Chapter 5. Frequency Analysis: The Fourier Transform

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CONTENTS

Introduction

- We continue the frequency analysis of signals.
  - **Generalization of the Fourier series**: The frequency representation of signals and the frequency response of systems are **tools of great importance in signal processing, communications, and control theory**.
    - The **Fourier transform** measures the **frequency content of a signal** and unifies the **representation of periodic and aperiodic signals**.
  - **Laplace and Fourier transform**: The **connection between Laplace and Fourier transforms** will be highlighted for **computational and analytical reasons**.
    - The **Fourier transform** is the case of **Laplace transform for signals of which the ROC includes the $j\Omega$ axis**.
  - **Basis of filtering**: **Filtering** is an important application of Fourier transform.
  - **Modulation and communications**: The **idea of changing the frequency content of a signal via modulation** is basic in **analog communications**.

Derivation of Fourier Transform

An **aperiodic or non-periodic signal $x(t)$** can be thought of a **periodic signal $\tilde{x}(t)$** with an **infinite period**. Using the Fourier series representation of this signal and a limiting process we obtain a pair

$$x(t) \Leftrightarrow X(\Omega)$$

where the **Fourier transform** is

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t}\ dt$$

while **inverse Fourier transform** is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t}\ d\Omega$$

Existence of the Fourier Transform

The Fourier transform

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\, dt$$

of a signal $x(t)$ exists provided

– $x(t)$ is **absolutely integrable**

– $x(t)$ has a **finite number of maxima, minima,** and **discontinuities**.

- Signals of practical interest have Fourier transforms and their spectra can be displayed using a **spectrum analyzer**.

Fourier and Laplace Transform

If the **ROC of $X(s) = \mathcal{L}[x(t)]$ contains $j\Omega$ axis**, so that $X(s)$ can be obtained for $s = j\Omega$, then

$$\mathcal{F}[x(t)] = \mathcal{L}[x(t)]|_{s=j\Omega} = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\, dt = X(s)|_{s=j\Omega}$$

- Rule of thumb for computing the Fourier transform.
  - If $x(t)$ **has a finite time support**, its **Fourier transform exists**.
  - If $x(t)$ **has a Laplace transform $X(s)$ with a ROC including $j\Omega$ axis**, its **Fourier transform is $X(s)|_{s=j\Omega}$**.
  - If $x(t)$ **is periodic of infinite energy but finite power**, its Fourier transform obtained from **its Fourier series using delta functions**.
  - If $x(t)$ **is none of the above**, use **properties of the Fourier transform**.

Example

[Ex 5.1] Discuss whether it is possible to obtain the Fourier transform of the following signals using their Laplace transforms:

$$x_1(t) = u(t)$$
$$x_2(t) = e^{-2t}u(t)$$
$$x_3(t) = e^{-|t|}$$

Linearity of Fourier Transform

- The **linearity and duality between time and frequency of the Fourier transform** will help us to determine the **transform of signals that do not satisfy the Laplace transform condition**.

If $\mathcal{F}[x(t)] = X(\Omega)$ and $\mathcal{F}[y(t)] = Y(\Omega)$, for constant α and β, we have that

$$\boldsymbol{\mathcal{F}[\alpha x(t) + \beta y(t)] = \alpha\mathcal{F}[x(t)] + \beta\mathcal{F}[y(t)] = \alpha X(\Omega) + \beta Y(\Omega)}$$

[Ex 5.2] Suppose you create a periodic sine

$$x(t) = \sin \Omega_0 t \qquad -\infty < t < \infty$$

by adding a causal sine $v(t) = \sin \Omega_0 t\, u(t)$ and an anti-causal sine $y(t) = \sin \Omega_0 t\, u(-t)$, for each of which you can find Laplace transform $V(s)$ and $Y(s)$. Discuss what would be wrong with this approach to find the Fourier transform of $x(t)$ by letting $s = j\Omega$.

Inverse Proportionality of Time/Frequency

- The **frequency is inversely proportional to time**, and that as such, time and frequency signal characterizations are complementary.
  - Fourier transform of the **impulse signal** $x_1(t) = \delta(t)$ is
    $$X_1(\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{j\Omega t}\, dt = e^{j\Omega 0} \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$
  - Fourier transform of **dc signal** $x_2(t) = A$ is $X_2(\Omega) = 2\pi A\, \delta(\Omega)$.
  - Fourier transform of the **rectangular signal** $x_3(t) = A[u(t + \tau/2) - u(t - \tau/2)]$ is
    $$X_3(\Omega) = X(s)|_{s=j\Omega} = A \frac{e^{j\Omega\tau/2} - e^{-j\Omega\tau/2}}{j\Omega} = A\tau \frac{\sin(\Omega\tau/2)}{\Omega\tau/2}$$

If $x(t)$ has a Fourier transform $X(\Omega)$, we have the pair

$$x(\alpha t) \Leftrightarrow 1/|\alpha| X(\Omega/\alpha)$$

Duality

[Ex 5.3] Consider a pulse $x(t) = u(t) - u(t-1)$. Find the Fourier transform of $x_1(t) = x(2t)$.

To the **Fourer transform pair $x(t) \Leftrightarrow X(\Omega)$** corresponds the following **dual-Fourier transform pair**

$$\boldsymbol{X(t) \Leftrightarrow 2\pi x(-\Omega)}$$

[Ex 5.5] Use the duality property to find the Fourier transform of the sinc signal

$$x(t) = A\frac{\sin(0.5t)}{0.5t} = A \operatorname{sinc}(0.5t) \qquad -\infty < t < \infty$$

[Ex 5.6] Find the Fourier transform of $x(t) = \cos(\Omega_0 t)$ using duality.

Signal Modulation

Frequency shift: if $X(\Omega)$ **is the Fourier transform of** $x(t)$, then we have the pair

$$x(t)e^{j\Omega_0 t} \Leftrightarrow X(\Omega - \Omega_0)$$

Modulation: The Fourier transform of the **modulated signal** $x(t)\cos(\Omega_0 t)$ is given by

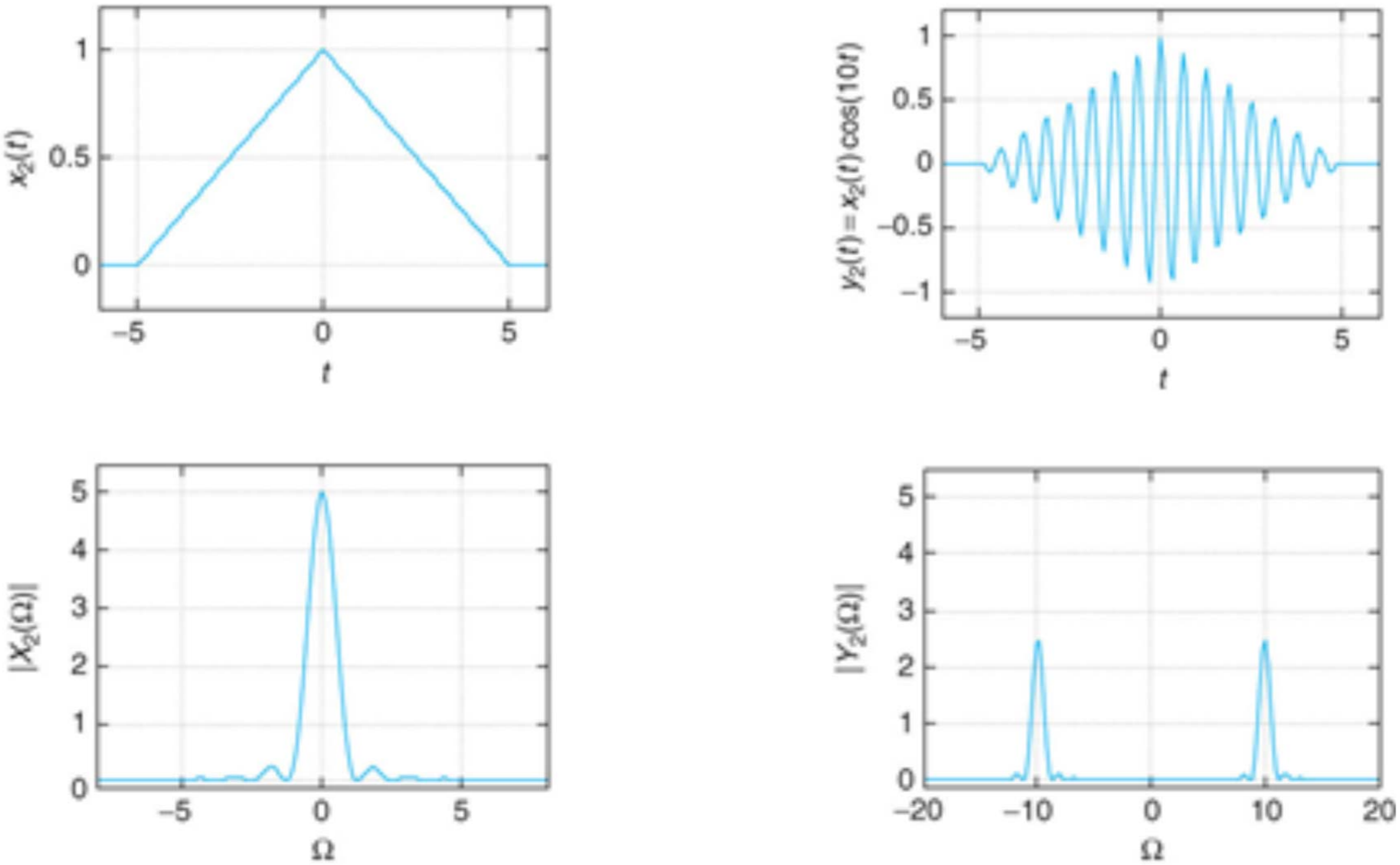
$$0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$$

- In communications, the **message** $x(t)$ modulates the **carrier** $\cos(\Omega_0 t)$ to obtain the **modulated signal** $x(t)\cos(\Omega_0 t)$.
- Modulation using a sine **changes the phase of the Fourier transform** of the incoming signal,

$$x(t)\sin(\Omega_0 t) \Leftrightarrow 0.5[-jX(\Omega - \Omega_0) + jX(\Omega + \Omega_0)]$$

- According to the eigenfunction property of LTI systems, **modulation systems are not LTI**.

Examples of Signal Modulation



Fourier Transform of Periodic Signals

For a **periodic signal $x(t)$ of period T_0**, we have the Fourier pair

$$x(t) = \sum_k X_k e^{jk\Omega_0 t} \Leftrightarrow X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$$

obtained by representing $x(t)$ by its Fourier series.

– Line spectrum displays the Fourier series coefficients at their corresponding frequencies, while the spectrum from the Fourier transform displays the **concentration of the power at the harmonic frequencies** by means of **delta function with amplitudes of 2π times of the Fourier series coefficients**.

[Ex 5.8] Find the Fourier transform of a periodic signal $x(t)$ with a period $x_1(t) = r(t) - 2r(t-0.5) + r(t-1)$.

Parseval's Energy Conservation

- For aperiodic signals of finite energy, an energy version of **Parseval's result** indicate **how the signal energy is distributed over frequencies**.

For a **finite-energy signal $\boldsymbol{x(t)}$ with Fourier transform $\boldsymbol{X(\Omega)}$**, its **energy is conserved** when going from the time to the frequency domain, or

$$\boldsymbol{E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 \, d\Omega}$$

Thus $|\boldsymbol{X(\Omega)}|^2$ is an **energy density** indicating the **amount of energy at each of the frequencies $\boldsymbol{\Omega}$**. The plot $|\boldsymbol{X(\Omega)}|^2$ versus $\boldsymbol{\Omega}$ is called the **energy spectrum** of $x(t)$.

Examples

[Ex 5.9] Parseval's result helps us to understand better the nature of an impulse $\delta(t)$. It is clear from its definition that the area under an impulse is unity, which means $\delta(t)$ is absolutely integrable, but does it have finite energy? Show how Parseval's result can help resolve this issue.

[Ex 5.10] Consider a pulse $p(t) = u(t+1) - u(t-1)$. Use its Fourier transform $\mathrm{P}(\Omega)$ and Parseval's result to show that

$$\int_{-\infty}^{\infty} \left(\frac{\sin \Omega}{\Omega}\right)^2 \, d\Omega = \pi$$

Symmetry of Spectral Representation

If $\boldsymbol{X(\Omega)}$ **is the Fourier transform of a real-valued signal** $\boldsymbol{x(t)}$, periodic or aperiodic, the **magnitude** $|\boldsymbol{X(\Omega)}|$ is an **even function** of $\boldsymbol{\Omega}$: $|\boldsymbol{X(\Omega)}| = |\boldsymbol{X(-\Omega)}|$

and the **phase** $\angle\boldsymbol{X(\Omega)}$ is an **odd function** of $\boldsymbol{\Omega}$:

$$\angle\boldsymbol{X(\Omega)} = -\angle\boldsymbol{X(-\Omega)}$$

- In reality, only positive frequencies exist and can be measured, **negative frequencies** must be understood as **necessary to generate real-valued signal**.
- The **bandwidth of a signal** $\boldsymbol{x(t)}$ is the **support of its Fourier transform** $\boldsymbol{X(\Omega)}$.

[Ex 5.12] Consider the signals (a) $x(t) = 0.5e^{-|t|}$ and (b) $y(t) = e^{-|t|}\cos(\Omega_0 t)$, find their Fourier transforms.

Convolution and Filtering

- The **convolution property** is basic in the **analysis and design of filters**.

If the **input $x(t)$ to a stable LTI system has a Fourier transform $X(\Omega)$** and the system has **frequency response $H(j\Omega) =$** $\mathcal{F}[h(t)]$ where $h(t)$ is the impulse response, the **output of the LTI system** is the **convolution integral,**

$$y(t) = (x * h)(t) \Leftrightarrow Y(\Omega) = X(\Omega)H(j\Omega)$$

If the **input $x(t)$** is **periodic**, the **output** is also **periodic with Fourier transform**

$$Y(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(jk\Omega_0)\delta(\Omega - k\Omega_0)$$

Basics of Filtering

- **Filtering** consists in getting rid of **undesirable components of a signal**.
  - The problem is to **design a filter that will get rid of the noise as much as possible**.
  - **Frequency-discriminating filters keeps the frequency components of a signal** in a certain frequency band and **attenuate the rest**.

$$Y(\Omega) = X(\Omega)H(j\Omega)$$

Examples

[Ex 5.13] Consider how to obtain a dc source using a full-wave rectifier and a low-pass filter. Let the full-wave rectified signal $x(t)$ be the input of the filter and let the output of the filter be $y(t)$. We want to $y(t) = 1$ volt. The rectifier and the low-pass filter constitute a system that converts alternating into direct voltage.

[Ex 5.14] Windowing is a time-domain process by which we select a part of a signal. This is done by multiplying the signal by a “window” signal $w(t)$. Consider the rectangular window

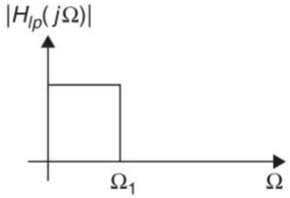
$$w(t) = u(t + \Delta) - u(t - \Delta) \qquad \Delta > 0$$

For a given signal $x(t)$, the windowed signal is given by

$$y(t) = x(t)w(t)$$

Discuss how windowing relates to the convolution property.

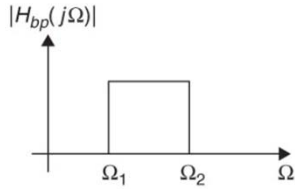
Ideal Filters



- Frequency response of **ideal low-pass filter**:

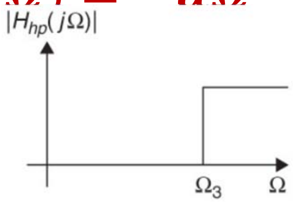
$$|H_{lp}(j\Omega)| = \begin{cases} 1, & -\Omega_1 < \Omega < \Omega_1 \\ 0 & \text{otherwise} \end{cases} \qquad \angle H_{lp}(j\Omega) = -\alpha\Omega$$

  - Ω_1: **cut-off frequency** of the low-pass filter.



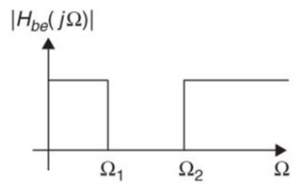
- Frequency response of **ideal band-pass filter**:

$$|H_{bp}(j\Omega)| = \begin{cases} 1, & \Omega_1 < \Omega < \Omega_2 \text{ and } -\Omega_2 < \Omega < -\Omega_1 \\ 0 & \text{otherwise} \end{cases} \qquad \angle H_{bp}(j\Omega) = -\alpha\Omega$$



- Frequency response of **ideal high-pass filter**:

$$|H_{hp}(j\Omega)| = \begin{cases} 1, & \Omega \geq \Omega_2 \text{ and } \Omega \leq -\Omega_2 \\ 0 & \text{otherwise} \end{cases} \qquad \angle H_{hp}(j\Omega) = -\alpha\Omega$$



- Frequency response of **ideal band-stop filter**:

$$|H_{bs}(j\Omega)| = 1 - |H_{bp}(j\Omega)|$$

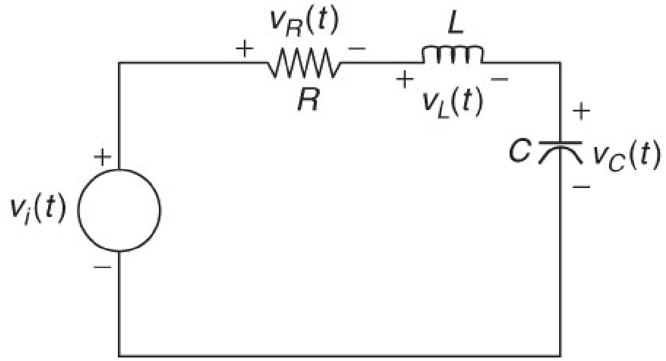
Remarks to Ideal Filters

- If $h_{lp}(t)$ is the impulse response of a **low-pass filter (LPF)**, then $\boldsymbol{h_{lp}(t)\cos(\Omega_0 t)}$ corresponds to the impulse response of a **band-pass filter (BPF) centered around $\boldsymbol{\Omega_0}$**.
- A **zero-phase ideal LPF $\boldsymbol{H_{lp}(j\Omega) = u(\Omega+\Omega_1) - u(\Omega-\Omega_1)}$** has **a sinc function with an infinite support** as an impulse response.
  - To make it causal, we approximate by using
    $$\boldsymbol{h_1(t) = h_{lp}(t)w(t), \qquad w(t) = u(t+\tau) - u(t-\tau)}$$
    where $\boldsymbol{|h_1(t-\tau)| \approx |h_{lp}(t)|}$ and $\boldsymbol{\angle h_1(t-\tau) = -\tau\Omega}$.
- A **measure of attenuation** is given by the loss function in **decibels**, i.e. $\boldsymbol{\alpha(\Omega) = -20\log_{10}|H(j\Omega)|}$ dB.

Examples

[Ex 5.15] The Gibb's phenomenon consists in ringing around the discontinuities. To see this, consider a periodic train of square pulses $x(t)$ of period T_0 displaying discontinuities at $kT_0/2$, for $k = \pm 1, \pm 2, \ldots$. Show how the Gibb's phenomenon is due to ideal low-pass filtering.

[Ex 5.16] Obtain different filters from an RLC circuit by choosing different outputs. Let the input be a voltage source with Laplace transform $V_i(s)$. For simplicity, let $R = 1\ \Omega$, $L = 1$ H, and $C = 1$ F, and assume the initial conditions to be zero.



Frequency Response from Poles and Zeros

For a **filter with a transfer function**

$$H(s) = \frac{\prod_i (s - z_i)}{\prod_k (s - p_k)}$$

where vectors $\overrightarrow{Z_i}(\Omega) = j\Omega - z_i$ and $\overrightarrow{P_k}(\Omega) = j\Omega - p_k$. Then, the **frequency response of this filter** is

$$H(j\Omega) = H(s)|_{s=j\Omega} = \frac{\prod_i \overrightarrow{Z_i}(\Omega)}{\prod_k \overrightarrow{P_k}(\Omega)} = \frac{\prod_i |\overrightarrow{Z_i}(\Omega)|}{\prod_k |\overrightarrow{P_k}(\Omega)|} e^{j\left[\sum_i \angle \overrightarrow{Z_i}(\Omega) - \sum_k \angle \overrightarrow{P_k}(\Omega)\right]}$$

- **Poles** create **hills** at frequencies in the $j\Omega$ axis in front of imaginary parts of the poles.
- **Zeros** create **valleys** at frequencies in the $j\Omega$ axis in front of the imaginary parts of the zeros.

Example

[Ex 5.17] Consider series RC circuit with a voltage source $v_i(t)$. Choose the output to obtain low-pass and high-pass filters and use the poles and zeros of the transfer functions to determine their frequency responses. Let $R = 1\ \Omega$, $C = 1$ F, and the initial conditions be zero.

Time Shifting

If $x(t)$ **has a Fourier transform** $X(\Omega)$, then

$$x(t - t_0) \Leftrightarrow X(\Omega)e^{-j\Omega t_0}$$
$$x(t + t_0) \Leftrightarrow X(\Omega)e^{j\Omega t_0}$$

The **effect of the time shift** is only in the **phase spectrum**.

[Ex 5.19] Consider computing the Fourier transform of $y(t) = \sin(\Omega_0 t)$ using the Fourier transform of the cosine signal $x(t) = \cos(\Omega_0 t)$.

Differentiation and Integration

If $x(t)$ **has a Fourier transform** $X(\Omega)$, then

$$\frac{d^N x(t)}{dt^N} \Leftrightarrow (j\Omega)^N X(\Omega)$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \Leftrightarrow \frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$$

[Ex 5.20] Suppose a system is represented by a second-order differential equation with constant coefficients:

$$2y(t) + 3\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} = x(t)$$

and that the initial conditions are zero. Let $x(t) = \delta(t)$. Find $y(t)$.

Examples

[Ex 5.21] Find the Fourier transform of the triangular pulse

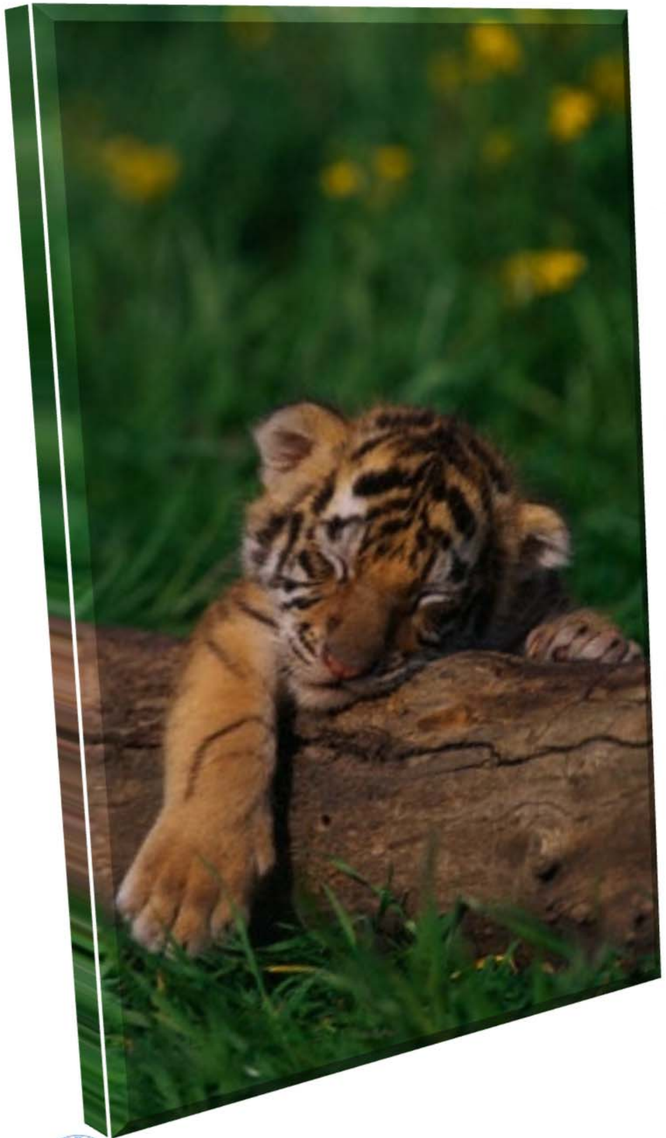
$$x(t) = r(t) - 2r(t-1) + r(t-2)$$

which is piecewise linear, using the derivative property.

[Ex 5.22] Consider the integral

$$y(t) = \int_{-\infty}^{t} x(\tau)\ d\tau \qquad\qquad -\infty < t < \infty$$

where $x(t) = u(t+1) - u(t-1)$. Find the Fourier transform $Y(\Omega)$ directly and from the integration property.



Thank You