Chapter 4. Time Response

4.1 Introduction

- Analysis of system transient response
- Step response of the first and second-order systems
- Poles and zeros of a transfer function

4.2 Poles, Zeros and System Response

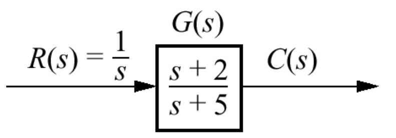
- In a transfer function:

$$G(s) = \frac{n(s)}{d(s)} \qquad Zeros:\ n(s)=0 \qquad Poles:\ d(s)=0 \qquad \boxed{\frac{\text{numerator}}{\text{denominator}}}$$

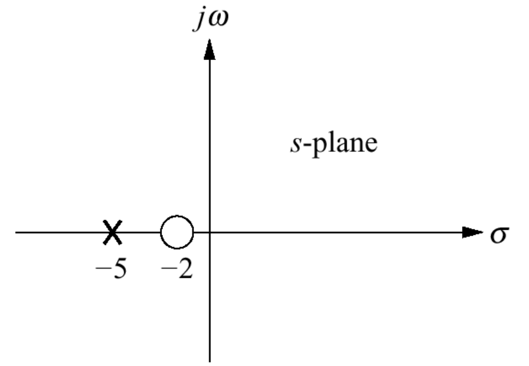
- Poles of a transfer function: The values of p ($pole$) that cause $G(p)=\infty$
- Zeros of a transfer function: The values of z ($zero$) that cause $G(z)=0$

• Poles and zeros of a first-order system: An example

Figure 4.1
(*a*) System showing input and output;
(*b*) pole-zero plot of the system;
(*c*) evolution of a system response.



(a)



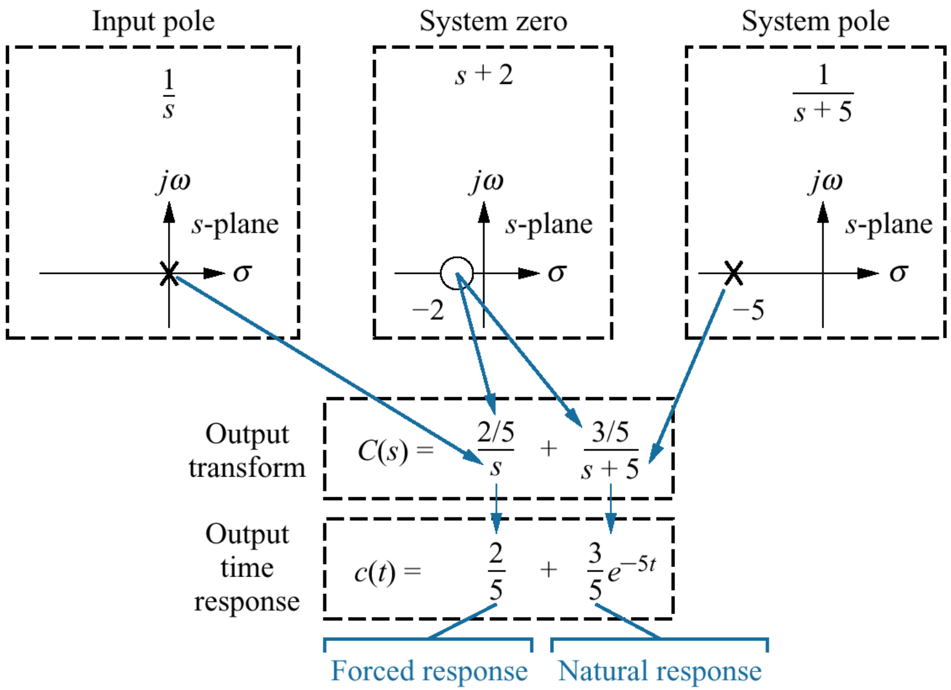
(b)

$$C(s)=\frac{(s+2)}{s(s+5)}=\frac{A}{s}+\frac{B}{s+5}=\frac{2/5}{s}+\frac{3/5}{s+5}$$

$$A=sC(s)\Big|_{s\to 0}=\frac{(s+2)}{(s+5)}\Bigg|_{s\to 0}=\frac{2}{5}$$

$$B=(s+5)C(s)\Big|_{s\to -5}=\frac{(s+2)}{s}\Bigg|_{s\to -5}=\frac{3}{5}$$

$$c(t)=\frac{2}{5}+\frac{3}{5}e^{-5t}$$



(c)

Example 4.1: Evaluating response using poles

For the given system, write the output, $c(t)$, in general terms.
Specify the forced and natural parts of the solution.

$$R(s) = \frac{1}{s} \longrightarrow \boxed{\frac{(s+3)}{(s+2)(s+4)(s+5)}} \xrightarrow{C(s)}$$

- By inspection, each system pole generates an exponential as part of the natural response.
- The input's pole generates the forced response.

$$C(s) = \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural response}}$$

- Taking the inverse Laplace transform:

$$c(t) = \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}}$$

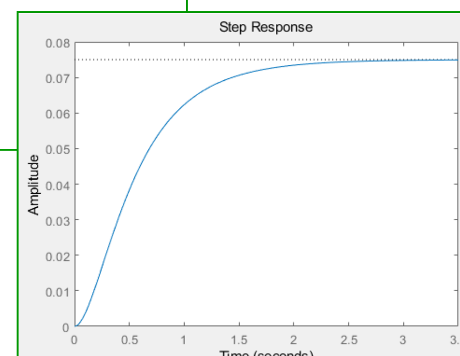
```
clc, clear all
numg = poly([-3]);
deng = poly([-2 -4 -5]);
G=tf(numg, deng)

step(G)
%[y,t] = step(sys);

%'Factored form, Tzpk(s)'
Tzpk=zpk(G)
```

```
G =
              s + 3
   ------------------------
   s^3 + 11 s^2 + 38 s + 40


Tzpk =
          (s+3)
   -----------------
   (s+5) (s+4) (s+2)
```



4.3 first-Order Systems (page 166)

```
>> [exp(-1), 1-exp(-1)]
ans =
     0.3679    0.6321
```

$$C(s) = R(s)G(s)$$

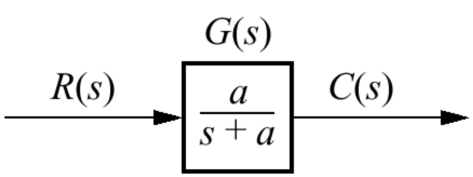
$$= \left(\frac{a}{s+a}\right)\frac{1}{s} = \frac{1}{s} - \frac{1}{s+a} \Rightarrow c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

when $t = \dfrac{1}{a}$

$$e^{-at}\Big|_{t=\frac{1}{a}} = e^{-1} = 0.37$$

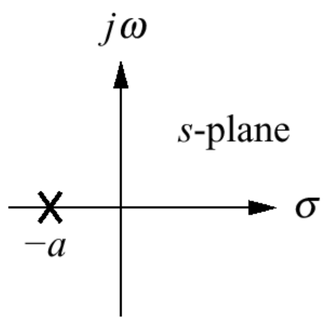
or

$$c(t)\Big|_{t=\frac{1}{a}} = 1 - e^{-at}\Big|_{t=\frac{1}{a}} = 0.63$$



(a)

First-order system



(b)

pole plot

$\dfrac{1}{a}$: *time constant* of the response

- Time constant:
  - decay to 37% of its initial value
  - rise to 63% of its final value

- Time constant: the time to rise to 63% of its final value, $\tau = \dfrac{1}{a}$

- Rise Time: the time to go from 10% to 90% of its final value

$$1 - e^{-at_1} = 0.9, \quad at_1 = -\ln(0.1) = 2.31, \quad t_1 = \frac{2.31}{a}$$

$$1 - e^{-at_2} = 0.1, \quad at_2 = -\ln(0.9) = 0.11, \quad t_2 = \frac{0.11}{a}$$

$$\rightarrow \quad T_r = t_1 - t_2 = \boxed{\frac{2.2}{a}}$$

- Settling Time: the time to reach and stay within 2% of its final value

$$1 - e^{-at_1} = 0.98, \qquad e^{-at_1} = 0.02$$

$$at_1 = -\ln(0.02) = 4$$

```
>> log(0.02)
ans = -3.9120
```

$$t_1 = \frac{4}{a}$$

$$T_s = \boxed{\frac{4}{a}}$$

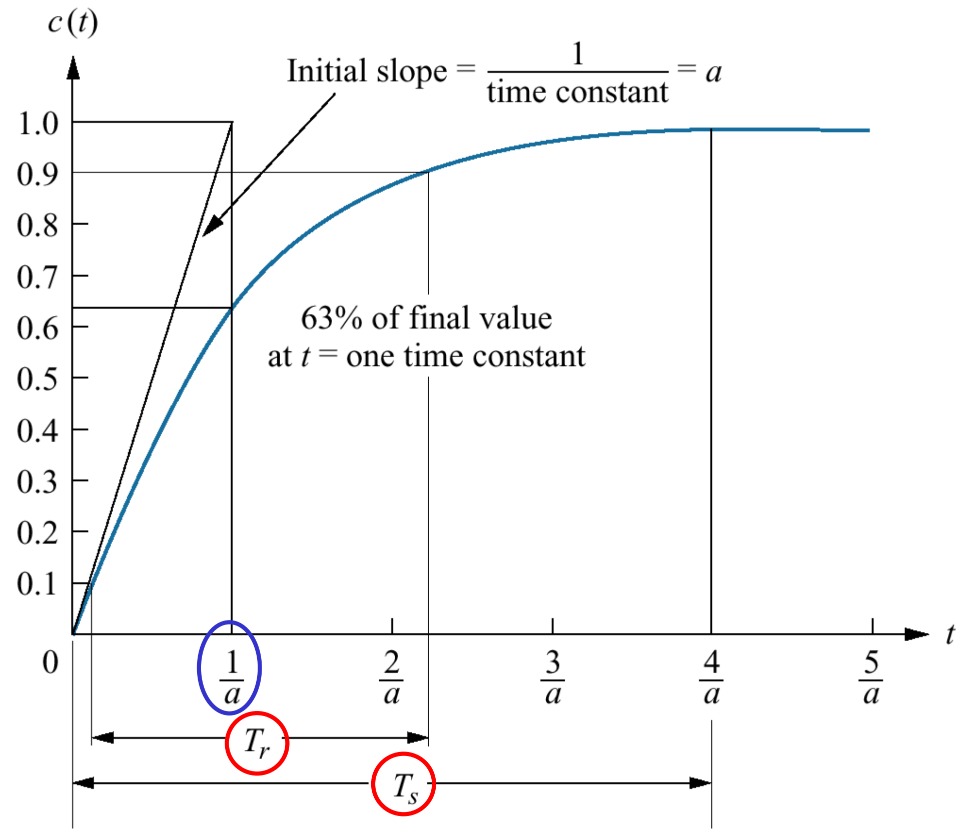


Figure 4.5:
First-order system response to a unit step

• First-Order Transfer Function via Testing

Example:

Assume the unit step response given in Figure 4.6.

- No overshoot
- Nonzero initial slope

Consider a first-order system that has the final value of 0.72.

The unit step response of $G(s) = K/(s+a)$:

$$C(s) = G(s)\frac{1}{s} = \frac{K}{s(s+a)} = \frac{K}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

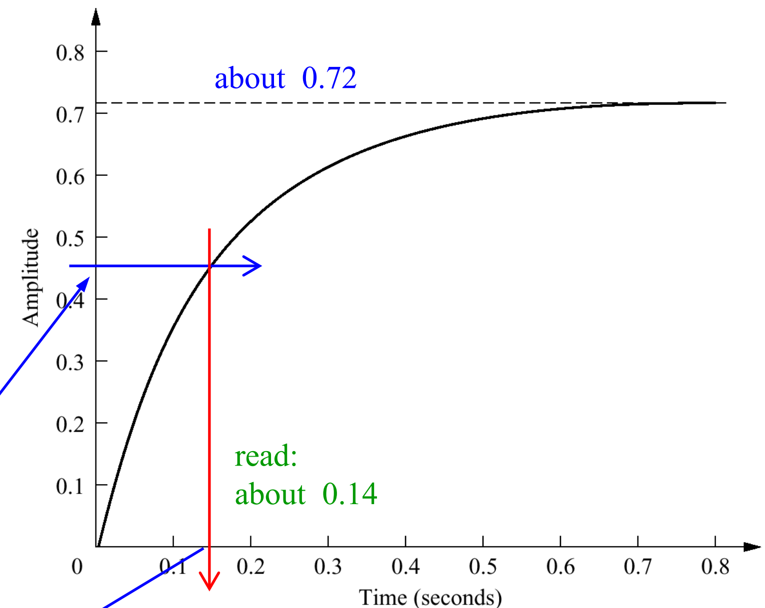


Figure 4.6:
Laboratory results of a system step response test

- Time constant:
  - decay to 37% of its initial value
  - rise to 63% of its final value

$$0.72 \times 63\% = 0.4536 \quad \rightarrow$$

read:
about 0.14

$$\boxed{\tau = \frac{1}{a}}$$

$$\tau = \frac{1}{a} = 0.14 \quad \rightarrow \quad a = 7.14$$

Actually, the figure was generated using $G(s) = 5/(s+7)$.

Find K using steady-state value: $\boxed{\lim_{s\to 0} sC(s) = \lim_{s\to 0} G(s) = \frac{K}{a} \quad \rightarrow \quad \frac{K}{a} = 0.72 \quad \rightarrow \quad K = 0.72a = 5.14}$

Thus, the transfer function is: $G(s) = \boxed{\frac{5.14}{(s+7.14)}}$

4.4 Second-Order Systems: Introduction (page 168)

System	Pole-zero Plot	Response

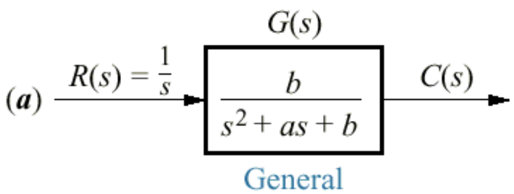
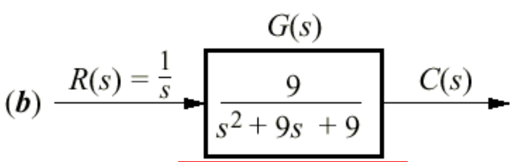
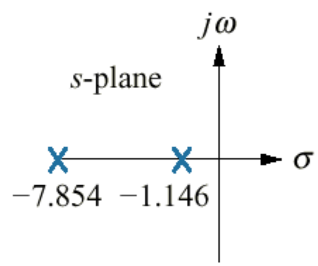
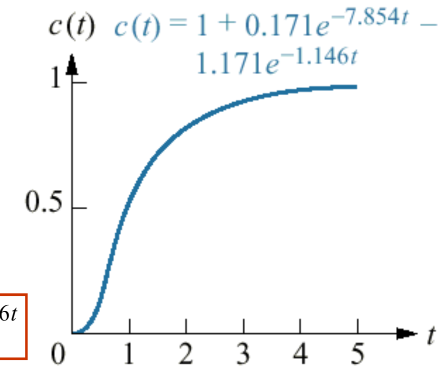


Figure 4.7
Second-order systems, pole plots, and step responses



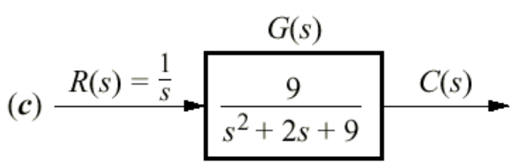
Overdamped



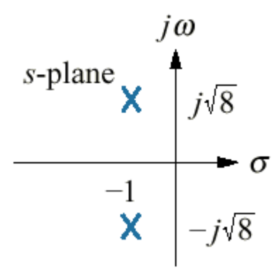


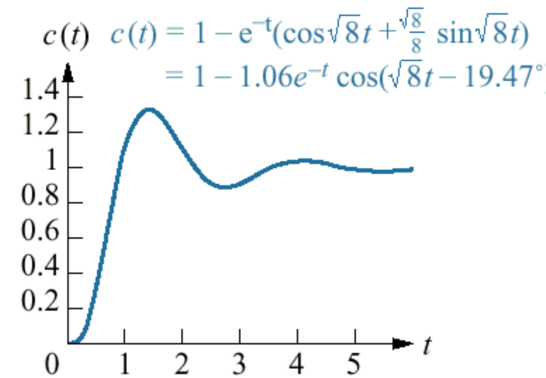

```
>> roots([1 9 9])
ans =
   -7.8541
   -1.1459
```

$$c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$$



Underdamped

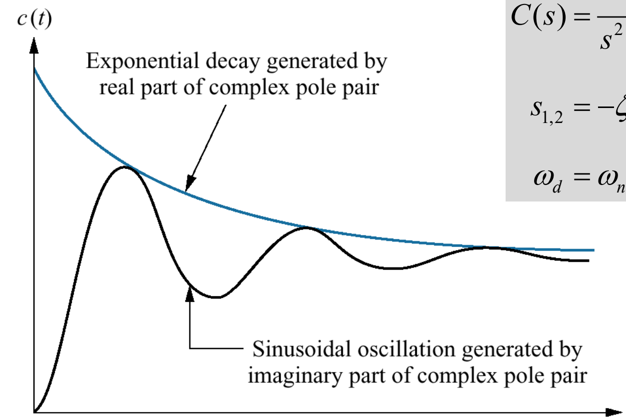





```
>> roots([1 2 9])
ans =
  -1.0000 + 2.8284i
  -1.0000 - 2.8284i
```

$$c(t) = K_1 + e^{-t}(K_2 \cos\sqrt{8}t + K_3 \sin\sqrt{8}t)$$

- *Underdamped response*:
  - Second-order step response components generated by complex poles
  - Sinusoidal freq.: damped frequency of oscillation, ω_d

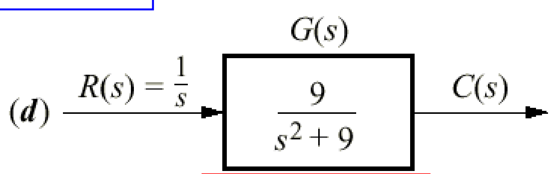


$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

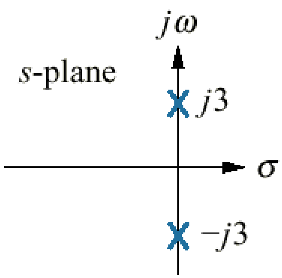
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm \omega_d$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

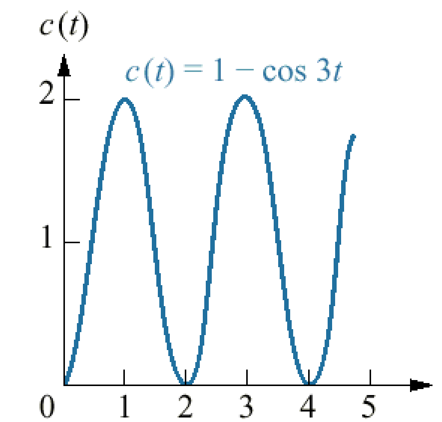
```
>> roots([1 0 9])
ans =
   0.0000 + 3.0000i
   0.0000 - 3.0000i
```

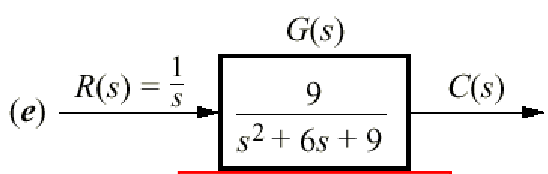


Undamped

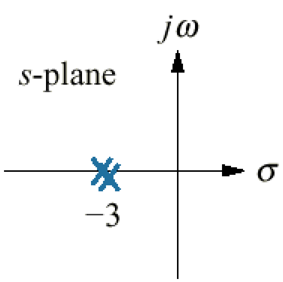


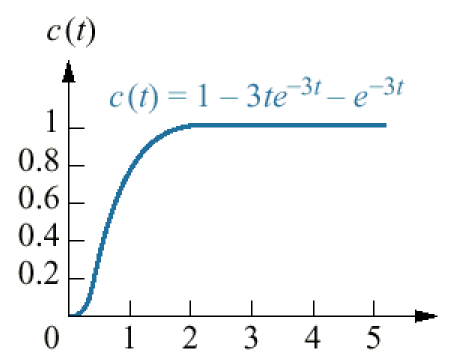
$c(t) = K_1 + K_2 \cos 3t + K_3 \sin 3t$





Critically damped






```
>> roots([1 6 9])
ans =
  -3.0000 + 0.0000i
  -3.0000 - 0.0000i
```

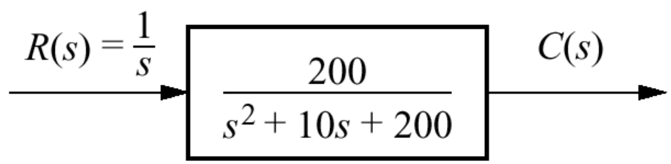
$c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$

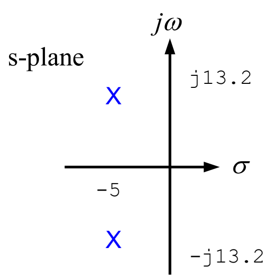
Example 4.2: Form of underdamped response using poles (page 171)

By inspection, write the form of the step response of the following system

```
>> roots([1 10 200])

ans =
   -5.0000 +13.2288i
   -5.0000 -13.2288i
```





$\rightarrow$ *underdamped*

Poles: $s=-5\pm j13.23$

real part: -5

imaginary part: 13.23

$$c(t)=K_1+e^{-5t}(K_2\cos 13.23t+K_3\sin 13.23t)$$

$$=K_1+K_4e^{-5t}\cos(13.23t-\phi)$$

where, $\phi=\tan^{-1}\left(\frac{K_3}{K_2}\right)$, $K_4=\sqrt{K_2^2+K_3^2}$

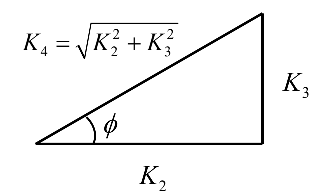
$\sin(A+B)=\sin(A)\cdot\cos(B)+\cos(A)\cdot\sin(B)$

$\sin(A-B)=\sin(A)\cdot\cos(B)-\cos(A)\cdot\sin(B)$

$\cos(A+B)=\cos(A)\cdot\cos(B)-\sin(A)\cdot\sin(B)$

$\cos(A-B)=\cos(A)\cdot\cos(B)+\sin(A)\cdot\sin(B)$

$$\cos(13.23t-\phi)$$
$$=\cos(13.23t)\cdot\cos(\phi)+\sin(13.23t)\cdot\sin(\phi)$$
$$=\cos(13.23t)\cdot\frac{K_2}{K_4}+\sin(13.23t)\cdot\frac{K_3}{K_4}$$
$$=\frac{1}{K_4}\{K_2\cos(13.23t)+K_3\sin(13.23t)\}$$



$$\cos(\phi)=\frac{K_2}{K_4}$$

$$\sin(\phi)=\frac{K_3}{K_4}$$

• Natural responses and their characteristics

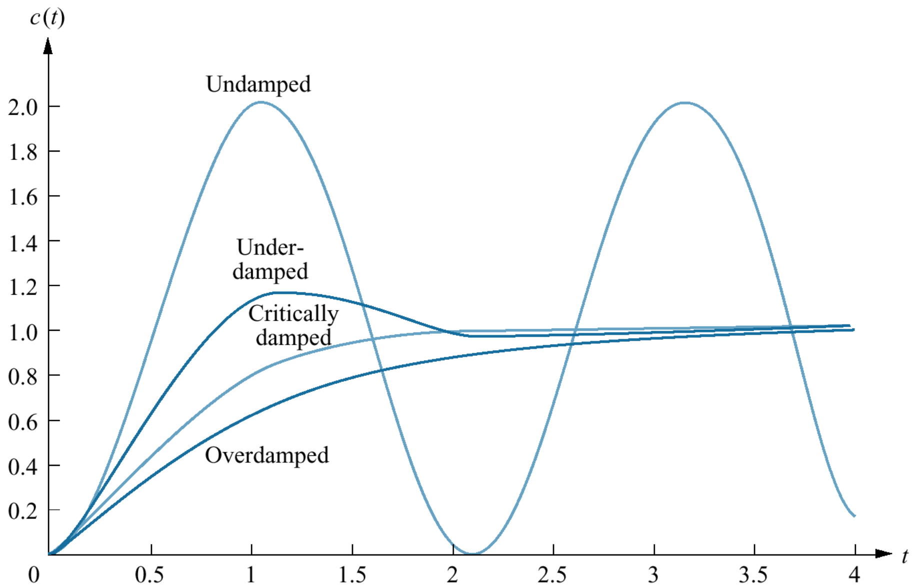


Figure 4.10:
Step responses for second-order system damping cases

1. *Overdamped responses*:

   Poles: Two real at $-\sigma_1, -\sigma_2$

   $$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

2. *Underdamped responses*:

   Poles: Two complex at $-\sigma_d \pm jw_d$

   $$c(t) = Ae^{-\sigma_d t}\cos(w_d t - \phi)$$

3. *Undamped responses*:

   Poles: Two imaginary at $\pm jw_1$

   $$c(t) = A\cos(w_1 t - \phi)$$

4. *Critically damped responses*:

   Poles: Two real at σ_1

   $$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$

4.5 The General Second-Order System (page 175)

$$G(s)=\frac{b}{s^2+as+b}$$

$$s=\frac{-a\pm\sqrt{a^2-4b}}{2}$$

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

ζ : *damping ratio*

ω_n : *natural frequency*

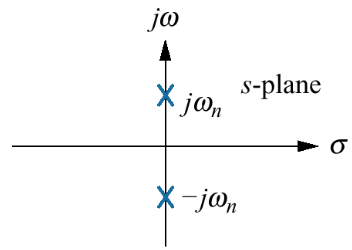
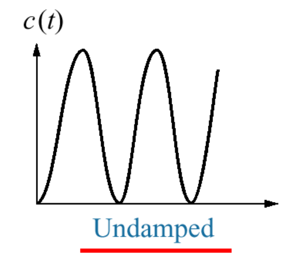
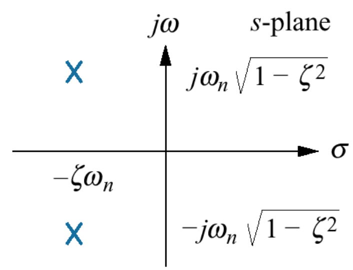
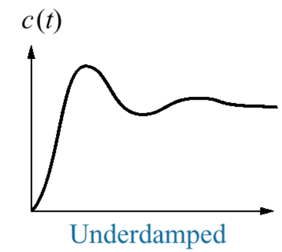
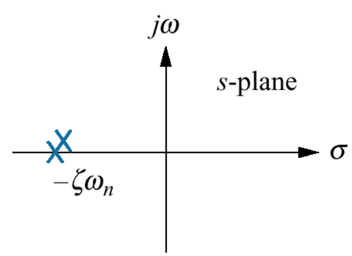
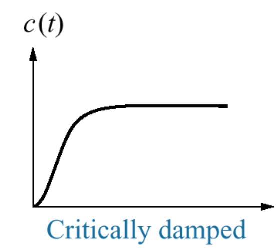
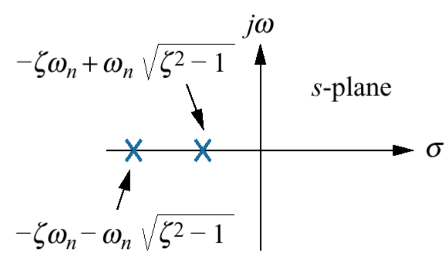
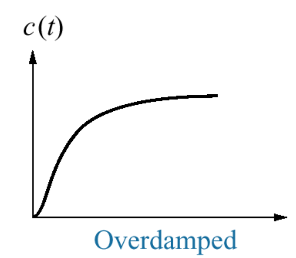
- Poles of $G(s)$:

$$s_{1,2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$

$$\Downarrow$$

$$b=\omega_n^2$$
$$a=2\zeta\omega_n$$

Example 4.4

ζ	Poles	Step response
0		
$0 < \zeta < 1$		
$\zeta = 1$		
$\zeta > 1$		

Example 4.4: Characterizing response from the value of ζ (page 176)

Find the value of ζ and report the kind of response expected

$$R(s) \longrightarrow \boxed{\dfrac{12}{s^2+8s+12}} \longrightarrow C(s)$$

(*a*)

$$R(s) \longrightarrow \boxed{\dfrac{16}{s^2+8s+16}} \longrightarrow C(s)$$

(*b*)

$$R(s) \longrightarrow \boxed{\dfrac{20}{s^2+8s+20}} \longrightarrow C(s)$$

(*c*)

$$G(s) = \frac{b}{s^2 + as + b}$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$a = 2\zeta\omega_n, \quad b = \omega_n^2$$

$$\zeta = \frac{a}{2\omega_n} = \frac{a}{2\sqrt{b}}$$

$$(a)\ Poles:\ -6,\ -2\ \rightarrow \zeta = \frac{a}{2\sqrt{b}} = 1.155 \rightarrow \zeta > 1\ \rightarrow\ overdamped$$

$$(b)\ Poles:\ -4,\ -4\ \rightarrow \zeta = \frac{a}{2\sqrt{b}} = 1 \rightarrow \zeta = 1\ \rightarrow\ critically\ damped$$

$$(c)\ Poles:\ -4 \pm 2i\ \rightarrow \zeta = \frac{a}{2\sqrt{b}} = 0.894 \rightarrow 0 < \zeta < 1\ \rightarrow\ underdamped$$

4.6 Underdamped Second-Order Systems (page 177)

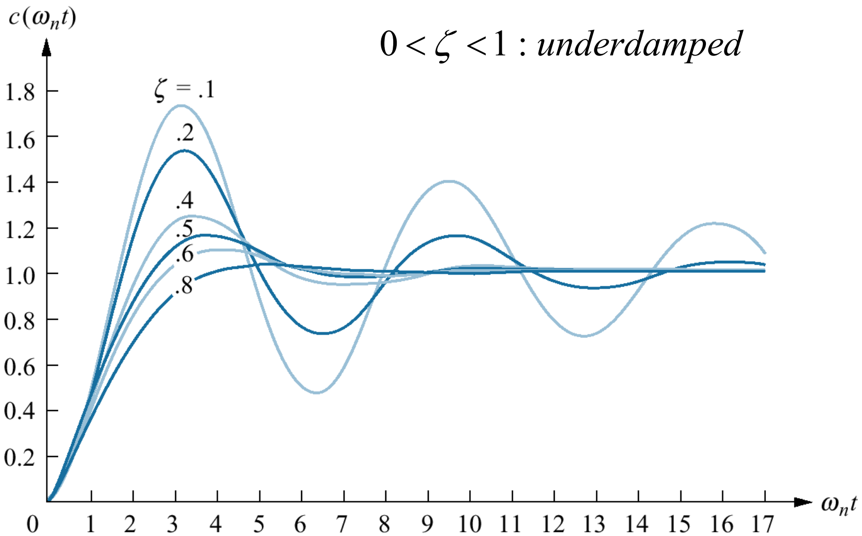


Figure 4.13:
Second-order underdamped responses for damping ratio values

$$C(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2},\quad (0<\zeta<1)$$

$$s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}=-\zeta\omega_n \pm j\omega_d$$

$$\omega_d=\omega_n\sqrt{1-\zeta^2}$$

Damped natural frequency
Frequency of transient oscillation

- For $0<\zeta<1$

$$C(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{K_1}{s}+\frac{K_2 s+K_3}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$=\frac{1}{s}-\frac{(s+\zeta\omega_n)+\dfrac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$\Downarrow$ Inverse Laplace Transform

$$c(t)=1-e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right)$$

$$=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t-\phi\right)$$

where $\phi=\tan^{-1}\left(\dfrac{\zeta}{\sqrt{1-\zeta^2}}\right)$

• Underdamped Step Response $(0<\zeta<1)$

$$C(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$=\frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\zeta^2\omega_n^2+\omega_n^2-\zeta^2\omega_n^2}$$

$$=\frac{1}{s}-\frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$=\frac{1}{s}-\frac{(s+\zeta\omega_n)+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$=\frac{1}{s}-\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$=\frac{1}{s}-\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}-\frac{\frac{\zeta\omega_n}{\sqrt{\omega_n^2(1-\zeta^2)}}\cdot\sqrt{\omega_n^2(1-\zeta^2)}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$=\frac{1}{s}-\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}-\frac{\frac{\zeta}{\sqrt{(1-\zeta^2)}}\cdot\sqrt{\omega_n^2(1-\zeta^2)}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$C(s)=\frac{K_1}{s}+\frac{K_2 s+K_3}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$=\frac{K_1(s^2+2\zeta\omega_n s+\omega_n^2)+s(K_2 s+K_3)}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$=\frac{(K_1+K_2)s^2+(2\zeta\omega_n+K_3)s+K_1\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$\begin{aligned}&K_1+K_2=0 & & K_1=1\\ &2\zeta\omega_n+K_3=0 \quad \Rightarrow & & K_2=-1\\ &K_1=1 & & K_3=-2\zeta\omega_n\end{aligned}$$

$$\rightarrow C(s)=\frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$L\{f(t)\}=F(s)$$

$$L\{f(t)e^{-\alpha t}\}=F(s+\alpha)$$

$$L\{\sin at\}=\frac{a}{s^2+a^2}$$

$$L\{\cos at\}=\frac{s}{s^2+a^2}$$

- Applying the inverse Laplace transform:

$$C(s)=\frac{1}{s}-\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}-\frac{\zeta}{\sqrt{(1-\zeta^2)}}\cdot\frac{\sqrt{\omega_n^2(1-\zeta^2)}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$c(t)=L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}\right]-\frac{\zeta}{\sqrt{(1-\zeta^2)}}\cdot L^{-1}\left[\frac{\sqrt{\omega_n^2(1-\zeta^2)}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}\right]$$

$$=1-e^{-\zeta\omega_n t}\cdot\cos\omega_n\sqrt{(1-\zeta^2)}t-\frac{\zeta}{\sqrt{(1-\zeta^2)}}e^{-\zeta\omega_n t}\cdot\sin\omega_n\sqrt{(1-\zeta^2)}t$$

$$=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left[\sqrt{(1-\zeta^2)}\cdot\cos\omega_n\sqrt{(1-\zeta^2)}t+\zeta\cdot\sin\omega_n\sqrt{(1-\zeta^2)}t\right]$$

$$L\{f(t)\}=F(s)$$

$$L\{f(t)e^{-\alpha t}\}=F(s+\alpha)$$

$$L\{\sin at\}=\frac{a}{s^2+a^2}$$

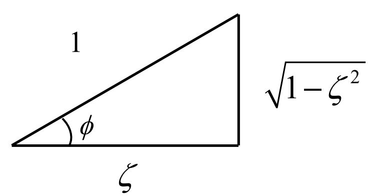
$$L\{\cos at\}=\frac{s}{s^2+a^2}$$

$$\boxed{\sin(A+B)=\sin(A)\cdot\cos(B)+\cos(A)\cdot\sin(B)}$$

$$\boxed{\cos(A-B)=\cos(A)\cdot\cos(B)+\sin(A)\cdot\sin(B)}$$

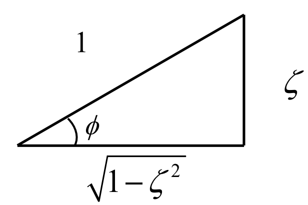
$\cos\phi=\zeta$

$\sin\phi=\sqrt{1-\zeta^2}$



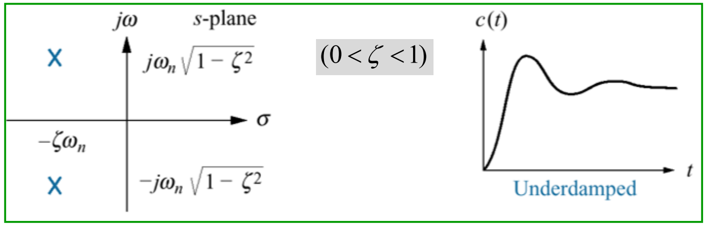
$\sin\phi=\zeta$

$\cos\phi=\sqrt{1-\zeta^2}$



From $c(t)$,

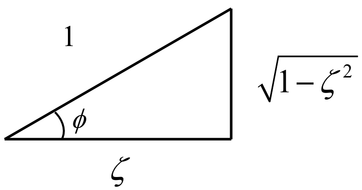
$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left[\sqrt{(1-\zeta^2)} \cdot \cos\omega_n\sqrt{(1-\zeta^2)}t + \zeta \cdot \sin\omega_n\sqrt{(1-\zeta^2)}t\right]$$



Case 1: $\sin(A+B) = \sin(A)\cdot\cos(B) + \cos(A)\cdot\sin(B)$

$\cos\phi = \zeta$

$\sin\phi = \sqrt{1-\zeta^2}$



$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left[\sin\phi \cdot \cos\omega_n\sqrt{(1-\zeta^2)}t + \cos\phi \cdot \sin\omega_n\sqrt{(1-\zeta^2)}t\right]$$

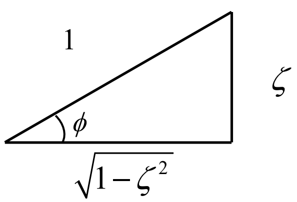
$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\sin(\omega_n\sqrt{(1-\zeta^2)}t + \phi)$$

with $\phi = \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}$

Case 2: $\cos(A-B) = \cos(A)\cdot\cos(B) + \sin(A)\cdot\sin(B)$

$\sin\phi = \zeta$

$\cos\phi = \sqrt{1-\zeta^2}$



$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left[\sin\phi \cdot \cos\omega_n\sqrt{(1-\zeta^2)}t + \cos\phi \cdot \sin\omega_n\sqrt{(1-\zeta^2)}t\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\cos(\omega_n\sqrt{(1-\zeta^2)}t - \phi)$$

with $\phi = \tan^{-1}\dfrac{\zeta}{\sqrt{1-\zeta^2}}$

• Undamped Step Response $(\zeta = 0)$

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

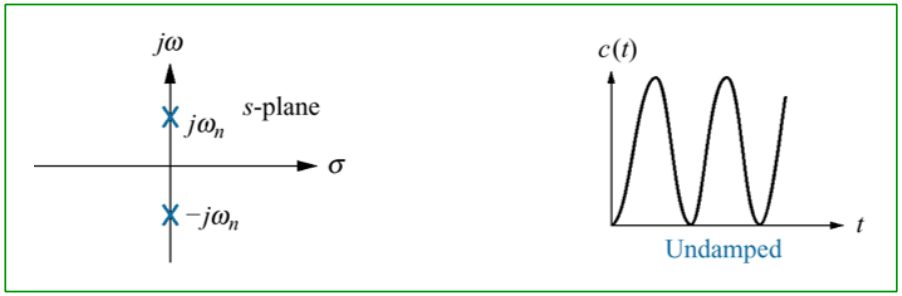
$$s_{1,2} = \pm j\omega_n$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left[\sqrt{(1-\zeta^2)}\cdot\cos\omega_n\sqrt{(1-\zeta^2)}t + \zeta\cdot\sin\omega_n\sqrt{(1-\zeta^2)}t\right]$$

When damping ratio is zero, $\zeta = 0$

$$c(t) = 1 - \frac{e^{-0\omega_n t}}{\sqrt{(1-0^2)}}\left[\sqrt{(1-0^2)}\cdot\cos\omega_n\sqrt{(1-0^2)}t + 0\cdot\sin\omega_n\sqrt{(1-0^2)}t\right]$$

$$= 1 - \cos\omega_n t$$



$$c(t) = 1 - \cos\omega_n t \quad \text{for} \quad \zeta = 0$$

→ no exponential term is in the $c(t)$, the time response of the control system is undamped for unit step input function with zero damping ratio.

$$C(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$s_{1,2}=$ two real at $-\zeta\omega_n$,

• Critically Damped Step Response $(\zeta=1)$

$$c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left[\sqrt{(1-\zeta^2)}\cdot\cos\omega_n\sqrt{(1-\zeta^2)}t+\zeta\cdot\sin\omega_n\sqrt{(1-\zeta^2)}t\right]$$

When damping ratio is unity, $\zeta=1$

$$c(t)=\lim_{\zeta\to1}\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left\{\sqrt{(1-\zeta^2)}\cdot\underline{\cos\omega_n\sqrt{(1-\zeta^2)}t}+\zeta\cdot\underline{\sin\omega_n\sqrt{(1-\zeta^2)}t}\right\}\right]$$

$$\lim_{\zeta\to1}\left[\cos\omega_n\sqrt{(1-\zeta^2)}t\right]\to1$$

$$\lim_{\zeta\to1}\left[\sin\omega_n\sqrt{(1-\zeta^2)}t\right]\to\omega_n\sqrt{(1-\zeta^2)}t$$

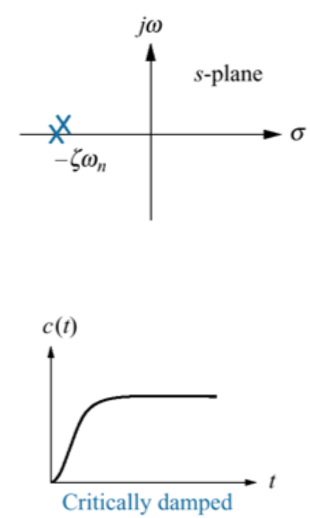
$$c(t)=\lim_{\zeta\to1}\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left\{\sqrt{(1-\zeta^2)}\cdot1+\zeta\cdot\omega_n\sqrt{(1-\zeta^2)}t\right\}\right]$$

$$=1-e^{-\omega_n t}\left\{\lim_{\zeta\to1}\frac{\sqrt{(1-\zeta^2)}\cdot1}{\sqrt{(1-\zeta^2)}}+\lim_{\zeta\to1}\frac{\zeta\cdot\omega_n\sqrt{(1-\zeta^2)}t}{\sqrt{(1-\zeta^2)}}\right\}$$

$$=1-e^{-\omega_n t}(1+\omega_n t)$$

$$c(t)=1-e^{-\omega_n t}(1+\omega_n t)$$

for $\zeta=0$

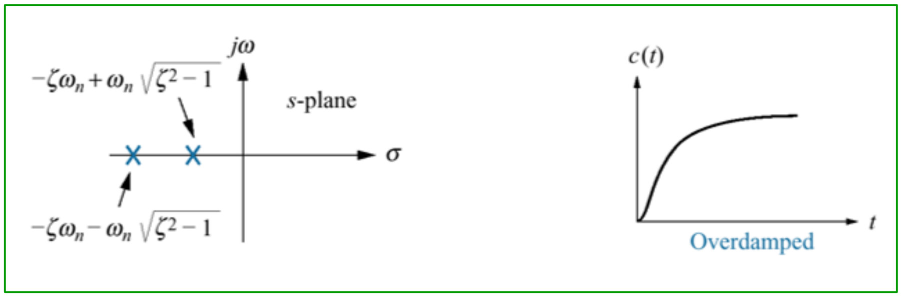


• Overdamped Step Response ($\zeta > 1$)

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

When $\zeta > 1$, the response of the unit step input given to the system, does not exhibit oscillating part in the output. This is called overdamped response.



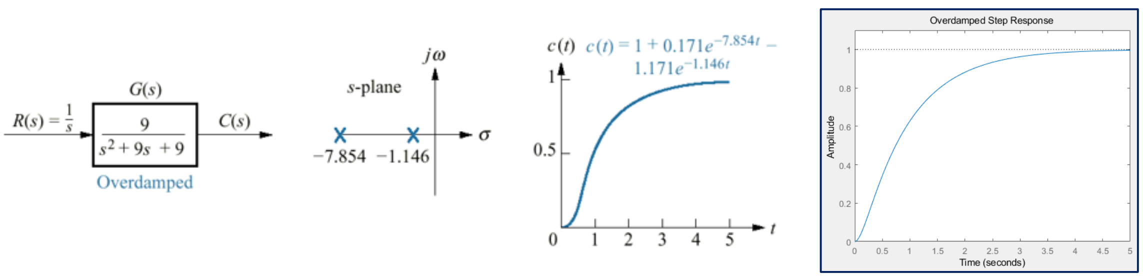
$$C(s) = \frac{9}{s(s^2+9s+9)} = \frac{9}{s(s+p_1)(s+p_2)} = \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{s+p_2}$$

$$p_1 = -7.854, \quad p_2 = -1.146$$

$$K_1 = sG(s)\big|_{s\to 0} = \left.\frac{9}{(s+p_1)(s+p_2)}\right|_{s\to 0} = \frac{9}{p_1 p_2} = \frac{9}{9} = 1$$

$$K_2 = (s+p_1)G(s)\big|_{s\to -p_1} = \left.\frac{9}{s(s+p_2)}\right|_{s\to -p_1} = \frac{9}{-p_1(-p_1+p_2)} = 0.171$$

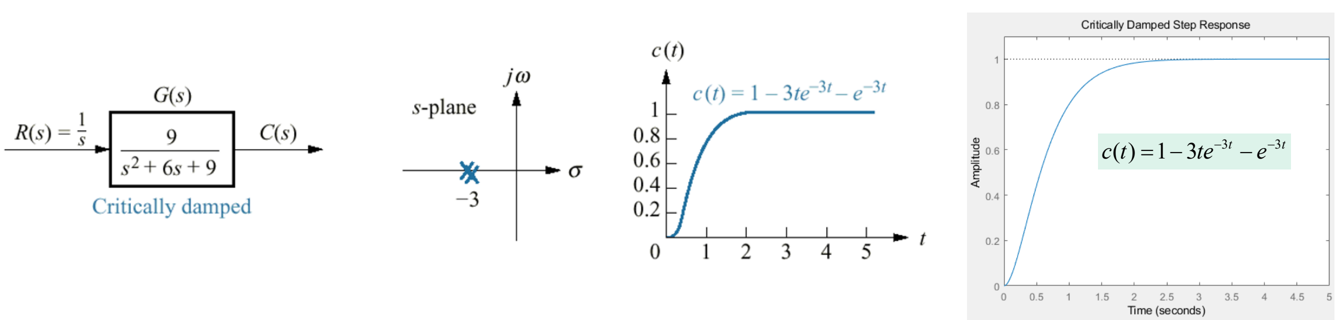
$$K_3 = (s+p_2)G(s)\big|_{s\to -p_2} = \left.\frac{9}{s(s+p_1)}\right|_{s\to -p_2} = \frac{9}{-p_2(-p_2+p_1)} = -1.171$$



- Taking the inverse Laplace transform:

$$c(t) = K_1 + K_2 e^{p_1 t} + K_3 e^{p_2 t}$$

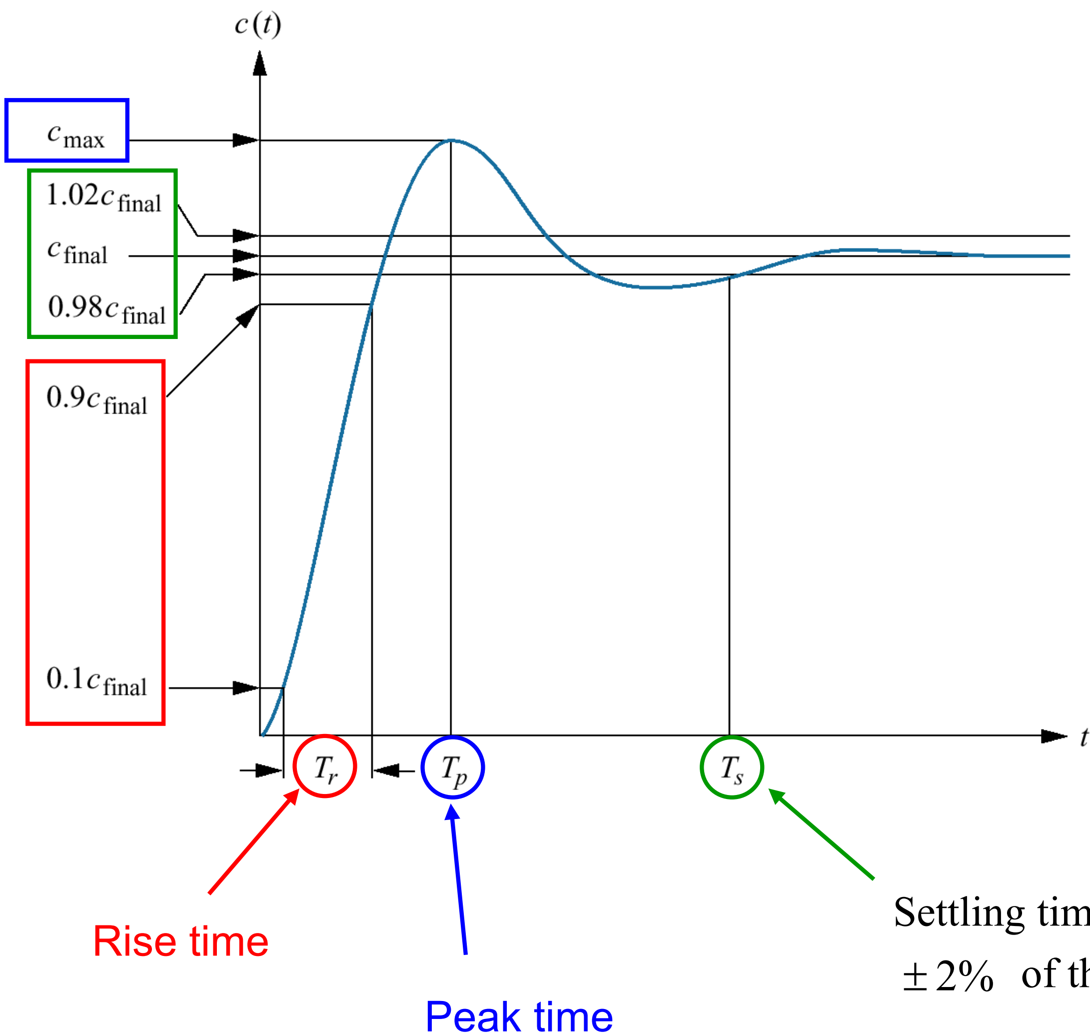
$$c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}$$




```
clc, clear all
numg = [9];
deng = [1 6 9];
G=tf(numg, deng)

step(G)
```

• Second-order underdamped response specifications



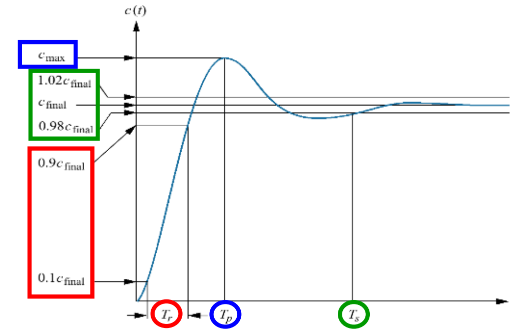
% overshoot:

$$\%OS = \frac{c_{\max} - c_{\text{final}}}{c_{\text{final}}} \times 100$$

Settling time:

$\pm 2\%$ of the steady-state value

- Evaluation of T_p



$$c(t) \Rightarrow \dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}t\right) \Rightarrow \dot{c}(t) = 0$$

$$\omega_n\sqrt{1-\zeta^2}t = n\pi \Rightarrow t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \Rightarrow \boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

$$L(\dot{c}(t)) = sC(s) = s\cdot\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\zeta^2\omega_n^2+\omega_n^2-\zeta^2\omega_n^2}$$

$$= \frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$= \frac{\left(\frac{\omega_n}{\sqrt{1-\zeta^2}}\right)\cdot\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}t$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

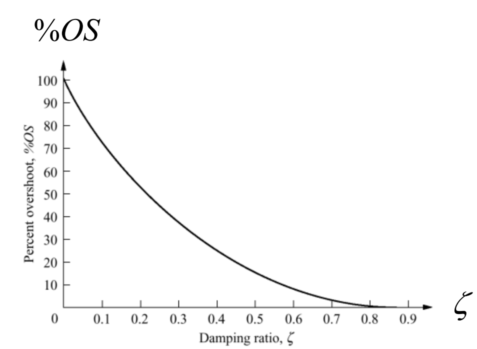
$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

- Evaluation of %OS

$$\%OS = \frac{c_{\max}-c_{\text{final}}}{c_{\text{final}}}\times 100, \quad \text{at} \quad c_{\text{final}} = 1$$

$$c_{\max} = c(T_p) = 1 - e^{-(\zeta\pi/\sqrt{1-\zeta^2})}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) = 1 + e^{-(\zeta\pi/\sqrt{1-\zeta^2})}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})}\times 100 \quad \text{or} \quad \boxed{\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}}$$



- Evaluation of T_s

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1-\zeta^2}\, t - \phi\right),$$

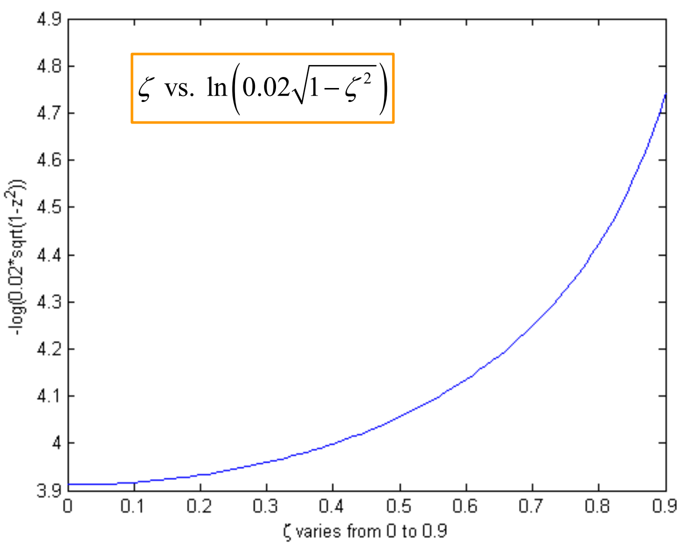
At the settling time, assume that

$$\cos(\omega_n \sqrt{1-\zeta^2}\, t - \phi) = 1$$

$$\Rightarrow \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T_s} = 0.02 \quad \Rightarrow \quad \boxed{T_s = \frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}}$$

$$e^{-\zeta\omega_n T_s} = 0.02\sqrt{1-\zeta^2}$$

$$-\zeta\omega_n T_s = \ln\left(0.02\sqrt{1-\zeta^2}\right)$$

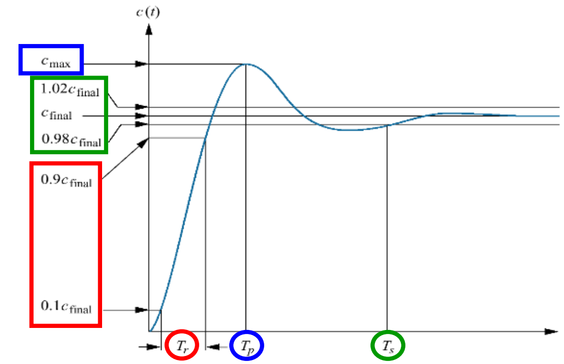


- ζ varies from 0 to 0.9

$$\Rightarrow -\ln\left(0.02\sqrt{1-\zeta^2}\right) \text{ varies from 3.91 to 4.74}$$

$$\boxed{T_s = \frac{4}{\zeta\omega_n}}$$

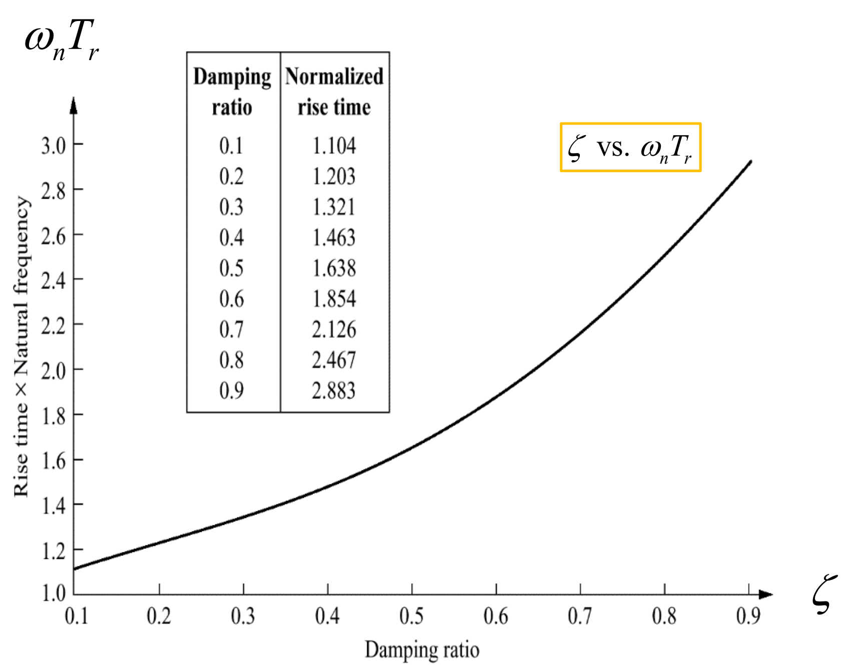
- Evaluation of T_r



$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t - \phi\right)$$

- Solve for the values of $\omega_n t$ that yield the $c(t) = 0.9$ and $c(t) = 0.1$ $\Rightarrow \zeta$ vs. $\omega_n T_r$



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Example 4.5: Finding T_p, $\%OS$, T_s, and T_r from a transfer function (page 182)

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

$$T_s = \frac{4}{\zeta\omega_n}$$

$$G(s) = \frac{100}{s^2 + 15s + 100} \quad \Rightarrow \quad \omega_n = 10, \quad \zeta = 0.75$$

$$T_p = \frac{\pi}{10\sqrt{1-0.75^2}} = 0.475\,\text{sec}$$

$$\%OS = e^{\frac{-0.75\pi}{\sqrt{1-0.75^2}}} \times 100 = 2.838\%$$

$$T_s = \frac{4}{0.75 \times 10} = 0.533\,\text{sec}$$

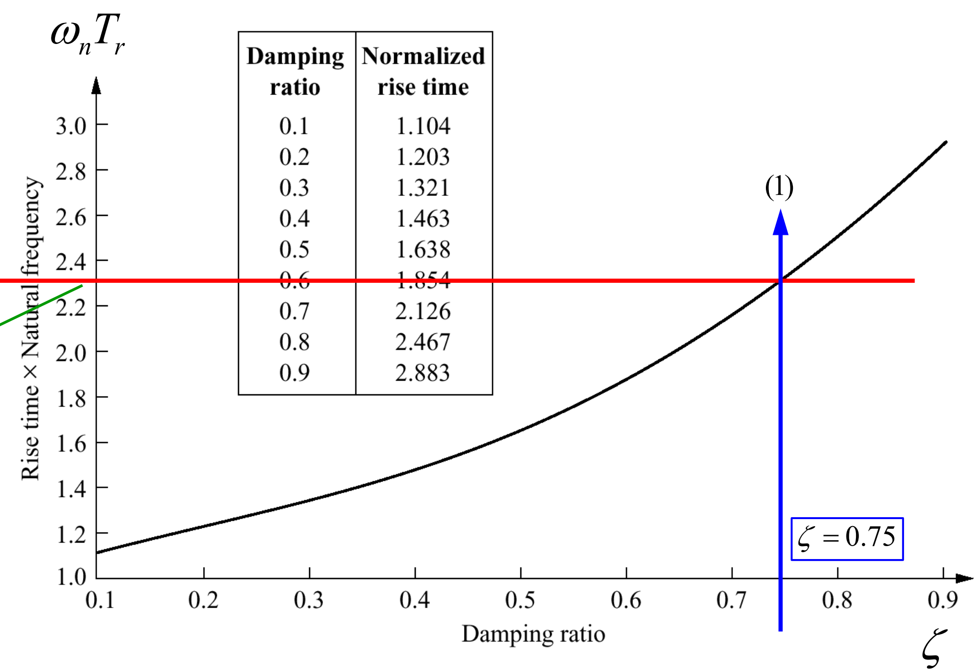
$$T_r = ?$$

(3)

$$\omega_n T_r \cong 2.3$$

$$T_r \cong 0.23$$

(2)



Damping ratio	**Normalized rise time**
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

ζ : damping ratio

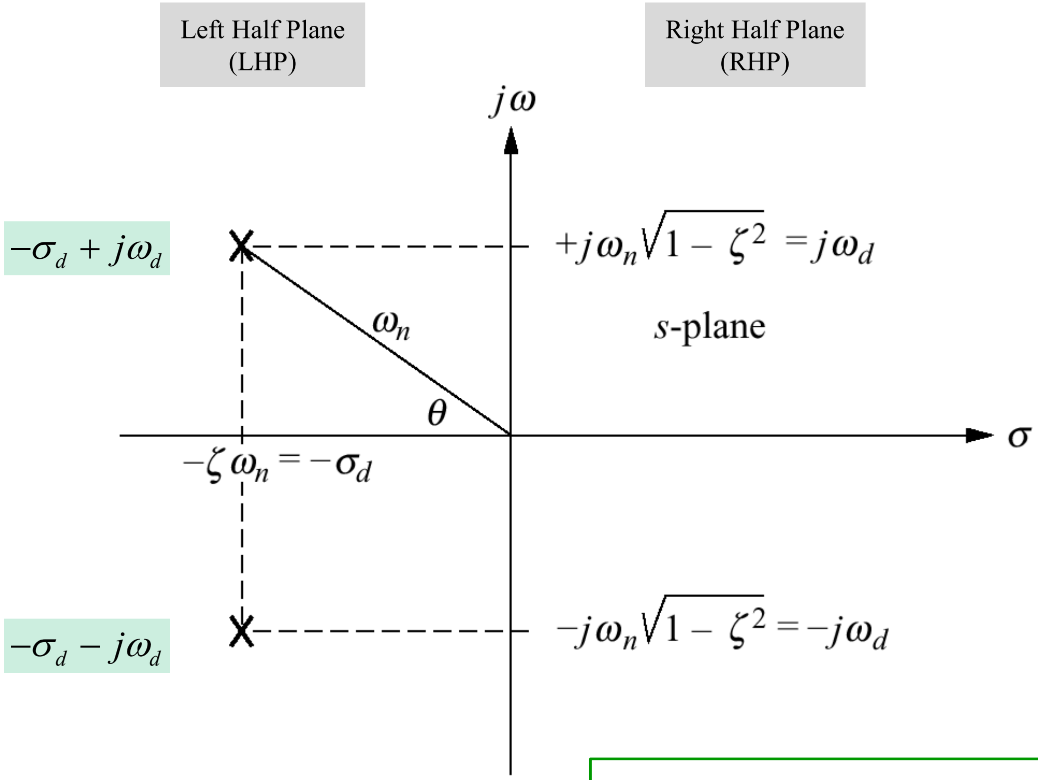
ω_n : natural frequency

$$T_p=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\zeta=\frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}$$

$$T_s=\frac{4}{\zeta\omega_n}$$

- Pole plot for an underdamped second-order system



$$T_p=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}=\frac{\pi}{\omega_d}$$

$$T_s=\frac{4}{\zeta\omega_n}=\frac{4}{\sigma_d}$$

$$C(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2},\quad (0<\zeta<1)$$

$$s_{1,2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}=-\zeta\omega_n\pm j\omega_d$$

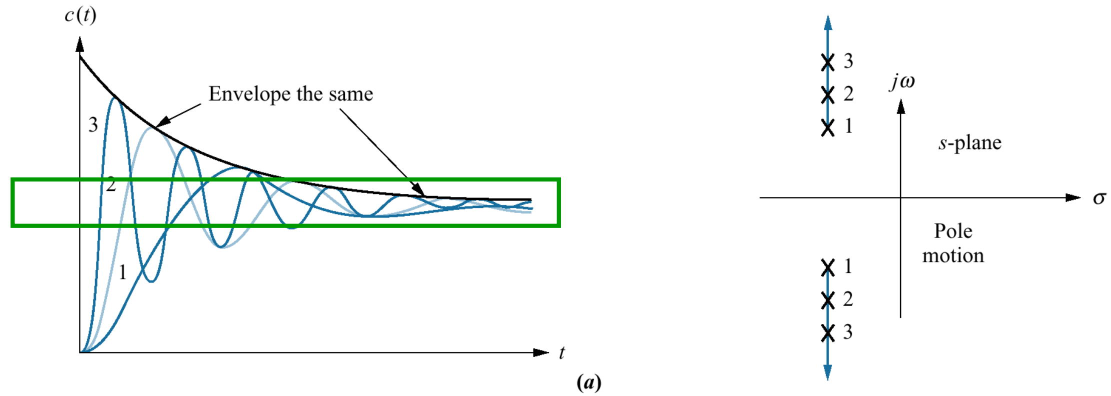
$$\omega_d=\omega_n\sqrt{1-\zeta^2}$$

$$s_{1,2}=-\sigma_d\pm j\omega_d$$

• Step responses of second-order underdamped systems

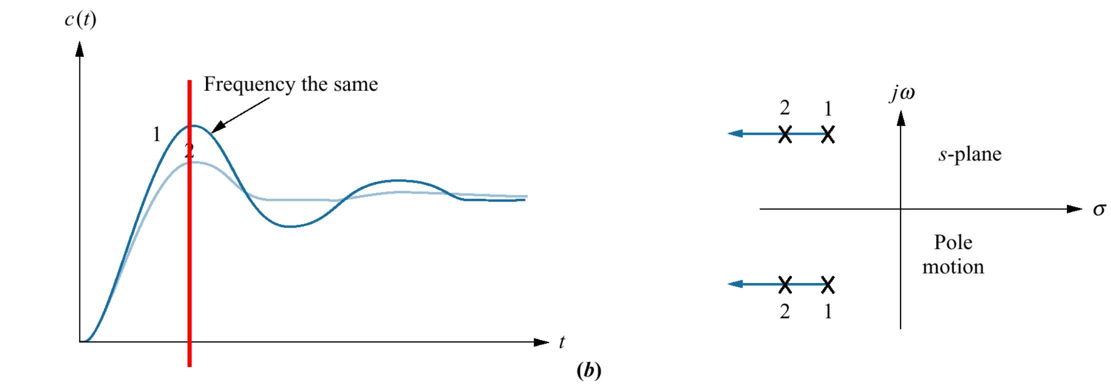
Poles move:
(a) with constant real part.

Ts is the same for all waveforms.



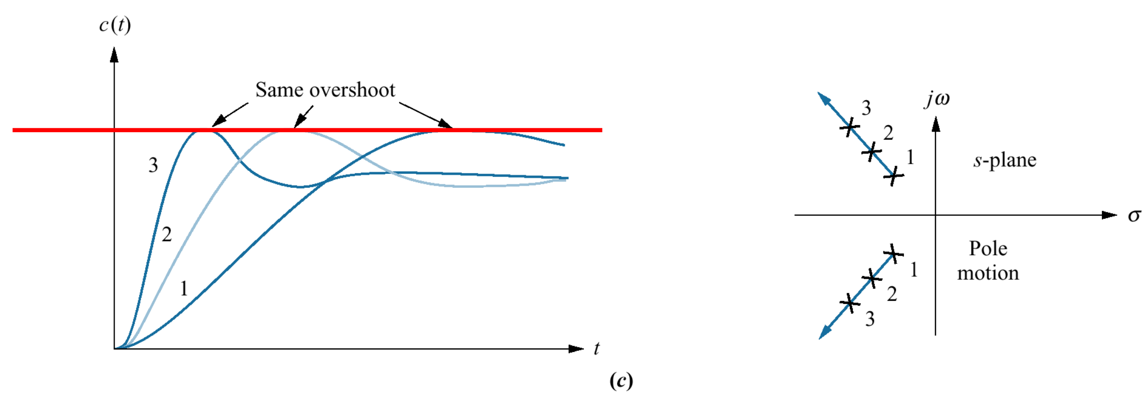
(b) with constant imaginary part.

Tp is the same for all waveforms.



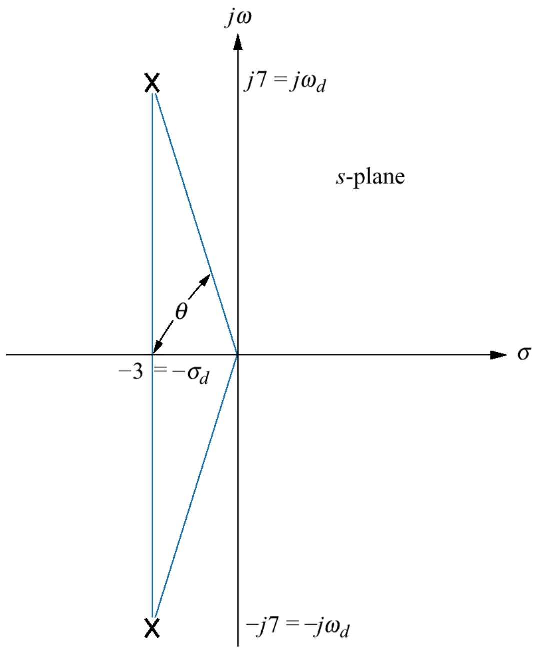
(c) with constant damping ratio.

%OS remains the same.



Example 4.6: Finding T_p, %Os, and T_s from pole location (page 184)

Find ζ, w_n, T_p, %OS, and T_s.



$$\zeta = \cos\theta = \frac{3}{\sqrt{3^2+7^2}} = 0.394$$

$$\omega_n = \sqrt{3^2+7^2} = 7.616$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449\,\text{sec}$$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 26\%$$

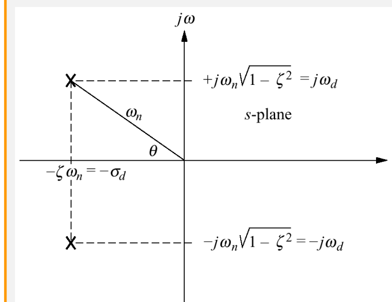
$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333\,\text{sec}$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

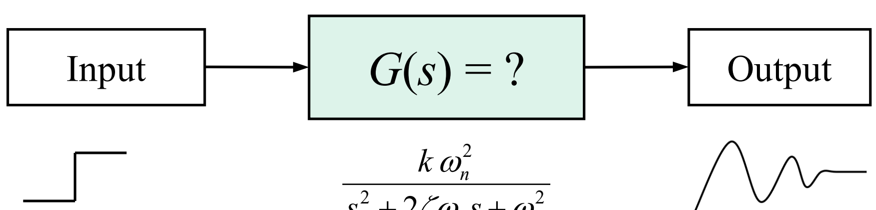
$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

$$T_s = \frac{4}{\zeta\omega_n}$$



- Second-Order Transfer Functions via Testing



$$\frac{k\,\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

4.7 System Response with Additional Poles (page 186)

The Cybermotion SR3 security robot on patrol. The robot navigates by ultrasound and path programs transmitted from a computer, eliminating the need for guide strips on the floor. It has video capabilities as well as temperature, humidity, fire, intrusion, and gas sensors.

$\Downarrow$

More than two poles or with zeros

$\Downarrow$

Approximate to two complex dominant poles

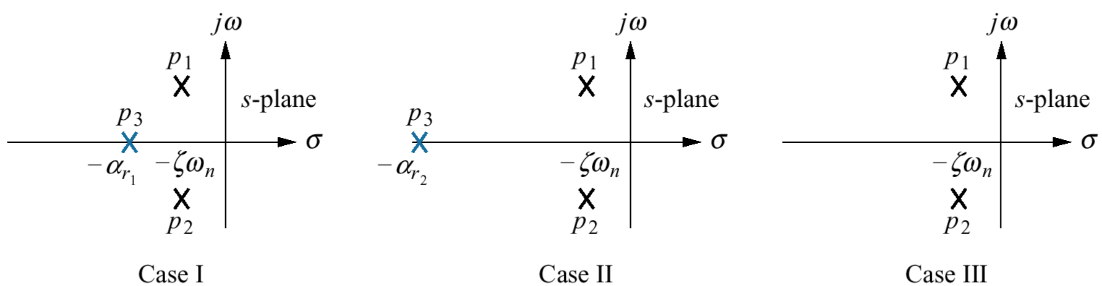
$\Downarrow$

The effect of an *additional pole* on the second-order response?

The effect of *adding a zero* to a two-pole system?

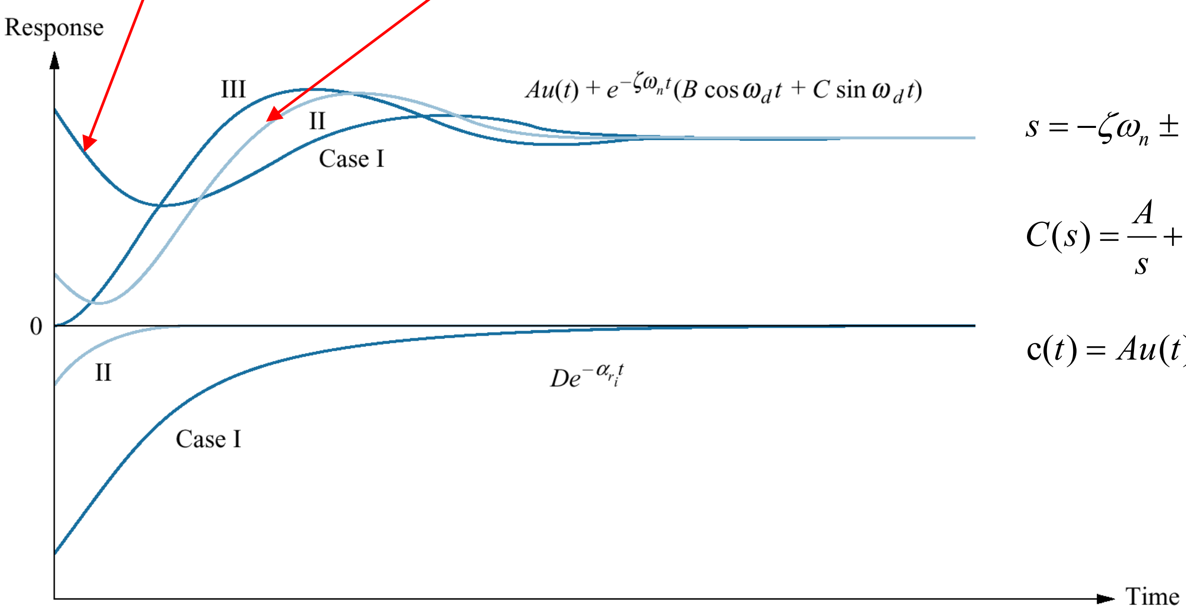
• Component responses of a three-pole system

(a) pole plot
component responses: nondominant pole is near dominant second-order pair (Case I), far from the pair (Case II), and at infinity (Case III)



$$C(s) = \frac{bc}{s(s^2 + as + b)(s + c)}$$
$$= \frac{A}{s} + \frac{Bs + C}{s^2 + as + b} + \frac{D}{s + c}$$

(a)



$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r}$$

$$\mathrm{c}(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\cos\omega_d t) + De^{-\alpha_r t}$$

(b)

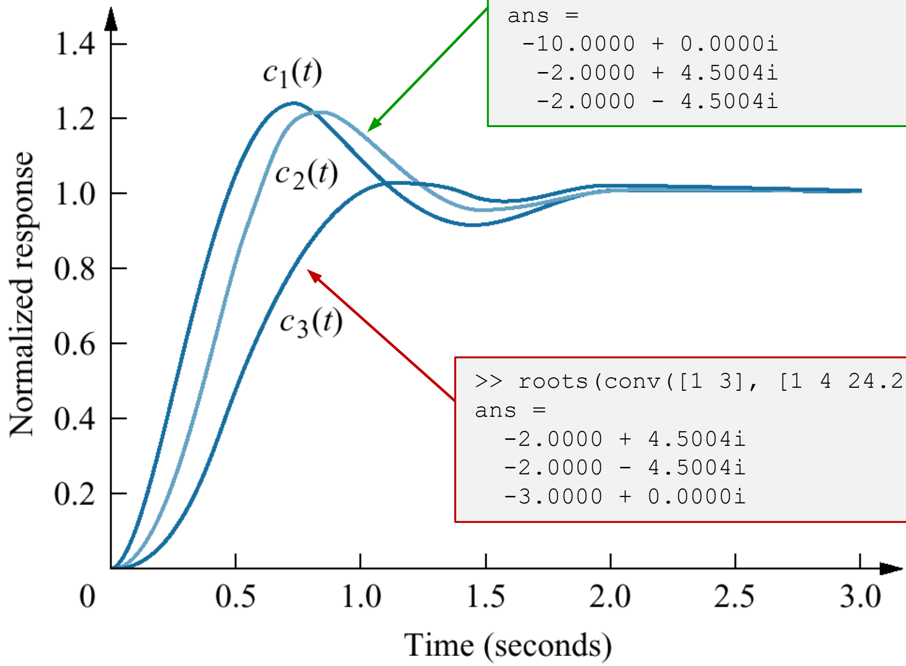
Example 4.8: Comparing responses of three-pole systems (page 189)

```
>> roots([1 4 24.254])
ans =
  -2.0000 + 4.5004i
  -2.0000 - 4.5004i
```

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.254}$$

$$T_2(s) = \frac{245.42}{(s+10)(s^2 + 4s + 24.254)}$$

$$T_3(s) = \frac{73.626}{(s+3)(s^2 + 4s + 24.254)}$$




```
>> roots(conv([1 10], [1 4 24.254]))
ans =
 -10.0000 + 0.0000i
  -2.0000 + 4.5004i
  -2.0000 - 4.5004i
```

```
>> roots(conv([1 3], [1 4 24.254]))
ans =
  -2.0000 + 4.5004i
  -2.0000 - 4.5004i
  -3.0000 + 0.0000i
```

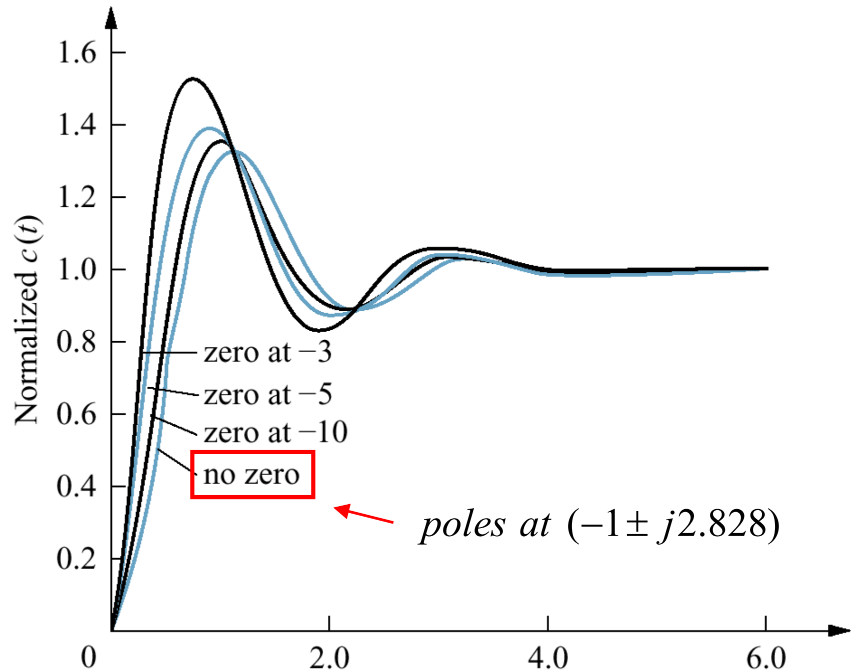
- $c_2(t)$, with its third pole at -10 and farthest from the dominant poles, is better approximation of $c_1(t)$
- $c_3(t)$ with a third pole close to the dominant poles, yield the most error

4.8 System Response with Zeros (page 191)

- Effect of *adding a zero* to a two-pole system:

  The closer the zero is to the dominant poles, the greater its effect on the transient response.

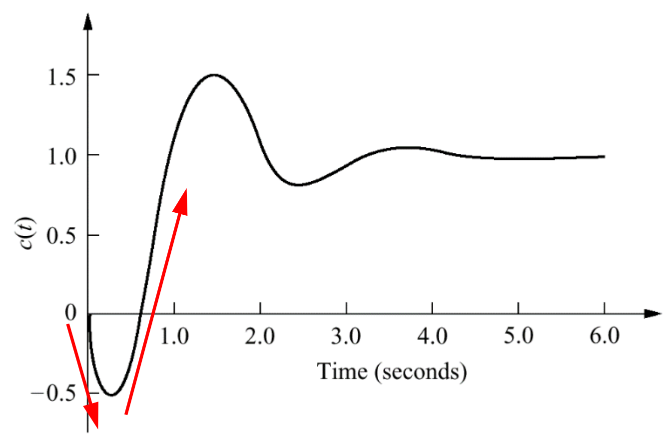
  As the zero moves away from the dominant poles, the response approaches that of the two-pole system



poles at $(-1 \pm j2.828)$

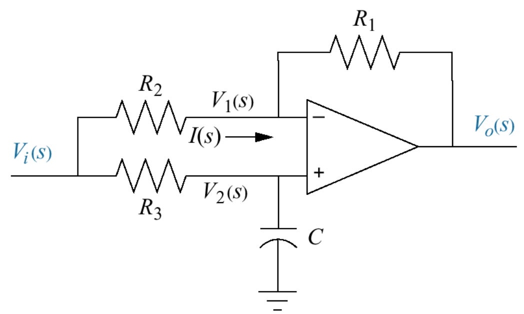
$$G(s) = \frac{bs + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \rightarrow \quad zero: -\frac{\omega_n^2}{b}$$

- Step response of a *nonminimum-phase* system



If a transfer function has poles and/or zeros in the right half s-plane then this system shows *non-minimum phase* behavior.

Example 4.9: Transfer function of a nonminimum-phase system (page 192)



- Nonminimum-phase electrical circuit

$A = \infty,\ R_1 = R_2,\ R_3 C = 1/10$

$\Rightarrow\ V_o(s)/V_i(s) = ?$

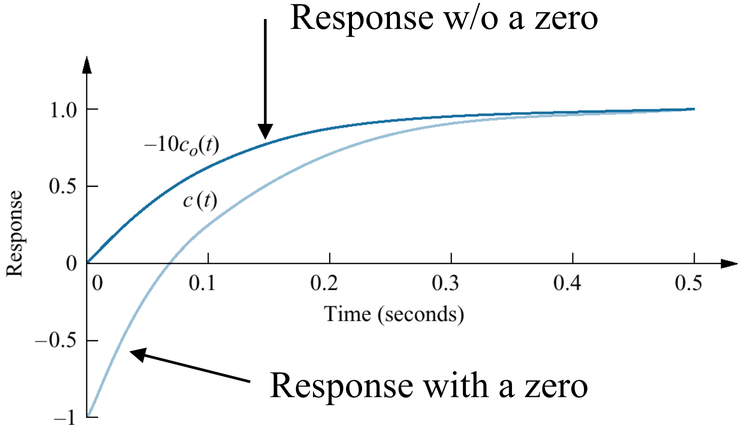
$\Rightarrow\ C(s) = ?$

$$\frac{V_i - V_1}{R_2} = \frac{V_1 - V_o}{R_1} \xrightarrow[(R_1 = R_2)]{} V_o = 2V_1 - V_i$$

$$V_1 = V_2 = V_i \frac{1/sC}{R_3 + 1/sC} = V_i \frac{1}{1 + sR_3C}$$

$$V_o = 2V_i \frac{1}{1 + sR_3C} - V_i = V_i\left(\frac{2}{1 + sR_3C} - 1\right) = V_i\left(\frac{1 - sR_3C}{1 + sR_3C}\right)$$

$$= V_i\left(\frac{1 - s\frac{1}{10}}{1 + s\frac{1}{10}}\right) = V_i\left(\frac{-s + 10}{s + 10}\right)$$



$$c(0+) = \lim_{s \to \infty} sC(s) = -1$$

- For a step input, $\quad (s + a)C(s) = sC(s) + aC(s)$

$$C(s) = -\frac{(s - 10)}{s(s + 10)} = -\frac{1}{(s + 10)} + 10\frac{1}{s(s + 10)}$$

$$= sC_o(s) - 10C_o(s)$$

where

$$C_o(s) = -\frac{1}{s(s + 10)}$$

System with minimum or nonminimum phase behavior

Stable systems without dead time, which are described by the transfer function

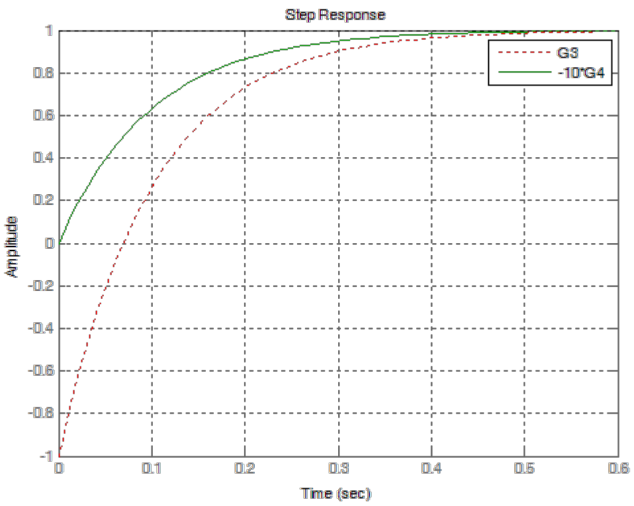
$$G(s) = \frac{N(s)}{D(s)}$$

and which do not have zeros in the right half plane, are called *minimum phase systems*. If a transfer function has poles and/or zeros in the right half s-plane then this system shows *non-minimum phase* behavior.

Example 4.9: Transfer function of a nonminimum-phase system

```
% For page 192 in textbook
G3=tf([-1 10], [1 10]);
G4=tf(-1, [1 10]);
step(G3,'r:', -10*G4,'g')
legend('G3', '-10*G4')
```

$$G_3(s) = \frac{-s+10}{s+10} \qquad G_4(s) = \frac{-10}{s+10}$$



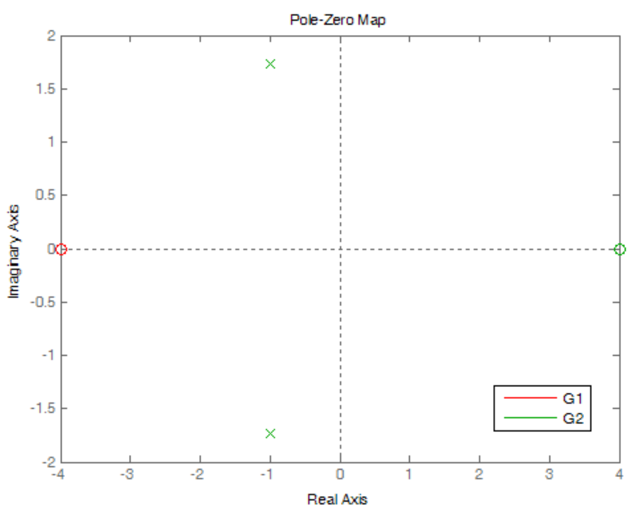
Example:

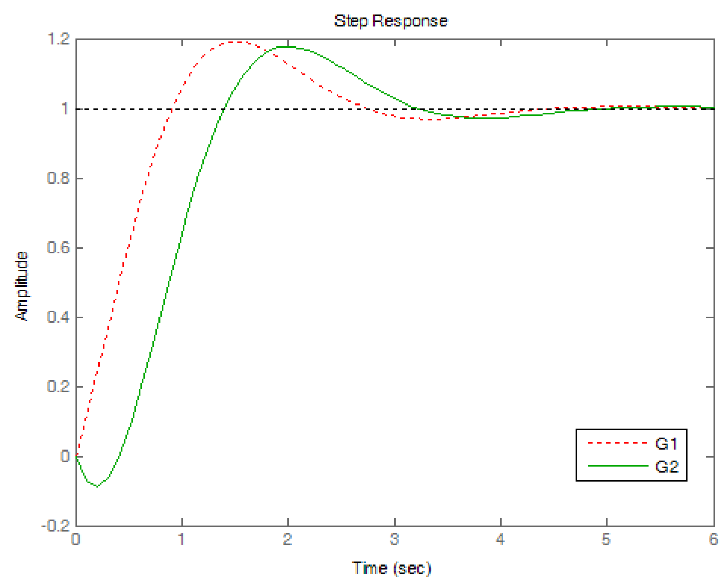
(a) Minimum phase system: The transfer function of a second order system with one zero in left hand plane is chosen as below:

$$G_1(s) = \frac{s+4}{s^2+2s+4}$$

(b) Nonminimum phase system: The transfer function of a second order system with one zero in right hand plane is chosen as below:

$$G_2(s) = \frac{-s+4}{s^2+2s+4}$$





The step response plots

4.10 Laplace Transform Solution of State Equations (page 199)

State equation: $\dot{x} = Ax + Bu$

Output equation: $y = Cx + Du$ $\longrightarrow$ $Y(s) = CX(s) + DU(s)$

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$(sI - A)A(s) = x(0) + BU(s)$$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

$$= \frac{\mathrm{adj}(sI - A)}{\det(sI - A)}\left[x(0) + BU(s)\right]$$

- *Eigenvalues of the system matrix, A.*
- *The system poles*

$$\det(sI - A) = 0$$

Let $x(0) = 0$: $\dfrac{Y(s)}{U(s)} = C\left[\dfrac{\mathrm{adj}(sI - A)}{\det(sI - A)}BU(s)\right] + DU(s)$

$$= \frac{C\,\mathrm{adj}(sI - A)B + D\det(sI - A)}{\det(sI - A)} = C(sI - A)^{-1}B + D$$

Transfer function

4.11 Time Domain Solution of State Equations (page 203)

(1) Homogeneous state equation: $\dot{x}(t) = Ax(t) \Rightarrow x(t) = e^{At}x(0)$

Appendix I. Derivation of the time domain solution of state equations

$e^{At} \Rightarrow$ State-transition matrix with the initial time $t_0 = 0$

$x(t) = e^{A(t-t_0)}x(t_0) \Rightarrow$ In the case of the initial time $t_0 \neq 0$

(1) Nonhomogeneous state equation: $\dot{x}(t) = Ax(t) + Bu(t)$

$$e^{-At}[\dot{x}(t) - Ax(t)] = e^{-At}Bu(t)$$

$$\frac{d}{dt}[e^{-At}x(t)] = e^{-At}Bu(t)$$

$$[e^{-At}x(t)]\Big|_0^t = e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \underbrace{\Phi(t)x(0)}_{\text{zero input response}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}_{\text{convolution integral}} \quad \text{where, } \Phi(t) \equiv e^{At}$$

$$\mathrm{X}(s) = \underline{(s\mathrm{I}-\mathrm{A})^{-1}\mathrm{x}(0)} + (s\mathrm{I}-\mathrm{A})^{-1}\mathrm{BU}(s)$$

$$\mathrm{x}(t) = \underline{\Phi(t)\mathrm{x}(0)} + \int_0^t \Phi(t-\tau)\mathrm{Bu}(\tau)d\tau$$

$$\mathrm{X}(s) = \underline{(s\mathrm{I}-\mathrm{A})^{-1}\mathrm{x}(0)} + (s\mathrm{I}-\mathrm{A})^{-1}\mathrm{BU}(s)$$

response for unforced system

zero input response: $\Phi(t)\mathrm{x}(0)$

- For the unforced system

$$L[\mathrm{x}(t)] = L[\Phi(t)\mathrm{x}(0)] = (s\mathrm{I}-\mathrm{A})^{-1}\mathrm{x}(0)$$

$$L^{-1}\left[(sI-A)^{-1}\right] = L^{-1}\left[\frac{\mathrm{adj}\,(s\mathrm{I}-\mathrm{A})}{\det(s\mathrm{I}-\mathrm{A})}\right] = \Phi(t)$$

$$\begin{aligned}\mathrm{X}(s) &= (s\mathrm{I}-\mathrm{A})^{-1}\mathrm{x}(0) + (s\mathrm{I}-\mathrm{A})^{-1}\mathrm{BU}(s)\\ &= \frac{\mathrm{adj}(s\mathrm{I}-\mathrm{A})}{\det(s\mathrm{I}-\mathrm{A})}\left[\mathrm{x}(0)+\mathrm{BU}(s)\right]\end{aligned}$$

$$\boxed{L^{-1}\left[(sI-A)^{-1}\right] = \Phi(t) = e^{At}}$$

$(sI-A)^{-1}$: Laplace transform of the state transition matrix, $\Phi(t)$

System poles: roots of the denominator in $(sI-A)^{-1}$ $\rightarrow$ $\boxed{\det(sI-A)=0}$

$$e^{At} = \mathrm{I} + \mathrm{A}t + \frac{1}{2}\mathrm{A}^2t^2 + \cdots + \frac{1}{k!}\mathrm{A}^k t^k + \frac{1}{(k+1)!}\mathrm{A}^{k+1}t^{k+1} + \cdots$$

Example 4.13: State-transition matrix via Laplace transform (page 206)

Find the state-transition matrix and then solve for x(t) under a unit step input.

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ -8 & -6\end{bmatrix}x(t)+\begin{bmatrix}0\\ 1\end{bmatrix}u(t),\quad x(0)=\begin{bmatrix}1\\ 0\end{bmatrix}\quad\Rightarrow$$

$$\boxed{\begin{gathered}\Phi(t)=L^{-1}\left[(sI-A)^{-1}\right]=e^{At}\\ \mathrm{x}(t)=\Phi(t)\mathrm{x}(0)+\int_0^t\Phi(t-\tau)\mathrm{B}\mathrm{u}(\tau)d\tau\end{gathered}}$$

$$(sI-A)=\begin{bmatrix}s & -1\\ 8 & (s+6)\end{bmatrix}$$

$$(sI-A)^{-1}=\frac{\begin{bmatrix}s+6 & 1\\ -8 & s\end{bmatrix}}{s^2+6s+8}=\begin{bmatrix}\dfrac{s+6}{s^2+6s+8} & \dfrac{1}{s^2+6s+8}\\ \dfrac{-8}{s^2+6s+8} & \dfrac{s}{s^2+6s+8}\end{bmatrix}$$

$$=\begin{bmatrix}\left(\dfrac{2}{s+2}-\dfrac{1}{s+4}\right) & \left(\dfrac{1/2}{s+2}-\dfrac{1/2}{s+4}\right)\\ \left(\dfrac{-4}{s+2}+\dfrac{4}{s+4}\right) & \left(\dfrac{-1}{s+2}+\dfrac{2}{s+4}\right)\end{bmatrix}$$

$$\boxed{\mathrm{x}(t) = \Phi(t)\mathrm{x}(0) + \int_0^t \Phi(t-\tau)\mathrm{Bu}(\tau)d\tau}$$

(1) $$\Phi(t) = \begin{bmatrix} \left(2e^{-2t} - e^{-4t}\right) & \left(\frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t}\right) \\ \left(-4e^{-2t} + 4e^{-4t}\right) & \left(-e^{-2t} + 2e^{-4t}\right) \end{bmatrix}$$

$$\Rightarrow \quad \Phi(t)x(0) = \begin{bmatrix} \left(2e^{-2t} - e^{-4t}\right) & \left(\frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t}\right) \\ \left(-4e^{-2t} + 4e^{-4t}\right) & \left(-e^{-2t} + 2e^{-4t}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ -4e^{-2t} + 4e^{-4t} \end{bmatrix}$$

(2) $$\Phi(t-\tau)B = \begin{bmatrix} \frac{1}{2}e^{-2(t-\tau)} - \frac{1}{2}e^{-4(t-\tau)} \\ -e^{-2(t-\tau)} + 2e^{-4(t-\tau)} \end{bmatrix} \qquad \Rightarrow$$

$$\int_0^t \Phi(t-\tau)\mathrm{Bu}(\tau)d\tau = \begin{bmatrix} \frac{1}{2}e^{-2t}\int_0^t e^{2\tau}d\tau - \frac{1}{2}e^{-4t}\int_0^t e^{4\tau}d\tau \\ -2e^{-2t}\int_0^t e^{2\tau}d\tau + 2e^{-4t}\int_0^t e^{4\tau}d\tau \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{8} - \frac{1}{4}e^{-2t} + \frac{1}{8}e^{-4t} \\ \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix}$$

$$\mathrm{x}(t) = \Phi(t)\mathrm{x}(0) + \int_0^t \Phi(t-\tau)\mathrm{Bu}(\tau)d\tau$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ -4e^{-2t} + 4e^{-4t} \end{bmatrix} + \begin{bmatrix} \frac{1}{8} - \frac{1}{4}e^{-2t} + \frac{1}{8}e^{-4t} \\ \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix} = \begin{bmatrix} \frac{1}{8} + \frac{7}{4}e^{-2t} - \frac{7}{8}e^{-4t} \\ -\frac{7}{2}e^{-2t} + \frac{7}{2}e^{-4t} \end{bmatrix}$$

[End of example 4.12 & 13]

Example 4.12: State-transition matrix (page 204)

Find the state-transition matrix

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$$

$$\boxed{\begin{aligned} \Phi(0) &= I \\ \dot{\Phi}(0) &= A \end{aligned}}$$

(1) Find eigenvalues: $(sI - A) = \begin{bmatrix} s & -1 \\ 8 & (s+6) \end{bmatrix} = 0 \Rightarrow s = -2 \text{ and } -4$

$$\Phi(t) = \begin{bmatrix} \left(K_1 e^{-2t} + K_2 e^{-4t}\right) & \left(K_3 e^{-2t} + K_4 e^{-4t}\right) \\ \left(K_5 e^{-2t} + K_6 e^{-4t}\right) & \left(K_7 e^{-2t} + K_8 e^{-4t}\right) \end{bmatrix}$$

(2) From $\Phi(0) = I:$ $\quad K_1 + K_2 = 1, \; K_3 + K_4 = 0, \; K_5 + K_6 = 0, \; K_7 + K_8 = 1$

(3) From $\dot{\Phi}(0) = A:$ $\quad -2K_1 - 4K_2 = 0, \; -2K_3 - 4K_4 = 1,$

$$-2K_5 - 4K_6 = -8, \; -2K_7 - 4K_8 = -6$$

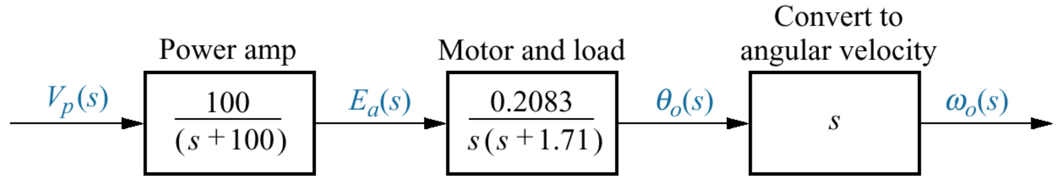
(4) We can find $K_i, \; i = 1, \cdots, 8.$

• Case Studies:

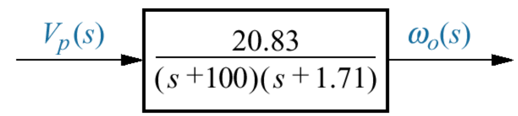
(1) Antenna azimuth position control system for angular velocity:

a. forward path

b. equivalent forward path



(a)



(b)

(2) Unmanned Free-Swimming Submersible (UFSS) vehicle:

Pitch control loop for the UFSS vehicle