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Chapter 8. Root Locus Techniques

Objectives

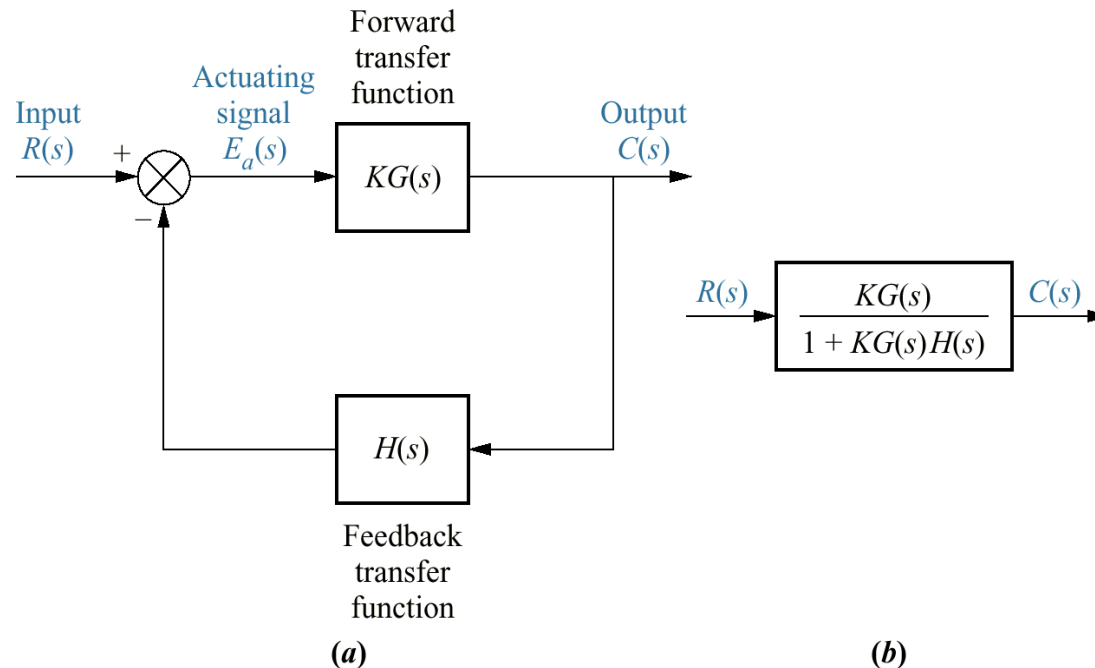
- The *definition* of a root locus
- How to *sketch* a root locus
- How to *refine* your sketch of a root locus
- How to use the root locus to *find the poles* of a closed-loop system
- How to use the root locus to *design a parameter value* to meet a transient response specification

```
s=tf('s');  
sys=1/(s+1)  
controlSystemDesigner(sys)
```

```
s=tf('s');  
sys=1/(s+1)^3  
pidtool(sys)
```

8.1 Introduction

- Root locus (Evans, 1948:1950): a powerful method of analysis and design for stability and transient response



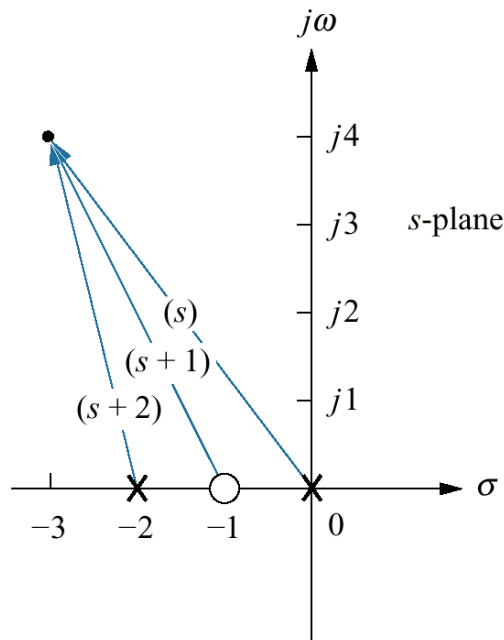
$$G(s) = \frac{(s+1)}{s(s+2)}, \quad H(s) = \frac{(s+3)}{(s+4)} \Rightarrow T(s) = \frac{K(s+1)(s+4)}{s^3 + (6+K)s^2 + (8+4K)s + 3K}$$

- The root locus will be used to trace the poles of $T(s)$ as K varies.

Example 8.1: Evaluation of a complex function via vectors

Find $F(s)$ at the point $s = -3 + j4$.

$$F(s) = \frac{(s+1)}{s(s+2)}$$

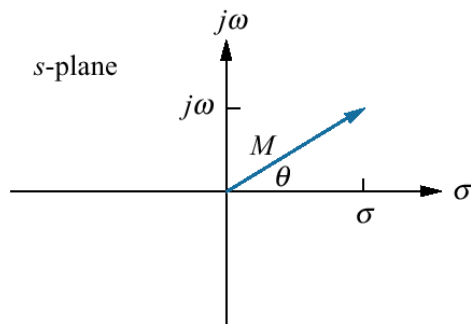


```
% ch8_1.m
s = -3+4j;
G = (s+1) / (s^2+2*s);
Theta = (180/pi) * angle(G);
M = abs(G)
```

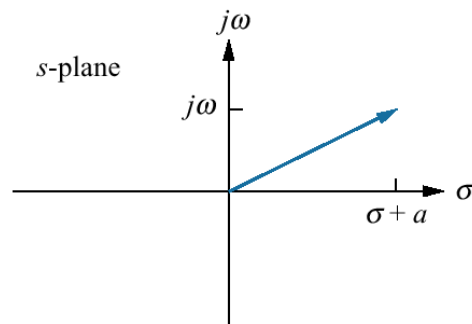
$$\begin{aligned}
 F(s) &= \frac{[s - (-1)]}{[s - 0][s - (-2)]} \\
 &= \frac{\sqrt{20} \angle 116.6^\circ}{(5 \angle 126.9^\circ)(\sqrt{17} \angle 104.0^\circ)} \\
 &= \frac{\sqrt{20}}{5\sqrt{17}} \angle 116.6^\circ - 126.9^\circ - 104.0^\circ \\
 &= 0.217 \angle -114.3^\circ = M \angle \theta
 \end{aligned}$$

Use magnitude and phase angle

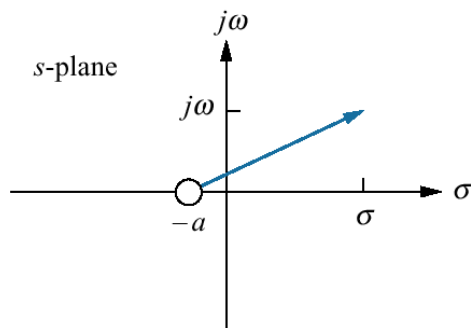
• Vector Representation of Complex Numbers



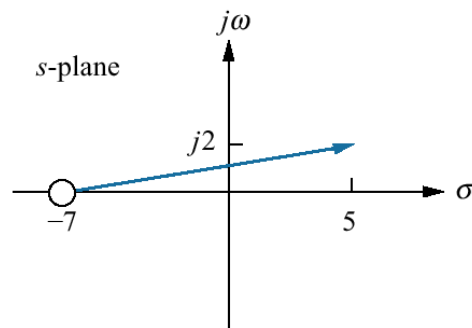
(a)



(b)



(c)



(d)

$$F(s) = \frac{\prod_{i=1}^m (s + z_i)}{\prod_{j=1}^n (s + p_j)}$$

$$M = \frac{\prod_{i=1}^m |s - (-z_i)|}{\prod_{j=1}^n |s - (-p_i)|} = \frac{\Pi(\text{zero lengths})}{\Pi(\text{pole lengths})}$$

$$\theta = \boxed{\sum \text{Zero angles} - \sum \text{Pole angles}}$$

$$= \sum_{i=1}^m \angle(s - (-Z_i)) - \sum_{j=1}^n \angle(s - (-P_j))$$

(a) $s = \sigma + j\omega$,

(b) $(s + a)$;

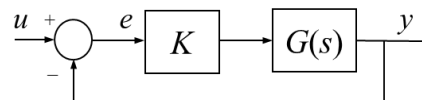
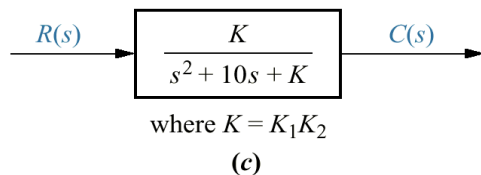
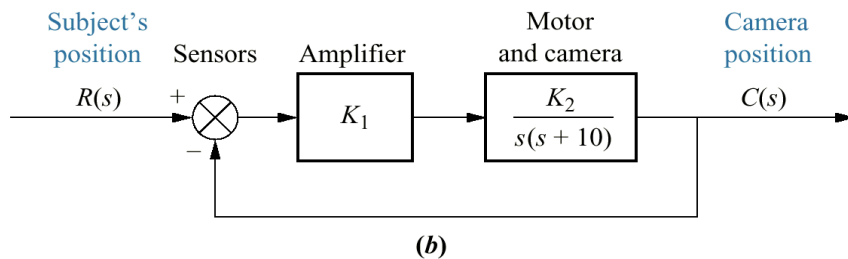
(c) Alternate representation of $(s + a)$;

(d) $(s + 7)|_{s \rightarrow 5 + j2}$

8.2 Defining the Root Locus



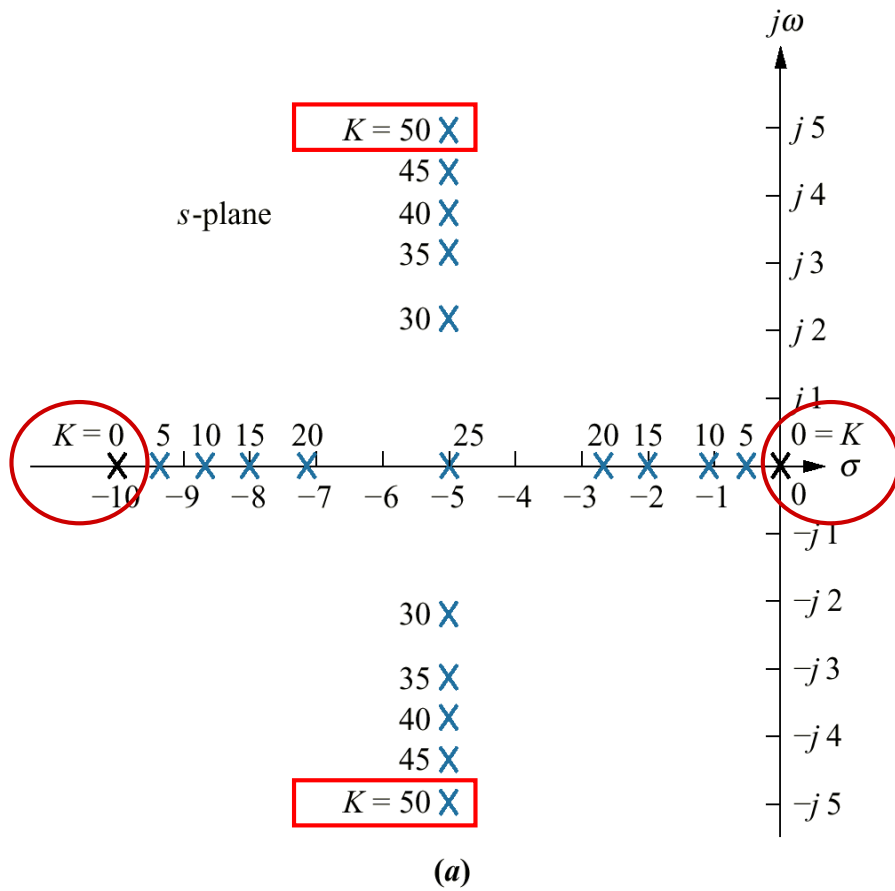
CameraMan: Courtesy of ParkerVision



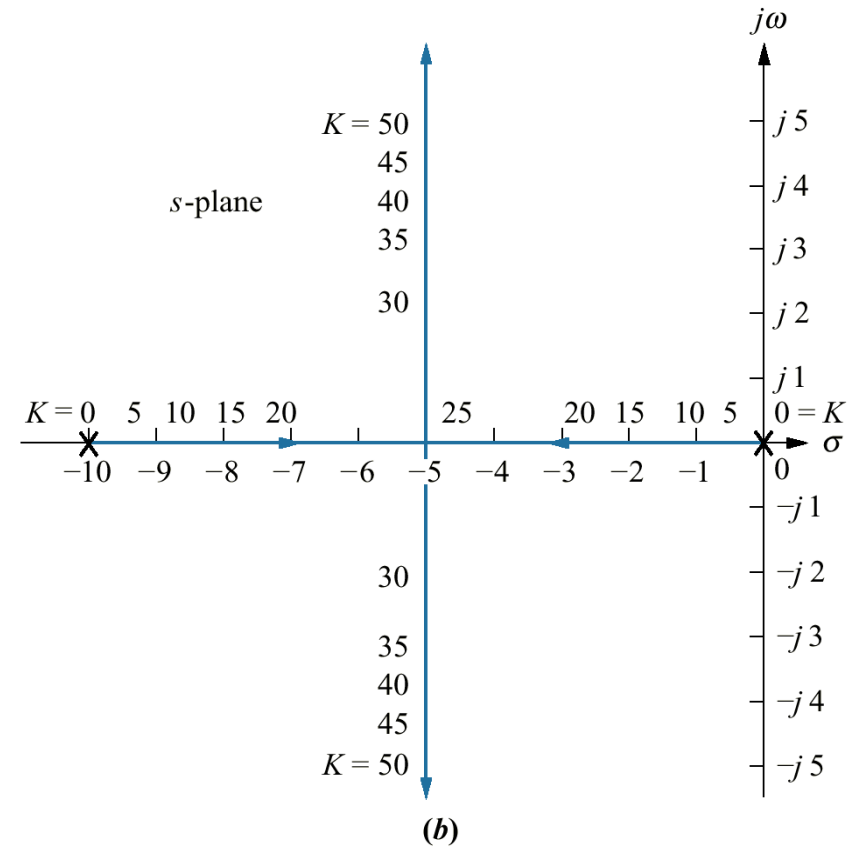
- Pole location as a function of gain for the system

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

```
% ch8_2.m
clc, clear all
numg=[1];
deng=[1 10 0];
% deng=poly([0 -10]);
G=tf(numg, deng);
rlocus(G)
```



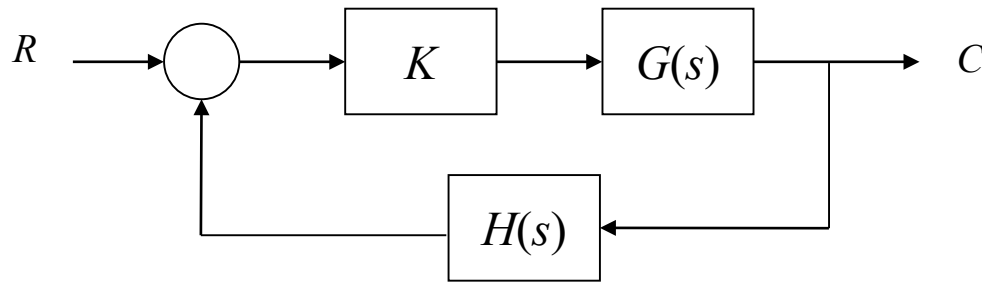
Pole plot



Root locus

overdamped ($0 \leq K < 25$), *critically damped* ($K = 25$), and *underdamped* ($K > 25$)

8.3 Properties of the Root Locus



$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

Closed-loop pole: $1 + KG(s)H(s) = 0$

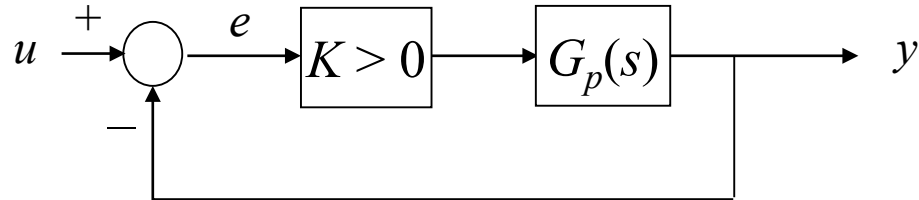
$$KG(s)H(s) = -1$$

$$= 1 \angle (2k+1)180^\circ, \quad k = 0, \pm 1, \pm 2$$

$$|KG(s)H(s)| = 1 \quad \text{and}$$

$$\angle G(s)H(s) = (2k+1)180^\circ, \quad K > 0$$

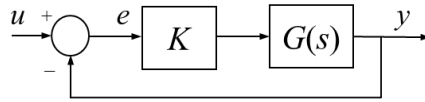
Root Locus



- Plot the poles of the closed-loop system $T(s)$, as a function of K .

$$T(s) = \frac{KG_P(s)}{1 + KG_P(s)}$$

Poles: $1 + KG_P(s) = 0$



Example:

$$G_P(s) = \frac{1}{(s+2)^2}$$

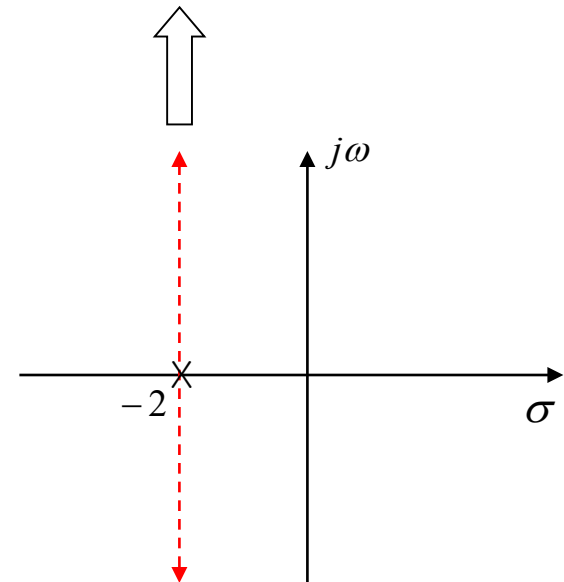
$$1 + KG_P(s) = 0, \quad 1 + \frac{K}{(s+2)^2} = 0$$

$$s^2 + 4s + (K+4) = 0$$

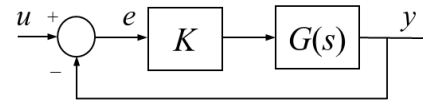
$$s = -2 \pm \sqrt{4 - (K+4)} = -2 \pm \sqrt{-K}$$

$$s = -2 \pm j\sqrt{K}$$

System is stable for all K



```
clc, clear all
numg = [1];
deng = poly([-2 -2]);
G=tf(numg, deng)
rlocus(G)
```



Example: $G_P(s) = \frac{s-1}{(s+1)^2}$

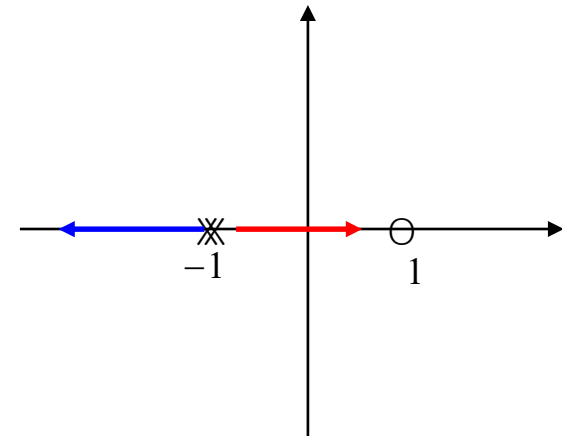
$$1 + KG_P(s) = 0, \quad (s+1)^2 + K(s-1) = 0$$

$$s^2 + (2+K)s + (1-K) = 0$$

$$s = \frac{-(2+K) \pm \sqrt{(2+K)^2 - 4(1-K)}}{2}$$

$$= \frac{-(2+K) \pm \sqrt{K^2 + 4K + 4 - 4 + 4K}}{2},$$

$$s = \frac{-2-K \pm \sqrt{K^2 + 8K}}{2}$$



```
clc, clear all
numg = poly([1]);
deng = poly([-1 -1]);
G=tf(numg, deng)
rlocus(G)
```

Example (continued):

$$s = \frac{-2 - K \pm \sqrt{K^2 + 8K}}{2} = \frac{-2 + (-K \pm \sqrt{K^2 + 8K})}{2}$$

$$+ : -K + \sqrt{K^2 + 8K} = \frac{(-K + \sqrt{})(K + \sqrt{})}{K + \sqrt{}} = \frac{-K^2 + K^2 + 8K}{K + \sqrt{K^2 + 8K}} = \frac{8K}{K \left(1 + \sqrt{1 + \frac{8}{K}} \right)}$$

$$\rightarrow \lim_{K \rightarrow \infty} \frac{8K}{K \left(1 + \sqrt{1 + \frac{8}{K}} \right)} = \lim_{K \rightarrow \infty} \frac{8K}{K (1 + \sqrt{1 + 0})} = \frac{8}{2} \rightarrow 4 \Rightarrow K \rightarrow \infty, s \rightarrow \frac{-2 + 4}{2} = 1$$

$$- : -K - \sqrt{K^2 + 8K} = \frac{(-K - \sqrt{})(-K + \sqrt{})}{-K + \sqrt{}} = \frac{(K^2 - K^2 - 8K)}{-K + \sqrt{K^2 + 8K}} = \frac{-8K}{K \left(-1 + \sqrt{1 + \frac{8}{K}} \right)}$$

$$\rightarrow \lim_{K \rightarrow \infty} \frac{-8K}{K \left(-1 + \sqrt{1 + \frac{8}{K}} \right)} = \lim_{K \rightarrow \infty} \frac{-8K}{K (-1 + \sqrt{1 + 0})} = \frac{-8}{0} \rightarrow -\infty$$

$$\Rightarrow K \rightarrow \infty, s \rightarrow \frac{-2 - \infty}{2} = -\infty$$

$$s^2 + (2 + \bar{K})s + (1 - \bar{K}) \Big|_{s=0} = 0 \Rightarrow \boxed{\bar{K} = 1}$$

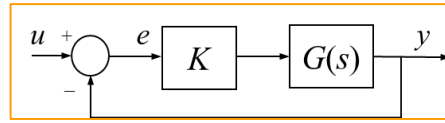
Stability: $K < 1$

If you have higher order polynomials,
it can be impossible to plot the roots in this way.

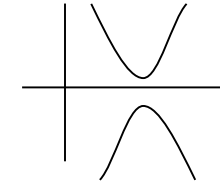
⇒ You need systematic plotting methods.

$$\text{Let } G_p(s) = A \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}, \quad m \leq n$$

Rules for Root Locus



Rule 1. Symmetry about the real axis



Rule 2. Each branch starts at a pole of $G_p(s)$ and terminates at a zero of $G_p(s)$, including zeros at ∞ .

Explanation \rightarrow Let $G_p(s) = \frac{s-1}{s^3+4s+2}$

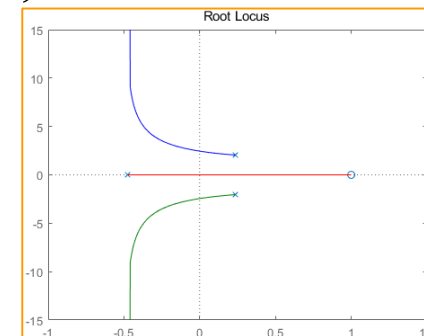
```
numg = poly([1]);
deng = [1 0 4 2];
G=tf(numg, deng)
rlocus(G)
```

$G_p(s)$ has one zero and three poles.

At ∞ , namely as $s \rightarrow \infty$, $G_p(s)$ behaves like $\frac{1}{s^2}$,
i.e. it has two zeros at ∞ .

\Rightarrow How do the *branches* approach to ∞ ?

$$G = \frac{s - 1}{s^3 + 4s + 2}$$



Rule 3. $n-m$ *branches* approach ∞ like asymptotes

with angles $\psi_K = \frac{180 + 360K}{n-m}$, $K = 0, 1, \dots, n-m-1$.

The asymptotes intersect at the real axis at

$$\sigma = \frac{\sum \text{Poles} - \sum \text{Zeros}}{n-m}$$

```
numg = [1];
deng = poly([-2 -2]);
G=tf(numg, deng)
rlocus(G)
```

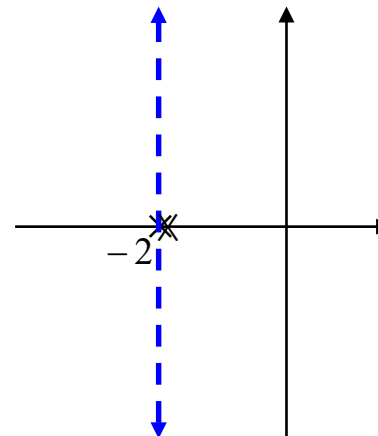
Example:

$$G_P(s) = \frac{1}{(s+2)^2}, \quad n=2, \quad m=0$$

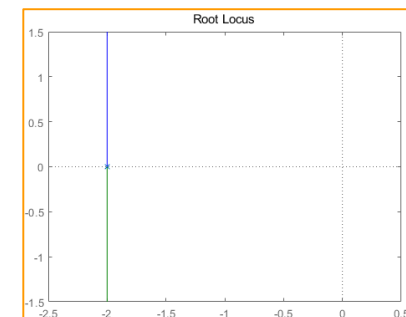
$$\sigma = (-4)/2 = -2,$$

$$\psi_0 = 180/2 = 90,$$

$$\psi_1 = (180 + 360)/2 = -90$$

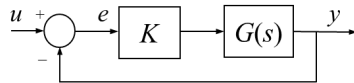


$$G = \frac{1}{s^2 + 4s + 4}$$



Rule 4. If a branch is on the real axis, then the number of real poles + zeros of $G_P(s)$ to the right of it must be odd.

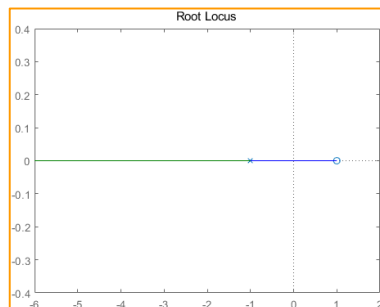
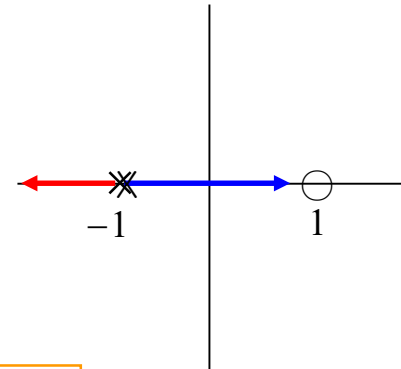
Example:



$$G_P(s) = \frac{s-1}{(s+1)(s+1)}$$

$$n - m = 1$$

$\Psi_0 = 180 \rightarrow$ you have a branch at $-\infty$



$$G = \frac{s-1}{s^2 + 2s + 1}$$

```
numg = poly([1]);
deng = poly([-1 -1]);
G=tf(numg, deng)
rlocus(G)
```



```
numg=[1 -2];
deng = [1 6 5];
G=tf(numg, deng)
rlocus(G)
```

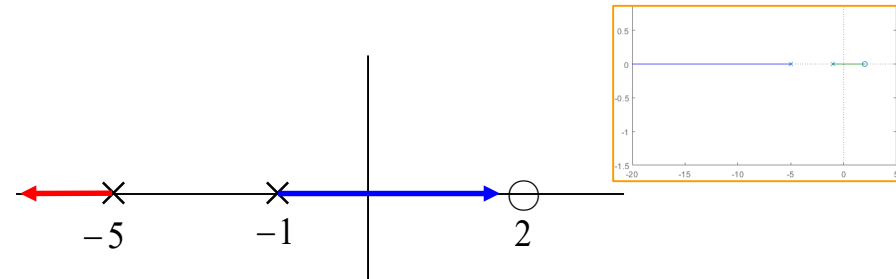
```
numg = poly([2]);
deng = poly([-1 -5]);
G=tf(numg, deng)
rlocus(G)
```

Example: $G_p(s) = \frac{s-2}{(s+1)(s+5)}$

$$n - m = 1$$

$$\Psi_0 = 180 \quad \text{Stability : } K < \bar{K}$$

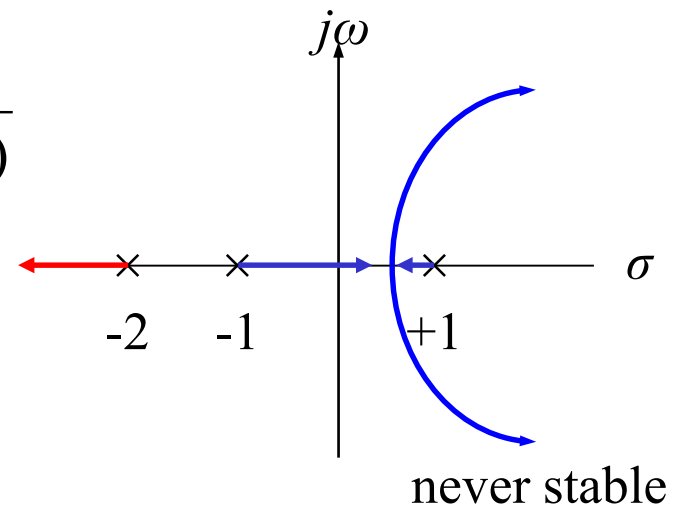
$$1 + \bar{K} \frac{s-2}{(s+1)(s+5)} \Big|_{s=0} = 0, \quad \bar{K} = 2.5, \quad \text{Stability: } K < 2.5$$



Example: $G_p(s) = \frac{1}{(s+2)(s-1)(s+1)}$

$$1 + KG_p(s) = 0$$

$$s^3 + 2s^2 - s - 2 + K = 0$$



Cross $j\omega$ axis: $\Rightarrow 1 + KG_p(s) \Big|_{s=j\omega} = 0$

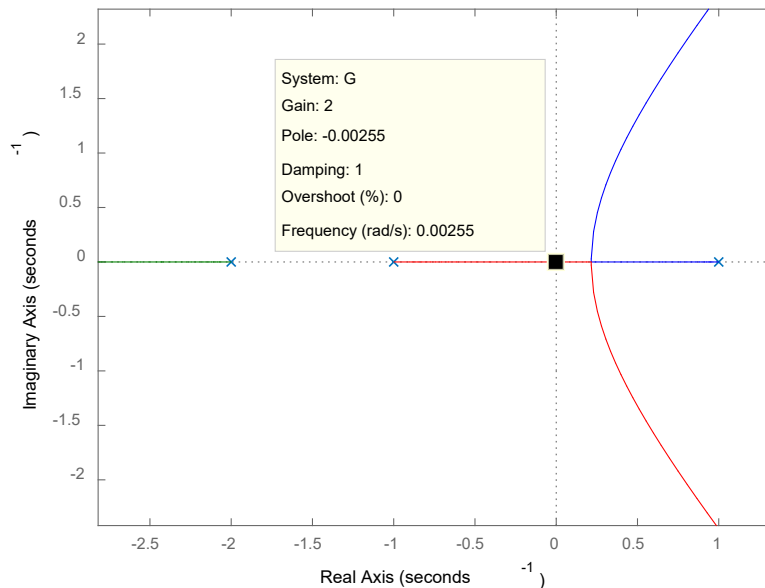
$$-j\omega^3 - 2\omega^2 - j\omega - 2 + K = 0$$

$$(K - 2\omega^2 - 2) - j\omega(1 + \omega^2) = 0$$

$$\neq 0$$

$$K=2$$

$\omega^2 + 1 = 0$ or $\omega = 0$ is the only solution



```
numg = poly([]);  
deng = poly([-2 1 -1]);  
G=tf(numg, deng)  
rlocus(G)
```

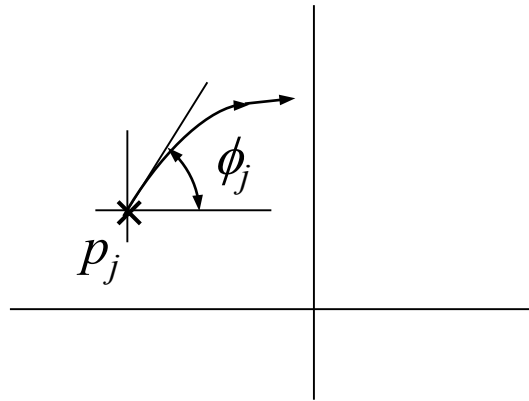
Rule 5. Angle rule

Let p_1, \dots, p_n be the poles of G_p

Let z_1, \dots, z_m be the zeros of G_p

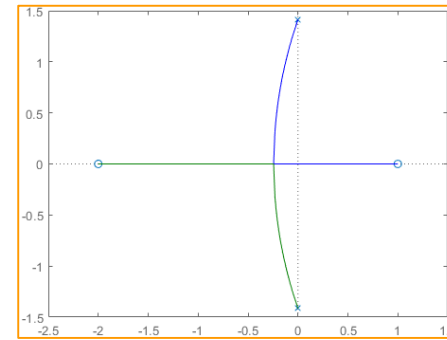
Simple pole: Let p_i be a simple pole.

Let ϕ_j = angle at which the branch starts from p_j

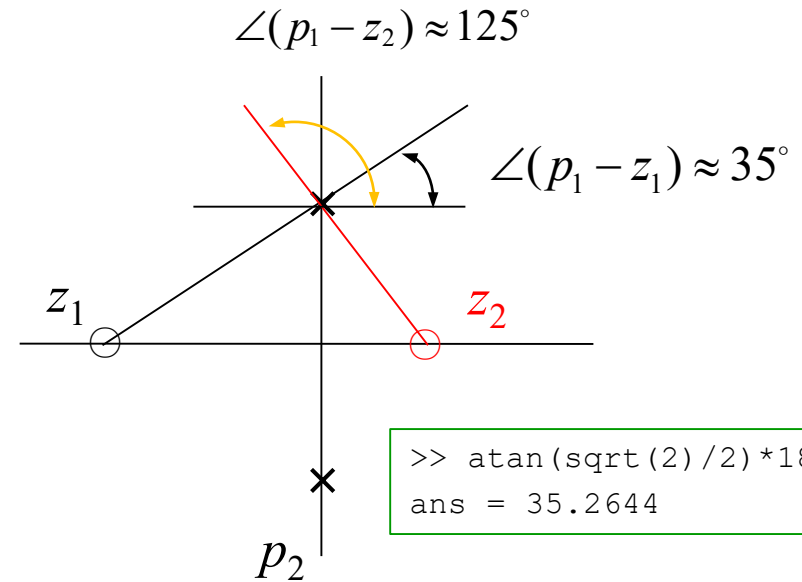
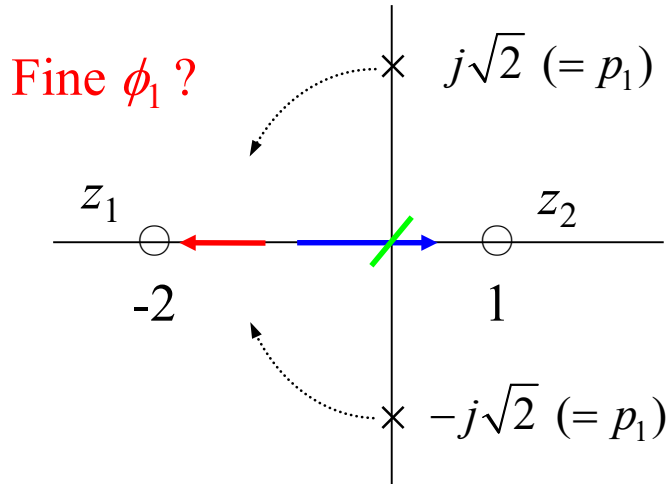


$$\phi_j = \sum_{k=1}^m \angle(p_j - z_k) - \sum_{\substack{i=1 \\ i \neq j}}^n \angle(p_j - p_i) + r \cdot 180 \quad (r : \text{odd})$$

Example: $G_p(s) = \frac{(s-1)(s+2)}{s^2 + 2}$



```
numg = poly([1 -2]);
deng = [1 0 2];
G=tf(numg, deng)
rlocus(G)
```

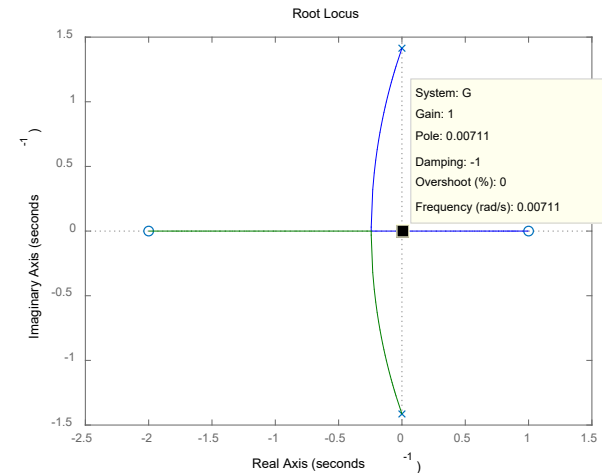


```
>> atan(sqrt(2)/2)*180/pi
ans = 35.2644
```

$$\phi_1 = \angle(p_1 - z_1) + \angle(p_1 - z_2) - \angle(p_1 - p_2) + (1) \cdot 180$$

$$\phi_1 \approx 35^\circ + 125^\circ - 90^\circ + 180^\circ = 250^\circ = -110^\circ$$

- Crossing of '0': $1 + K \frac{(s-1)(s+2)}{s^2 + 2} = 0$
 $s = 0, 1 + K \frac{-2}{2} = 0, K = 1$
 stability: $K < 1$



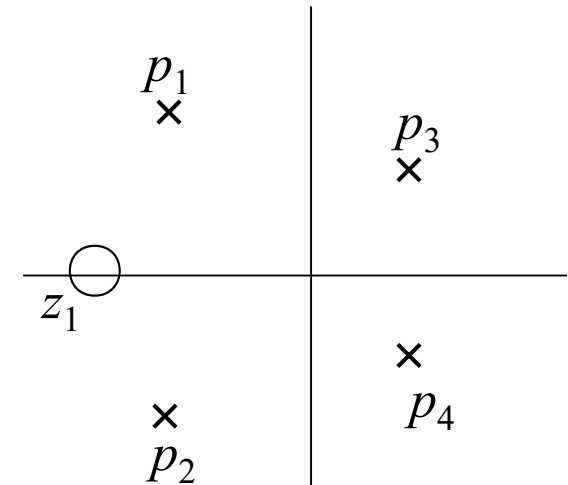
- Explanation of rule 5: For every s on a branch,

$$T(s) = \frac{KG_P(s)}{1 + KG_P(s)}$$

$$1 + K \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)} = 0$$

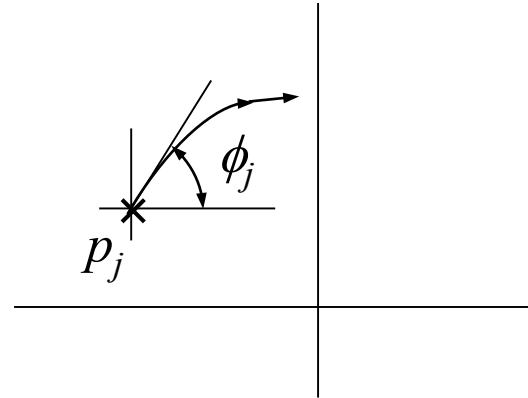
$$K \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)} = -1$$

$$\sum_{k=1}^m \angle(s - z_k) - \sum_{i=1}^n \angle(s - p_i) = r \cdot 180 \quad (r : \text{odd})$$



- Explanation of rule 5(continue):

now: $\theta_j = \lim_{s \rightarrow p_j} \angle(s - p_j)$



Then, $\theta_j = \lim_{s \rightarrow p_j} \angle(s - p_j)$

$$= \lim_{s \rightarrow p_j} \sum_{k=1}^m \angle(s - z_k) - \lim_{s \rightarrow p_j} \sum_{\substack{i=1 \\ i \neq j}}^n \angle(s - p_i) + r \cdot 180 \quad (r : \text{odd})$$

$$= \sum_{k=1}^m \angle(p_j - z_k) - \sum_{\substack{i=1 \\ i \neq j}}^n \angle(p_j - p_i) + r \cdot 180$$

$$\sum_{k=1}^m \angle(s - z_k) - \sum_{i=1}^n \angle(s - p_i) = r \cdot 180 \quad (r : \text{odd})$$

$$\sum_{k=1}^m \angle(s - z_k) - \sum_{i=1}^n \angle(s - p_i) = r \cdot 180 \quad (r : \text{odd})$$

- Angle rule for double pole

Let $p_1 = p_2$, for all s on a branch

$$\sum_{k=1}^m \angle(s - z_k) - \sum_{i=1}^n \angle(s - p_i) + r \cdot 180$$

$$\angle(s - p_1) + \angle(s - p_2) = \sum_{k=1}^m \angle(s - z_k) - \sum_{i=3}^n \angle(s - p_i) + r \cdot 180$$

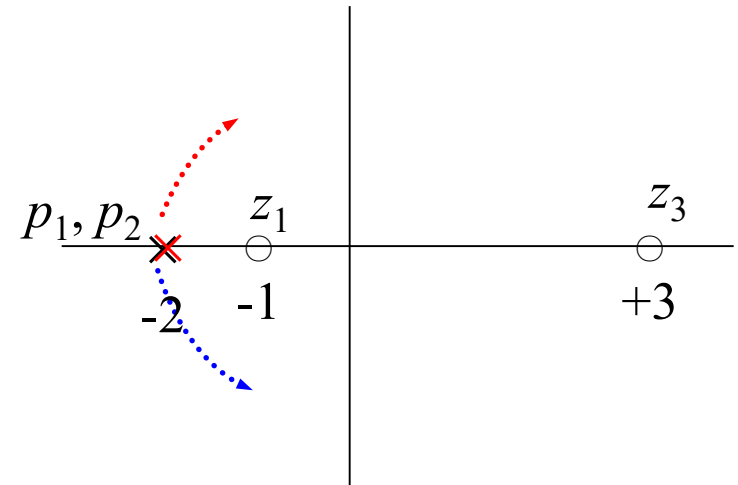
but $p_1 = p_2 \rightarrow \angle(s - p_1) = \angle(s - p_2)$

$$\Rightarrow 2\angle(s - p_1) = \sum_{k=1}^m \angle(s - z_k) - \sum_{i=3}^n \angle(s - p_i) + r \cdot 180$$

$$\Rightarrow \varphi_1, \varphi_2 = \left[\sum_{k=1}^m \underset{\downarrow p_j}{\angle(s - z_k)} - \sum_{i=3}^n \underset{\downarrow p_j}{\angle(s - p_i)} + r \cdot 180 \right] / 2$$

Example: $G_p(s) = \frac{(s+1)(s-3)}{(s+2)^2}$

$$\begin{cases} z_1 = -1, & z_2 = 3 \\ p_1 = p_2 = -2 \end{cases}$$



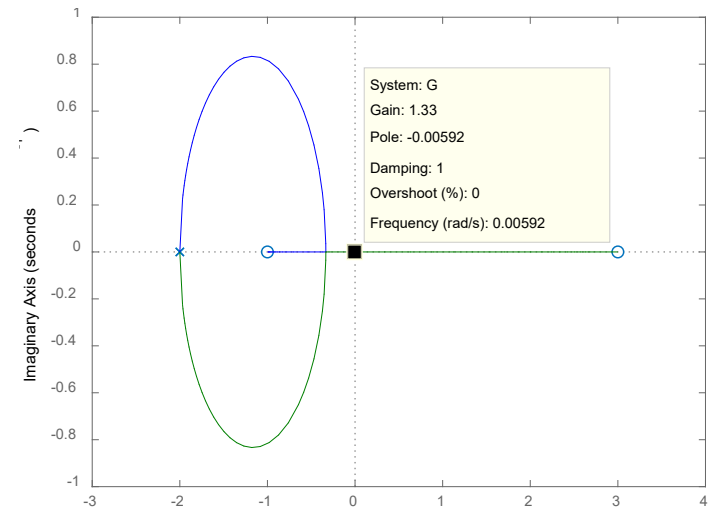
$$\angle(p_1 - z_1) = \angle(p_1 - z_2) = 180$$

$$\rightarrow \varphi_1, \varphi_2 = [+180 + 180 - 0 + 180r] / 2 = 90r$$

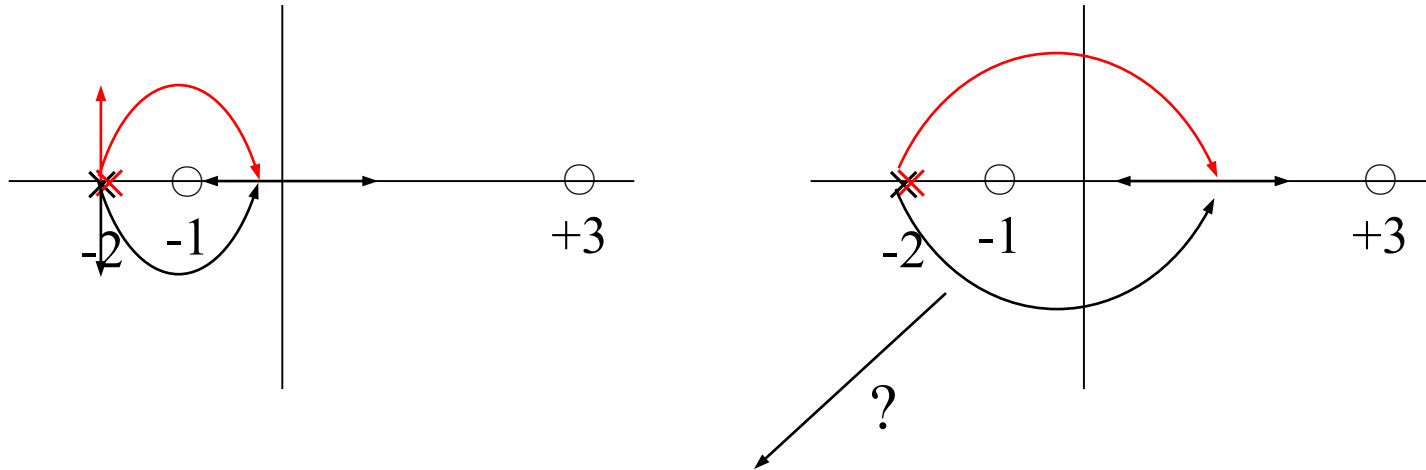
$$r = 1 : \varphi_1 = 90^\circ$$

$$r = 3 : \varphi_2 = 270^\circ = -90^\circ$$

```
numg = poly([-1 3]);
deng = poly([-2 -2]);
G=tf(numg, deng)
rlocus(G)
```



Example (continue):



$$G_p(s) = \frac{s^2 - 2s - 3}{s^2 + 4s + 4}, \quad 1 + KG_p(s) = 0$$

$$s^2 + 4s + 4 + K(s^2 - 2s - 3) = 0$$

Example (continue):

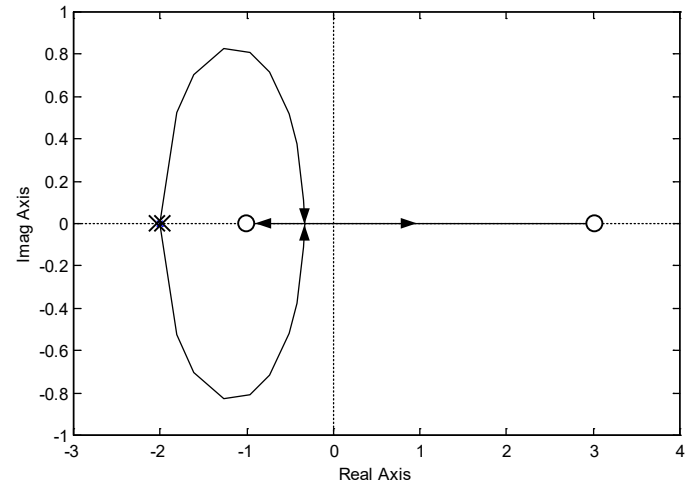
$$(1+K)s^2 + (4-2K)s + (4-3K) = 0 \quad \text{at } j\omega\text{-axis} \rightarrow s = j\omega$$

$$-\omega^2(1+K) + j\omega(4-2K) + (4-3K) = 0$$

$$K = 2,$$

$$-\omega^2(3) + (4-6) = 0$$

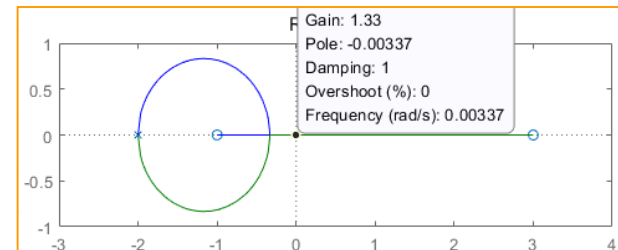
$$3\omega^2 + 2 \neq 0 : \text{impossible} \quad \therefore \Rightarrow$$



$$\text{At } s = 0 \rightarrow 1 + KG_p(s) = 0 \rightarrow 1 + K\left(-\frac{3}{4}\right) = 0 \rightarrow K = \frac{4}{3} = 1.333$$

$$(1+K)s^2 + (4-2K)s + (4-3K) = 0$$

$$\therefore \text{ stability } K < \frac{4}{3}$$



Pole Sensitivity

Case 1: Very small changes in gain

High Sensitivity. \longrightarrow

Very large changes
- in pole location
- in performance

Case 2: Low Sensitivity.

\Rightarrow We prefer systems with *low sensitivity* to changes in gain

• Root sensitivity

Sensitivity of a closed-loop pole:

Current pole location \nearrow $S_{s:K} = \frac{K}{s} \cdot \frac{\delta s}{\delta K}$ \nwarrow Current gain

$$S_{s:K} = \frac{K}{s} \cdot \frac{\Delta s}{\Delta K}$$

\Rightarrow

$$\Delta s = s(S_{s:K}) \frac{\Delta K}{K}$$

Change in the pole location

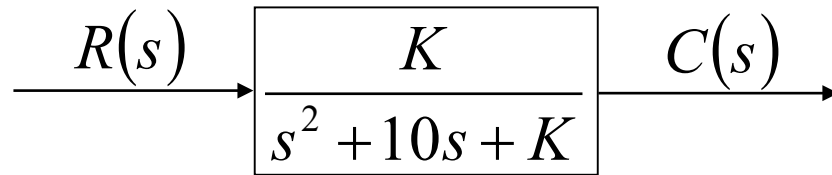
$$S_{F:P} = \frac{P}{F} \cdot \frac{\delta F}{\delta P}$$

Function \nearrow \nwarrow Parameter

(ref: chapter 7.7)

$$S_{F:P} = \lim_{\Delta P \rightarrow 0} \frac{\frac{\Delta F}{F}}{\frac{\Delta P}{P}} = \lim_{\Delta P \rightarrow 0} \frac{P}{F} \frac{\Delta F}{\Delta P}$$

Example:



- ① Find the root sensitivity of the system at $s = -9.47$ and $s = -5 + j5$
- ② Calculate the change in the pole location for a 10% change in K .

\Rightarrow Characteristic equation

$$s^2 + 10s + K = 0$$

Example (continued):

$$\Rightarrow \text{Characteristic equation}$$

$$s^2 + 10s + K = 0$$

$$\frac{\delta s}{\delta K} = 2s \frac{\delta s}{\delta K} + 10 \frac{\delta s}{\delta K} + 1 = 0$$

$$\frac{\delta s}{\delta K} = \frac{-1}{2s + 10}$$

$$S_{s:K} = \frac{K}{s} \cdot \frac{\delta s}{\delta K} = \frac{K}{s} \cdot \frac{-1}{2s + 10} \dots\dots\dots (1)$$

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
\vdots	\vdots	\vdots
50	-5+j5	-5-j5

i) $s = -9.47 \rightarrow K = 5 \rightarrow (1)$

$$S_{s:K} = -0.059$$

$$\Delta s = s(S_{s:K}) \frac{\Delta K}{K} \longrightarrow$$

$$= 0.056$$

The pole will move to the right by 0.056 units for a 10% change in K at $K=5$ or $s = -9.47$

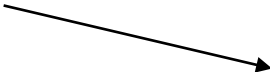
Example (continue):

ii) For $s = -5 + j5 \rightarrow K = 50 \rightarrow (1)$;

$$S_{s:k} = \frac{K}{s} \cdot \frac{-1}{2s+10} = \frac{1}{2} - j\frac{1}{2} = \sqrt{\frac{1}{2}} \angle -45^\circ$$

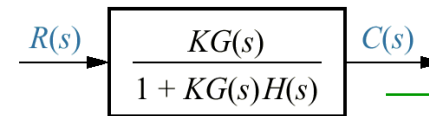
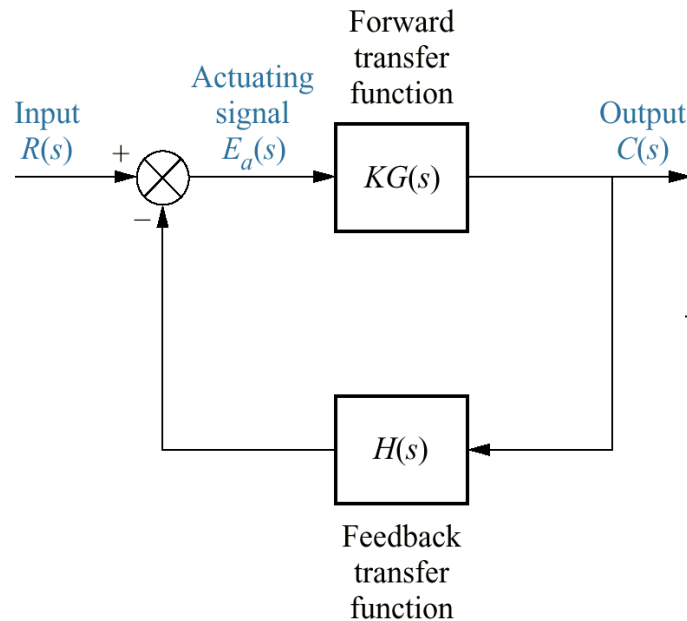
Find the change in the pole location at $\Delta K / K = 10\%$

$$\begin{aligned}\Delta s &= s(S_{s:K}) \frac{\Delta K}{K} \\ &= (-5 + j5) \left(\frac{1}{2} - j\frac{1}{2} \right) (0.1) \\ &= j0.5\end{aligned}$$



The pole will move vertically by 0.5 unit for a 10% change in K

8.3 Properties of the Root Locus



$$1 + KG(s)H(s) = 0$$

$$\sum_{k=1}^m \angle(s - z_k) - \sum_{i=1}^n \angle(s - p_i) = r \cdot 180^\circ \quad (r: \text{odd})$$

$$1 + KG(s)H(s) = 0$$

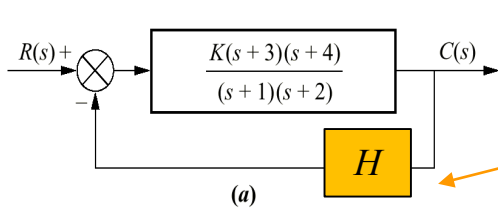
$$KG(s)H(s) = -1 = 1 \angle (2k + 1)180^\circ, \quad k = 0, \pm 1, \pm 2, \dots$$

$$|KG(s)H(s)| = 1 \Rightarrow K = \frac{1}{|G(s)H(s)|}$$

$$\angle G(s)H(s) = (2k + 1)180^\circ \quad (K > 0)$$



A point s on the root locus should be satisfied these conditions.

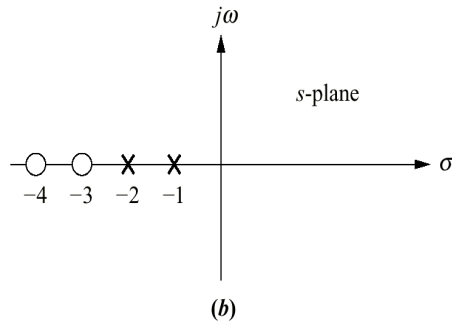
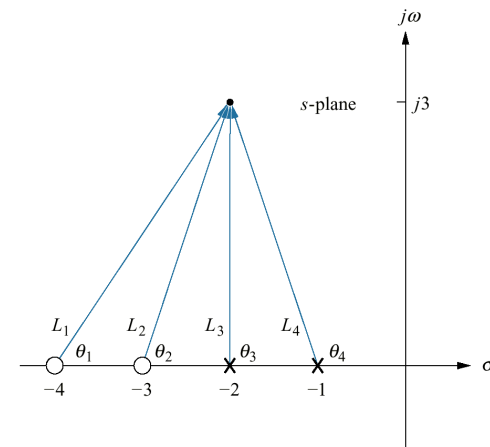


$$KG(s)H(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)}$$

(1) In the case of $s = -2 + j3$

$$\begin{aligned}\angle G(s)H(s) &= \angle G(s) = \theta_1 + \theta_2 - \theta_3 - \theta_4 \\ &= 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ \\ &= -70.55^\circ \neq \pm 180^\circ\end{aligned}$$

Therefore, $s = -2 + j3$ is not a point on the root locus.



(2) In the case of $s = -2 + j\frac{1}{\sqrt{2}}$

$$\begin{aligned}\angle G(s)H(s) &= \theta_1 + \theta_2 - \theta_3 - \theta_4 \\ &= 19.47^\circ + 35.26^\circ - 90^\circ - 144.74^\circ \\ &= -180^\circ = \pm 180^\circ\end{aligned}$$



Therefore, $s = -2 + j\frac{1}{\sqrt{2}}$ is a point on the root locus for some value of K .

Let's find the K !



$$K = \frac{1}{|G(s)||H(s)|} = \frac{1}{M} = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}}$$

$$K = \frac{l_3 l_4}{l_1 l_2} = \frac{\frac{1}{\sqrt{2}}(1.22)}{(2.12)(1.22)} = 0.33 \quad \text{and} \quad M = 3$$

$$\sum_{k=1}^m \angle(s - z_k) - \sum_{i=1}^n \angle(s - p_i) = r \cdot 180^\circ \quad (r: \text{odd})$$

$$\begin{aligned}GH &\rightarrow \frac{Z}{P} \\ \frac{1}{GH} &\rightarrow \frac{P}{Z}\end{aligned}$$

A point s on the root locus should be satisfied the following conditions.



$$|KG(s)H(s)| = 1 \Rightarrow K = \frac{1}{|G(s)H(s)|}$$

$$\angle G(s)H(s) = (2k+1)180^\circ \quad (K > 0)$$


```

numg = poly([-3 -4]);
deng = poly([-1 -2]);
G=tf(numg, deng)
rlocus(G)

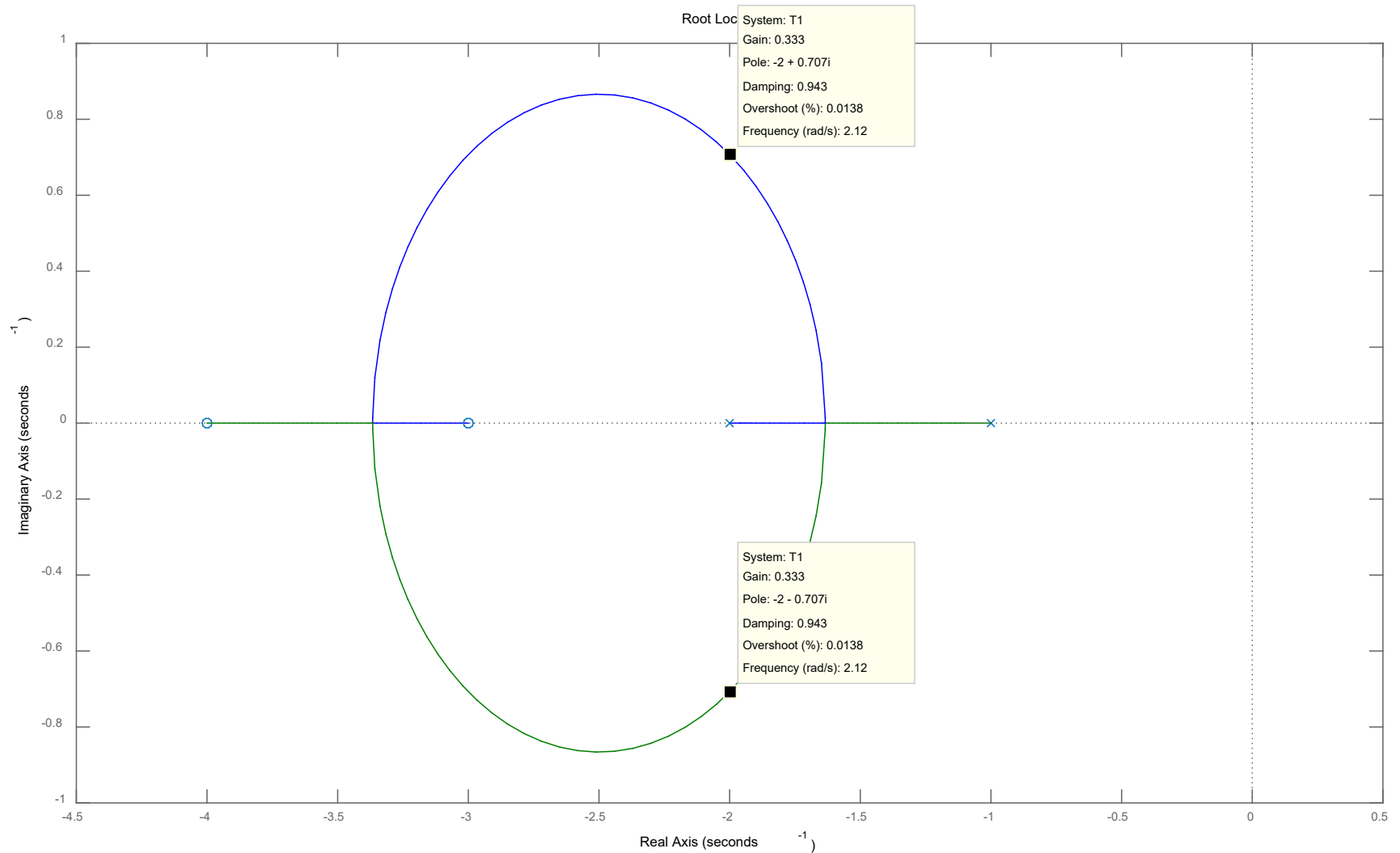
```

```

>> a=1/sqrt(2); s=-2+a*j;
G=(s^2+7*s+12)/(s^2+3*s+2);
Theta=(180/pi)*angle(G)
M=abs(G)

```

Theta = 180
 M = 3



8.4 Sketching the Root Locus

(1) Number of branches :

$$\# \text{ of branches} = \# \text{ of closed-loop poles}$$

(2) Symmetry

(3) Real-axis segments

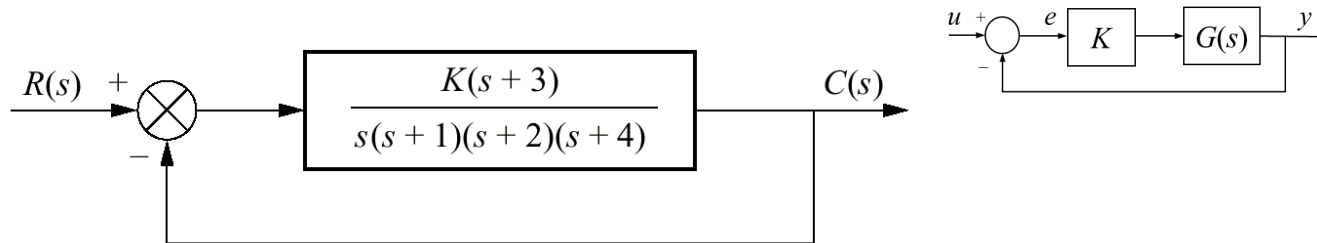
(4) Starting and ending points: ($K=0$, poles), ($K=\infty$, zeros)

(5) Behavior at infinity: asymptote

$$\text{Real-axis intercept: } \sigma_a = \frac{\Sigma \text{ finite poles} - \Sigma \text{ finite zeros}}{\# \text{ of finite poles} - \# \text{ of finite zeros}}$$

$$\text{Angle: } \theta_a = \frac{(2k+1)\pi}{\# \text{ of finite poles} - \# \text{ of finite zeros}}$$

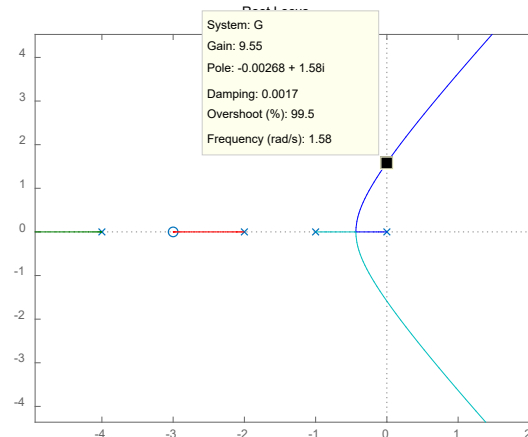
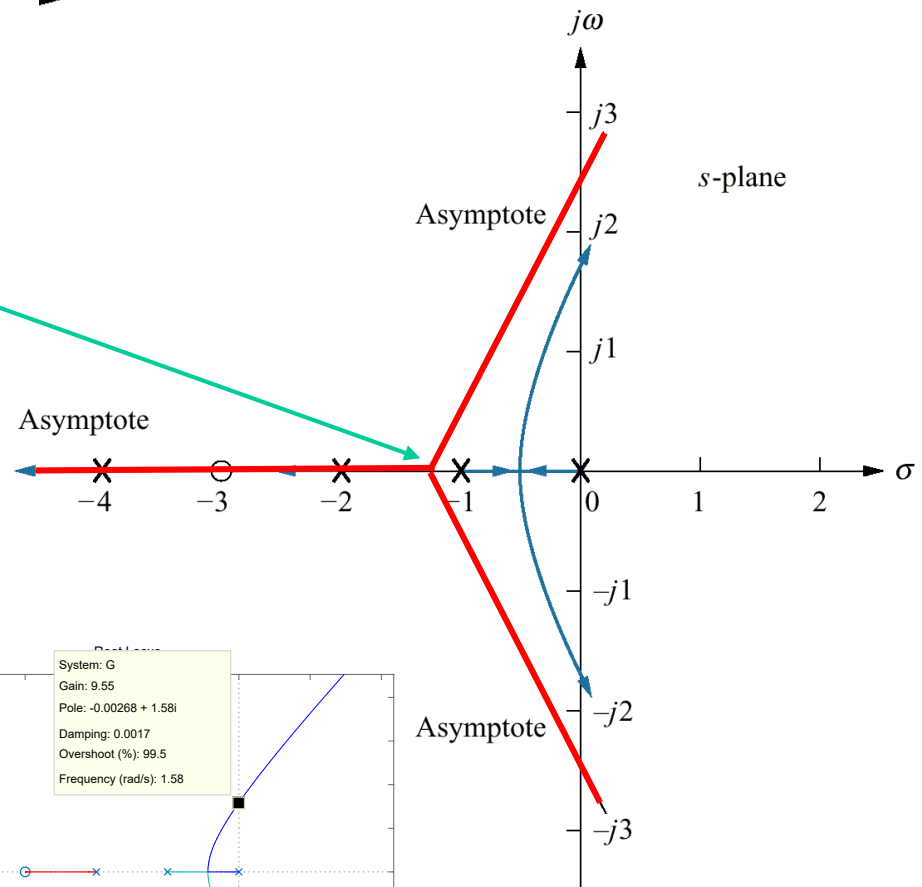
Example 8.2 Sketching a root locus with asymptotes



$$\sigma_a = \frac{(0-1-2-4)-(-3)}{(4-1)} = \frac{-7+3}{3} = -\frac{4}{3}$$

$$\theta_k = \frac{(2k+1)\pi}{(4-1)} = \frac{(2k+1)\pi}{3}$$

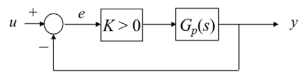
$$= \frac{\pi}{3}, \pi, \frac{5}{3}\pi, \text{ for } k = 0, 1, 2.$$



```
numg = poly([-3]);
deng = poly([0 -1 -2 -4]);
G=tf(numg, deng)
rlocus(G)
```

8.5 Refining the Sketch

(1) Root locus example showing real-axis breakaway (σ_1) and break-in points (σ_2)



- Plot the poles of the closed-loop system $T(s)$, as a function of K .

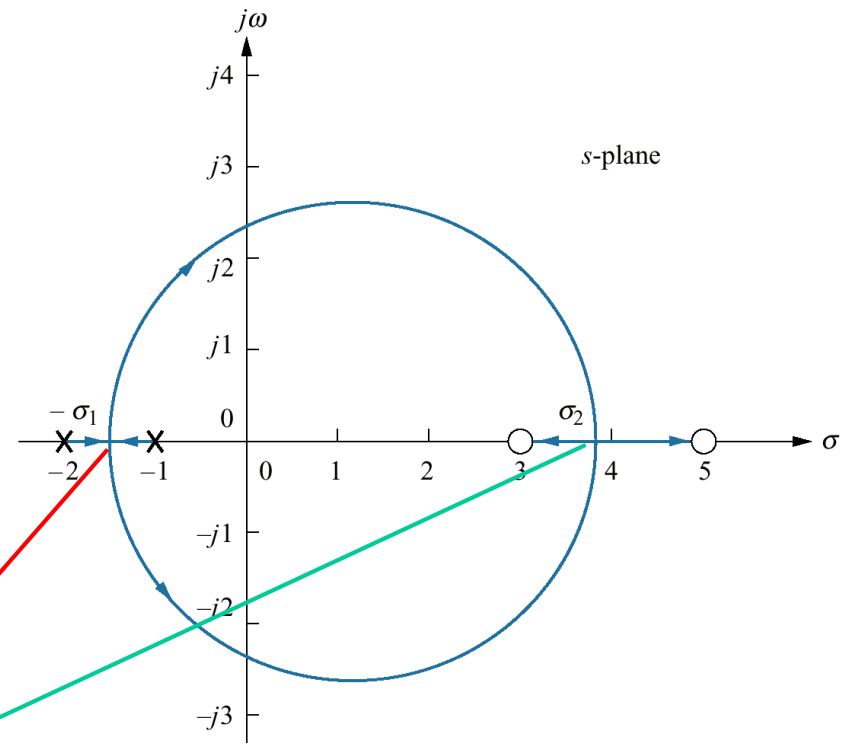
$$T(s) = \frac{KG_p(s)}{1 + KG_p(s)}$$

Poles: $1 + KG_p(s) = 0$

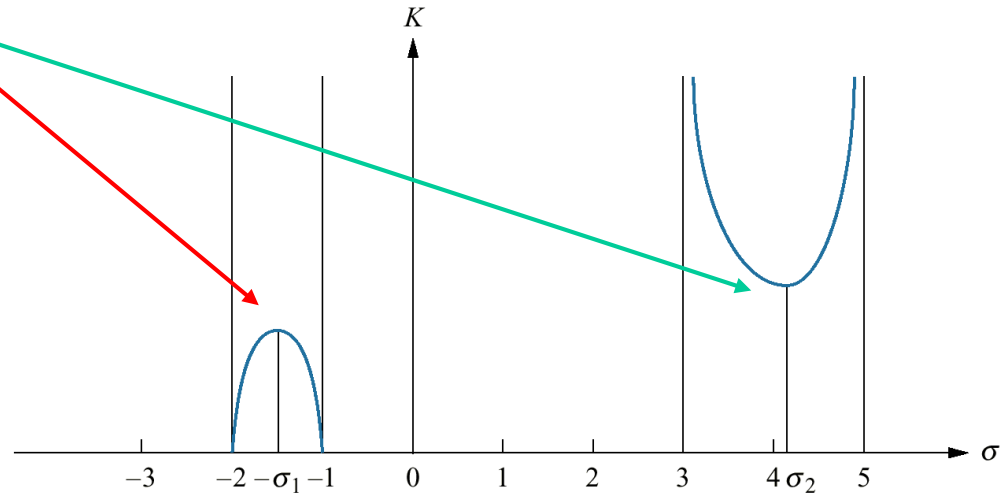
$$1 + KG(s)H(s) = 0$$

$$K = -\frac{1}{G(s)H(s)}$$

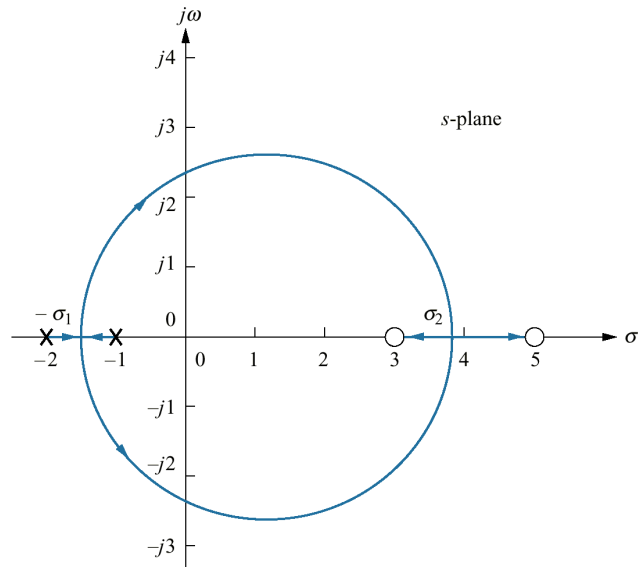
$$K(\sigma) = -\frac{1}{G(\sigma)H(\sigma)}$$



$$\frac{dK}{d\sigma} = 0$$



Example 8.3 Breakaway and break-in points via differentiation



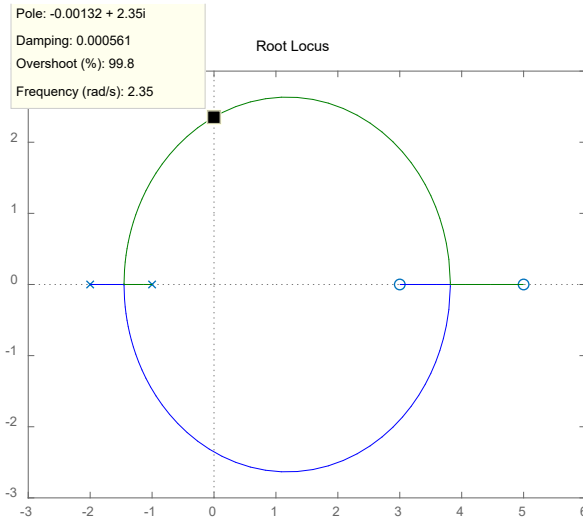
$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2 - 8s + 15)}{s^2 + 3s + 2}$$

$$\frac{K(\sigma^2 - 8\sigma + 15)}{\sigma^2 + 3\sigma + 2} = -1$$

$$K = \frac{-(\sigma^2 + 3\sigma + 2)}{(\sigma^2 - 8\sigma + 15)}$$

$$\frac{dK}{d\sigma} = \frac{11\sigma^2 - 26\sigma - 61}{\sigma^2 - 8\sigma + 15} = 0$$

$$\Rightarrow \sigma = -1.453, \quad 3.817$$

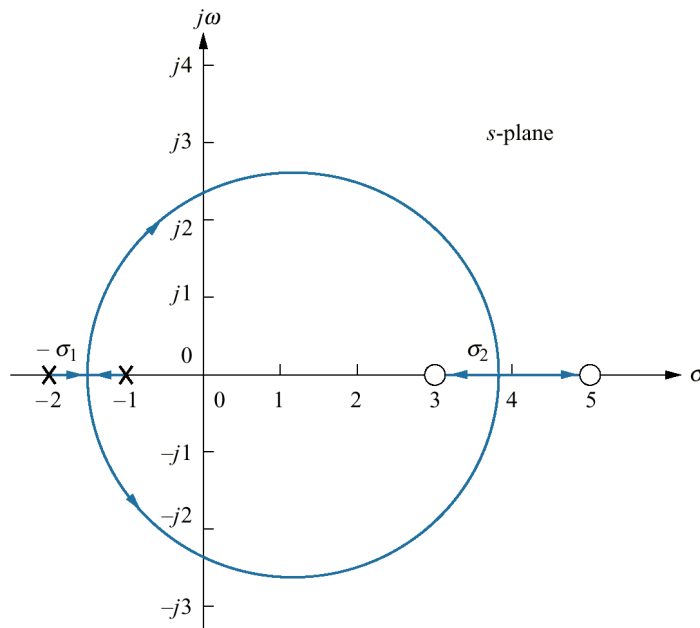


```
numg = poly([3 5]);
deng = poly([-1 -2]);
G=tf(numg, deng)
rlocus(G)
```

Example 8.4 Breakaway and break-in points without differentiation

- Breakaway and break-in points satisfy the relationship:
$$\sum_{i=1}^m \frac{1}{\sigma + z_i} = \sum_{i=1}^n \frac{1}{\sigma + p_i}$$

where z_i and p_i are the zero and pole values, respectively, of $G(s)H(s)$



$$\frac{1}{\sigma - 3} + \frac{1}{\sigma - 5} = \frac{1}{\sigma + 1} + \frac{1}{\sigma + 2}$$

$$11\sigma^2 - 26\sigma - 61 = 0$$

$$\Rightarrow \sigma = -1.453, \quad 3.817$$

- The $j\omega$ -Axis Crossings
- Angle of Departure and Arrival
- Plotting and Calibrating the Root Locus

8.7 Transient Response design via Gain Adjustment

(Find K for t_s , t_p , e_{ss} , overshoot)

8.8 Generalized Root Locus

How can we obtain a root locus for variations of the value of p_1 ?

8.9 Root Locus for Positive-Feedback Systems

8.10 Pole Sensitivity