

Chapter 5. Reduction of Multiple Subsystems

PREFACE, ix

1. INTRODUCTION, 1

- 1.1 Introduction, 2
- 1.2 A History of Control Systems, 4
- 1.3 System Configurations, 7
- 1.4 Analysis and Design Objectives, 10
 - Case Study, 12
- 1.5 The Design Process, 15
- 1.6 Computer-Aided Design, 20
- 1.7 The Control Systems Engineer, 21
 - Summary, 23
 - Review Questions, 23
 - Problems, 24
 - Cyber Exploration Laboratory, 30
 - Bibliography, 31

2. MODELING IN THE FREQUENCY DOMAIN, 33

- 2.1 Introduction, 34
- 2.2 Laplace Transform Review, 35
- 2.3 The Transfer Function, 44
- 2.4 Electrical Network Transfer Functions, 47
- 2.5 Translational Mechanical System Transfer Functions, 61
- 2.6 Rotational Mechanical System Transfer Functions, 69
- 2.7 Transfer Functions for Systems with Gears, 74
- 2.8 Electromechanical System Transfer Functions, 79
- 2.9 Electric Circuit Analogs, 84
- 2.10 Nonlinearities, 88
- 2.11 Linearization, 89
 - Case Studies, 94
 - Summary, 97
 - Review Questions, 97

- Problems, 98
- Cyber Exploration Laboratory, 112
- Bibliography, 115

3. MODELING IN THE TIME DOMAIN, 117

- 3.1 Introduction, 118
- 3.2 Some Observations, 119
- 3.3 The General State-Space Representation, 123
- 3.4 Applying the State-Space Representation, 124
- 3.5 Converting a Transfer Function to State Space, 132
- 3.6 Converting from State Space to a Transfer Function, 139
- 3.7 Linearization, 141
 - Case Studies, 144
 - Summary, 148
 - Review Questions, 149
 - Problems, 149
 - Cyber Exploration Laboratory, 157
 - Bibliography, 159

4. TIME RESPONSE, 161

- 4.1 Introduction, 162
- 4.2 Poles, Zeros, and System Response, 162
- 4.3 First-Order Systems, 166
- 4.4 Second-Order Systems: Introduction, 168
- 4.5 The General Second-Order System, 173
- 4.6 Underdamped Second-Order Systems, 177
- 4.7 System Response with Additional Poles, 186
- 4.8 System Response With Zeros, 191
- 4.9 Effects of Nonlinearities Upon Time Response, 196

- 4.10 Laplace Transform Solution of State Equations, 199
- 4.11 Time Domain Solution of State Equations, 203
 - Case Studies, 207
 - Summary, 213
 - Review Questions, 214
 - Problems, 215
 - Cyber Exploration Laboratory, 228
 - Bibliography, 232

5. REDUCTION OF MULTIPLE SUBSYSTEMS, 235

- 5.1 Introduction, 236
- 5.2 Block Diagrams, 236
- 5.3 Analysis and Design of Feedback Systems, 245
- 5.4 Signal-Flow Graphs, 248
- 5.5 Mason's Rule, 251
- 5.6 Signal-Flow Graphs of State Equations, 254
- 5.7 Alternative Representations in State Space, 256
- 5.8 Similarity Transformations, 266
 - Case Studies, 272
 - Summary, 278
 - Review Questions, 279
 - Problems, 280
 - Cyber Exploration Laboratory, 297
 - Bibliography, 299

6. STABILITY, 301

- 6.1 Introduction, 302
- 6.2 Routh-Hurwitz Criterion, 305
- 6.3 Routh-Hurwitz Criterion: Special Cases, 308
- 6.4 Routh-Hurwitz Criterion: Additional Examples, 314
- 6.5 Stability in State Space, 320
 - Case Studies, 323
 - Summary, 325
 - Review Questions, 325
 - Problems, 326

- Cyber Exploration Laboratory, 335
- Bibliography, 336

7. STEADY-STATE ERRORS, 339

- 7.1 Introduction, 340
- 7.2 Steady-State Error for Unity Feedback Systems, 343
- 7.3 Static Error Constants and System Type, 349
- 7.4 Steady-State Error Specifications, 353
- 7.5 Steady-State Error for Disturbances, 356
- 7.6 Steady-State Error for Nonunity Feedback Systems, 358
- 7.7 Sensitivity, 362
- 7.8 Steady-State Error for Systems in State Space, 364
 - Case Studies, 368
 - Summary, 371
 - Review Questions, 372
 - Problems, 373
 - Cyber Exploration Laboratory, 384
 - Bibliography, 386

8. ROOT LOCUS TECHNIQUES, 387

- 8.1 Introduction, 388
- 8.2 Defining the Root Locus, 392
- 8.3 Properties of the Root Locus, 394
- 8.4 Sketching the Root Locus, 397
- 8.5 Refining the Sketch, 402
- 8.6 An Example, 411
- 8.7 Transient Response Design via Gain Adjustment, 415
- 8.8 Generalized Root Locus, 419
- 8.9 Root Locus for Positive-Feedback Systems, 421
- 8.10 Pole Sensitivity, 424
 - Case Studies, 426
 - Summary, 431
 - Review Questions, 432
 - Problems, 432
 - Cyber Exploration Laboratory, 450
 - Bibliography, 452

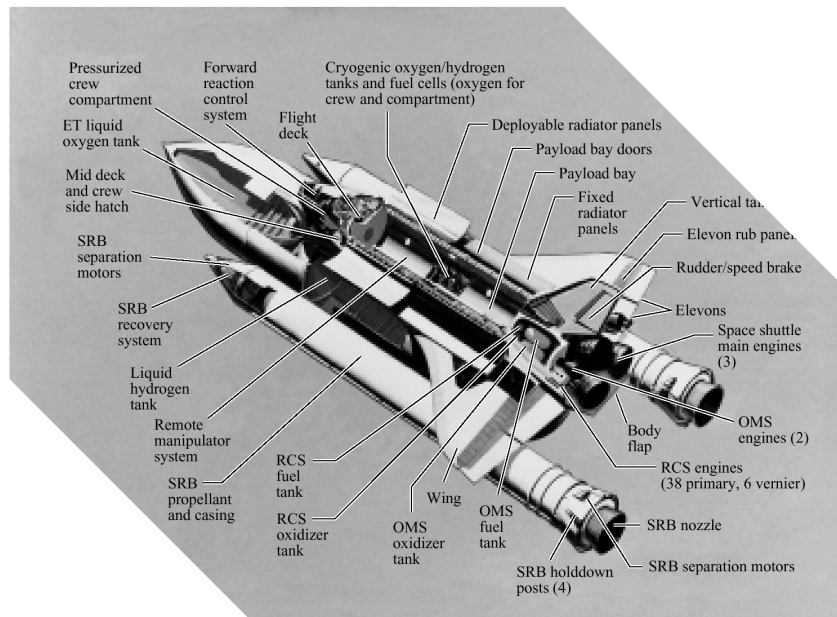
Chapter 5. Reduction of Multiple Subsystems

Objectives

- How to reduce a block diagram of multiple subsystems to a single block representing the transfer function from input to output
- How to analyze and design transient response for a system consisting of multiple subsystems
- How to representing in state space a system consisting of multiple subsystems
- How to convert between alternate representations of a system in state space

5.1 Introduction

- Interconnection of many subsystems
 - ⇒ frequency domain or state-space analysis and design
 - ⇒ block diagrams / signal-flow graphs
- Block diagrams \Leftarrow frequency domain analysis
- Signal-flow graphs \Leftarrow state-space analysis

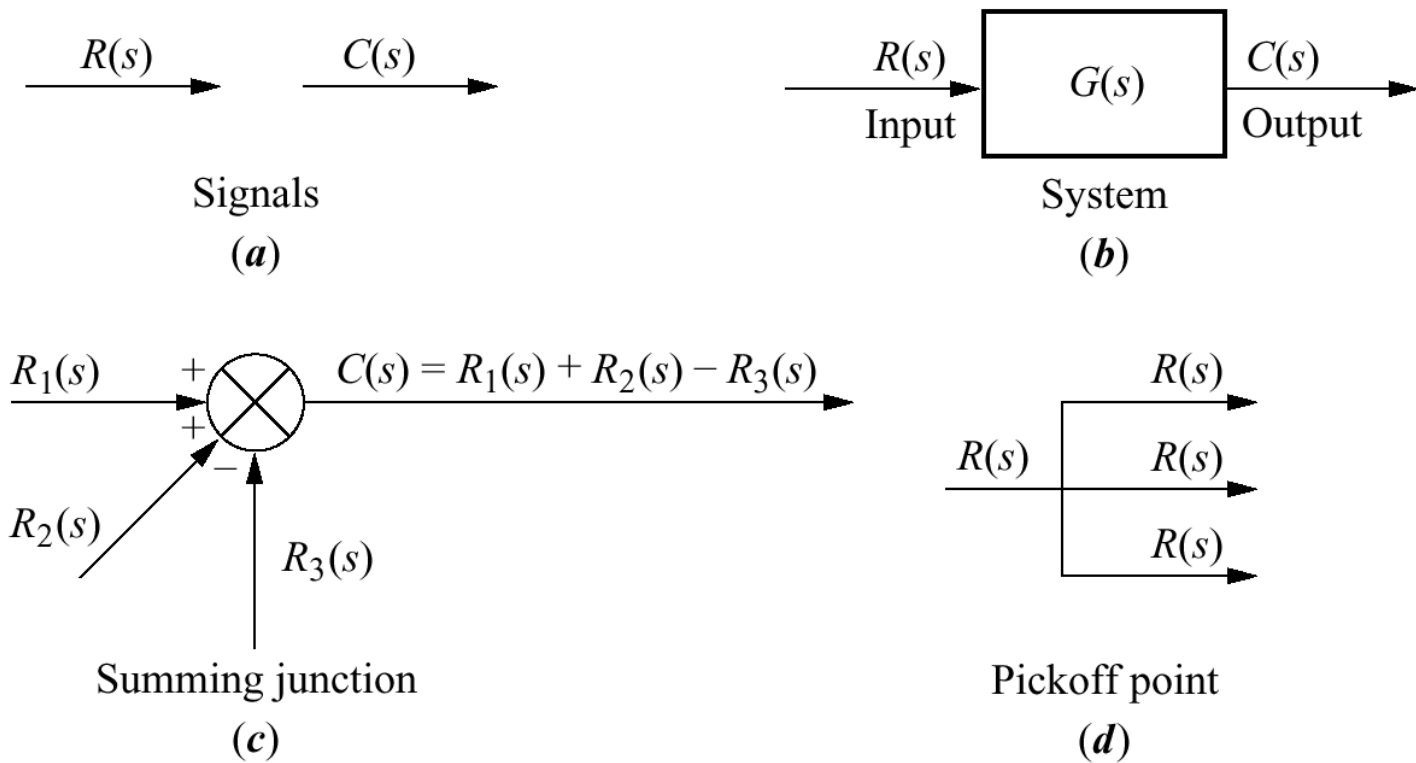


- The space shuttle consists of multiple subsystems.
- Can you identify those that are control systems, or parts of control systems?

5.2 Block Diagrams

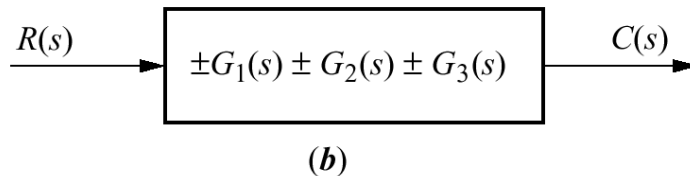
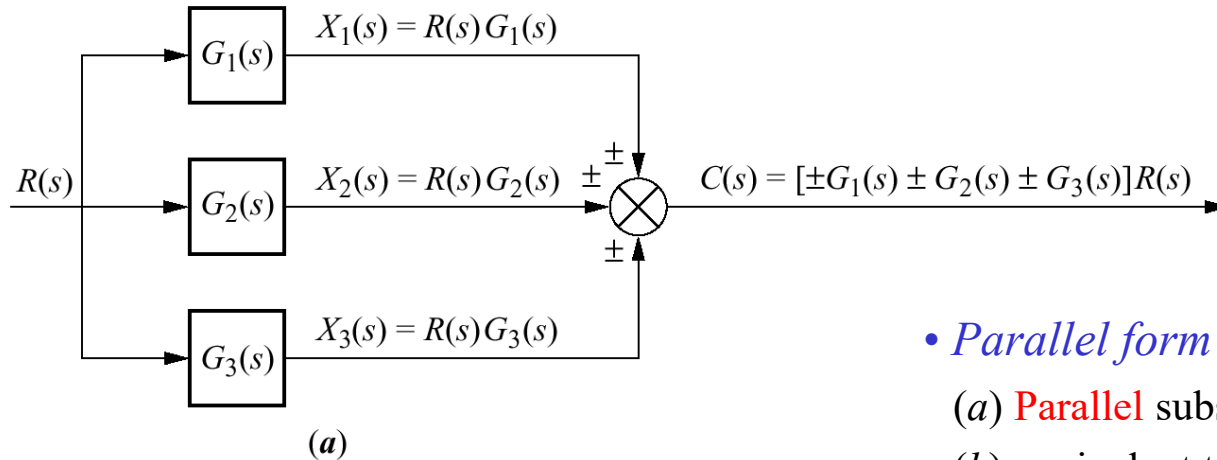
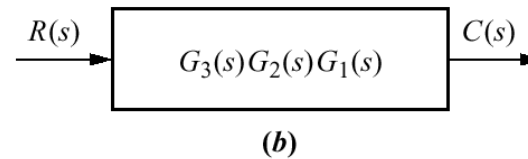
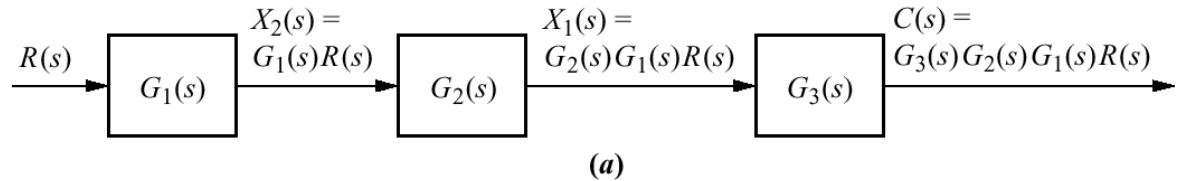
Figure 5.2

Components of a block diagram for a linear, time-invariant system



- *Cascade form*

- Cascaded** subsystems;
- equivalent transfer function



- *Parallel form*

- Parallel** subsystems;
- equivalent transfer function

• *Feedback form*

- a. **Feedback** control system;
- b. simplified model;
- c. equivalent transfer function

$$G_e(s) = \frac{C(s)}{R(s)} = ? \quad \Rightarrow \text{Closed-loop transfer function}$$

$$E(s) = R(s) \mp C(s)H(s)$$

$$C(s) = E(s)G(s) \Rightarrow E(s) = \frac{C(s)}{G(s)}$$

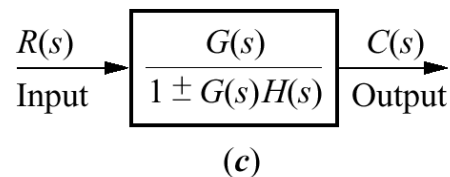
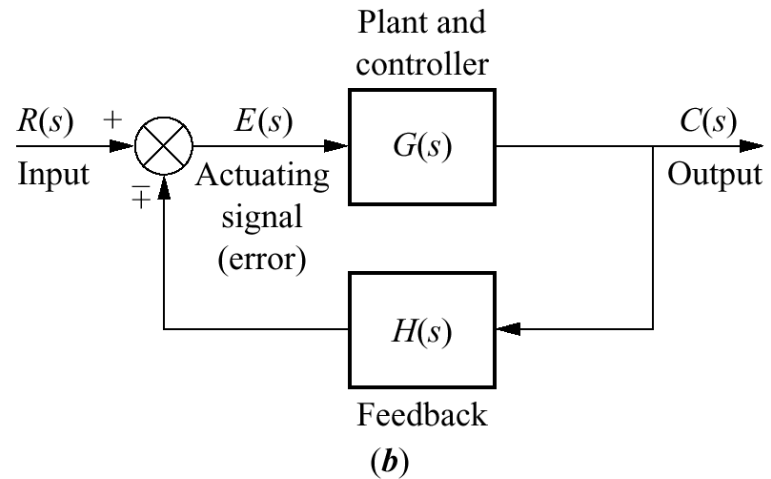
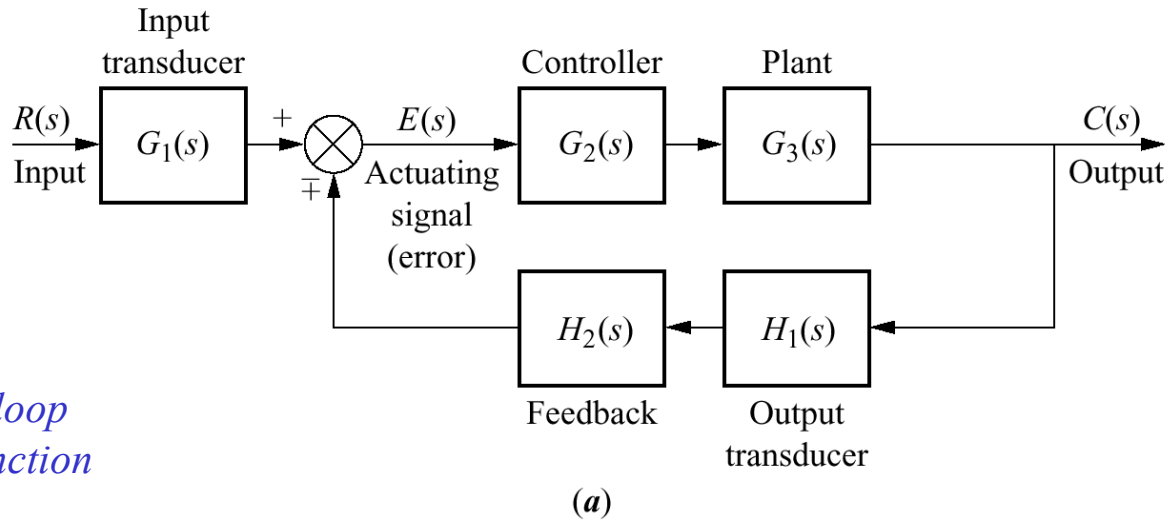
$$\frac{C(s)}{G(s)} = R(s) \mp C(s)H(s)$$

$$C(s) = G(s)R(s) \mp G(s)C(s)H(s),$$

$$C(s)[1 \pm G(s)H(s)] = G(s)R(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm \underline{G(s)H(s)}}$$

\Rightarrow *open-loop transfer function
or loop gain*



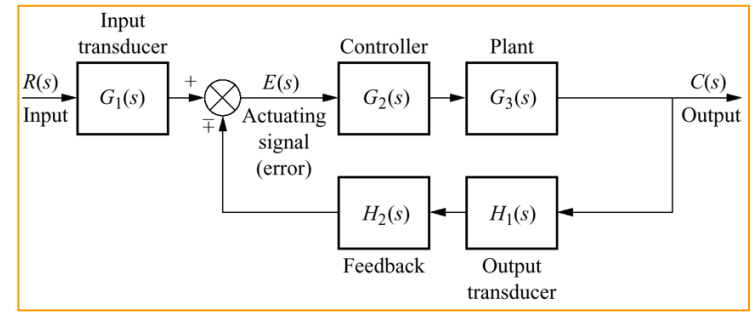
- *Moving blocks to create familiar forms*

Block diagram algebra for summing junctions

- equivalent forms for moving a block

(a) to the left past a summing junction;

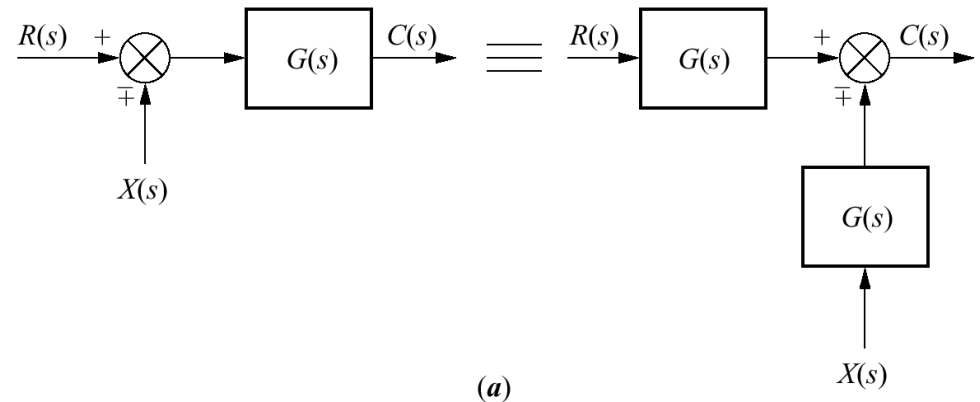
(b) to the right past a summing junction



$$[R(s) \mp X(s)]G(s) = C(s)$$



$$R(s)G(s) \mp X(s)G(s) = C(s)$$



$$R(s)G(s) \mp X(s) = C(s)$$



$$\left[R(s) \mp \frac{1}{G(s)} X(s) \right] G(s) = C(s)$$

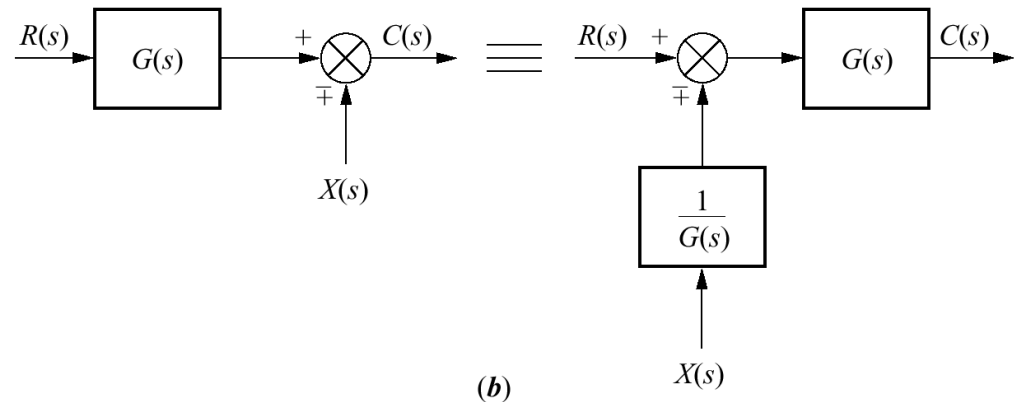
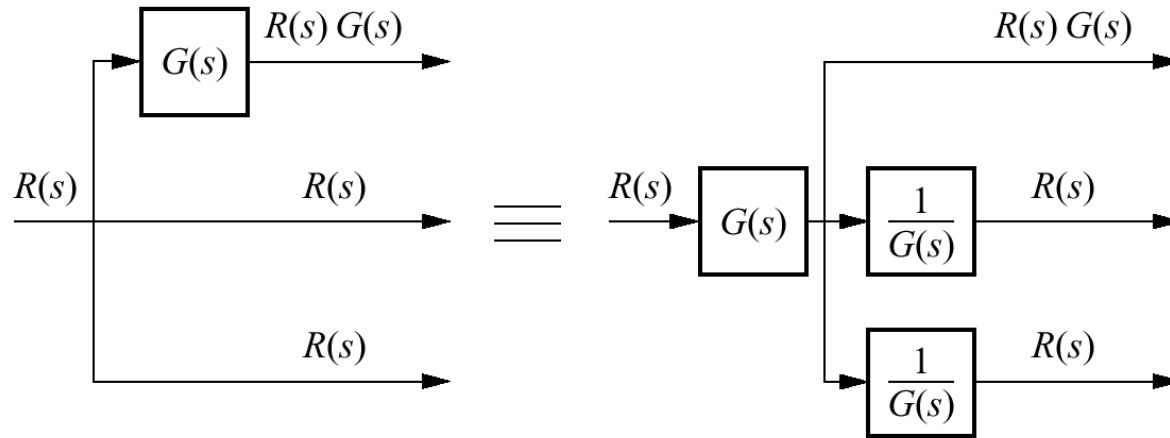


Figure 5.8

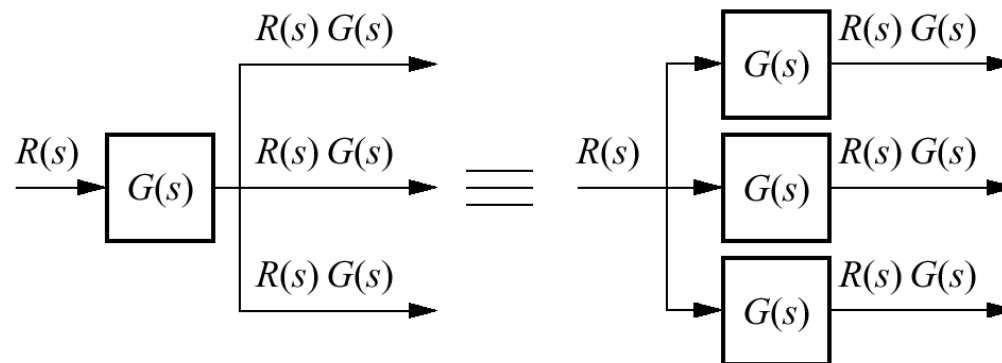
Block diagram algebra for *pickoff points* - equivalent forms for moving a block

(a) to the left past a pickoff point;

(b) to the right past a pickoff point



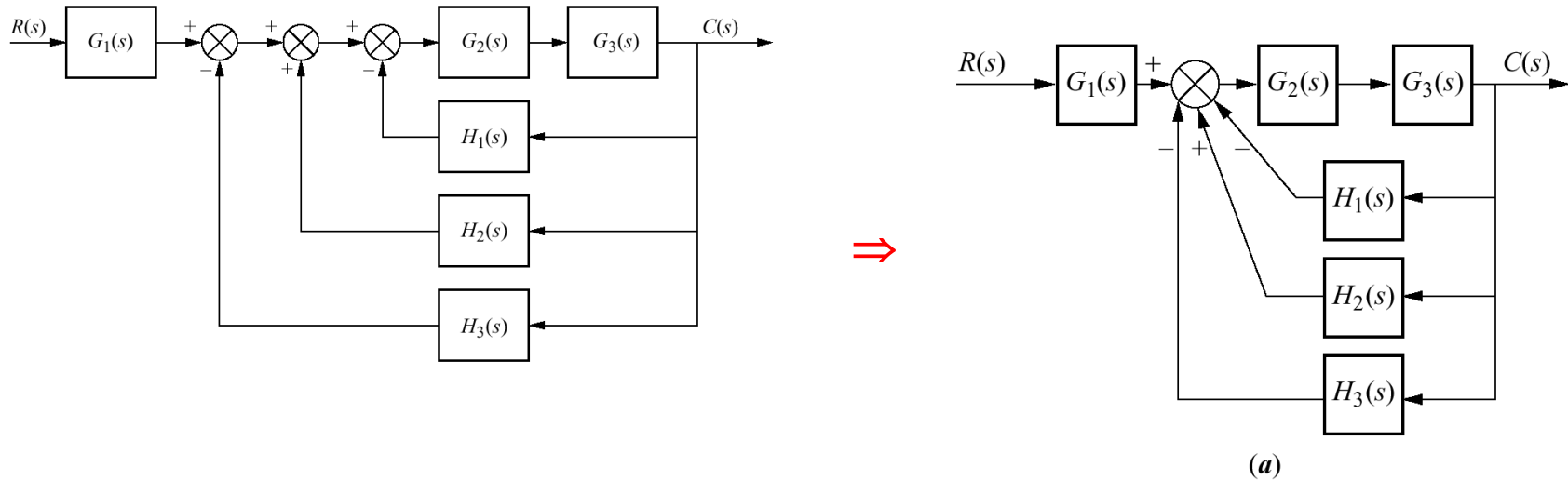
(a)



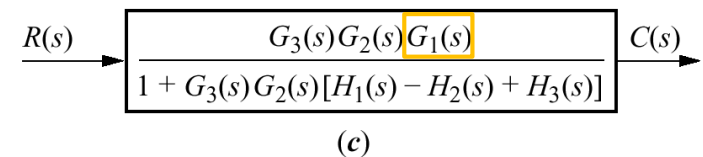
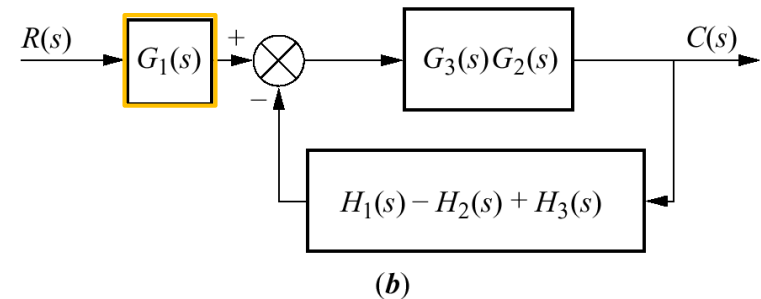
(b)

Example 5.1: Block diagram reduction via familiar forms (page 242)

Reduce the block diagram to a single transfer function



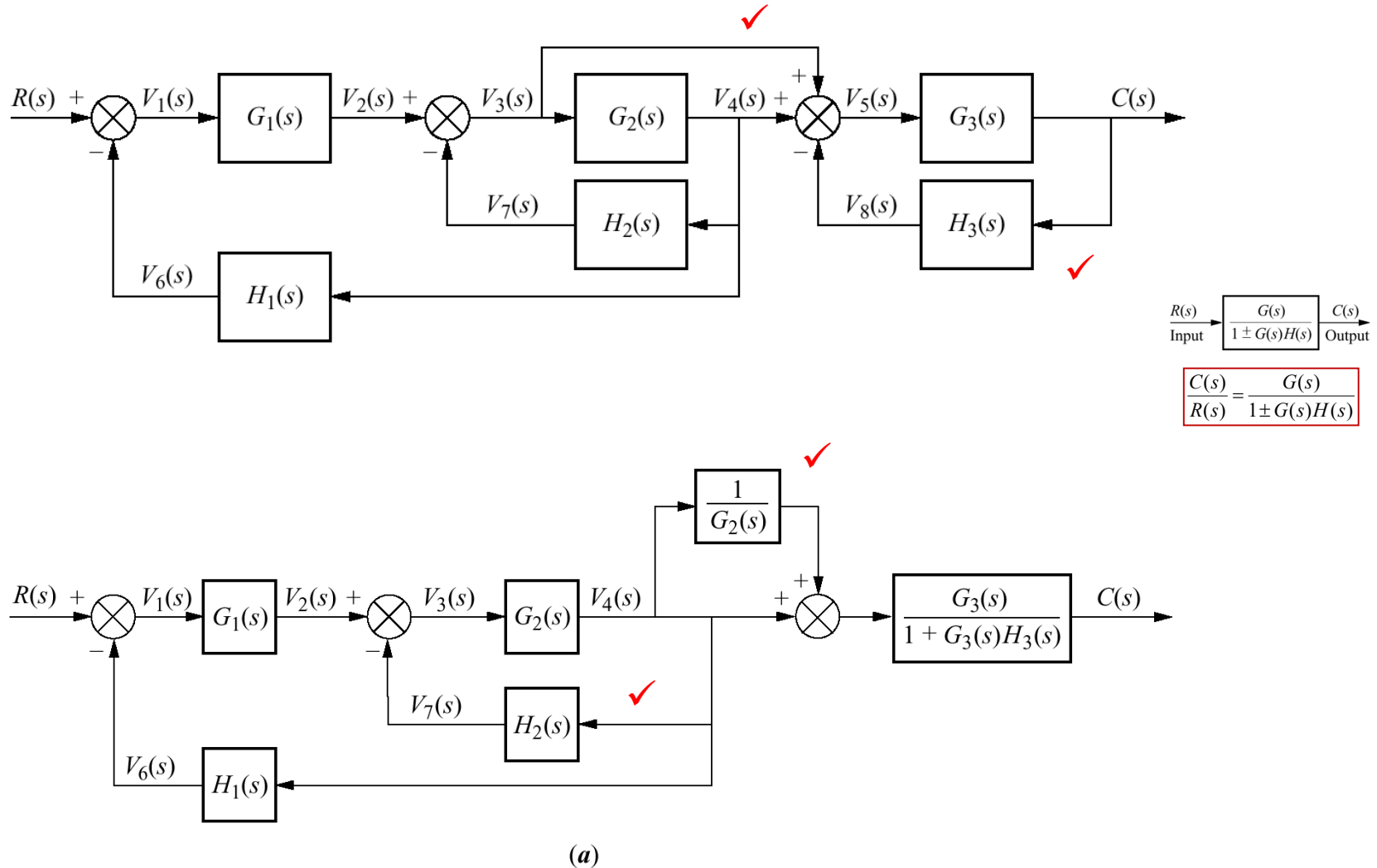
- (a) collapse summing junctions;
- (b) form equivalent cascaded system in the forward path and equivalent parallel system in the feedback path;
- (c) form equivalent feedback system and multiply by cascaded $G_1(s)$

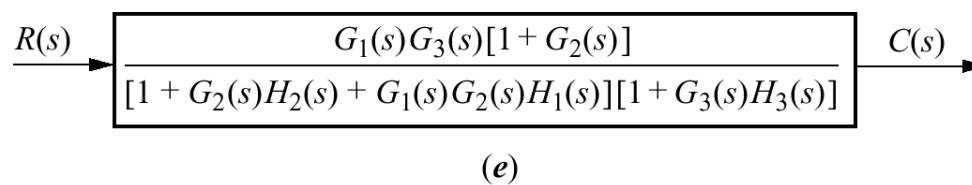
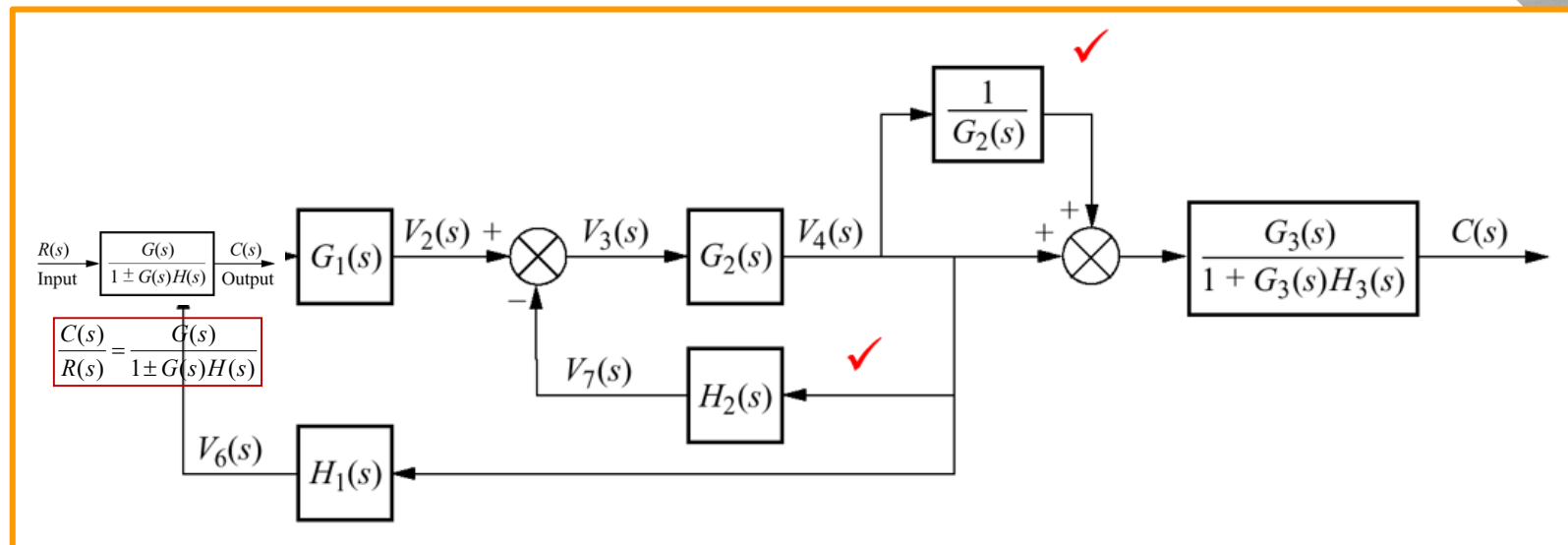
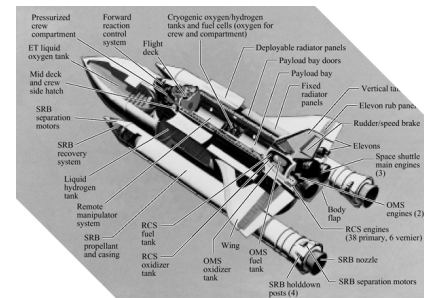
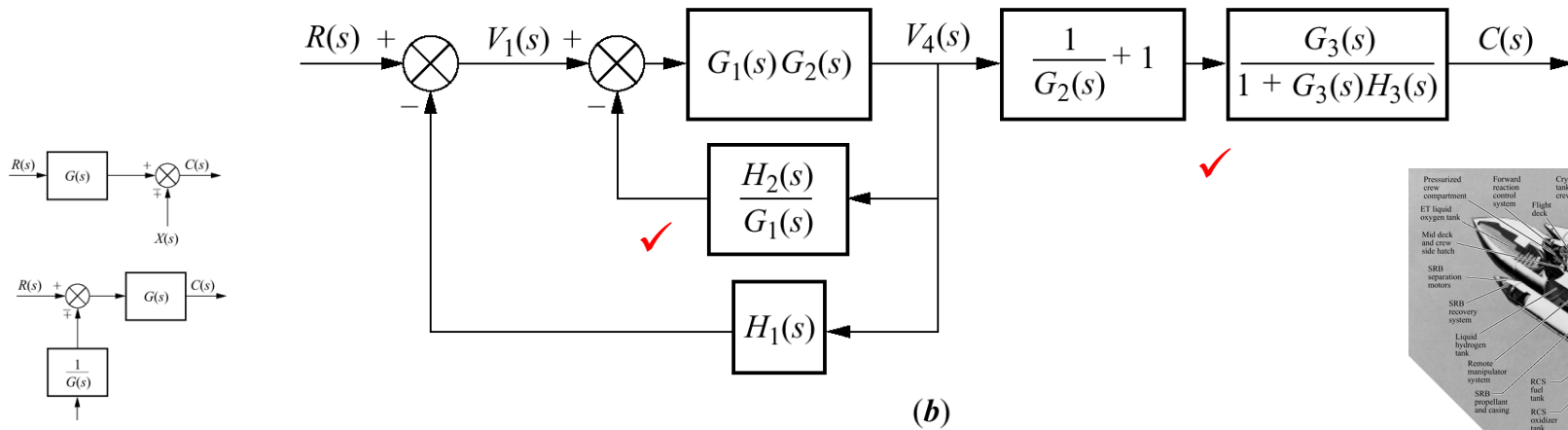


$$\begin{array}{c} \frac{R(s)}{\text{Input}} \rightarrow \boxed{\frac{G(s)}{1 \pm G(s)H(s)}} \rightarrow \frac{C(s)}{\text{Output}} \end{array} \quad \boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}}$$

Example 5.2: Block diagram reduction via moving blocks (page 243)

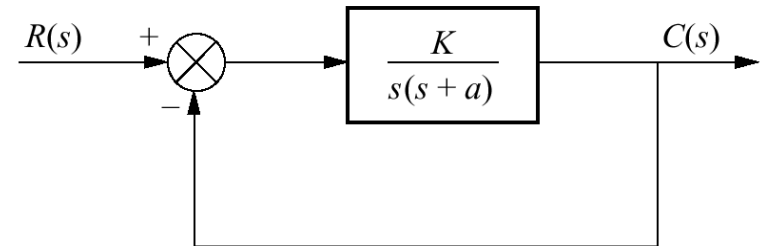
Reduce the system to a single transfer function





5.3 Analysis and Design of Feedback Systems

- Second-order feedback control system:



$$E(s) = R(s) - C(s)$$

$$C(s) = E(s)G(s) = R(s)G(s) - C(s)G(s) \Rightarrow$$

$$C(s)[1 + G(s)] = R(s)G(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \Rightarrow T(s) = \frac{\frac{K}{s(s+a)}}{1 + \frac{K}{s(s+a)}} = \frac{K}{s^2 + as + K}$$

$$\frac{R(s)}{\text{Input}} \rightarrow \left[\frac{G(s)}{1 \pm G(s)H(s)} \right] \rightarrow \frac{C(s)}{\text{Output}}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

Root Locus

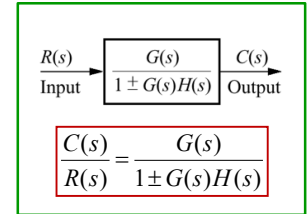
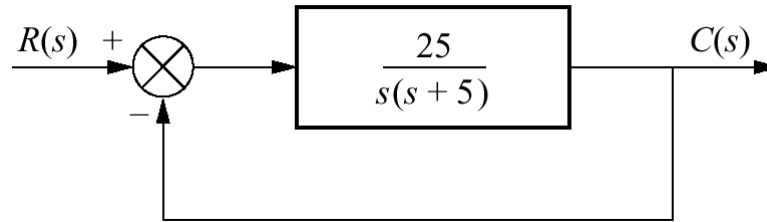
For $0 < K < \frac{a^2}{4}$, $s_{1,2} = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4K}}{2} \Rightarrow \boxed{\text{overdamped}}$

For $K = \frac{a^2}{4}$, $s_{1,2} = -\frac{a}{2} \Rightarrow \boxed{\text{critically damped}}$

For $K > \frac{a^2}{4}$, $s_{1,2} = -\frac{a}{2} \pm j\frac{\sqrt{4K - a^2}}{2} \Rightarrow \boxed{\text{underdamped}}$

Example 5.3: Finding transient response (page 246)

Find the peak time, % overshoot, and settling time.



$$T(s) = \frac{25}{s^2 + 5s + 25}$$

$$\omega_n = \sqrt{25} = 5, \quad 2\zeta\omega_n = 5, \Rightarrow \zeta = 0.5$$

$$\Rightarrow T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = 0.726 \text{ sec}$$

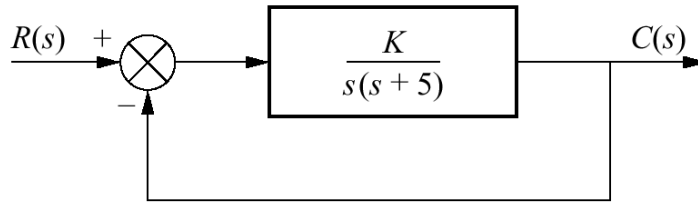
$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100\% = 16.30\%$$

$$T_s = \frac{4}{\zeta\omega_n} = 1.6 \text{ sec}$$

$$\begin{aligned} T(s) &= \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \\ &= \frac{\frac{25}{s(s+5)}}{1 + \frac{25}{s(s+5)}} = \frac{\frac{25}{s(s+5)}}{\frac{s(s+5) + 25}{s(s+5)}} \\ &= \frac{25}{s^2 + 5s + 25} \end{aligned}$$

Example 5.4: Gain design for transient response (page 247)

Design the gain K so that the system will respond with 10% overshoot.



We want: %OS = 10%

$$\zeta = \frac{-\ln(\%OS / 100)}{\sqrt{\pi^2 + \ln^2(\%OS / 100)}} = \frac{-\ln(0.1)}{\sqrt{\pi^2 + \ln^2(0.1)}} = 0.591$$

$$T(s) = \frac{K}{s^2 + 5s + K}$$

$$2\zeta\omega_n = 5, \quad \omega_n = \sqrt{K} \Rightarrow \zeta = \frac{5}{2\sqrt{K}}$$

$$2\zeta\sqrt{K} = 5$$

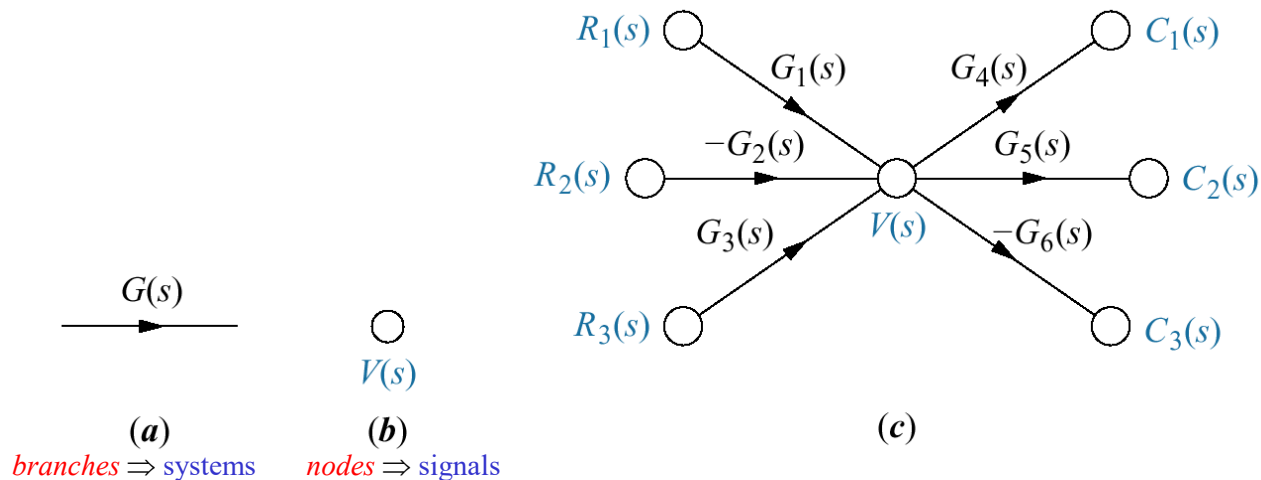
$$K = \left(\frac{5}{2\zeta} \right)^2$$

$$K = 17.89$$

$$\begin{aligned} T(s) &= \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \\ &= \frac{\frac{K}{s(s+5)}}{1 + \frac{K}{s(s+5)}} = \frac{\frac{K}{s(s+5)}}{\frac{s(s+5) + K}{s(s+5)}} \\ &= \frac{K}{s^2 + 5s + 25} \end{aligned}$$

5.4 Signal-Flow Graphs

- Block diagrams: blocks, signals, summing junctions, and pickoff points
- Signal-flow graph: *branches* \Rightarrow systems
nodes \Rightarrow signals



Signal-flow graph components:

(a) system

(b) signal

(c) interconnection of systems and signals

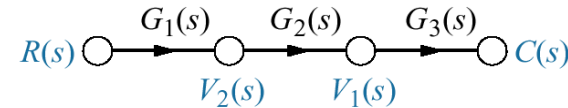
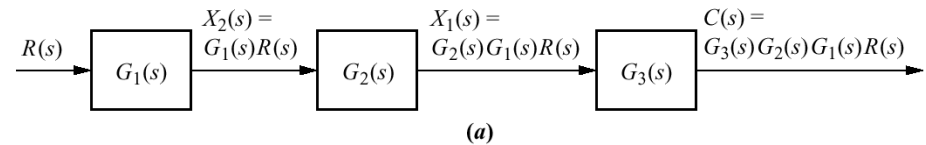
Example 5.5: Converting the block diagrams to signal-flow graphs (page 249)

Converting the block diagrams into signal-flow graphs.

Building signal-flow graphs:

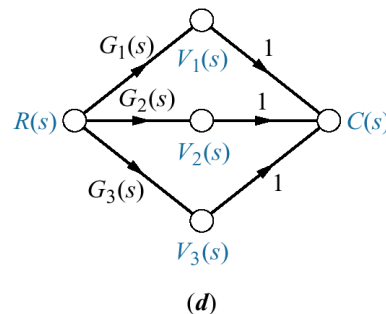
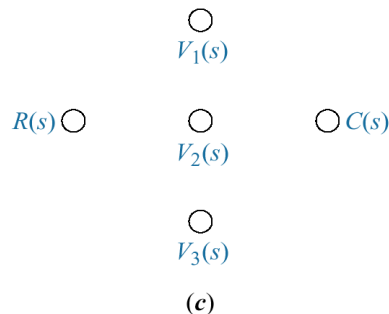
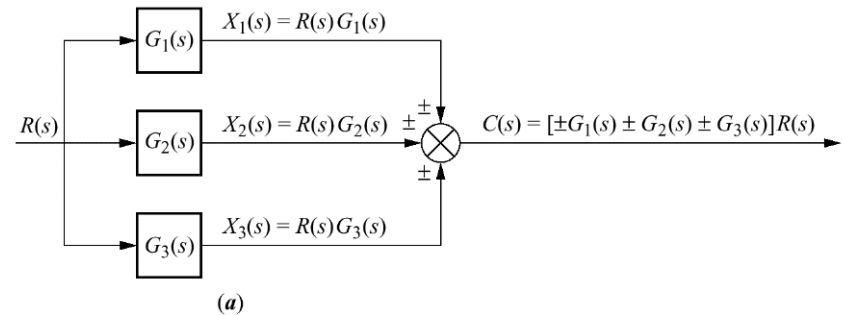
(a) cascaded system nodes (Figure 5.3(a));

(b) cascaded system signal-flow graph;



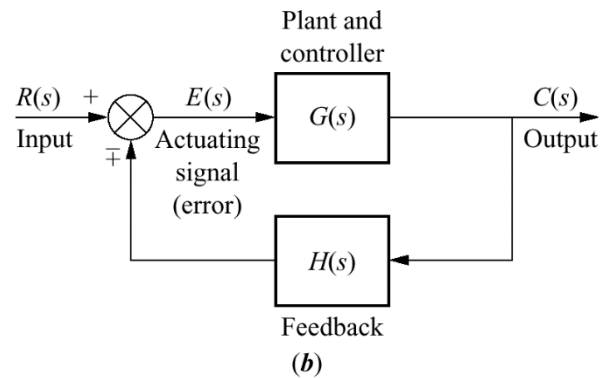
(c) parallel system nodes (Figure 5.5(a));

(d) parallel system signal-flow graph;

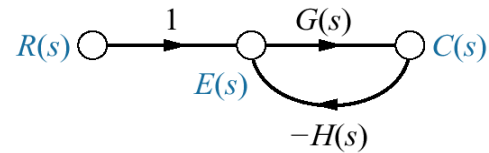


(e) feedback system nodes (from Figure 5.6(b));

(f) feedback system signal-flow graph



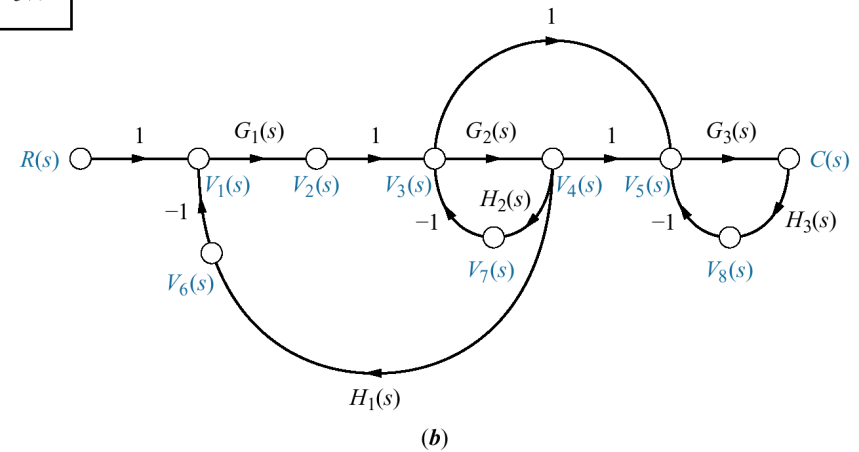
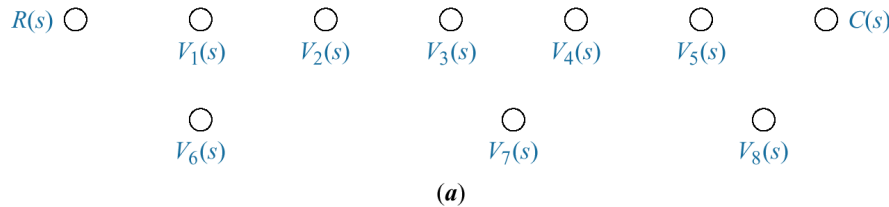
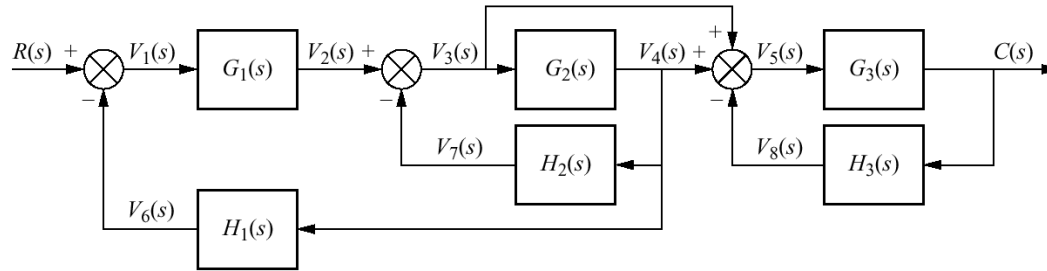
(e)



(f)

Example 5.6: Converting a block diagram to signal-flow graph (page 250)

Converting the block diagrams into signal-flow graphs.

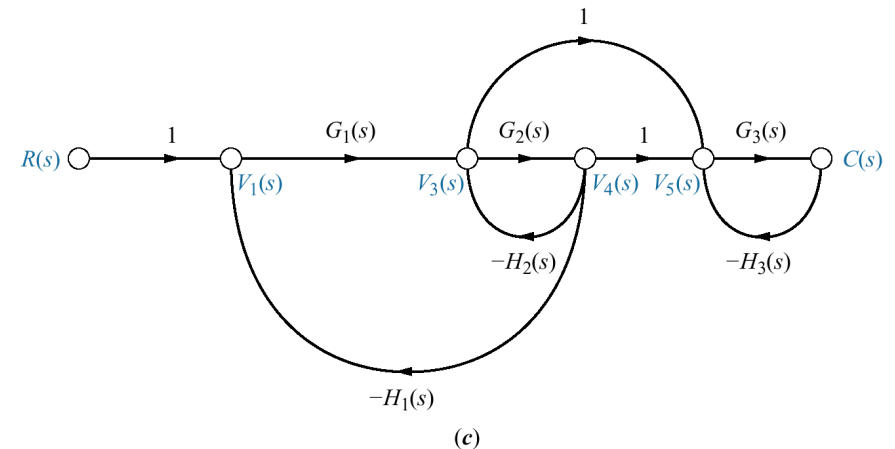


Signal-flow graph development:

(a) signal nodes;

(b) signal-flow graph;

(c) simplified signal-flow graph



5.5 Mason's Rule

k = Number of forward paths

T_k = The k^{th} forward-path gain

$\Delta = 1 - \Sigma(\text{loop gains}) + \Sigma(\text{nontouching-loop gains taken two at a time}) - \Sigma(\text{nontouching-loop gains taken three at a time}) + \Sigma \dots$

$\Delta_k = \Delta - \Sigma(\text{loop gain terms in } \Delta \text{ that touch the } k^{\text{th}} \text{ forward path})$

• Definition
$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

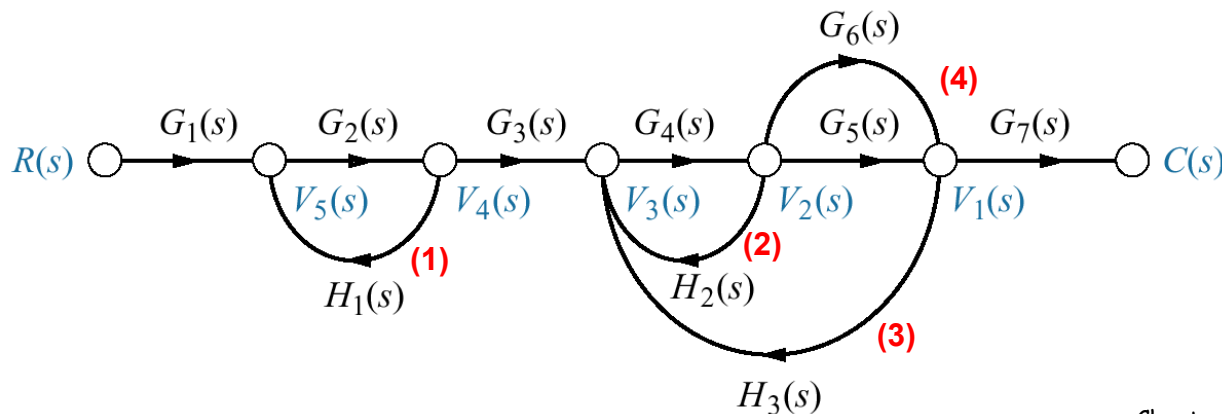
- Loop gain: $G_2(s)H_1(s)$, G_4H_2 , $G_4G_5H_3$, $G_4G_6H_3$

- Forward-path gain: $G_1G_2G_3G_4G_5G_7$, $G_1G_2G_3G_4G_6G_7$

- Nontouching loops: loop G_2H_1 does not touch loops G_4H_2 , $G_4G_5H_3$, $G_4G_6H_3$

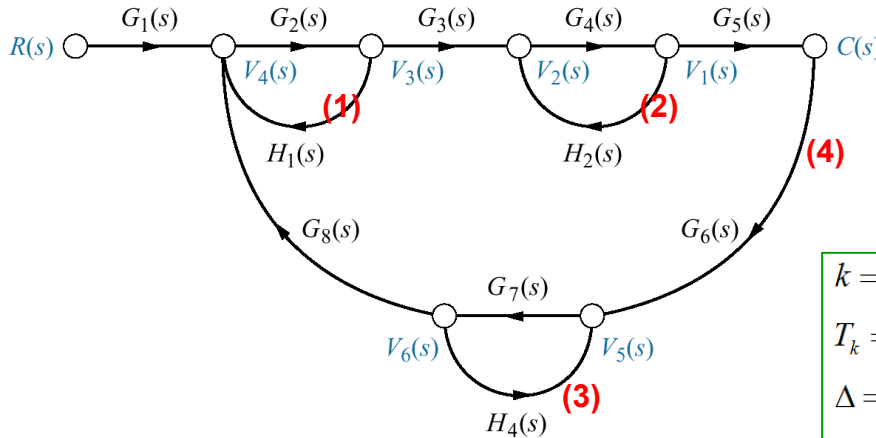
- Nontouching-loop gain: $\underline{G_2H_1} \underline{G_4H_2}$, $\underline{G_2H_1} \underline{G_4G_5H_3}$, $\underline{G_2H_1} \underline{G_4G_6H_3}$

\Rightarrow Nontouching-loop gains taken two at a time



Example 5.7: Transfer function via Mason's rule (page 252)

Find the transfer function for the signal-flow graphs.



$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

k = Number of forward paths

T_k = The k^{th} forward-path gain

$\Delta = 1 - \Sigma(\text{loop gains}) + \Sigma(\text{nontouching-loop gains taken two at a time}) - \Sigma(\text{nontouching-loop gains taken three at a time}) + \Sigma \dots$

$\Delta_k = \Delta - \Sigma(\text{loop gain terms in } \Delta \text{ that touch the } k^{\text{th}} \text{ forward path})$

- Forward-path gains: $G_1 G_2 G_3 G_4 G_5$

- Loop gains: (1) $G_2 H_1$, (2) $G_4 H_2$, (3) $G_7 H_4$, (4) $G_2 G_3 G_4 G_5 G_6 G_7 G_8$

- Nontouching loops taken two at a time:

$$(1, 2) G_2 H_1 G_4 H_2, (1, 3) G_2 H_1 G_7 H_4, (2, 3) G_4 H_2 G_7 H_4$$

- Nontouching loops taken three at a time: $(1, 2, 3) G_2 H_1 G_4 H_2 G_7 H_4$

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

k = Number of forward paths

T_k = The k^{th} forward-path gain

Δ = $1 - \Sigma(\text{loop gains}) + \Sigma(\text{nontouching-loop gains taken two at a time})$
 $- \Sigma(\text{nontouching-loop gains taken three at a time}) + \Sigma \dots$

$\Delta_k = \Delta - \Sigma(\text{loop gain terms in } \Delta \text{ that touch the } k^{\text{th}} \text{ forward path})$

$$\begin{aligned} \Delta = & 1 - [G_2 H_1 + G_4 H_2 + G_7 H_4 + G_2 G_3 G_4 G_5 G_6 G_7 G_8] \\ & + [G_2 H_1 G_4 H_2 + G_2 H_1 G_7 H_4 + G_4 H_2 G_7 H_4] \\ & - [G_2 H_1 G_4 H_2 G_7 H_4] \end{aligned}$$

$$T_1 = G_1 G_2 G_3 G_4 G_5$$

$$\Delta_1 = 1 - G_7 H_4$$

$$\Rightarrow G(s) = \frac{T_1 \Delta_1}{\Delta} = \frac{[G_1 G_2 G_3 G_4 G_5][1 - G_7 H_4]}{\Delta}$$

- Forward-path gains: $G_1 G_2 G_3 G_4 G_5$

- Loop gains: (1) $G_2 H_1$, (2) $G_4 H_2$, (3) $G_7 H_4$, (4) $G_2 G_3 G_4 G_5 G_6 G_7 G_8$

- Nontouching loops taken two at a time:

(1, 2) $G_2 H_1 G_4 H_2$, (1, 3) $G_2 H_1 G_7 H_4$, (2, 3) $G_4 H_2 G_7 H_4$

- Nontouching loops taken three at a time: (1, 2, 3) $G_2 H_1 G_4 H_2 G_7 H_4$

5.6 Signal-Flow Graphs of State Equations

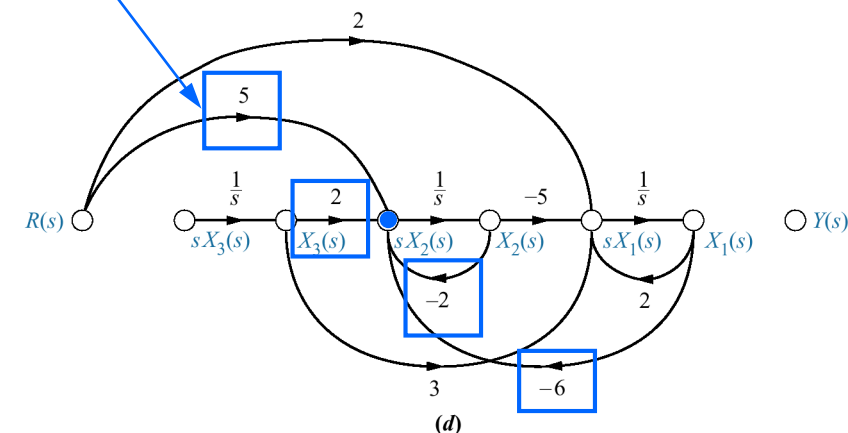
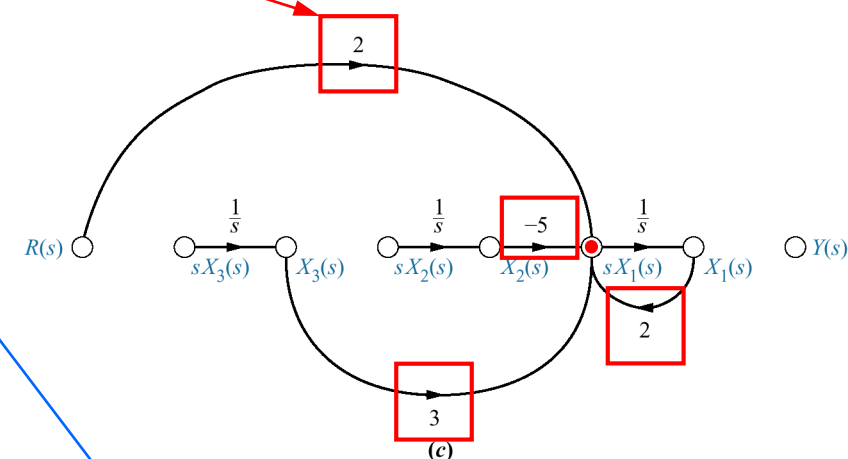
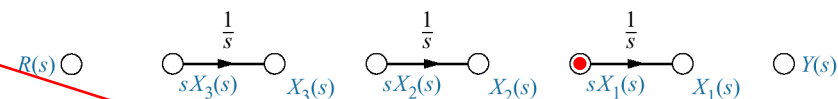
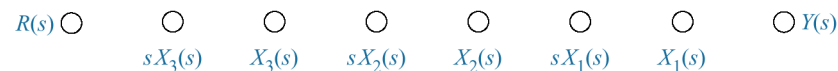
$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

$$y = -4x_1 + 6x_2 + 9x_3$$

- Block diagrams: blocks, signals, summing junctions, and pickoff points
- Signal-flow graph:
 - branches* \Rightarrow systems
 - nodes* \Rightarrow signals

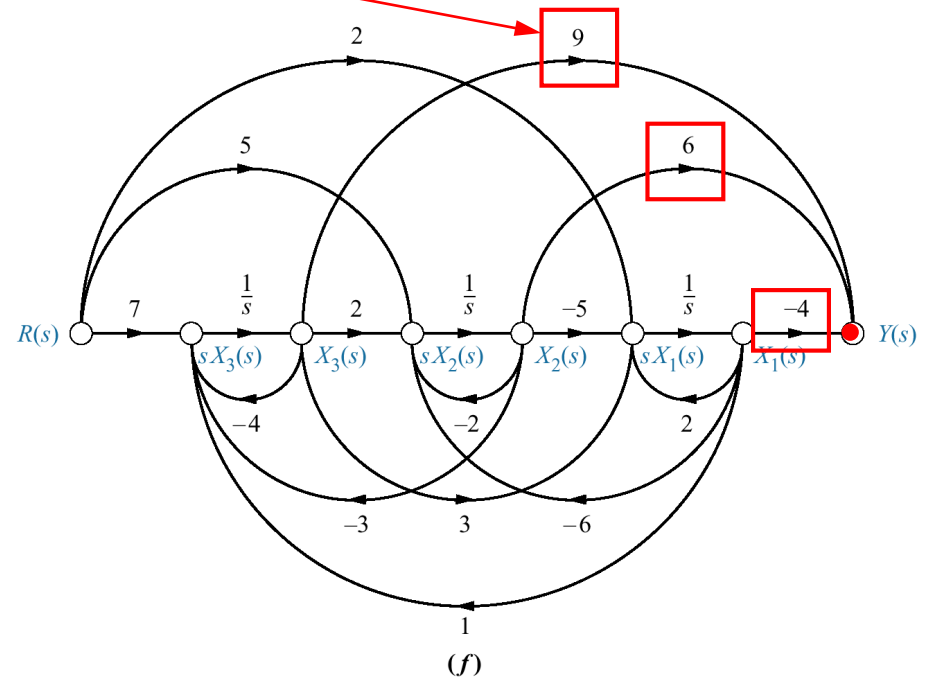
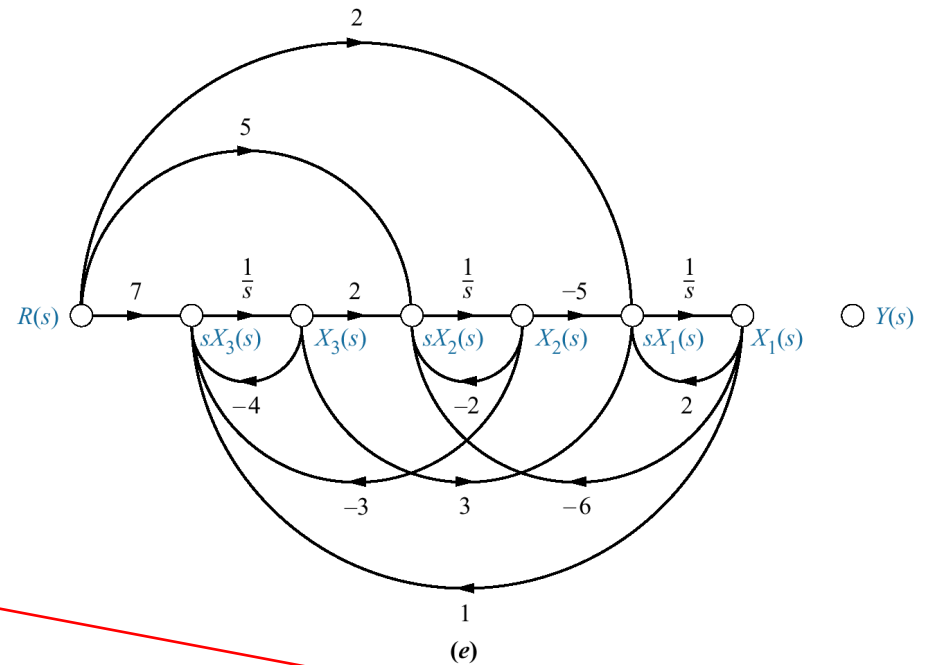


$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

$$y = -4x_1 + 6x_2 + 9x_3$$



- Block diagrams: blocks, signals, summing junctions, and pickoff points
- Signal-flow graph:
 - branches* \Rightarrow systems
 - nodes* \Rightarrow signals

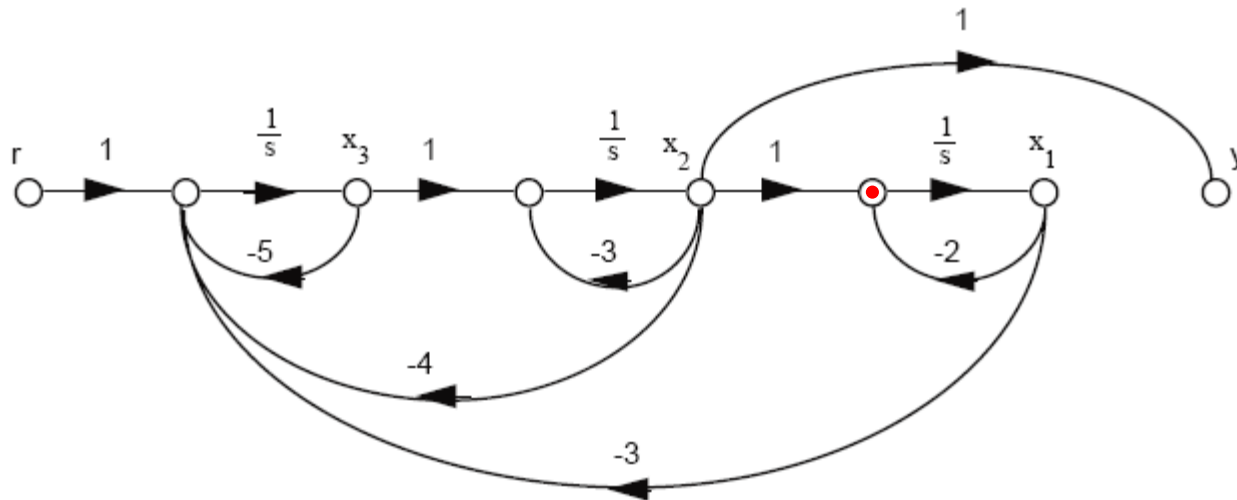
Exercise 5.5: Draw a signal-flow graph for the following state and output equations. (page 256)

$$\dot{\mathbf{x}} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r$$

$$y = (0 \quad 1 \quad 0) \mathbf{x}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = -2x_1 + x_2 \\ \dot{x}_2 = -3x_2 + x_3 \\ \dot{x}_3 = -3x_1 - 4x_2 - 5x_3 + r \\ y = x_2 \end{cases}$$

- Drawing the signal-flow diagram from the state equations:



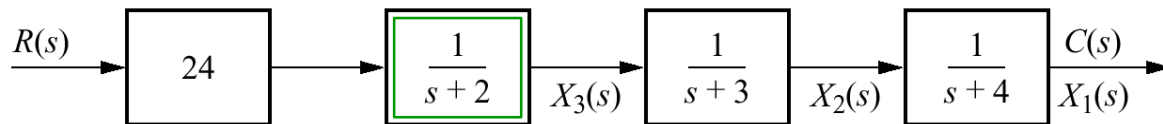
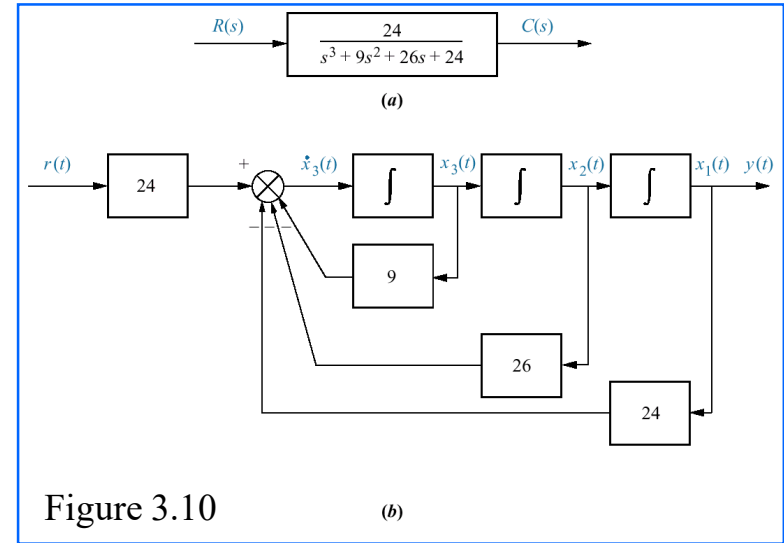
5.7 Alternate Representations in State Space

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24} = \frac{24}{(s+2)(s+3)(s+4)}$$

⇓

(1) Cascade form

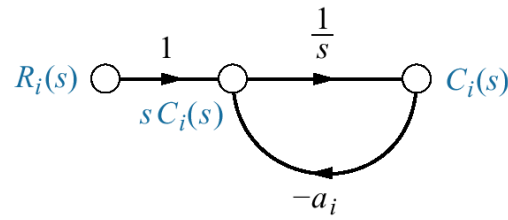
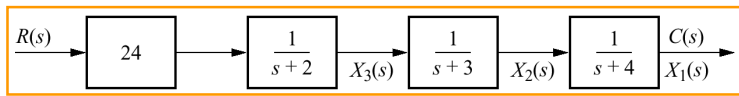
Representation of Figure 3.10 system as cascaded first-order systems



$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s + a_i)}$$

$$\Rightarrow (s + a_i)C_i(s) = R_i(s)$$

$$\Rightarrow \frac{d}{dt}c_i(t) = -a_i c_i(t) + r_i(t)$$

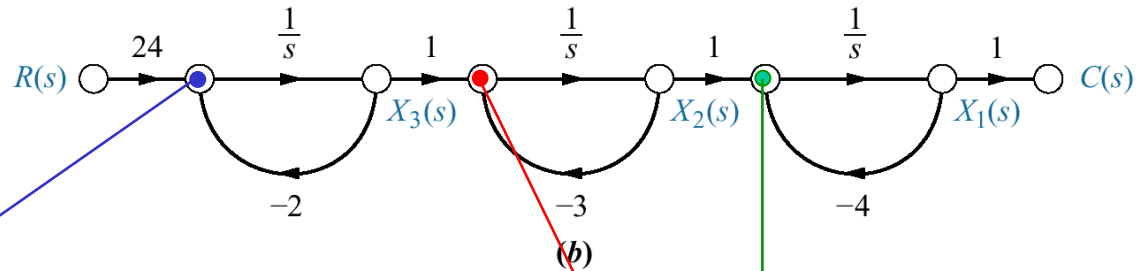


$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s + a_i)}$$

$$\Rightarrow \frac{d}{dt}c_i(t) = -a_i c_i(t) + r_i(t)$$

(a)

(a) first-order subsystem;
(b) signal-flow graph



(b)

$$\begin{cases} \dot{x}_1 = -4x_1 + x_2 \\ \dot{x}_2 = -3x_2 + x_3 \\ \dot{x}_3 = -2x_3 + 24r \\ y = x_1 = c(t) \end{cases}$$

\Rightarrow
state-space
representation

$$\dot{\mathbf{x}} = \begin{pmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ 24 \end{pmatrix} r$$

$$y = (1 \ 0 \ 0) \mathbf{x}$$

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24} = \frac{24}{(s+2)(s+3)(s+4)}$$

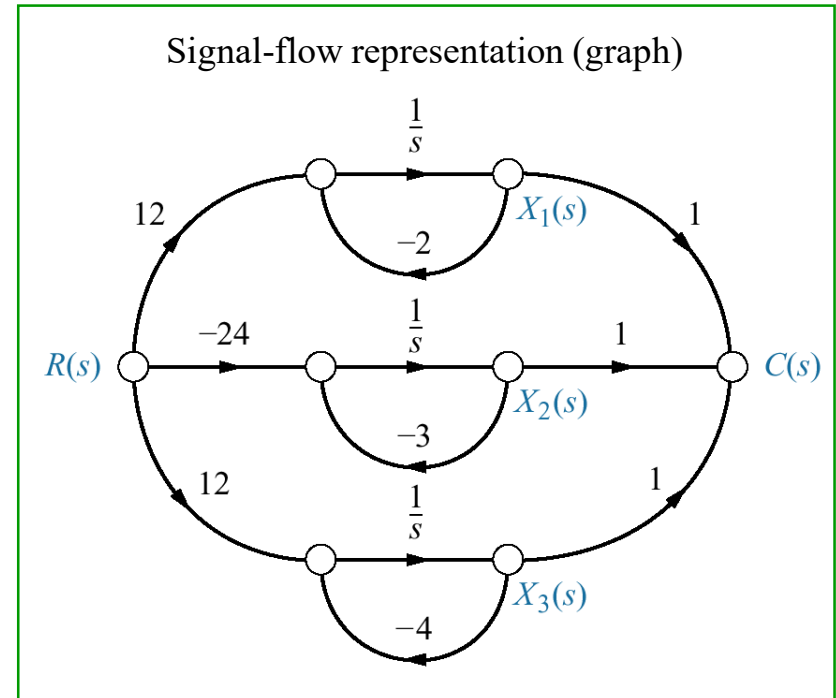
⇓

(2) Parallel form

Partial-fraction expansion of the transfer function. Sum of the first-order subsystems.

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{s+2} - \frac{24}{s+3} + \frac{12}{s+4}$$

$$C(s) = R(s) \frac{12}{(s+2)} - R(s) \frac{24}{(s+3)} + R(s) \frac{12}{(s+4)}$$



$$\begin{cases} \dot{x}_1 = -2x_1 & +12r \\ \dot{x}_2 = & -3x_2 & -24r \\ \dot{x}_3 = & & -4x_3 + 12r \\ y = c(t) = x_1 + x_2 + x_3 \end{cases} \Rightarrow \begin{matrix} \text{state-space} \\ \text{representation} \end{matrix}$$

$$\dot{\mathbf{x}} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 12 \\ -24 \\ 12 \end{pmatrix} r$$

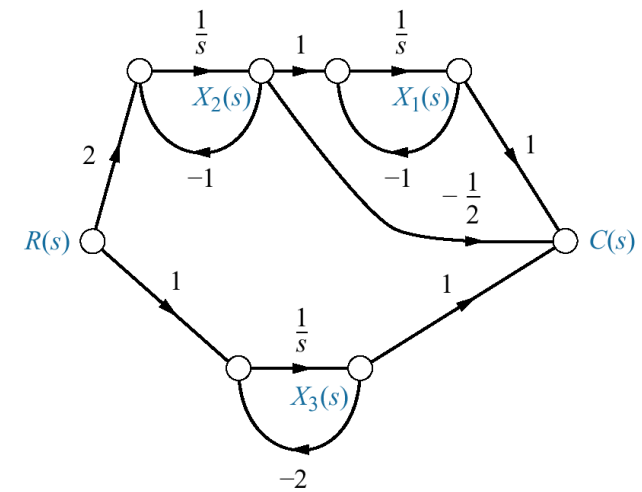
$$y = (1 \quad 1 \quad 1) \mathbf{x}$$

- Matrix A is a diagonal matrix \Rightarrow The equations are said to be *decoupled*.

- Jordan canonical form

$$\frac{C(s)}{R(s)} = \frac{s+3}{(s+1)^2(s+2)} = \frac{2}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+2}$$

- Repeated real roots
- Parallel form
- The system matrix, although not diagonal, has the *system poles along the diagonal*.
- Matrix A is called the *Jordan canonical form*.



$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_2 + 2r \\ \dot{x}_3 = -2x_3 + r \\ y = c(t) = x_1 - 0.5x_2 + x_3 \end{cases} \Rightarrow \begin{matrix} \text{state-space} \\ \text{representation} \end{matrix}$$

systems poles

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} r$$

$$y = (1 \quad -0.5 \quad 1) \mathbf{x}$$

Jordan canonical form (Non-repeated roots)

- In this form of realizing a TF the poles of the transfer function from a string along the main diagonal of the matrix A.
- In Jordan canonical form state space model will be like:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_0 \end{bmatrix} u$$

LTI system:

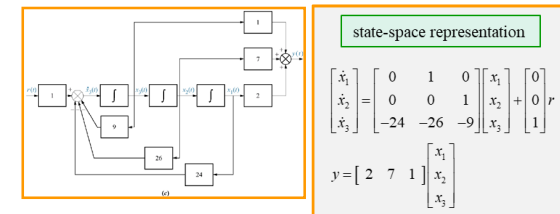
$$(1.1) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \end{cases} \quad \boxed{\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}} \quad (1.1), (1.2)$$

Definition 4.2. The linear system (1.2) is said to be (*completely*) controllable $\leftarrow (A, B)$ on $[t_0, t_f]$ if for all $x_0 \in \mathbb{R}^n$, there exists $u: \mathbb{T} \rightarrow \mathbb{R}^m$ such that the solution x of the state equation of (1.2) with $x(t_0) = x_0$ satisfies $x(t_f) = 0$.

Definition 5.1. The linear system (1.2) is said to be (*completely*) reachable on $[t_0, t_f]$ if for all $x_f \in \mathbb{R}^n$, there exists $u: \mathbb{T} \rightarrow \mathbb{R}^m$ such that the solution x of the state equation of (1.2) with $x(t_0) = 0$ satisfies $x(t_f) = x_f$.

Definition 6.2. The linear system (1.1) is said to be (*completely*) observable $\leftarrow (A, C)$ on $[t_0, t_f]$ if for all $u: \mathbb{T} \rightarrow \mathbb{R}^m$ and all $y: \mathbb{T} \rightarrow \mathbb{R}^r$, the linear system (1.1) has at most one solution x on $[t_0, t_f]$.

(3) Controller canonical form (page 260)

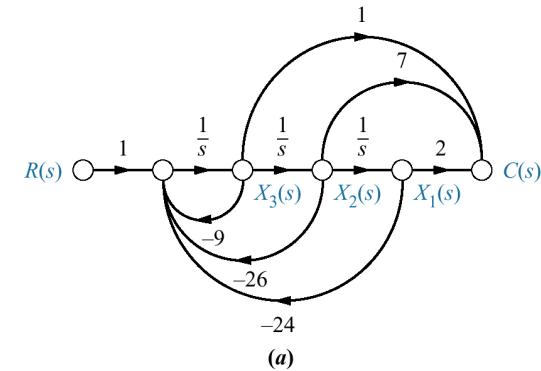


- Reordering the phase variables in the *reverse order* from the phase-variable form

Example: from example 3.5 in page 137. $\Rightarrow \frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$

- Phase-variable form:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r, \quad y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

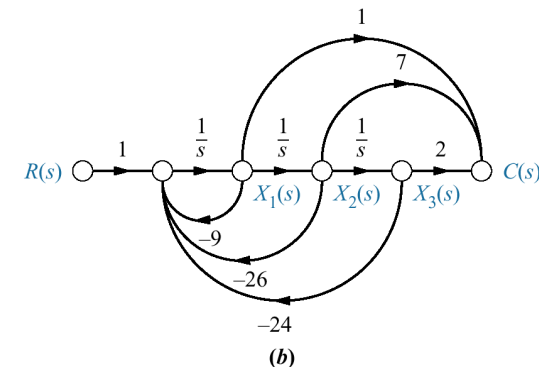


- The reverse order: $x_1 \rightarrow x_3, x_2 \rightarrow x_2, x_3 \rightarrow x_1$

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r, \quad y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$$

- Ascending numerical order \Rightarrow *controller canonical form*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -9 & -26 & -24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r, \quad y = \begin{bmatrix} 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



(4) Observer canonical form (page 262)

$$\left[\frac{1}{s} + \frac{7}{s^2} + \frac{2}{s^3} \right] R(s) - \left[\frac{9}{s} + \frac{26}{s^2} + \frac{24}{s^3} \right] C(s) = C(s)$$

$$\left(\frac{1}{s} R(s) - \frac{9}{s} C(s) \right) + \left(\frac{7}{s^2} R(s) - \frac{26}{s^2} C(s) \right) + \left(\frac{2}{s^3} R(s) - \frac{24}{s^3} C(s) \right) = C(s)$$

$$\frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24} = \frac{\frac{1}{s} + \frac{7}{s^2} + \frac{2}{s^3}}{1 + \frac{9}{s} + \frac{26}{s^2} + \frac{24}{s^3}} \Rightarrow \left[\frac{1}{s} + \frac{7}{s^2} + \frac{2}{s^3} \right] R(s) = \left[1 + \frac{9}{s} + \frac{26}{s^2} + \frac{24}{s^3} \right] C(s)$$

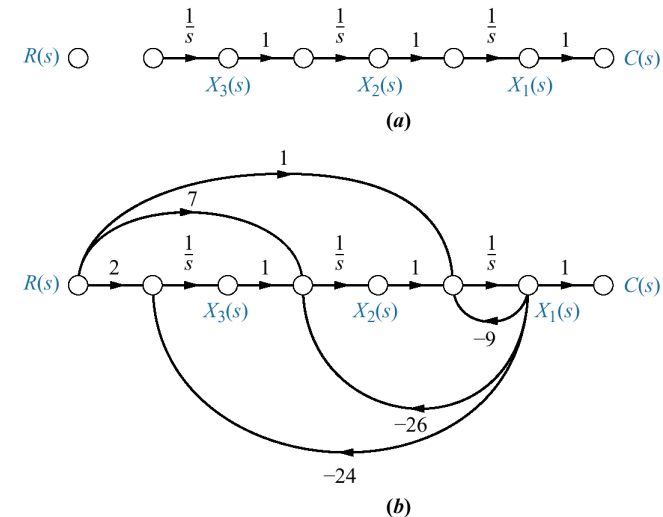
$$C(s) = \frac{1}{s} [R(s) - 9C(s)] + \frac{1}{s^2} [7R(s) - 26C(s)] + \frac{1}{s^3} [2R(s) - 24C(s)]$$

$$= \frac{1}{s} \left[[R(s) - 9C(s)] + \frac{1}{s} \left([7R(s) - 26C(s)] + \frac{1}{s} [2R(s) - 24C(s)] \right) \right]$$

$\underbrace{\hspace{10em}}_{x_3}$
 $\underbrace{\hspace{10em}}_{x_2}$
 $\underbrace{\hspace{10em}}_{x_1}$

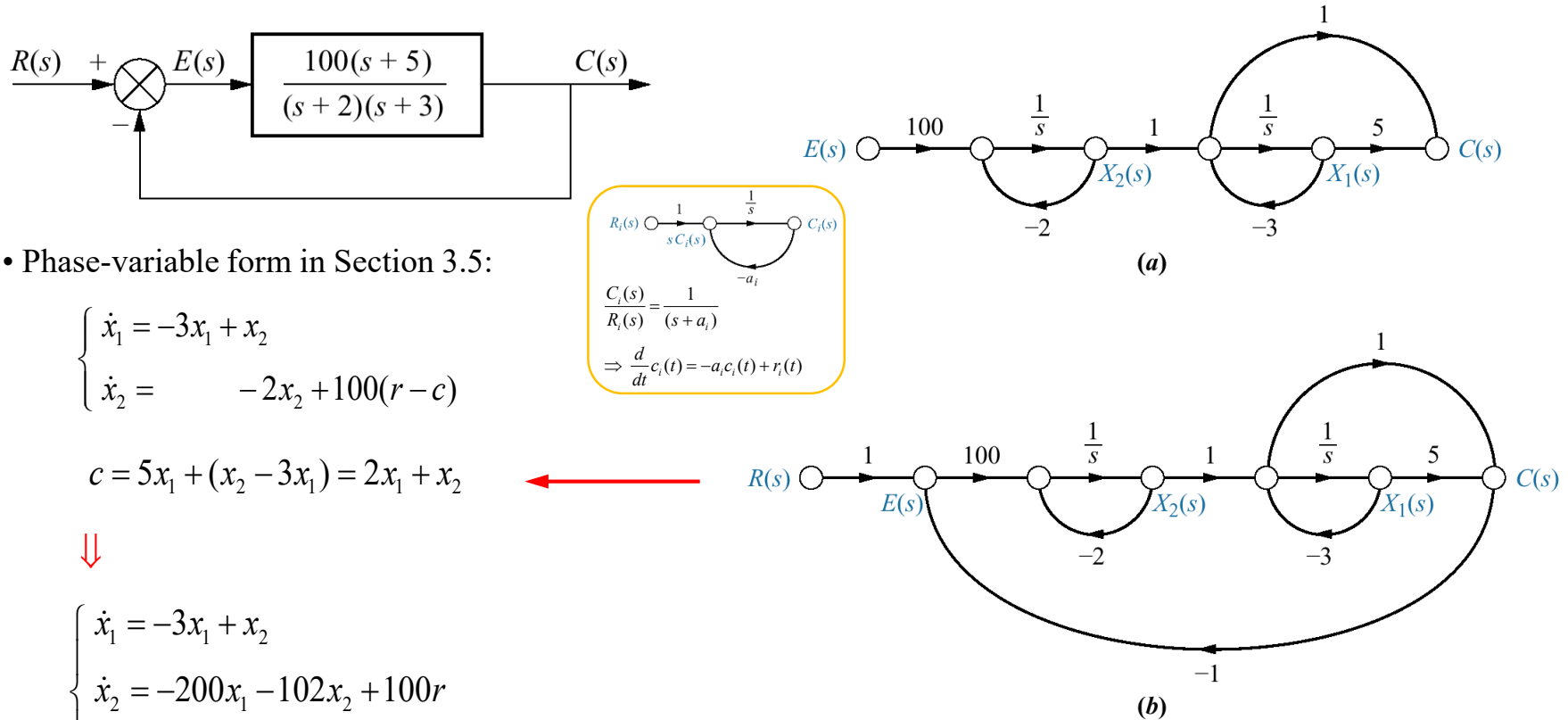
$$\begin{cases} \dot{x}_1 = -9x_1 + x_2 & +r \\ \dot{x}_2 = -26x_1 & +x_3 + 7r \\ \dot{x}_3 = -24x_1 & +2r \\ y = c(t) = x_1 \end{cases} \Rightarrow \dot{\mathbf{x}} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r$$

$$y = (1 \quad 0 \quad 0) \mathbf{x}$$



Example 5.8: State-space representation of feedback systems (page 263)

Represent the following feedback control system in state space.



- Phase-variable form in Section 3.5:

$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 \\ \dot{x}_2 = -2x_2 + 100(r - c) \end{cases}$$

$$c = 5x_1 + (x_2 - 3x_1) = 2x_1 + x_2$$

⇓

$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 \\ \dot{x}_2 = -200x_1 - 102x_2 + 100r \\ y = c(t) = 2x_1 + x_2 \end{cases}$$

⇓

$$\dot{\mathbf{x}} = \begin{pmatrix} -3 & 1 \\ -200 & -102 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 100 \end{pmatrix} r$$

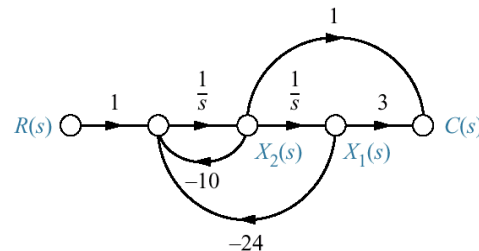
$$y = \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x}$$

(a) forward transfer function;
(b) complete system

Form	Transfer Function	Signal-Flow Diagram	State Equations
------	-------------------	---------------------	-----------------

Phase variable

$$\frac{1}{(s^2 + 10s + 24)} * (s + 3)$$



$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -24 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

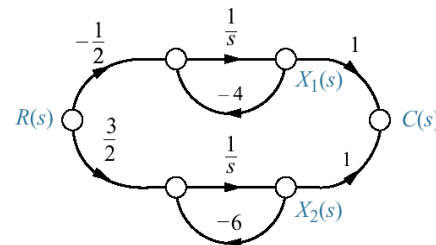
$$y = [3 \quad 1] \mathbf{x}$$

$$\frac{C(s)}{R(s)} = \frac{(s + 3)}{(s + 4)(s + 6)}$$

$$y = c(t)$$

Parallel

$$\frac{-1/2}{(s + 4)} + \frac{3/2}{s + 6}$$

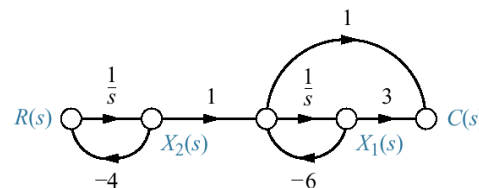


$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} r$$

$$y = [1 \quad 1] \mathbf{x}$$

Cascade

$$\frac{1}{(s + 4)} * \frac{(s + 3)}{(s + 6)}$$

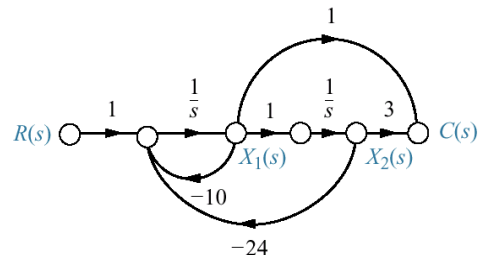


$$\dot{\mathbf{x}} = \begin{bmatrix} -6 & 1 \\ 0 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = [-3 \quad 1] \mathbf{x}$$

Controller canonical

$$\frac{1}{(s^2 + 10s + 24)} * (s + 3)$$

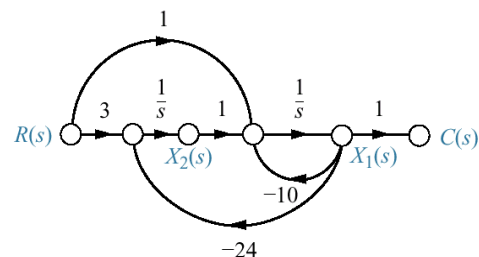


$$\dot{\mathbf{x}} = \begin{bmatrix} -10 & -24 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

$$y = [1 \quad 3] \mathbf{x}$$

Observer canonical

$$\frac{\frac{1}{s} + \frac{3}{s^2}}{1 + \frac{10}{s} + \frac{24}{s^2}}$$



$$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ -24 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} r$$

$$y = [1 \quad 0] \mathbf{x}$$

5.8 Similarity Transformations

Diagonalizing a system matrix
(Finding eigenvectors, page 269)