Chapter 8. Root Locus Techniques

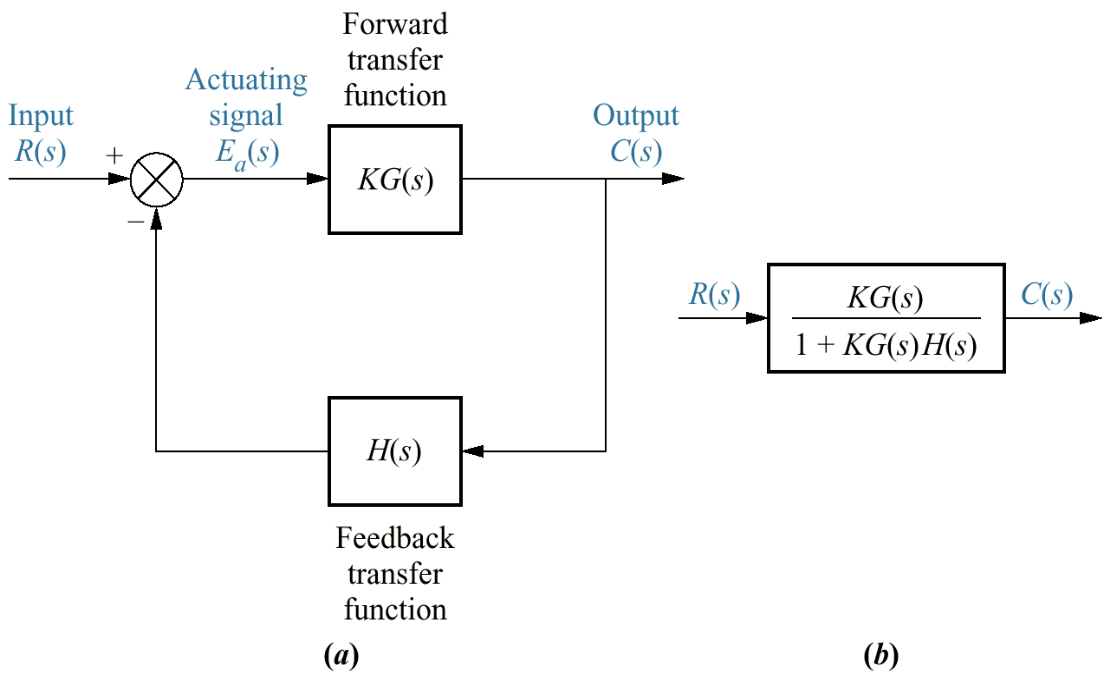
```
s=tf('s');
sys=1/(s+1)
controlSystemDesigner(sys)


s=tf('s');
sys=1/(s+1)^3
pidtool(sys)
```

Objectives

- The *definition* of a root locus
- How to *sketch* a root locus
- How to *refine* your sketch of a root locus
- How to use the root locus to *find the poles* of a closed-loop system
- How to use the root locus to *design a parameter value* to meet a transient response specification

8.1 Introduction

- Root locus (Evans, 1948:1950): a powerful method of analysis and design for stability and transient response



$$G(s)=\frac{(s+1)}{s(s+2)},\quad H(s)=\frac{(s+3)}{(s+4)} \Rightarrow T(s)=\frac{K(s+1)(s+4)}{s^3+(6+K)s^2+(8+4K)s+3K}$$

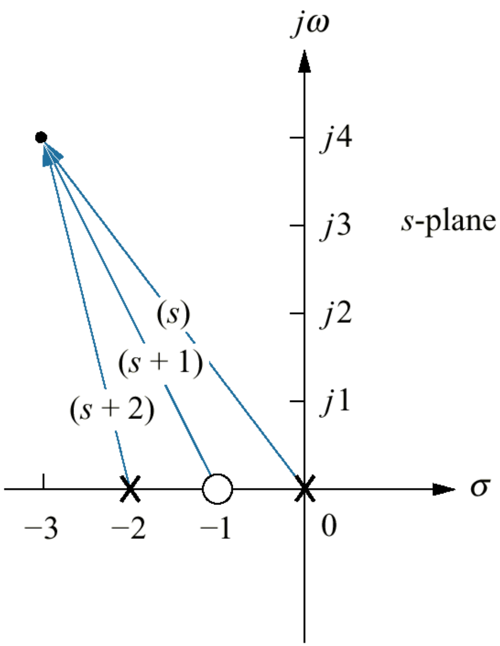
- The root locus will be used to trace the poles of $\boxed{T(s) \text{ as } K \text{ varies.}}$

Example 8.1: Evaluation of a complex function via vectors

Find $F(s)$ at the point $s = -3 + j4$.

$$F(s) = \frac{(s+1)}{s(s+2)}$$

```
% ch8_1.m
s = -3+4j;
G=(s+1)/(s^2+2*s);
Theta=(180/pi)*angle(G)
M=abs(G)
```



$$F(s) = \frac{[s-(-1)]}{[s-0][s-(-2)]}$$

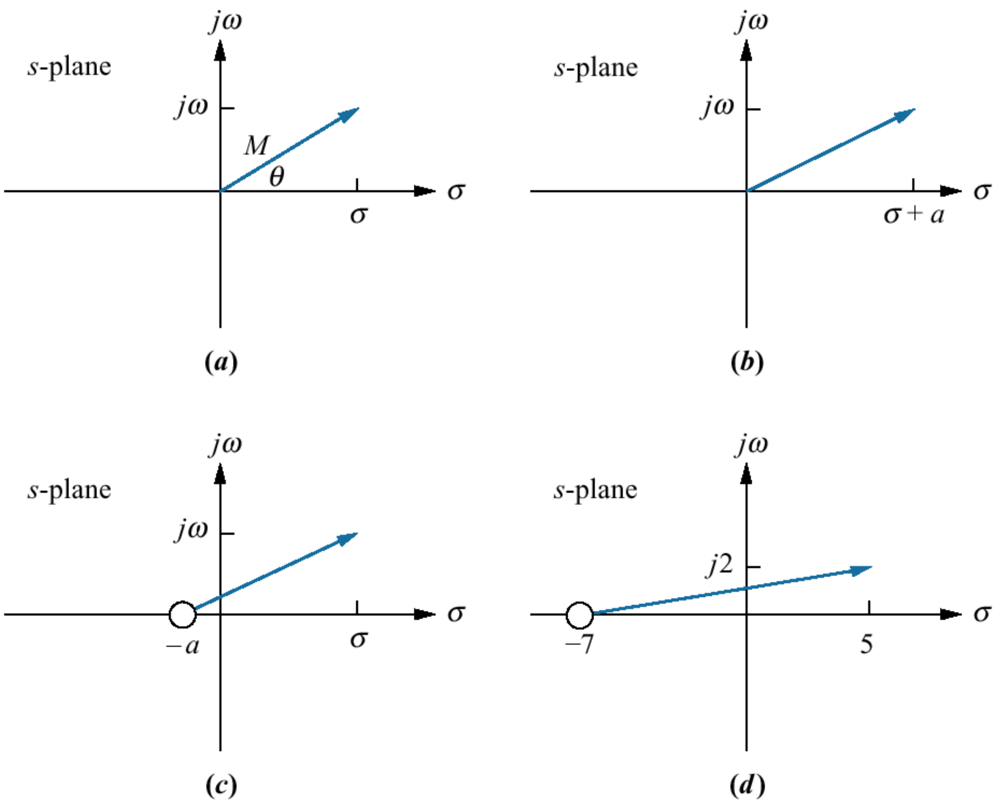
$$= \frac{\sqrt{20}\angle 116.6^\circ}{(5\angle 126.9^\circ)(\sqrt{17}\angle 104.0^\circ)}$$

$$= \frac{\sqrt{20}}{5\sqrt{17}}\angle 116.6^\circ - 126.9^\circ - 104.0^\circ$$

$$= 0.217\angle -114.3^\circ = \boxed{M\angle\theta}$$

Use magnitude and phase angle

• Vector Representation of Complex Numbers



$$F(s)=\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

$$M=\frac{\prod_{i=1}^{m}|s-(-z_i)|}{\prod_{j=1}^{n}|s-(-p_i)|}=\frac{\Pi(\text{zero lengths})}{\Pi(\text{pole lengths})}$$

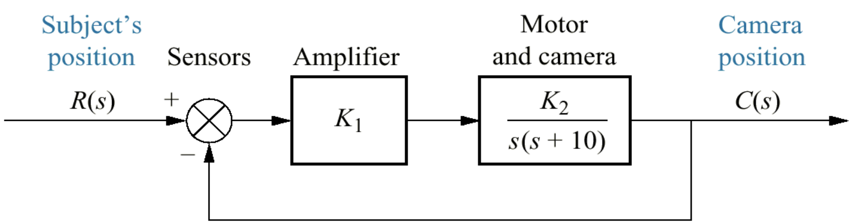
$$\theta=\boxed{\sum \textit{Zero angles}-\sum \textit{Pole angles}}$$

$$=\sum_{i=1}^{m}\angle(s-(-Z_i))-\sum_{j=1}^{n}\angle(s-(-P_j))$$

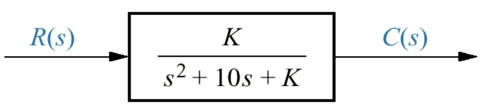
(a) s = $\sigma + j\omega$;
(b) $(s + a)$;
(c) Alternate representation of $(s + a)$;
(d) $(s + 7)|_{s\to 5+j2}$

8.2 Defining the Root Locus

CameraMan: Courtesy of ParkerVision



(b)

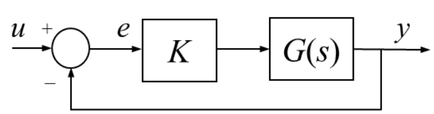


where $K = K_1K_2$

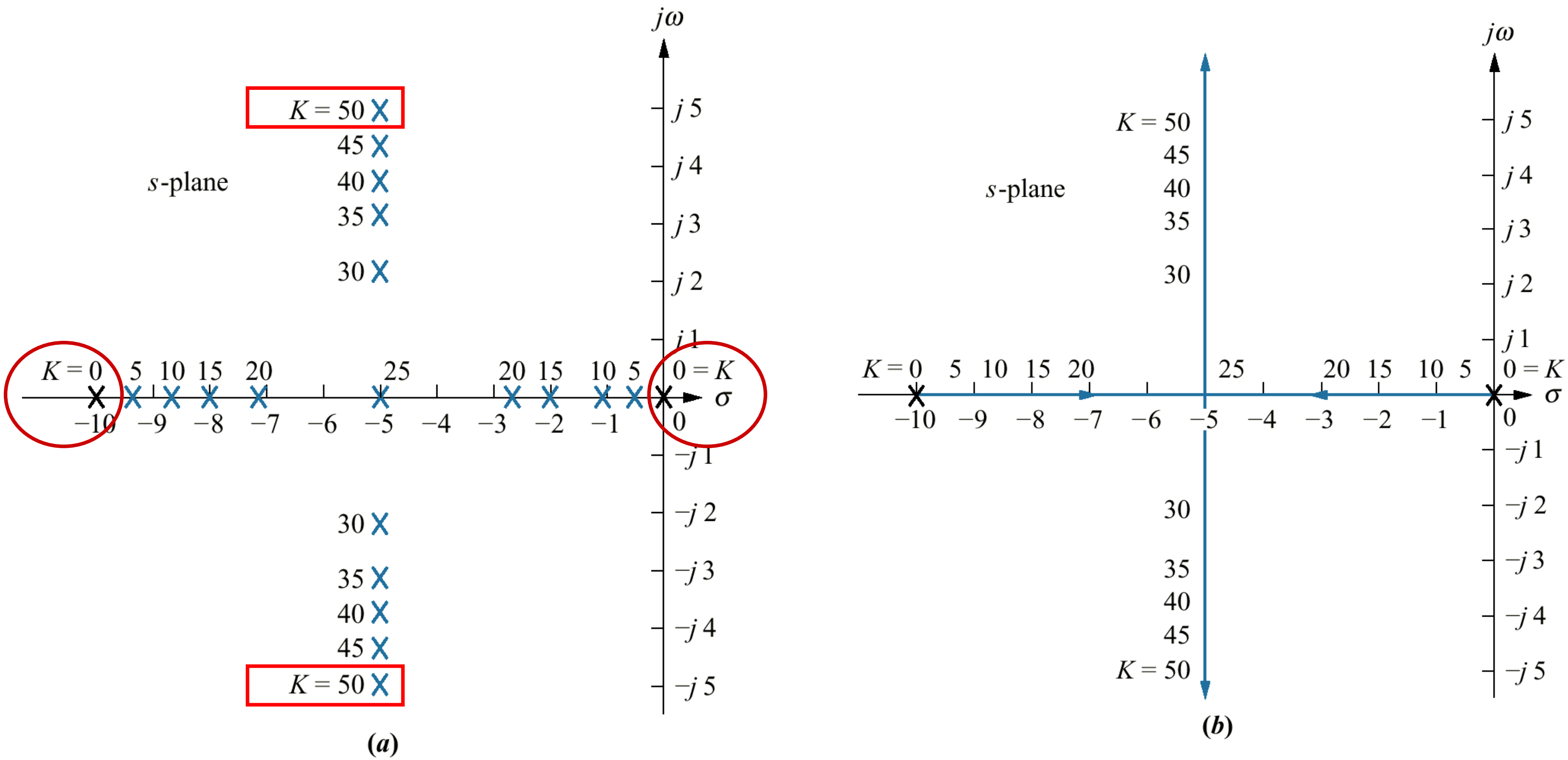
(c)

- Pole location as a function of gain for the system

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$




```
% ch8_2.m
clc, clear all
numg=[1];
deng=[1 10 0];
  % deng=poly([0 -10]);
G=tf(numg, deng);
rlocus(G)
```

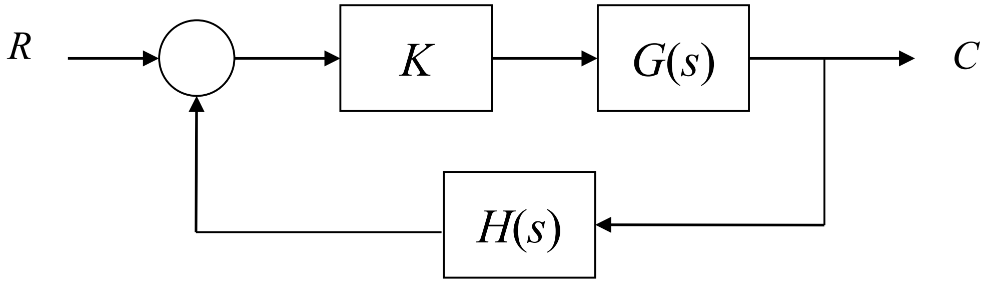


Pole plot

Root locus

$$overdamped\,(0 \le K < 25),\; critically\ damped\,(K = 25),\; and\ underdamped\,(K > 25)$$

8.3 Properties of the Root Locus



$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

$\boxed{\text{Closed-loop pole:}}$ $1 + KG(s)H(s) = 0$

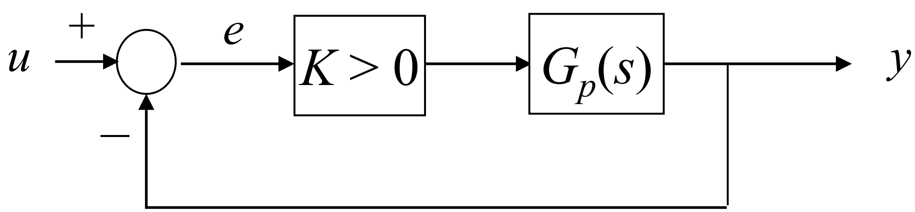
$$KG(s)H(s) = -1$$

$$= 1\angle(2k+1)180^\circ, \quad k = 0, \ \pm 1, \ \pm 2$$

$$|KG(s)H(s)| = 1 \quad \text{and}$$

$$\angle G(s)H(s) = (2k+1)180^\circ, \ K > 0$$

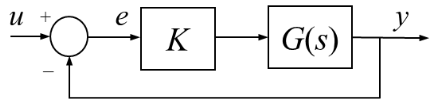
Root Locus



- Plot the poles of the closed–loop system $T(s)$, as a function of K.

$$T(s)=\frac{KG_P(s)}{1+KG_P(s)}$$

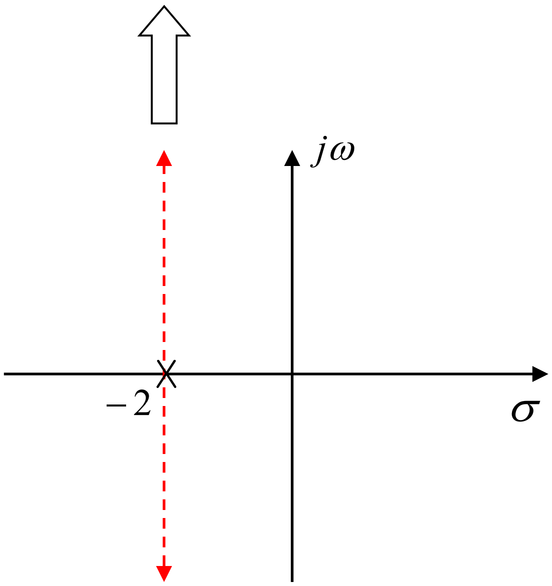
Poles: $\boxed{1+KG_P(s)=0}$



Example:

$$G_P(s) = \frac{1}{(s+2)^2}$$

System is stable for all K



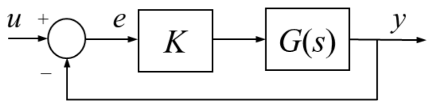
$$1 + KG_P(s) = 0, \;\; 1 + \frac{K}{(s+2)^2} = 0$$

$$s^2 + 4s + (K+4) = 0$$

$$s = -2 \pm \sqrt{4-(K+4)} = \boxed{-2 \pm \sqrt{-K}}$$

$$s = -2 \pm j\sqrt{K}$$

```
clc, clear all
numg = [1];
deng = poly([-2 -2]);
G=tf(numg, deng)
rlocus(G)
```



Example: $G_P(s) = \dfrac{s-1}{(s+1)^2}$

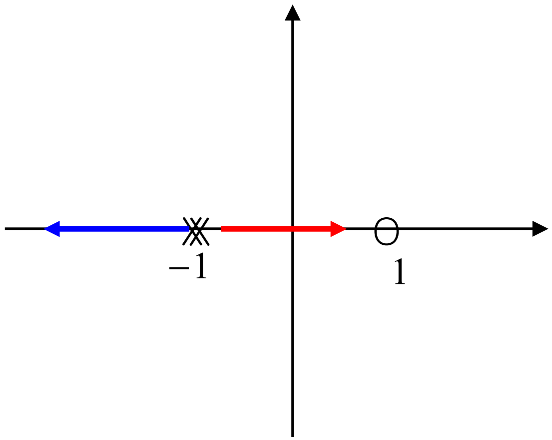
$$1 + KG_P(s) = 0, \quad (s+1)^2 + K(s-1) = 0$$

$$s^2 + (2+K)s + (1-K) = 0$$

$$s = \frac{-(2+K) \pm \sqrt{(2+K)^2 - 4(1-K)}}{2}$$

$$= \frac{-(2+K) \pm \sqrt{K^2 + 4K + 4 - 4 + 4K}}{2},$$

$$s = \boxed{\frac{-2-K \pm \sqrt{K^2 + 8K}}{2}}$$




```
clc, clear all
numg = poly([1]);
deng = poly([-1 -1]);
G=tf(numg, deng)
rlocus(G)
```

Example (continued):

$$s=\frac{-2-K\pm\sqrt{K^2+8K}}{2}=\frac{-2+(-K\pm\sqrt{K^2+8K})}{2}$$

$$+:-K+\sqrt{K^2+8K}=\frac{(-K+\sqrt{\ })(K+\sqrt{\ })}{K+\sqrt{\ }}=\frac{-K^2+K^2+8K}{K+\sqrt{K^2+8K}}=\frac{8K}{K\left(1+\sqrt{1+\frac{8}{K}}\right)}$$

$$\to\lim_{K\to\infty}\frac{8K}{K\left(1+\sqrt{1+\frac{8}{K}}\right)}=\lim_{K\to\infty}\frac{8K}{K\left(1+\sqrt{1+0}\right)}=\frac{8}{2}\to 4 \quad\Rightarrow K\to\infty\ ,\ s\to\frac{-2+4}{2}=1$$

$$-:-K-\sqrt{K^2+8K}=\frac{(-K-\sqrt{\ })(-K+\sqrt{\ })}{-K+\sqrt{\ }}=\frac{(K^2-K^2-8K)}{-K+\sqrt{K^2+8K}}=\frac{-8K}{K\left(-1+\sqrt{1+\frac{8}{K}}\right)}$$

$$\to\lim_{K\to\infty}\frac{-8K}{K\left(-1+\sqrt{1+\frac{8}{K}}\right)}=\lim_{K\to\infty}\frac{-8K}{K\left(-1+\sqrt{1+0}\right)}=\frac{-8}{0}\to-\infty$$

$$\Rightarrow K\to\infty,\ s\to\frac{-2-\infty}{2}=-\infty$$

$$s^2+(2+\overline{K})s+(1-\overline{K})\Big|_{s=0}=0\Rightarrow\boxed{\overline{K}=1}$$

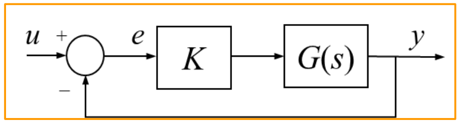
$$\boxed{\text{Stability: } K<1}$$

If you have higher order polynomials,
it can be impossible to plot the roots in this way.

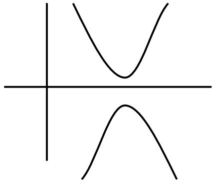
$\Rightarrow$ You need systematic plotting methods.

Let $\quad G_P(s) = A\dfrac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}, \quad m \le n$

Rules for Root Locus



Rule 1. Symmetry about the real axis

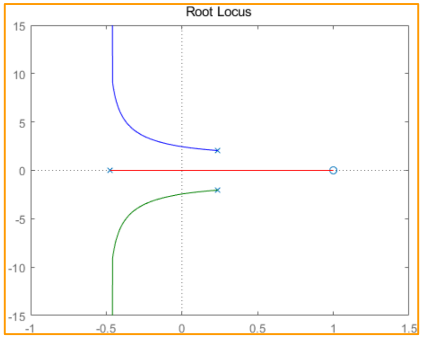


Rule 2. Each branch starts at a pole of $G_P(s)$ and terminates at a zero of $G_P(s)$, including zeros at ∞.

$\xrightarrow{\text{Explanation}}$ Let $G_P(s) = \dfrac{s-1}{s^3+4s+2}$

```
numg = poly([1]);
deng = [1 0 4 2];
G=tf(numg, deng)
rlocus(G)
```

```
G =
          s - 1
    -------------
    s^3 + 4 s + 2
```



$G_P(s)$ has one zero and three poles.

At ∞, namely as $s \to \infty$, $G_P(s)$ behaves like $\dfrac{1}{s^2}$,

i.e. it has two zeros at ∞.

$\Rightarrow$ How do the *branches* approach to ∞ ?

Rule 3. *n-m branches* approach ∞ like asymptotes

with angles $\Psi_K = \dfrac{180 + 360K}{n-m}$, $K = 0, 1, \cdots, n-m-1.$

The asymptotes intersect at the real axis at

$$\sigma = \boxed{\dfrac{\sum Poles - \sum Zeros}{n-m}}$$

```
numg = [1];
deng = poly([-2 -2]);
G=tf(numg, deng)
rlocus(G)
```

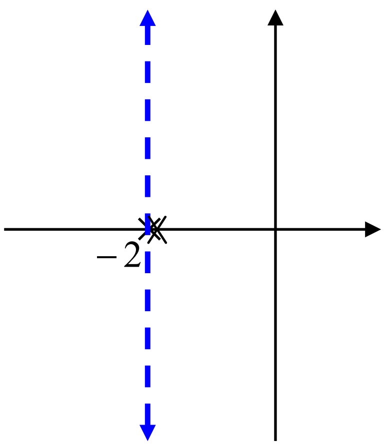
Example:

$$G_P(s) = \frac{1}{(s+2)^2}, \; n = 2, \; m = 0$$

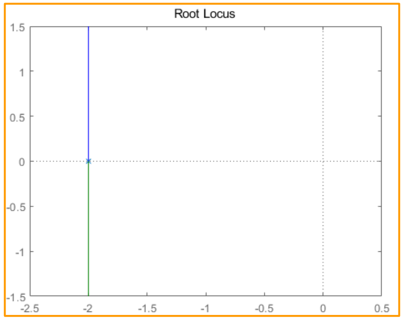
$$\sigma = (-4)/2 = -2,$$

$$\Psi_0 = 180/2 = 90,$$

$$\Psi_1 = (180+360)/2 = -90$$

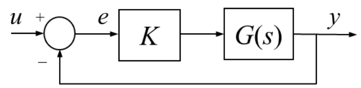



```
G =
 
         1
   -------------
   s^2 + 4 s + 4
```

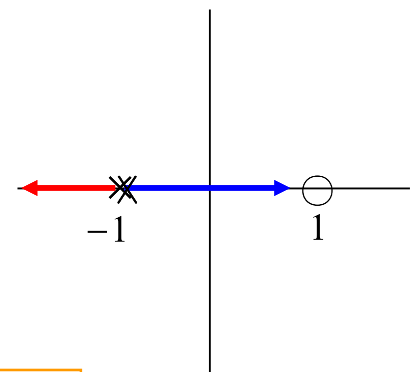


Rule 4. If a branch is on the real axis, then the number of real poles + zeros of $G_P(s)$ to the right of it must be odd.

Example:

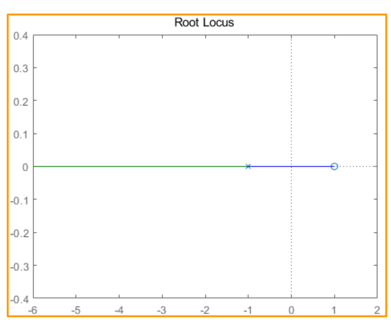


$$G_P(s) = \frac{s-1}{(s+1)(s+1)}$$



$$n - m = 1$$

$$\Psi_0 = 180 \rightarrow \text{you have a branch at } \boxed{-\infty}$$

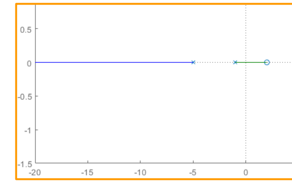



```
G =
 
        s - 1
  -------------
  s^2 + 2 s + 1
```

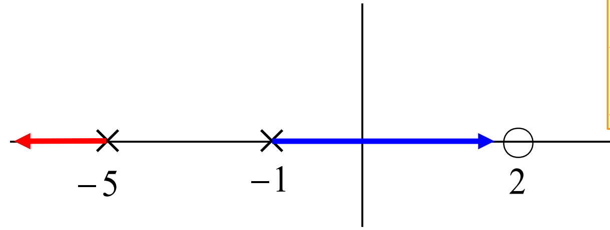
```
numg = poly([1]);
deng = poly([-1 -1]);
G=tf(numg, deng)
rlocus(G)
```

```
numg=[1 -2];
deng = [1 6 5];
G=tf(numg, deng)
rlocus(G)
```

```
numg = poly([2]);
deng = poly([-1 -5]);
G=tf(numg, deng)
rlocus(G)
```



Example: $G_P(s) = \dfrac{s-2}{(s+1)(s+5)}$



$n - m = 1$

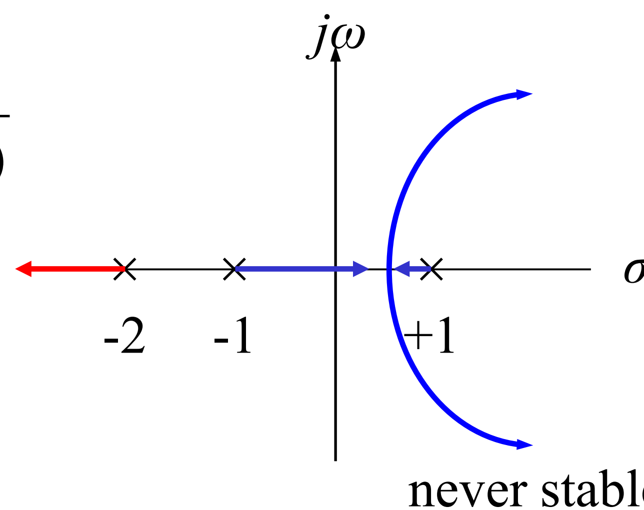
$\Psi_0 = 180 \quad \text{Stability} : K < \bar{K}$

$$1 + \bar{K}\left.\frac{s-2}{(s+1)(s+5)}\right|_{s=0} = 0, \quad \bar{K} = 2.5, \qquad \text{Stability: } \boxed{K < 2.5}$$

Example: $G_p(s) = \dfrac{1}{(s+2)(s-1)(s+1)}$

$1 + KG_p(s) = 0$

$s^3 + 2s^2 - s - 2 + K = 0$



Cross $j\omega$ axis: $\Rightarrow \boxed{1+KG_p(s)\Big|_{s=j\omega}=0}$

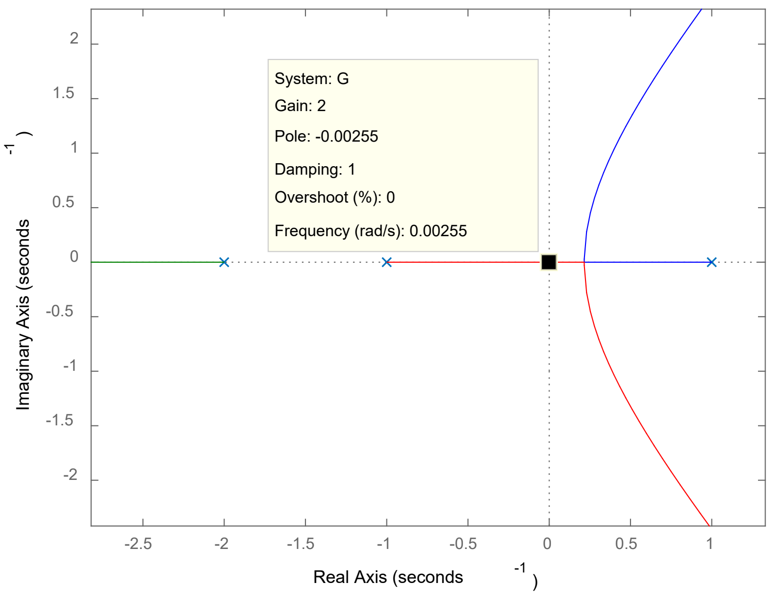
$$-j\omega^3-2\omega^2-j\omega-2+K=0$$

$$(K-2\omega^2-2)-j\omega(1+\omega^2)=0$$

$$\neq 0$$

$\boxed{K=2}$

$\omega^2+1=0$ or $\omega=0$ is the only solution

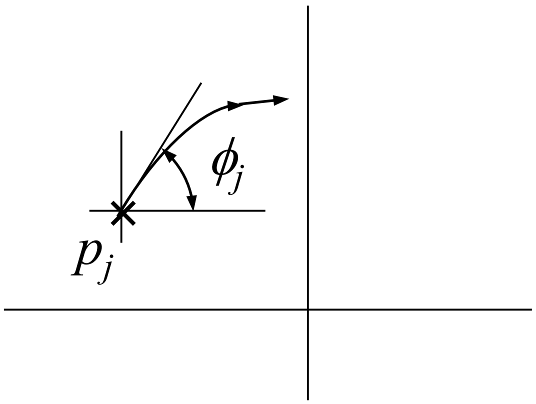



```
numg = poly([]);
deng = poly([-2 1 -1]);
G=tf(numg, deng)
rlocus(G)
```

Rule 5. Angle rule

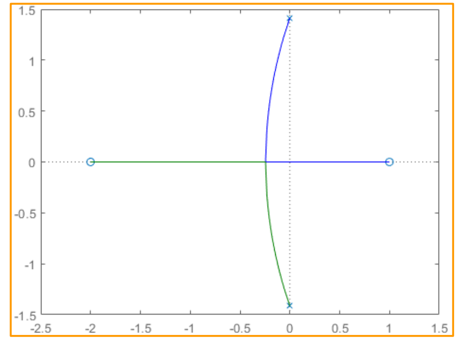
Let $p_1, \ldots, p_n$ be the poles of G_p
Let $z_1, \ldots, z_m$ be the zeros of G_p

Simple pole: Let p_i be a simple pole.
Let ϕ_j=angle at which the branch starts from p_j



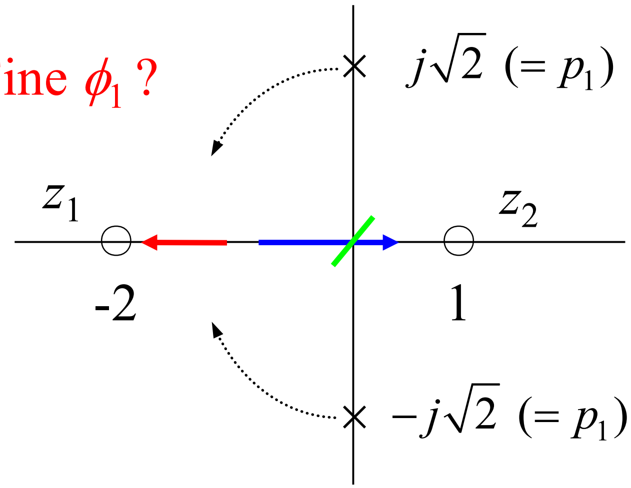
$$\phi_j = \sum_{k=1}^{m} \angle(p_j - z_k) - \sum_{\substack{i=1 \\ i \neq j}}^{n} \angle(p_j - p_i) + r \cdot 180 \quad (r : \text{odd})$$

Example: $G_p(s) = \dfrac{(s-1)(s+2)}{s^2+2}$

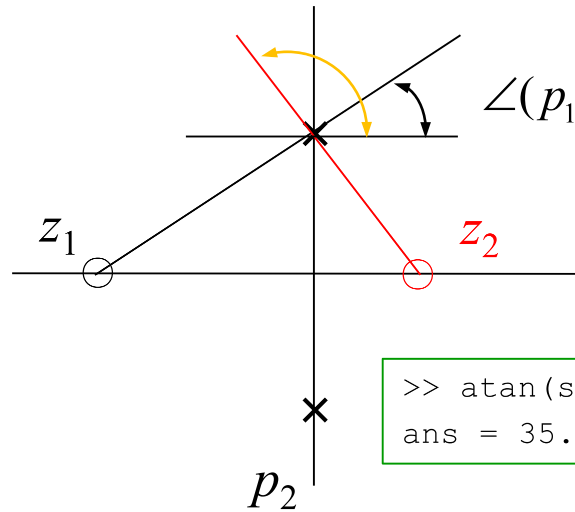


```
numg = poly([1 -2]);
deng = [1 0 2];
G=tf(numg, deng)
rlocus(G)
```

Fine ϕ_1 ?



$\angle(p_1 - z_2) \approx 125^\circ$




```
>> atan(sqrt(2)/2)*180/pi
ans = 35.2644
```

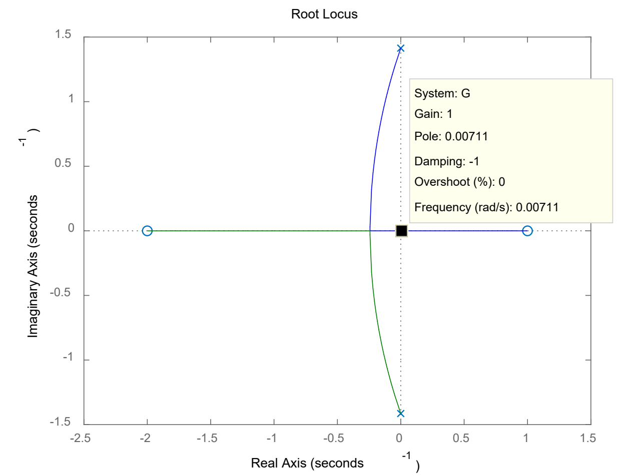
$$\phi_1 = \angle(p_1 - z_1) + \angle(p_1 - z_2) - \angle(p_1 - p_2) + (1)\cdot 180$$

$$\phi_1 \approx \boxed{35^\circ + 125^\circ - 90^\circ + 180^\circ = 250^\circ = -110^\circ}$$

- Crossing of ‘0’: $$1+K\frac{(s-1)(s+2)}{s^2+2}=0$$

$$s=0,\ 1+K\frac{-2}{2}=0,\ K=1$$

stability: $\boxed{K<1}$



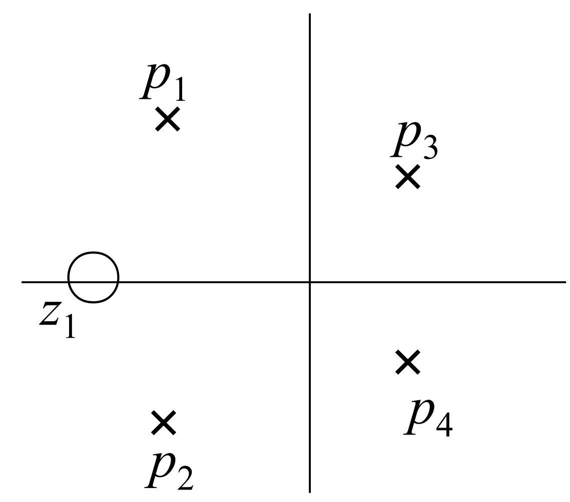
- Explanation of rule 5: For every s on a branch,

$$\boxed{T(s)=\frac{KG_P(s)}{1+KG_P(s)}}$$

$$1+K\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}=0$$

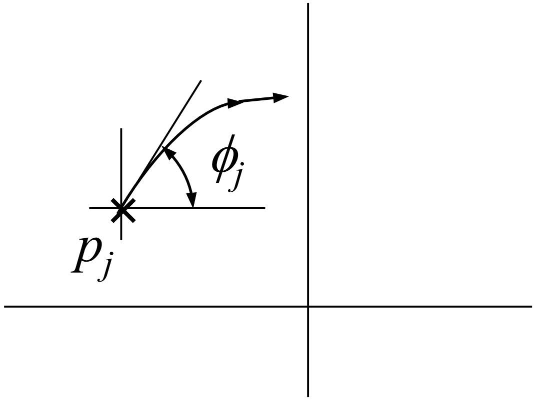
$$K\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}=-1$$

$$\sum_{k=1}^{m}\angle(s-z_k)-\sum_{i=1}^{n}\angle(s-p_i)=\boxed{r\cdot 180\ \ (r:\text{odd})}$$



- Explanation of rule 5(continue):

now: $\theta_j = \lim_{s \to p_j} \angle(s - p_j)$



Then, $\theta_j = \lim_{s \to p_j} \angle(s - p_j)$

$$= \lim_{s \to p_j} \sum_{k=1}^{m} \angle(s - z_k) - \lim_{s \to p_j} \sum_{\substack{i=1 \\ i \neq j}}^{n} \angle(s - p_i) + r \cdot 180 \quad (r : \text{odd})$$

$$= \sum_{k=1}^{m} \angle(p_j - z_k) - \boxed{\sum_{\substack{i=1 \\ i \neq j}}^{n} \angle(p_j - p_i) + r \cdot 180}$$

$$\boxed{\sum_{k=1}^{m} \angle(s - z_k) - \sum_{i=1}^{n} \angle(s - p_i) = r \cdot 180 \ \ (r : \text{odd})}$$

$$\boxed{\sum_{k=1}^{m}\angle(s-z_k)-\sum_{i=1}^{n}\angle(s-p_i)=r\cdot 180 \ \ (r:\text{odd})}$$

- Angle rule for double pole

Let $p_1 = p_2$, for all s on a branch

$$\sum_{k=1}^{m}\angle(s-z_k)-\sum_{i=1}^{n}\angle(s-p_i)+r\cdot 180$$

$$\angle(s-p_1)+\angle(s-p_2)=\sum_{k=1}^{m}\angle(s-z_k)-\sum_{i=3}^{n}\angle(s-p_i)+r\cdot 180$$

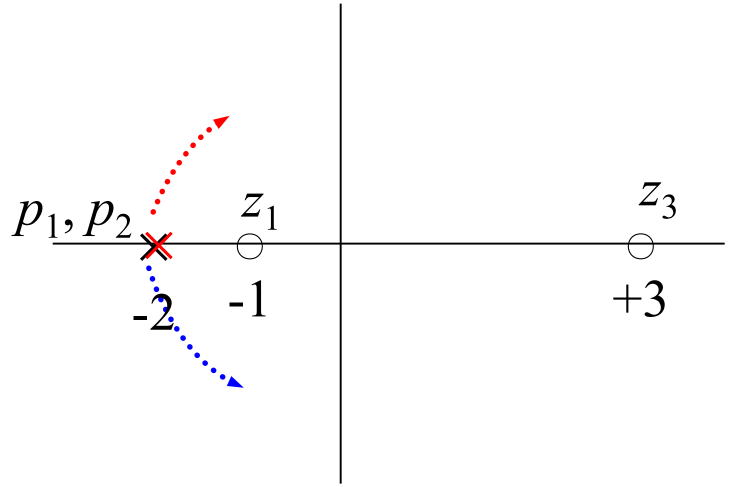
but $p_1 = p_2 \ \rightarrow \ \angle(s-p_1)=\angle(s-p_2)$

$$\Rightarrow 2\angle(s-p_1)=\sum_{k=1}^{m}\angle(s-z_k)-\sum_{i=3}^{n}\angle(s-p_i)+r\cdot 180$$

$$\Rightarrow \varphi_1,\varphi_2=\left[\sum_{k=1}^{m}\angle(\underset{\substack{\downarrow\\ p_j}}{s}-z_k)-\sum_{i=3}^{n}\angle(\underset{\substack{\downarrow\\ p_j}}{s}-p_i)+r\cdot 180\right]\Big/2$$

Example: $G_p(s) = \dfrac{(s+1)(s-3)}{(s+2)^2}$

$$\begin{cases} z_1 = -1, \quad z_2 = 3 \\ p_1 = p_2 = -2 \end{cases}$$



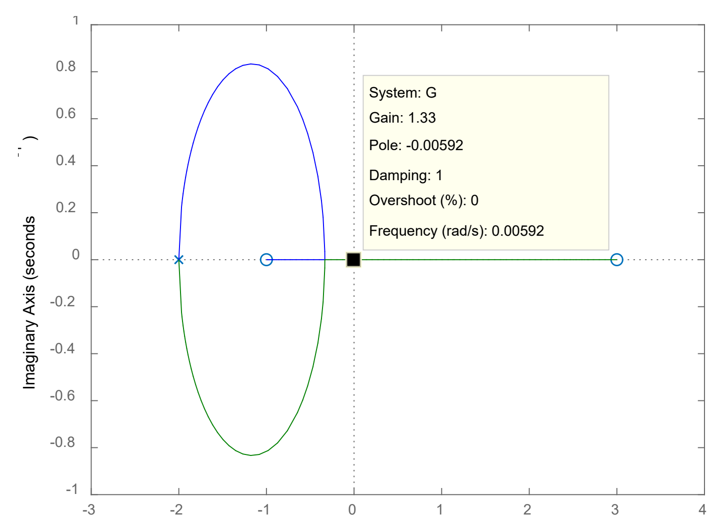
$$\angle(p_1 - z_1) \;=\; \angle(p_1 - z_2) \;= 180$$

$$\rightarrow \varphi_1, \varphi_2 \;=\; \left[+180 + 180 - 0 + 180r\right]/2 \;=\; 90r$$

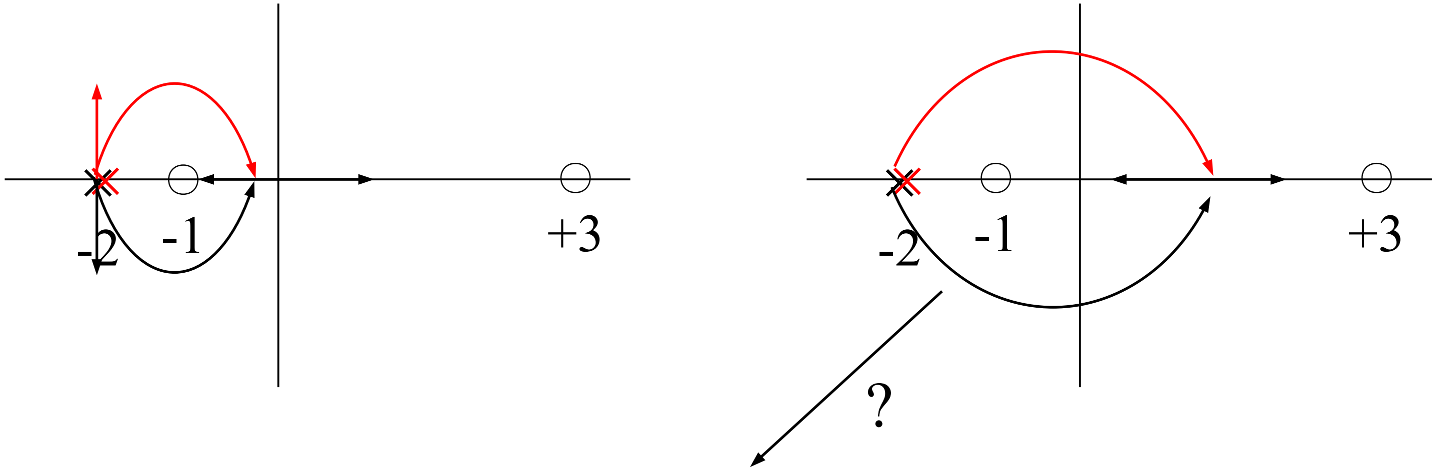
$$r = 1 \;:\; \varphi_1 = 90^\circ$$

$$r = 3 \;:\; \varphi_2 = 270^\circ = -90^\circ$$

```
numg = poly([-1 3]);
deng = poly([-2 -2]);
G=tf(numg, deng)
rlocus(G)
```



Example (continue):



$$G_p(s) = \frac{s^2 - 2s - 3}{s^2 + 4s + 4}, \qquad 1 + KG_p(s) = 0$$

$$s^2 + 4s + 4 + K(s^2 - 2s - 3) = 0$$

Example (continue):

$$\left(1+K\right)s^2+\left(4-2K\right)s+\left(4-3K\right)=0$$

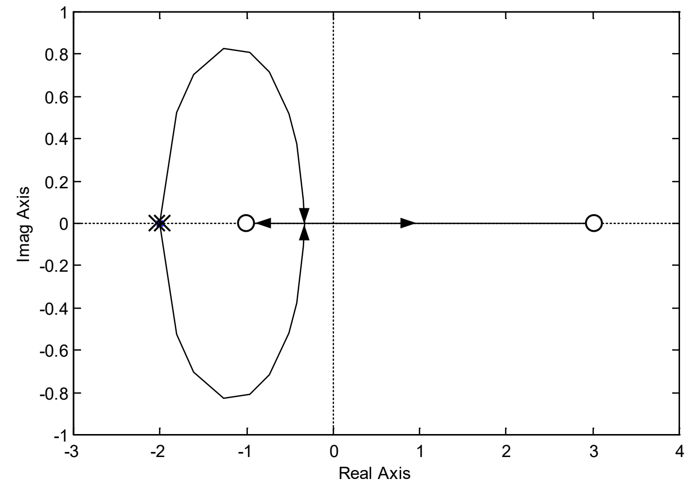
at $j\omega$-axis $\rightarrow$ $\boxed{s=j\omega}$

$$-\omega^2\left(1+K\right)+j\omega\left(4-2K\right)+\left(4-3K\right)=0$$

$$K=2,$$

$$-\omega^2\left(3\right)+\left(4-6\right)=0$$

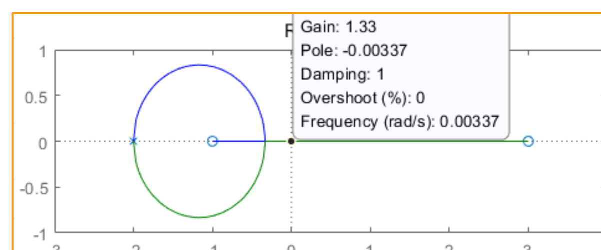
$3\omega^2+2\neq 0$: impossible $\quad \therefore \;\Rightarrow$



$$\text{At } s=0\rightarrow 1+KG_p(s)=0\rightarrow 1+K\left(-\frac{3}{4}\right)=0\rightarrow \boxed{K=\frac{4}{3}=1.333}$$

$$\left(1+K\right)s^2+\left(4-2K\right)s+\left(4-3K\right)=0$$

$\therefore$ stability $\boxed{K<\dfrac{4}{3}}$



Pole Sensitivity

Case 1: Very small changes in gain

High Sensitivity. → Very large changes
- in pole location
- in performance

Case 2: Low Sensitivity.

⇨ We prefer systems with *low sensitivity* to changes in gain

- Root sensitivity

Sensitivity of a closed-loop pole:

$$S_{s:K} = \boxed{\frac{K}{s} \cdot \frac{\delta s}{\delta K}}$$

Current pole location

Current gain

$$S_{F:P} = \frac{P}{F} \cdot \frac{\delta F}{\delta P}$$

Function

Parameter

⇩

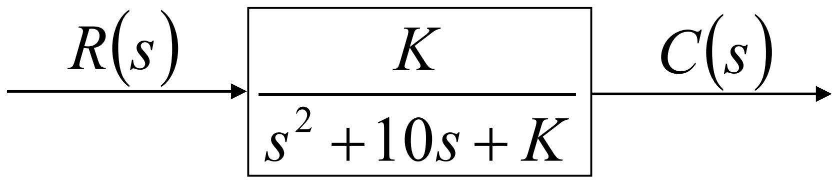
$$S_{s:K} = \frac{K}{s} \cdot \frac{\Delta s}{\Delta K} \Rightarrow \boxed{\Delta s = s\left(S_{S:K}\right)\frac{\Delta K}{K}}$$

Change in the pole location

(ref: chapter 7.7)

$$S_{F:P} = \lim_{\Delta P \to 0} \frac{\Delta F / F}{\Delta P / P} = \lim_{\Delta P \to 0} \frac{P}{F}\frac{\Delta F}{\Delta P}$$

Example:



① Find the root sensitivity of the system at $s = -9.47$ and $s = -5 + j5$

② Calculate the change in the pole location for a 10% change in K.

$\Rightarrow$ Characteristic equation

$$s^2 + 10s + K = 0$$

Example (continued):

$\Rightarrow$ Characteristic equation

$$s^2 + 10s + K = 0$$

$$\frac{\delta s}{\delta K} = 2s\frac{\delta s}{\delta K} + 10\frac{\delta s}{\delta K} + 1 = 0$$

$$\frac{\delta s}{\delta K} = \frac{-1}{2s+10}$$

$$S_{s:K} = \frac{K}{s}\cdot\frac{\delta s}{\delta K} = \frac{K}{s}\cdot\frac{-1}{2s+10} \cdots\cdots (1)$$

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
$\vdots$	$\vdots$	$\vdots$
50	$-5+j5$	$-5-j5$

i) $s = -9.47 \rightarrow K = 5 \rightarrow (1)$

$$S_{s:K} = -0.059$$

$$\Delta s = \boxed{s\left(S_{s:K}\right)\frac{\Delta K}{K}}$$

$$= 0.056$$

The pole will move to the right by 0.056 units for a 10% change in *K* at *K*=5 or s = –9.47

Example (continue):

ii) For $s = -5 + j5 \rightarrow K = 50 \rightarrow (1)$;

$$S_{s:k} = \frac{K}{s} \cdot \frac{-1}{2s+10} = \frac{1}{2} - j\frac{1}{2} = \sqrt{\frac{1}{2}} \ \angle -45^\circ$$

Find the change in the pole location at $\Delta K / K = 10\%$

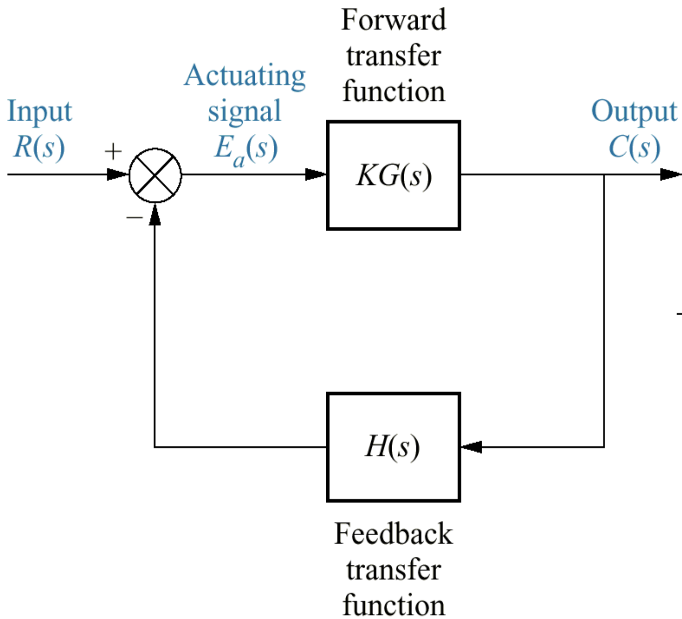
$$\Delta s = s\left(S_{s:K}\right)\frac{\Delta K}{K}$$

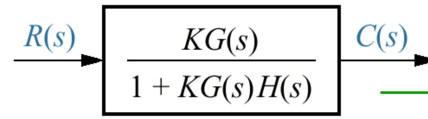
$$= (-5 + j5)\left(\frac{1}{2} - j\frac{1}{2}\right)(0.1)$$

$$= j0.5$$

The pole will move vertically by 0.5 unit for a 10% change in K

8.3 Properties of the Root Locus





$$1 + KG(s)H(s) = 0$$

$$\boxed{\sum_{k=1}^{m} \angle(s - z_k) - \sum_{i=1}^{n} \angle(s - p_i) = r \cdot 180 \quad (r : \text{odd})}$$

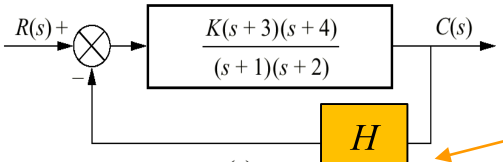
$$1 + KG(s)H(s) = 0$$

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ, \quad k = 0, \ \pm 1, \ \pm 2, \ldots$$

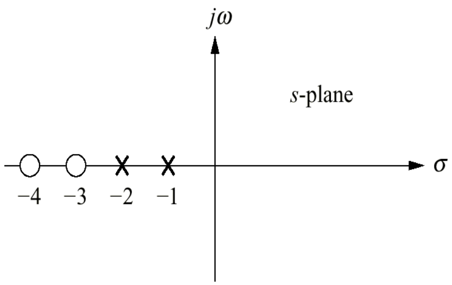
$$|KG(s)H(s)| = 1 \quad \Rightarrow \quad K = \frac{1}{|G(s)H(s)|}$$

$$\angle G(s)H(s) = (2k+1)180^\circ \quad (K > 0)$$

A point s on the root locus should be satisfied these conditions.



(a)

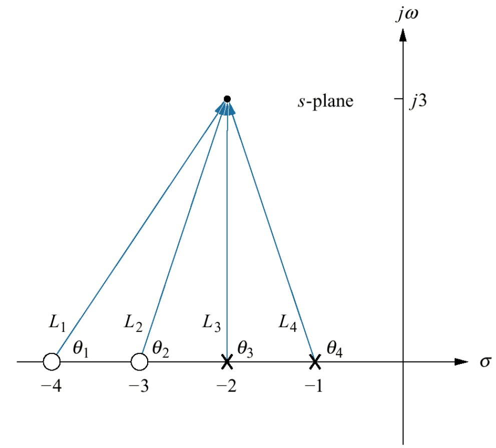


(b)

$H=1$

$$KG(s)H(s)=\frac{K(s+3)(s+4)}{(s+1)(s+2)}$$

(1) In the case of $s=-2+j3$



$$\angle G(s)H(s)=\angle G(s)=\theta_1+\theta_2-\theta_3-\theta_4$$

$$=56.31^\circ+71.57^\circ-90^\circ-108.43^\circ$$

$$=-70.55^\circ\neq\pm180^\circ$$

Therefore, $s=-2+j3$ is not a point on the root locus.

$$\boxed{\sum_{k=1}^{m}\angle(s-z_k)-\sum_{i=1}^{n}\angle(s-p_i)=r\cdot180\ \ (r:\text{odd})}$$

(2) In the case of $s=-2+j\frac{1}{\sqrt{2}}$

$$\angle G(s)H(s)=\theta_1+\theta_2-\theta_3-\theta_4$$

$$=19.47^\circ+35.26^\circ-90^\circ-144.74^\circ$$

$$=-180^\circ=\pm180^\circ$$

$\Rightarrow$

Therefore, $s=-2+j\frac{1}{\sqrt{2}}$ is a point on the root locus for some value of K.

$$\boxed{\begin{array}{l} GH\rightarrow\frac{Z}{P} \\ \frac{1}{GH}\rightarrow\frac{P}{Z}\end{array}}$$

Let's find the K!

$\Downarrow$

$$K=\frac{1}{|G(s)||H(s)|}=\frac{1}{M}=\frac{\Pi\text{ pole lengths}}{\Pi\text{ zero lengths}}$$

$$K=\frac{l_3l_4}{l_1l_2}=\frac{\frac{1}{\sqrt{2}}(1.22)}{(2.12)(1.22)}=0.33\quad\text{and}\quad M=3$$

A point s on the root locus should be satisfied the following conditions.

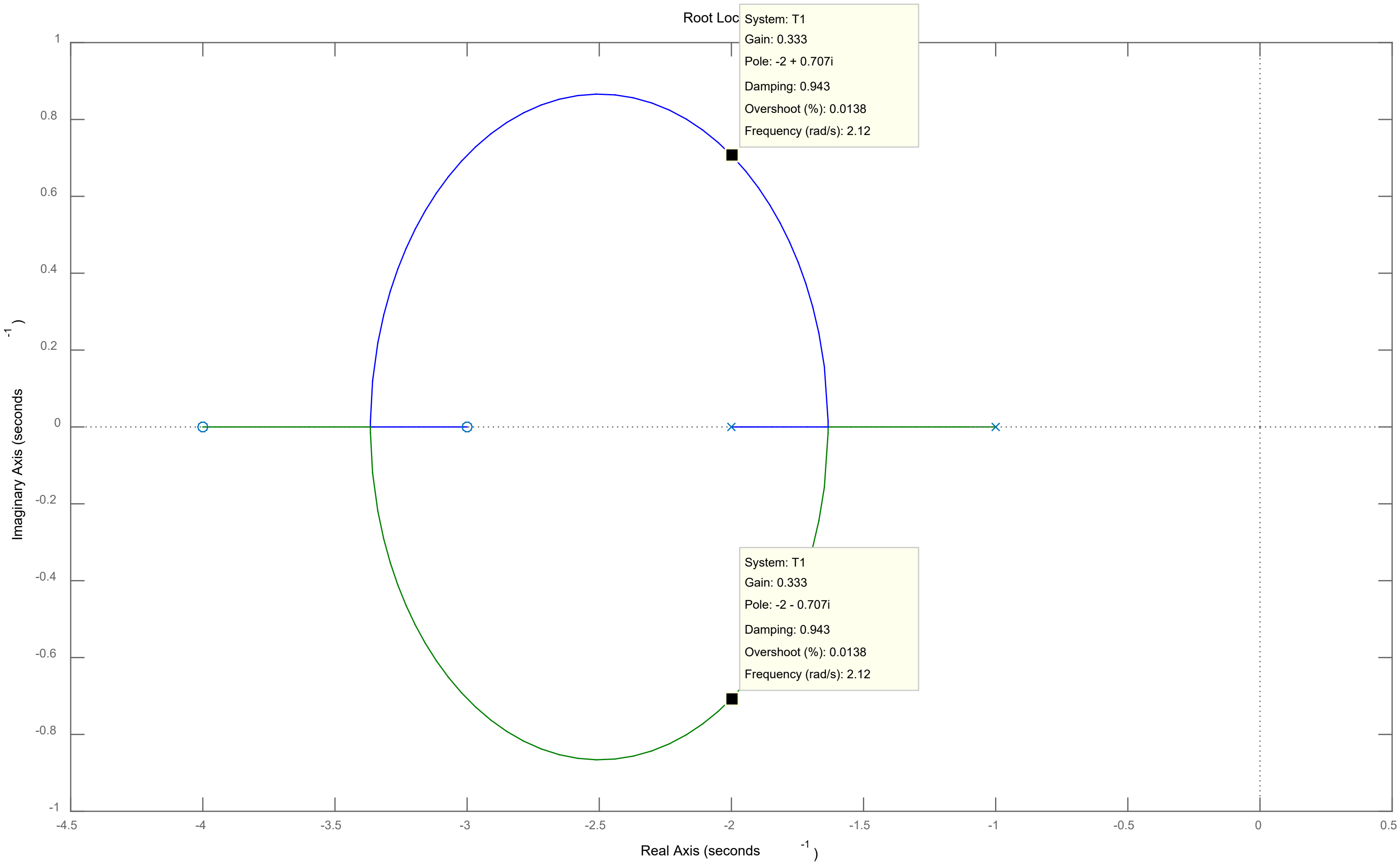
$\Downarrow$

$$|KG(s)H(s)|=1\quad\Rightarrow\quad K=\frac{1}{|G(s)H(s)|}$$

$$\angle G(s)H(s)=(2k+1)180^\circ\ \ (K>0)$$

```
numg = poly([-3 -4]);
deng = poly([-1 -2]);
G=tf(numg, deng)
rlocus(G)
```

```
>> a=1/sqrt(2); s=-2+a*j;
G=(s^2+7*s+12)/(s^2+3*s+2);
Theta=(180/pi)*angle(G)
M=abs(G)

Theta = 180
M = 3
```



8.4 Sketching the Root Locus

(1) Number of branches :

  # of branches = # of closed-loop poles

(2) Symmetry

(3) Real-axis segments

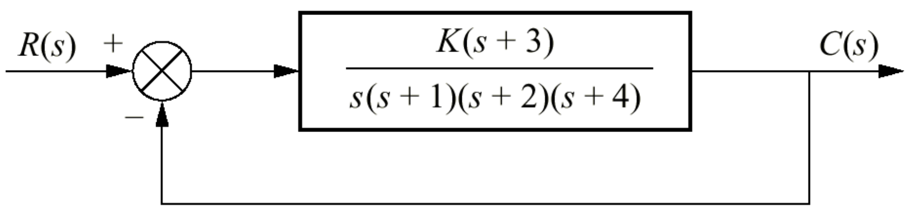
(4) Starting and ending points: (K=0, poles), (K=∞, zeros)

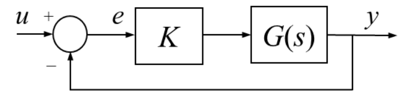
(5) Behavior at infinity: asymptote

$$\text{Real-axis intercept: } \sigma_a = \frac{\Sigma \text{ finite poles } - \Sigma \text{ finite zeros}}{\#\text{ of finite poles } - \#\text{ of finite zeros}}$$

$$\text{Angle: } \theta_a = \frac{(2k+1)\pi}{\#\text{ of finite poles } - \#\text{ of finite zeros}}$$

Example 8.2 Sketching a root locus with asymptotes

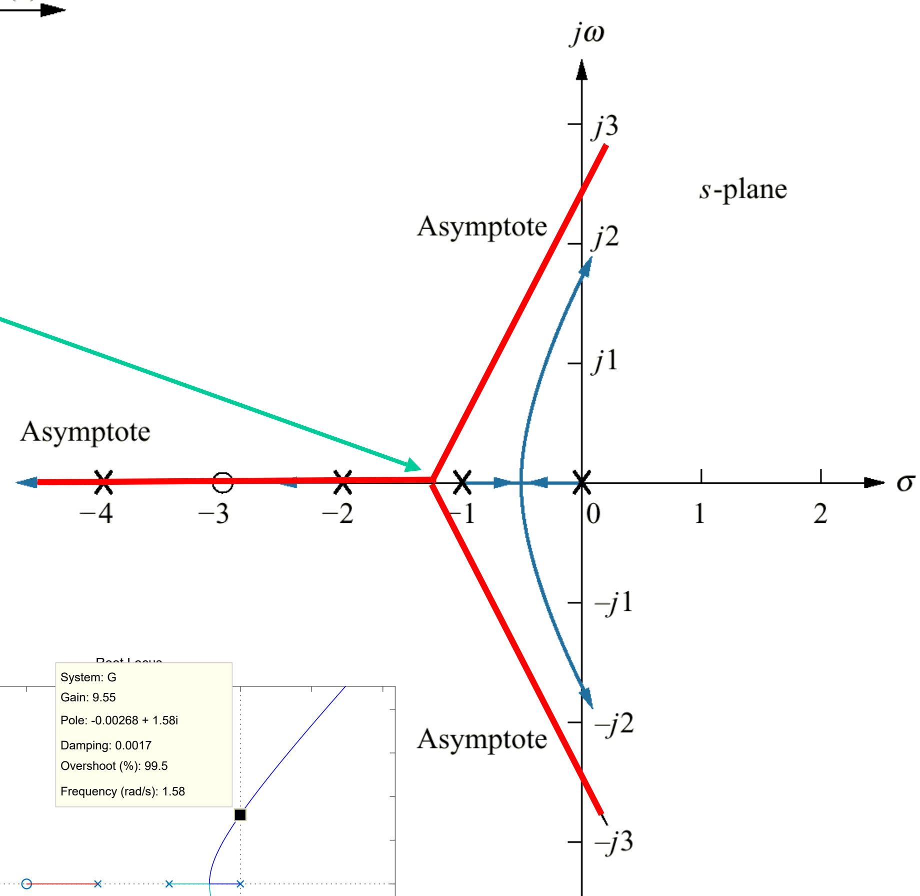


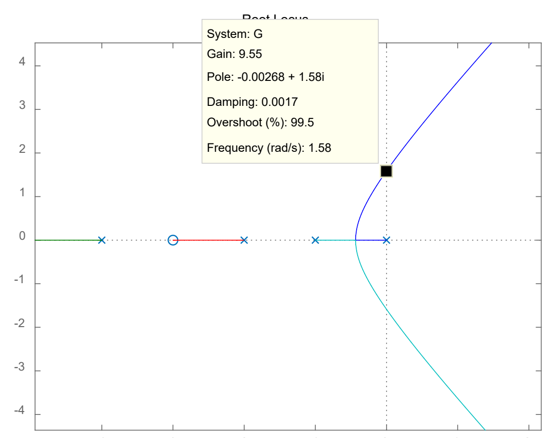


$$\sigma_a = \frac{(0-1-2-4)-(-3)}{(4-1)} = \frac{-7+3}{3} = -\frac{4}{3}$$

$$\theta_k = \frac{(2k+1)\pi}{(4-1)} = \frac{(2k+1)\pi}{3}$$

$$= \frac{\pi}{3},\ \ \pi,\ \ \frac{5}{3}\pi, \quad \text{for } k = 0,\ 1,\ 2.$$

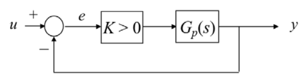





```
numg = poly([-3]);
deng = poly([0 -1 -2 -4]);
G=tf(numg, deng)
rlocus(G)
```

8.5 Refining the Sketch

(1) Root locus example showing real- axis breakaway (σ_1) and break-in points (σ_2)



- Plot the poles of the closed–loop system $T(s)$, as a function of K.

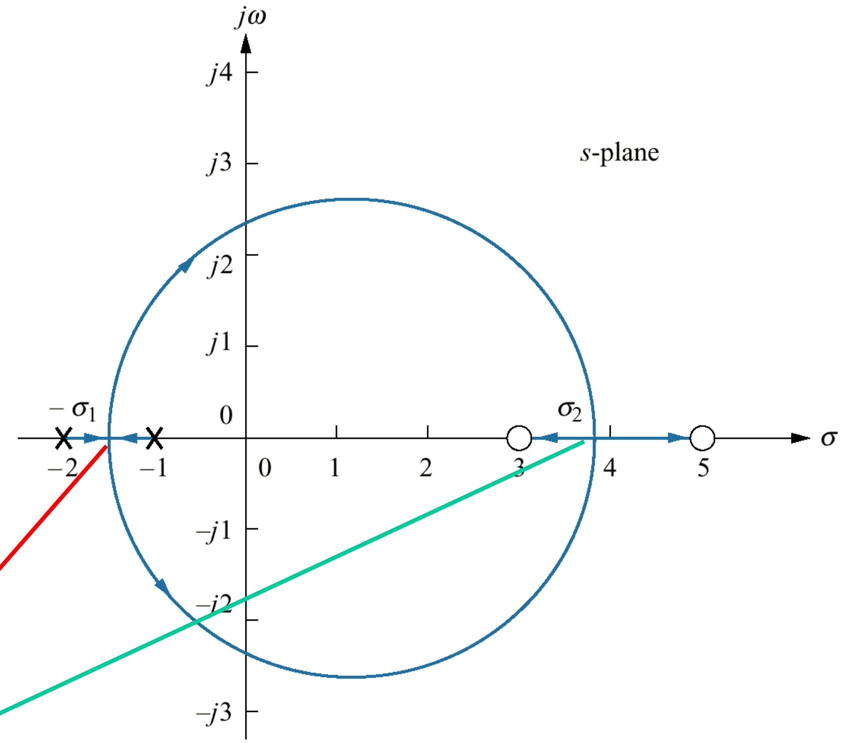
$$T(s)=\frac{KG_P(s)}{1+KG_P(s)}$$

Poles: $\boxed{1+KG_p(s)=0}$

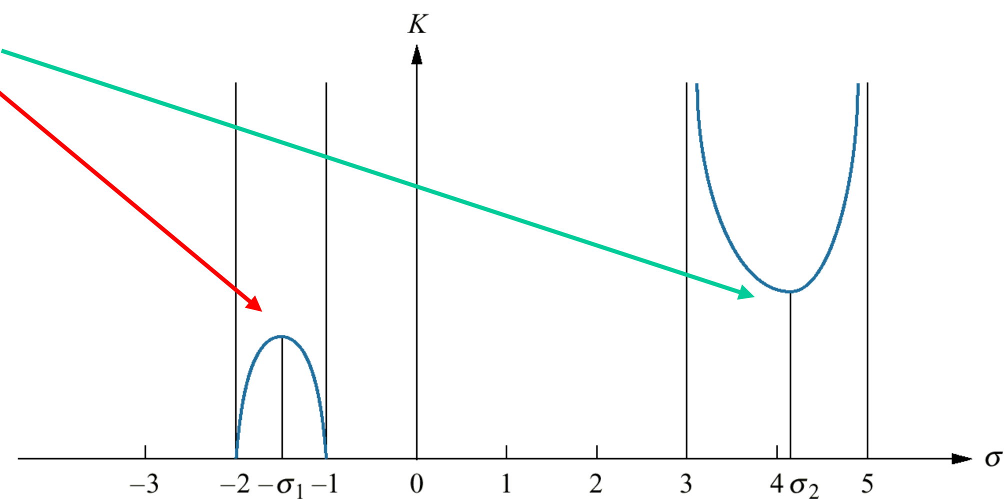
$$1+KG(s)H(s)=0$$

$$K=-\frac{1}{G(s)H(s)}$$

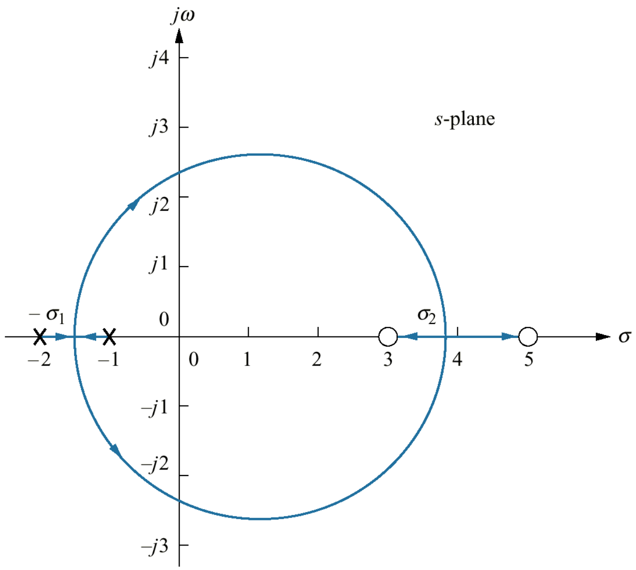
$$K(\sigma)=\boxed{-\frac{1}{G(\sigma)H(\sigma)}}$$



$$\frac{dK}{d\sigma}=0$$



Example 8.3 Breakaway and break-in points via differentiation



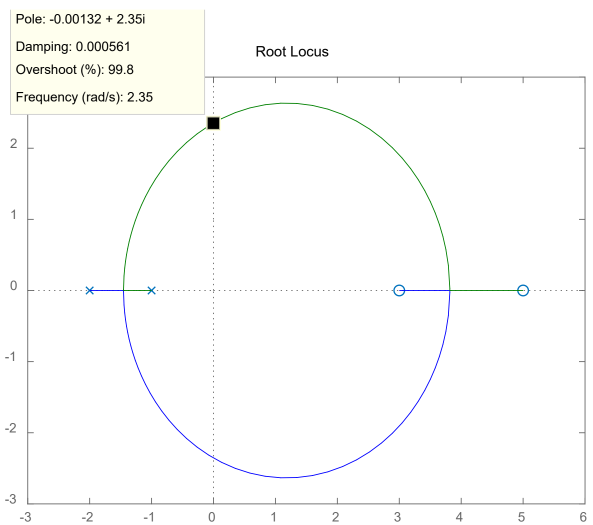
$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{s^2+3s+2}$$

$$\frac{K(\sigma^2-8\sigma+15)}{\sigma^2+3\sigma+2} = -1$$

$$K = \frac{-(\sigma^2+3\sigma+2)}{(\sigma^2-8\sigma+15)}$$

$$\frac{dK}{d\sigma} = \boxed{\frac{11\sigma^2-26\sigma-61}{\sigma^2-8\sigma+15} = 0}$$

$$\Rightarrow \ \sigma = -1.453, \quad 3.817$$

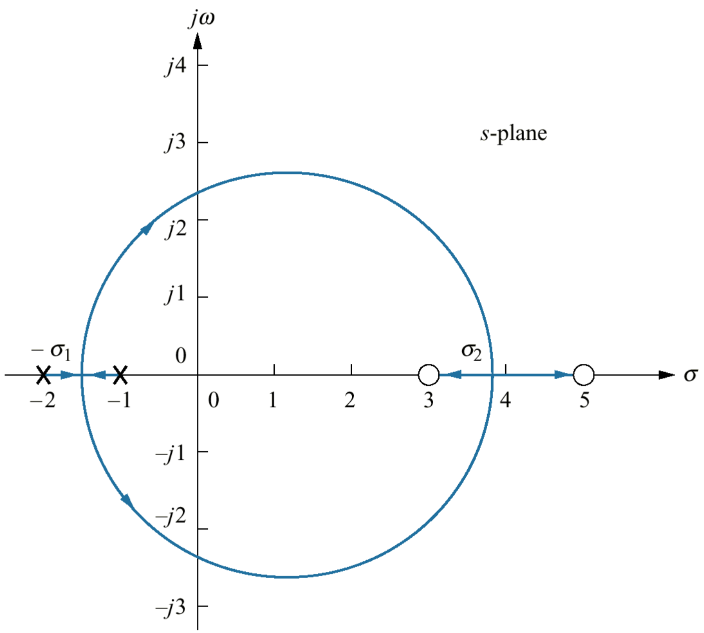



```
numg = poly([3 5]);
deng = poly([-1 -2]);
G=tf(numg, deng)
rlocus(G)
```

Example 8.4 Breakaway and break-in points without differentiation

- Breakaway and break-in points satisfy the relationship: $\sum_{i=1}^{m} \frac{1}{\sigma + z_i} = \sum_{i=1}^{n} \frac{1}{\sigma + p_i}$

  where z_i and p_i are the zero and pole values, respectively, of $G(s)H(s)$



$$\frac{1}{\sigma-3} + \frac{1}{\sigma-5} = \boxed{\frac{1}{\sigma+1} + \frac{1}{\sigma+2}}$$

$$11\sigma^2 - 26\sigma - 61 = 0$$

$$\Rightarrow \quad \sigma = -1.453, \quad 3.817$$

- The jw -Axis Crossings
- Angle of Departure and Arrival
- Plotting and Calibrating the Root Locus

8.7 Transient Response design via Gain Adjustment

(Find K for t_s, t_p, e_{ss}, overshoot)

8.8 Generalized Root Locus

How can we obtain a root locus for variations of the value of p_1?

8.9 Root Locus for Positive-Feedback Systems

8.10 Pole Sensitivity