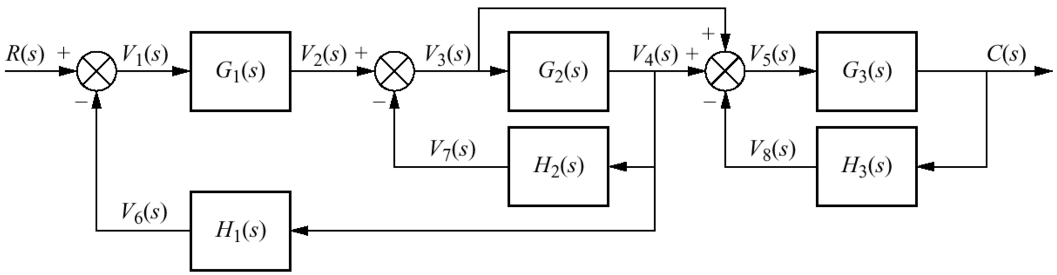
(*e*)

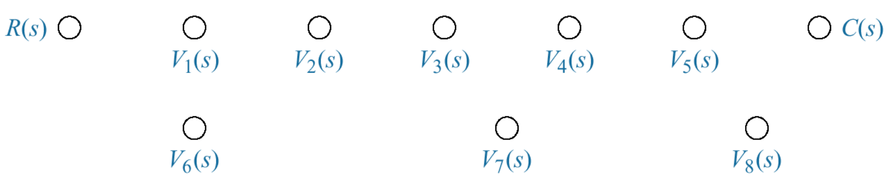


(*f*)

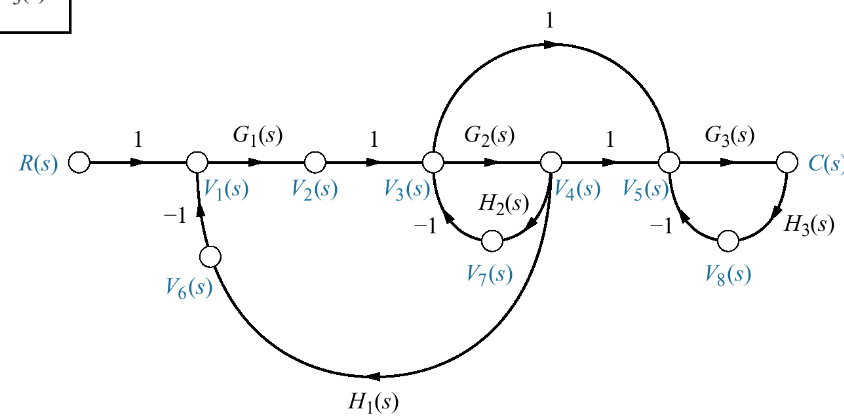
Example 5.6: Converting a block diagram to signal-flow graph (page 250)

Converting the block diagrams into signal-flow graphs.





(*a*)



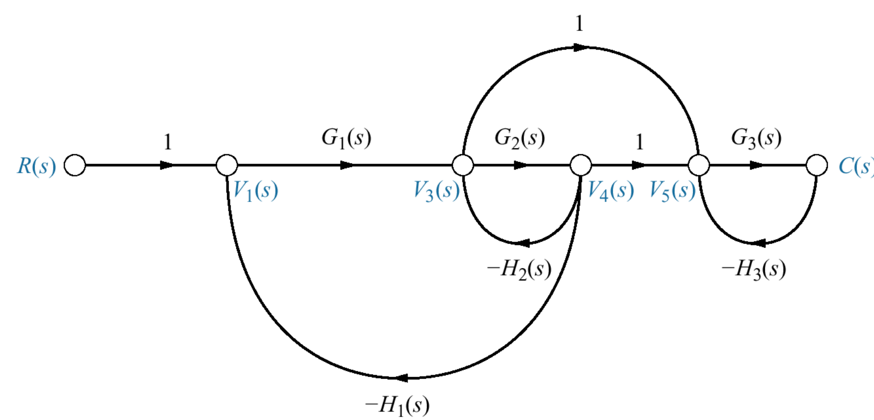
(*b*)

Signal-flow graph development:

(*a*) signal nodes;

(*b*) signal-flow graph;

(*c*) simplified signal-flow graph



(*c*)

5.5 Mason's Rule

$k =$ Number of forward paths

$T_k =$ The k^{th} forward-path gain

$\Delta =$ 1 – Σ(loop gains) + Σ(nontouching-loop gains taken two at a time) – Σ(nontouching-loop gains taken three at a time) + Σ ...

$\Delta_k = \Delta -$ Σ(loop gain terms in Δ that touch the k^{th} forward path)

- Definition $G(s) = \dfrac{C(s)}{R(s)} = \dfrac{\sum_k T_k \Delta_k}{\Delta}$

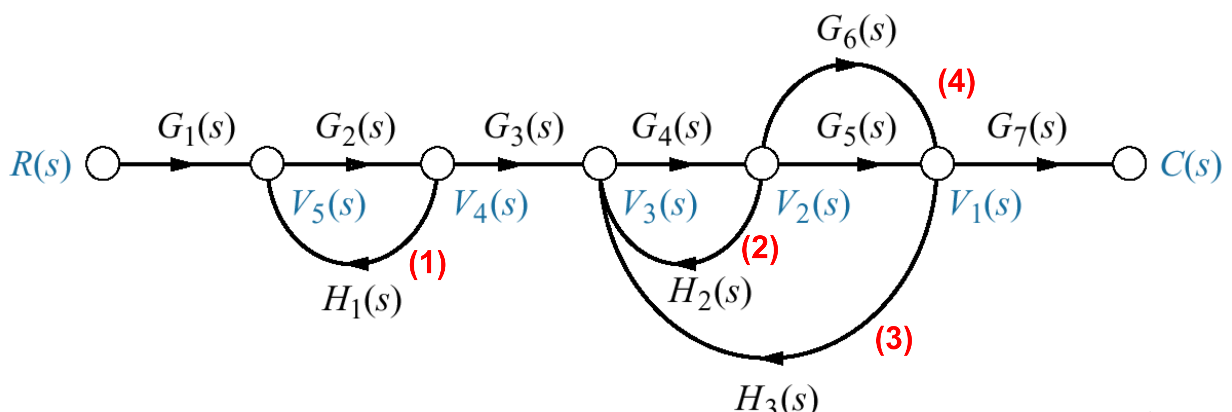
- Loop gain: $G_2(s)H_1(s),\ G_4H_2,\ G_4G_5H_3,\ G_4G_6H_3$

- Forward-path gain: $G_1G_2G_3G_4G_5G_7,\ G_1G_2G_3G_4G_6G_7$

- Nontouching loops: loop G_2H_1 does not touch loops $G_4H_2, G_4G_5H_3, G_4G_6H_3$

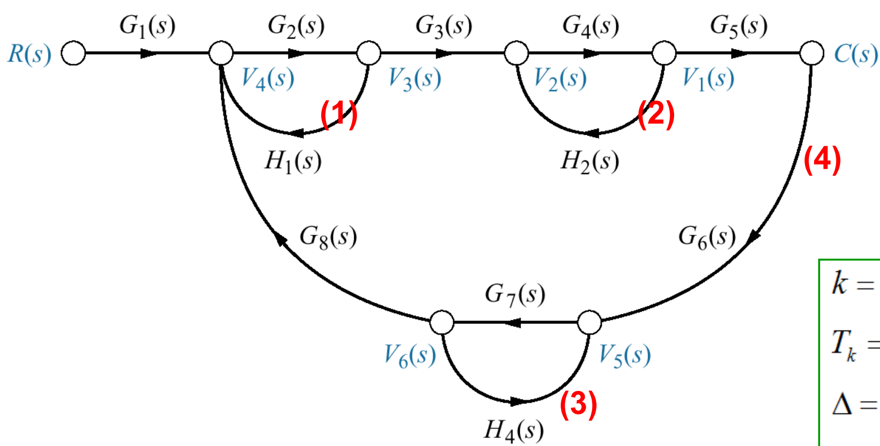
- Nontouching-loop gain: $\underline{G_2H_1}\ \underline{G_4H_2},\ \underline{G_2H_1}\ \underline{G_4G_5H_3},\ \underline{G_2H_1}\ \underline{G_4G_6H_3}$

$\Rightarrow$ *Nontouching-loop gains taken two at a time*



Example 5.7: Transfer function via Mason's rule (page 252)

Find the transfer function for the signal-flow graphs.



$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

$k =$ Number of forward paths

$T_k =$ The k^{th} forward-path gain

$\Delta =$ 1 – Σ(loop gains) + Σ(nontouching-loop gains taken two at a time) – Σ(nontouching-loop gains taken three at a time) + Σ ...

$\Delta_k = \Delta -$ Σ(loop gain terms in Δ that touch the k^{th} forward path)

- Forward-path gains: $G_1G_2G_3G_4G_5$

- Loop gains: $(1)\ G_2H_1, (2)\ G_4H_2, (3)\ G_7H_4, (4)\ G_2G_3G_4G_5G_6G_7G_8$

- Nontouching loops taken <u>two</u> at a time:

$$(1,2)\ G_2H_1G_4H_2,\ (1,3)\ G_2H_1G_7H_4, (2,3)\ G_4H_2G_7H_4$$

- Nontouching loops taken <u>three</u> at a time: $(1,2,3)\ G_2H_1G_4H_2G_7H_4$

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

$k =$ Number of forward paths

$T_k =$ The k^{th} forward-path gain

$\Delta =$ 1 – Σ(loop gains) + Σ(nontouching-loop gains taken two at a time) – Σ(nontouching-loop gains taken three at a time) + Σ ...

$\Delta_k = \Delta -$ Σ(loop gain terms in Δ that touch the k^{th} forward path)

$$\Delta = 1 - [G_2H_1 + G_4H_2 + G_7H_4 + G_2G_3G_4G_5G_6G_7G_8] + [G_2H_1G_4H_2 + G_2H_1G_7H_4 + G_4H_2G_7H_4] - [G_2H_1G_4H_2G_7H_4]$$

$$T_1 = G_1G_2G_3G_4G_5$$

$$\Delta_1 = 1 - G_7H_4$$

$$\Rightarrow \quad G(s) = \frac{T_1\Delta_1}{\Delta} = \frac{[G_1G_2G_3G_4G_5][1 - G_7H_4]}{\Delta}$$

- Forward-path gains: $G_1G_2G_3G_4G_5$
- Loop gains: (1) G_2H_1, (2) G_4H_2, (3) G_7H_4, (4) $G_2G_3G_4G_5G_6G_7G_8$
- Nontouching loops taken two at a time: (1, 2) $G_2H_1G_4H_2$, (1, 3) $G_2H_1G_7H_4$, (2, 3) $G_4H_2G_7H_4$
- Nontouching loops taken three at a time: (1, 2, 3) $G_2H_1G_4H_2G_7H_4$

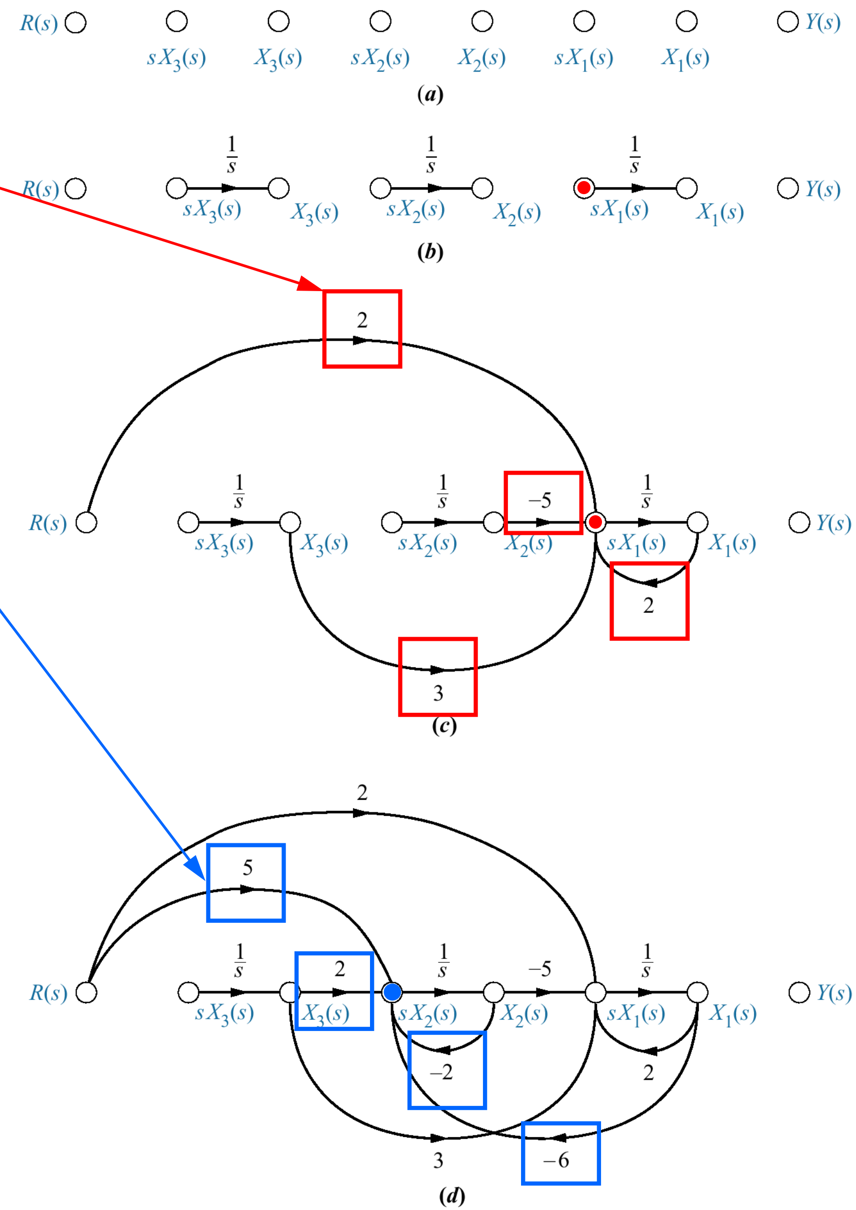
5.6 Signal-Flow Graphs of State Equations

$$\boxed{\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r}$$

$$\boxed{\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r}$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

$$y = -4x_1 + 6x_2 + 9x_3$$



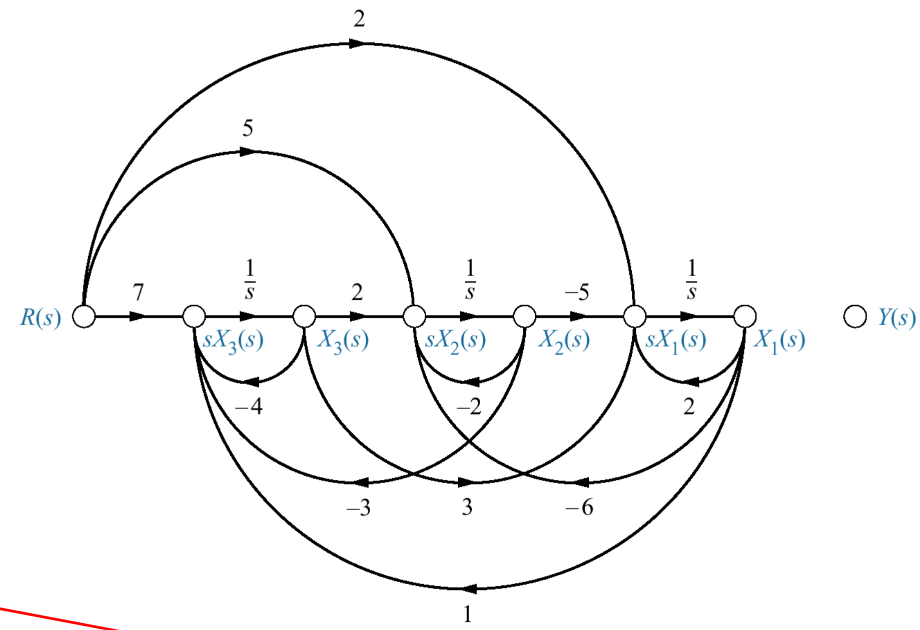
- Block diagrams: blocks, signals, summing junctions, and pickoff points
- Signal-flow graph:
  branches ⇒ systems
  nodes ⇒ signals

$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

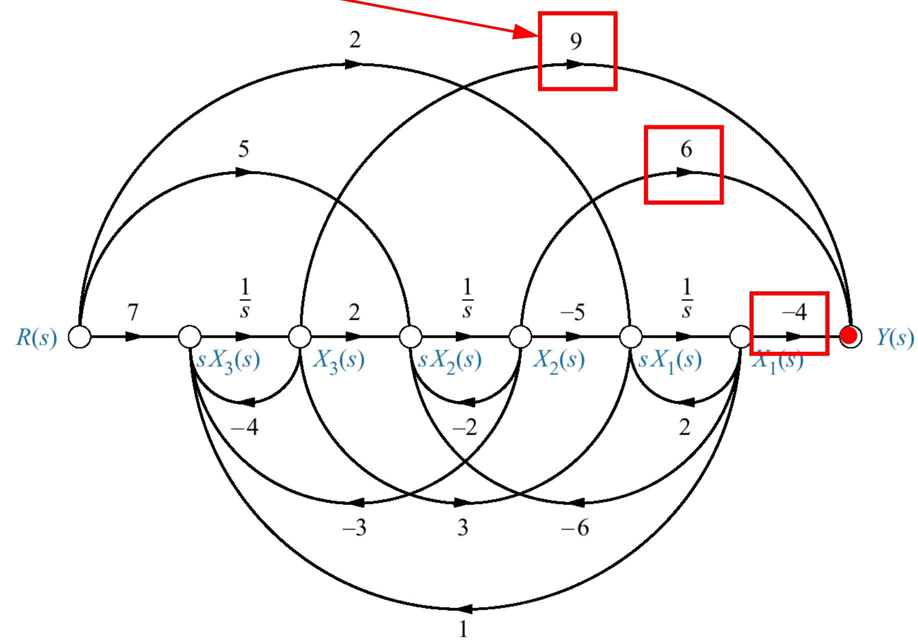
$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

$$\boxed{y = -4x_1 + 6x_2 + 9x_3}$$



(e)



(f)

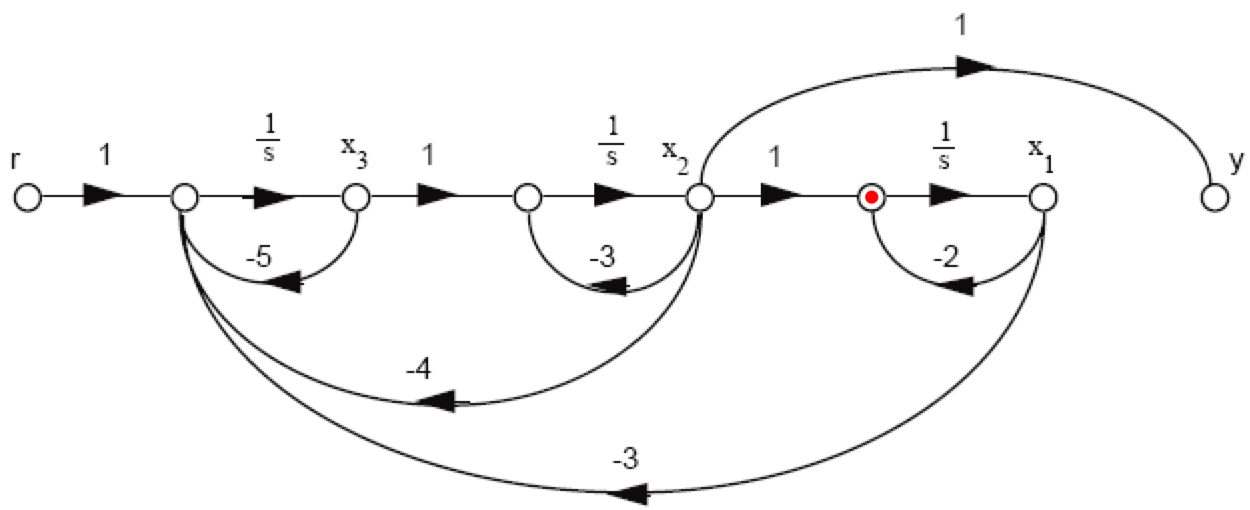
- Block diagrams: blocks, signals, summing junctions, and pickoff points
- Signal-flow graph:
  branches ⇒ systems
  nodes ⇒ signals

Exercise 5.5: Draw a signal-flow graph for the following state and output equations. (page 256)

$$\dot{\mathrm{x}} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{pmatrix} \mathrm{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r \qquad \Rightarrow \begin{cases} \dot{x}_1 = -2x_1 + x_2 \\ \dot{x}_2 = -3x_2 + x_3 \\ \dot{x}_3 = -3x_1 - 4x_2 - 5x_2 + r \\ y = x_2 \end{cases}$$

$$y = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \mathrm{x}$$

- Drawing the signal-flow diagram from the state equations:



5.7 Alternate Representations in State Space

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24} = \frac{24}{(s+2)(s+3)(s+4)}$$

$\Downarrow$

(1) Cascade form

Representation of Figure 3.10 system as *cascaded* first-order systems

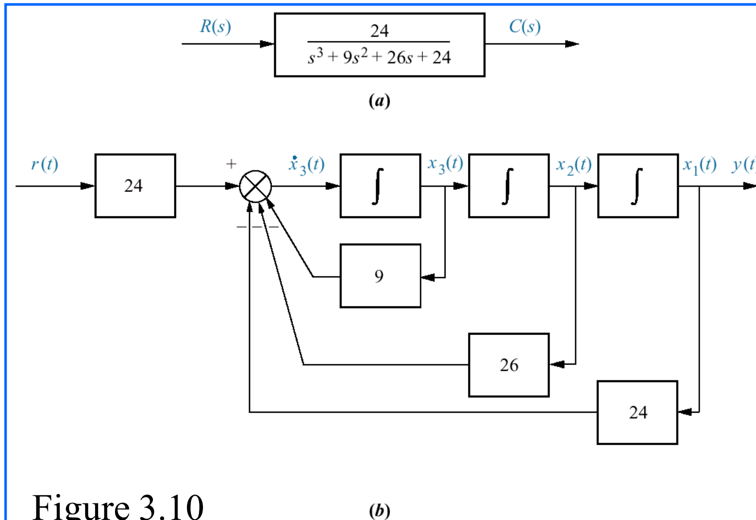
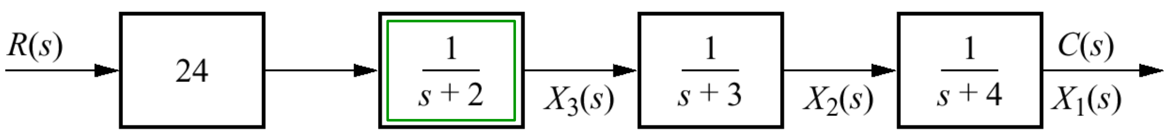


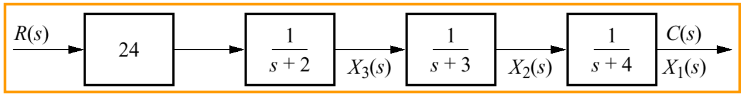
Figure 3.10

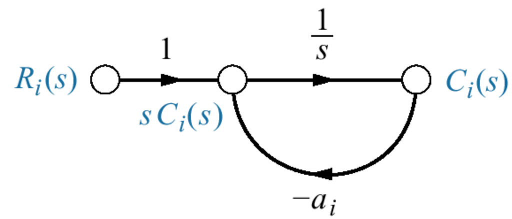


$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s+a_i)}$$

$$\Rightarrow (s+a_i)C_i(s) = R_i(s)$$

$$\Rightarrow \frac{d}{dt}c_i(t) = \boxed{-a_i c_i(t) + r_i(t)}$$



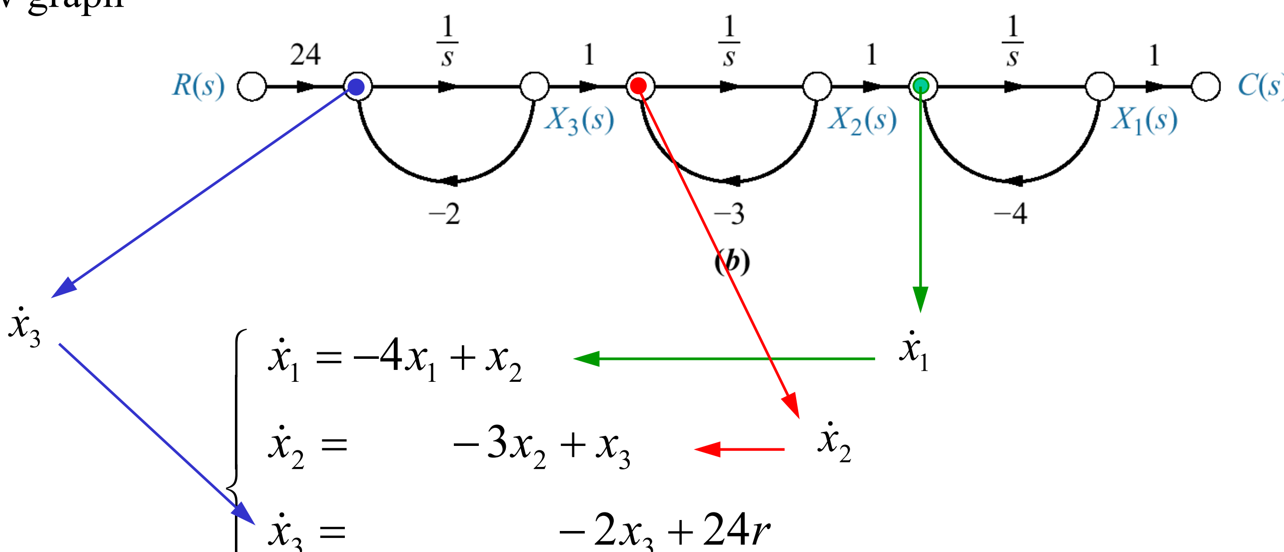


(*a*)

$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s+a_i)}$$

$$\Rightarrow \frac{d}{dt}c_i(t) = -a_i c_i(t) + r_i(t)$$

(a) first-order subsystem;
(b) signal-flow graph



(*b*)

$$\begin{cases} \dot{x}_1 = -4x_1 + x_2 \\ \dot{x}_2 = \quad -3x_2 + x_3 \\ \dot{x}_3 = \quad -2x_3 + 24r \\ y = x_1 = c(t) \end{cases}$$

$\Rightarrow$

state-space representation

$$\dot{\mathrm{x}} = \begin{pmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{pmatrix} \mathrm{x} + \begin{pmatrix} 0 \\ 0 \\ 24 \end{pmatrix} r$$

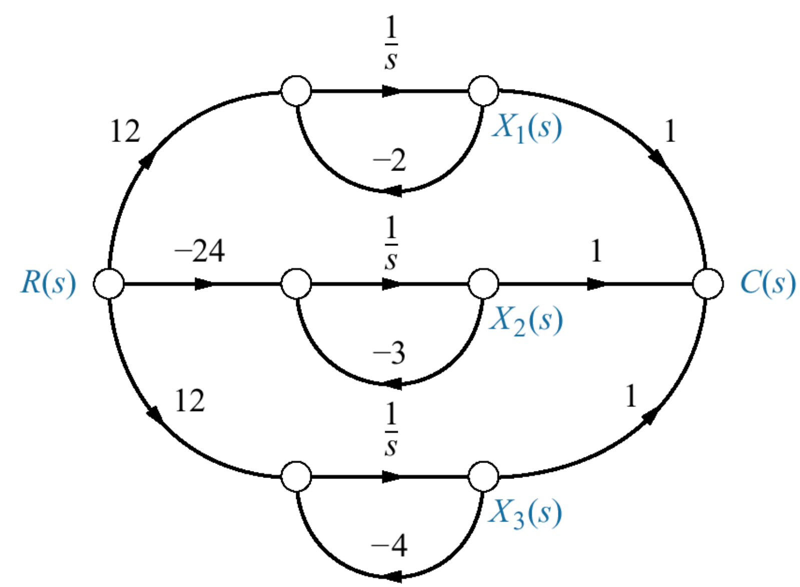
$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \mathrm{x}$$

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24} = \frac{24}{(s+2)(s+3)(s+4)}$$

$$\Downarrow$$

(2) Parallel form

Partial-fraction expansion of the transfer function. Sum of the first-order subsystems.



$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{s+2} - \frac{24}{s+3} + \frac{12}{s+4}$$

$$C(s) = R(s)\frac{12}{(s+2)} - R(s)\frac{24}{(s+3)} + R(s)\frac{12}{(s+4)}$$

$$\begin{cases} \dot{x}_1 = -2x_1 \qquad\qquad\qquad +12r \\ \dot{x}_2 = \qquad\quad -3x_2 \qquad\quad -24r \\ \dot{x}_3 = \qquad\qquad\qquad -4x_3 + 12r \\ y = c(t) = x_1 + x_2 + x_3 \end{cases}$$

$\Rightarrow$ state-space representation

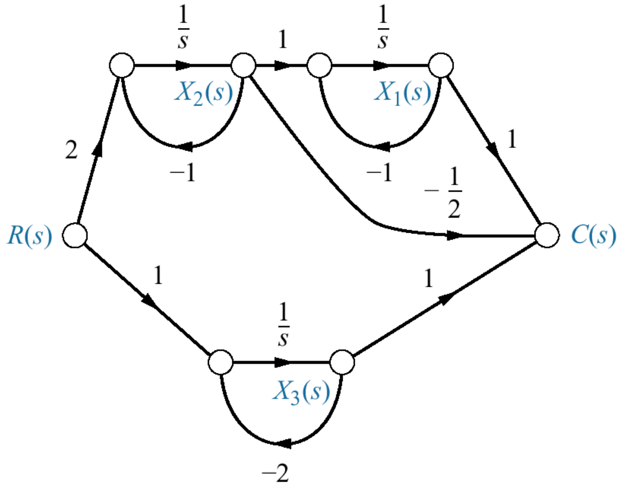
$$\dot{\mathrm{x}} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix} \mathrm{x} + \begin{pmatrix} 12 \\ -24 \\ 12 \end{pmatrix} r$$

$$y = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathrm{x}$$

• Matrix A is a diagonal matrix $\Rightarrow$ The equations are said to be *decoupled*.

- Jordan canonical form

$$\frac{C(s)}{R(s)} = \frac{s+3}{(s+1)^2(s+2)} = \frac{2}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+2}$$



– Repeated real roots
– Parallel form
– The system matrix, although not diagonal, has the *system poles along the diagonal*.
– Matrix A is called the *Jordan canonical form*.

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = \quad -x_2 \quad + 2r \\ \dot{x}_3 = \quad -2x_3 + r \\ y = c(t) = x_1 - 0.5x_2 + x_3 \end{cases}$$

$\Rightarrow$

state-space representation

systems poles

$$\dot{\mathrm{x}} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \mathrm{x} + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} r$$

$$y = \begin{pmatrix} 1 & -0.5 & 1 \end{pmatrix} \mathrm{x}$$

Jordan canonical form (Non-repeated roots)

- In this form of realizing a TF the <u>*poles of the transfer function*</u> from a string along the main diagonal of the matrix A.
- In Jordan canonical form state space model will be like:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + + \begin{bmatrix} d_0 \end{bmatrix} u$$

LTI system:

$$(1.1) \qquad \begin{cases} x^{\Delta}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \end{cases}$$

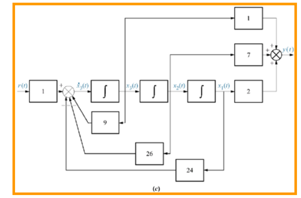
$$\boxed{\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}} \quad (1.1),\ (1.2)$$

Definition 4.2. The linear system (1.2) is said to be (*completely*) *controllable* on $[t_0, t_f]$ if for all $x_0 \in \mathbb{R}^n$, there exists $u\colon \mathbb{T} \to \mathbb{R}^m$ such that the solution x of the state equation of (1.2) with $x(t_0) = x_0$ satisfies $x(t_f) = 0$. $\leftarrow (A, B)$

Definition 5.1. The linear system (1.2) is said to be (*completely*) *reachable* on $[t_0, t_f]$ if for all $x_f \in \mathbb{R}^n$, there exists $u\colon \mathbb{T} \to \mathbb{R}^m$ such that the solution x of the state equation of (1.2) with $x(t_0) = 0$ satisfies $x(t_f) = x_f$.

Definition 6.2. The linear system (1.1) is said to be (*completely*) *observable* on $[t_0, t_f]$ if for all $u\colon \mathbb{T} \to \mathbb{R}^m$ and all $y\colon \mathbb{T} \to \mathbb{R}^r$, the linear system (1.1) has at most one solution x on $[t_0, t_f]$. $\leftarrow (A, C)$

(3) Controller canonical form (page 260)



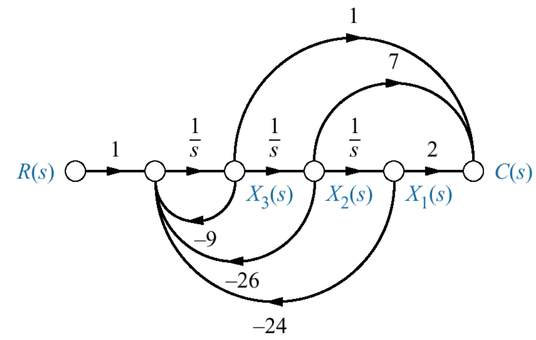
state-space representation

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-24&-26&-9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}r$$

$$y=\begin{bmatrix}2&7&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

- Reordering the phase variables in the *reverse order* from the phase-variable form
- Example: from example 3.5 in page 137. $\Rightarrow \dfrac{C(s)}{R(s)}=\dfrac{\boxed{s^2+7s+2}}{s^3\boxed{+9s^2+26s+24}}$
- Phase-variable form:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\-24&-26&-9\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}r,\qquad y=\begin{pmatrix}2&7&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$



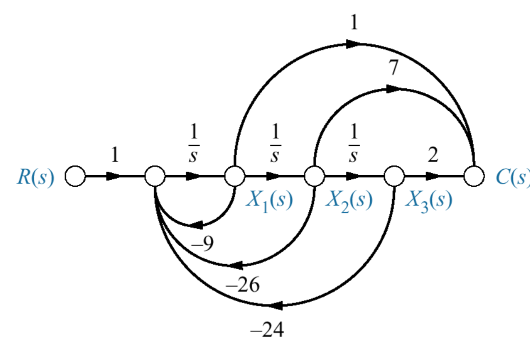
(a)

- The reverse order: $x_1\to x_3,\ x_2\to x_2,\ x_3\to x_1$

$$\begin{pmatrix}\dot{x}_3\\ \dot{x}_2\\ \dot{x}_1\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\-24&-26&-9\end{pmatrix}\begin{pmatrix}x_3\\x_2\\x_1\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}r,\qquad y=\begin{pmatrix}2&7&1\end{pmatrix}\begin{pmatrix}x_3\\x_2\\x_1\end{pmatrix}$$

- Ascending numerical order $\Rightarrow$ *controller canonical form*

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}=\begin{pmatrix}-9&-26&-24\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}1\\0\\0\end{pmatrix}r,\qquad y=\boxed{\begin{pmatrix}1&7&2\end{pmatrix}}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$



(b)

(4) Observer canonical form (page 262)

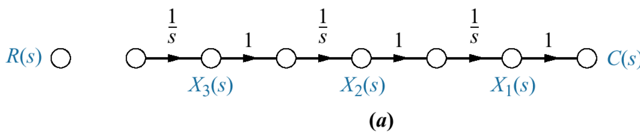
$$\left[\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}\right]R(s)-\left[\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}\right]C(s)=C(s)$$

$$\left(\frac{1}{s}R(s)-\frac{9}{s}C(s)\right)+\left(\frac{7}{s^2}R(s)-\frac{26}{s^2}C(s)\right)+\left(\frac{2}{s^3}R(s)-\frac{24}{s^3}C(s)\right)=C(s)$$

$$\frac{C(s)}{R(s)}=\frac{\boxed{s^2+7s+2}}{s^3\boxed{+9s^2+26s+24}}=\frac{\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}}{1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}} \Rightarrow \left[\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}\right]R(s)=\left[1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}\right]C(s)$$

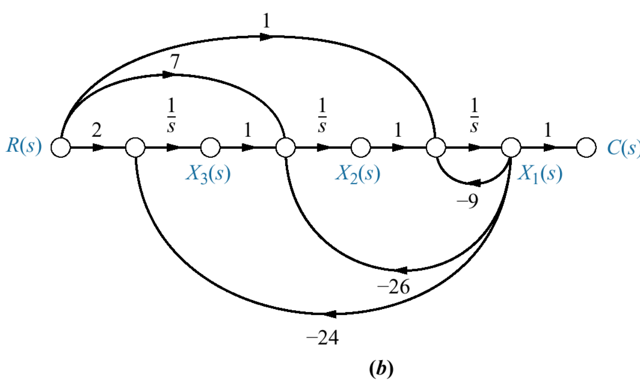
$$C(s)=\frac{1}{s}\left[R(s)-9C(s)\right]+\frac{1}{s^2}\left[7R(s)-26C(s)\right]+\frac{1}{s^3}\left[2R(s)-24C(s)\right]$$

$$=\underbrace{\frac{1}{s}\left[\left[R(s)-9C(s)\right]+\underbrace{\frac{1}{s}\left(\left[7R(s)-26C(s)\right]+\underbrace{\frac{1}{s}\left[2R(s)-24C(s)\right]}_{x_3}\right)}_{x_2}\right]}_{x_1}$$



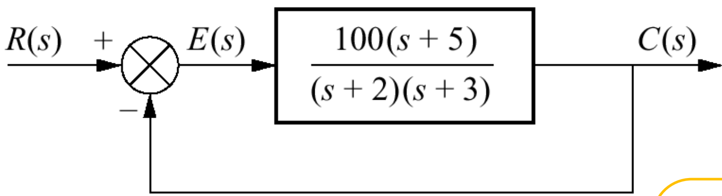
$$\begin{cases} \dot{x}_1 = -9x_1 + x_2 + r \\ \dot{x}_2 = -26x_1 + x_3 + 7r \\ \dot{x}_3 = -24x_1 + 2r \\ y = c(t) = x_1 \end{cases} \Rightarrow \quad \dot{\mathbf{x}} = \begin{pmatrix} \boxed{\begin{matrix} -9 \\ -26 \\ -24 \end{matrix}} & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \end{pmatrix}\mathbf{x} + \boxed{\begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}} r$$

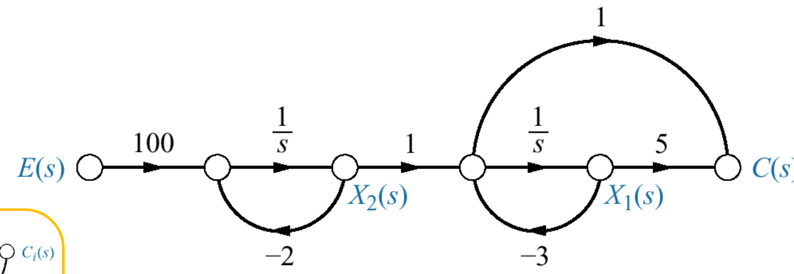
$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\mathbf{x}$$



Example 5.8: State-space representation of feedback systems (page 263)

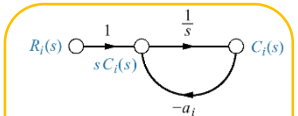
Represent the following feedback control system in state space.





(a)

- Phase-variable form in Section 3.5:

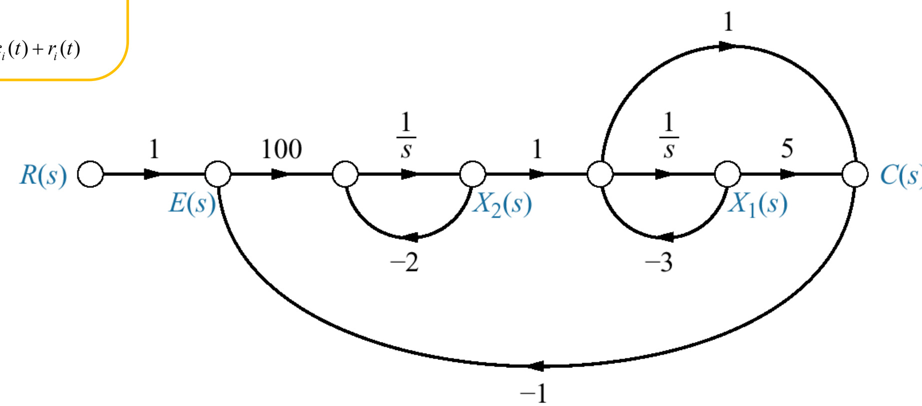


$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s+a_i)}$$

$$\Rightarrow \frac{d}{dt}c_i(t) = -a_i c_i(t) + r_i(t)$$

$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 \\ \dot{x}_2 = \qquad -2x_2 + 100(r-c) \end{cases}$$

$$c = 5x_1 + (x_2 - 3x_1) = 2x_1 + x_2$$



(b)

$\Downarrow$

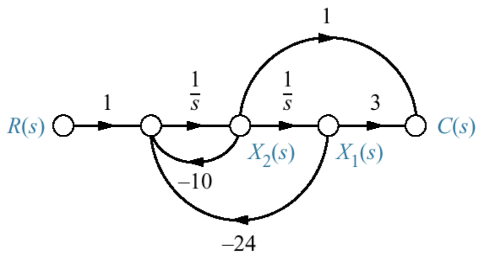
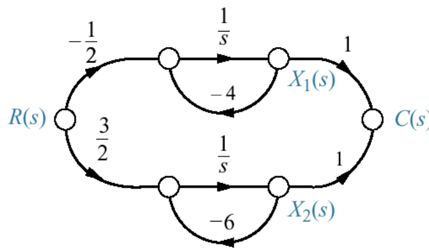
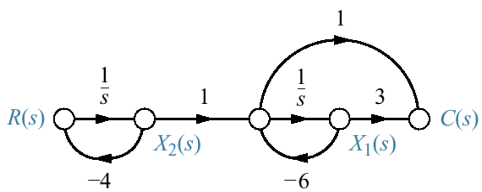
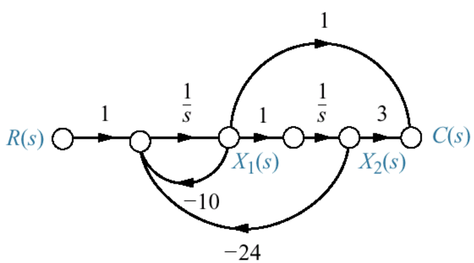
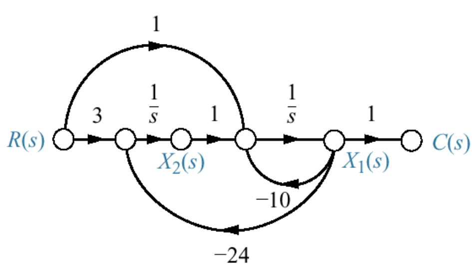
$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 \\ \dot{x}_2 = -200x_1 - 102x_2 + 100r \\ y = c(t) = 2x_1 + x_2 \end{cases}$$

(a) forward transfer function;
(b) complete system

$\Downarrow$

$$\dot{\mathrm{x}} = \begin{pmatrix} -3 & 1 \\ -200 & -102 \end{pmatrix}\mathrm{x} + \begin{pmatrix} 0 \\ 100 \end{pmatrix} r$$

$$y = \begin{pmatrix} 2 & 1 \end{pmatrix}\mathrm{x}$$

Form	Transfer Function	Signal-Flow Diagram	State Equations
Phase variable	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -24 & -10 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [3 \quad 1]\mathbf{x}$
Parallel	$\frac{-1/2}{(s+4)} + \frac{3/2}{s+6}$		$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} r$ $y = [1 \quad 1]\mathbf{x}$
Cascade	$\frac{1}{(s+4)} * \frac{(s+3)}{(s+6)}$		$\dot{\mathbf{x}} = \begin{bmatrix} -6 & 1 \\ 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [-3 \quad 1]\mathbf{x}$
Controller canonical	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & -24 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$ $y = [1 \quad 3]\mathbf{x}$
Observer canonical	$\frac{\frac{1}{s}+\frac{3}{s^2}}{1+\frac{10}{s}+\frac{24}{s^2}}$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ -24 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} r$ $y = [1 \quad 0]\mathbf{x}$

$$\frac{C(s)}{R(s)} = \frac{(s+3)}{(s+4)(s+6)}$$

$$y = c(t)$$

5.8 Similarity Transformations

Diagonalizing a system matrix
(Finding eigenvectors, page 269)