과제 3장)

3.3 $SST = \sum_{i=1}^{n}(y_i-\bar{y})^2 = \sum_{i=1}^{n} y_i^2 - 2\bar{y}\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\bar{y}^2 = \sum_{i=1}^{n} y_i^2 - n\bar{y}^2$

$= y^t y - n\frac{1}{n^2}(y^t 1)(1^t y) = y^t y - \frac{1}{n} y^t 1^t 1 y$

$= y^t(I - \frac{J}{n})y$

$T = (I-\frac{J}{n})$, $T^2 = (I-\frac{J}{n})(I-\frac{J}{n}) = I^2 - \frac{2}{n}J + \frac{1}{n^2}J^t J$

$= I^2 - \frac{2}{n}J + \frac{J}{n}$ $\quad \because J^2 = nJ$

$= I - \frac{J}{n}$ $\quad \therefore$ 멱등행렬.

$\because$ 멱등행렬의 성질에 의해 $rank(T) = tr(T) = tr(I-\frac{J}{n}) = n-1$

$SSE = \sum_{i=1}^{n}(y_i-\hat{y}_i) = (y-\hat{y})'(y-\hat{y}) = (y-X\hat{\beta})'(y-X\hat{\beta}) = y'y - 2y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}$

$\because \hat{\beta} = (X'X)^{-1}X'y$ $\quad = y'y - 2y'X(X'X)^{-1}X'y + (X'X)^{-1}X'yX'X(X'X)^{-1}X'y$

$= y'(I - X(X'X)^{-1}X')y = y'Ey$

$E = I - X(X'X)^{-1}X'$ $\quad E^2 = I^2 - 2X(X'X)^{-1}X' + X(X'X)^{-1}X'X(X'X)^{-1}X'$

$= I - X(X'X)^{-1}X' = E$ $\quad \therefore$ 멱등행렬

$\because rank(E) = tr(E) = tr(I - X(X'X)^{-1}X') = n - tr(X(X'X)^{-1}X')$

$= n - tr((X'X)^{-1}X'X) = n - tr(I_p) = n-p$

3.14 t검정) $H_0: \beta_j = 0$, $\beta_j \neq 0$

$$t = \frac{\hat{\beta}_j}{S.E(\hat{\beta}_j)} = \frac{\hat{\beta}_j}{s\sqrt{C_{j+1,j+1}}} = \frac{\hat{\beta}_j}{\sqrt{MSE/S_{xx}}}.$$

부분 F 검정) $R(\beta_j|\beta_0,\cdots,\beta_{p-1}) = SSR(\beta) - SSR(\beta_0,\cdots,\beta_{p-1})$

$= \Sigma(\hat{Y}_i - \bar{Y})^2 - \sum_{\beta_j=0}(\hat{Y}_i-\bar{Y})^2 = \hat{\beta}^2\Sigma(X_i-\bar{X})^2 - \hat{\beta}^2_{\beta_j=0}\Sigma(X_i-\bar{X})^2$

$= \hat{\beta}_j^2\Sigma(X_i-\bar{X})^2 = \hat{\beta}_j^2 S_{xx}$

$$\therefore F_0 = \frac{\hat{\beta}_j^2 S_{xx}}{MSE} \sim t^2.$$

과제 4장.

4.4 $Var(\hat{y}_{i(i)})$

$$y_i = e_i + \hat{y}_i$$

$$\hat{y}_i = \omega_i (e_i + \hat{y}_i) + (1-\omega_i)\hat{y}_{i(i)}$$

$$(1-\omega_i)\hat{y}_i = (1-\omega_i)\hat{y}_{i(i)} + \omega_i e_i$$

$$(1-\omega_i)(\hat{y}_i - \hat{y}_{i(i)}) = \omega_i e_i \quad \Rightarrow \quad (1-\omega_i)(x_i'(\hat{\beta} - \hat{\beta}_{(i)})) = \omega_i e_i$$

$$x_i'(\hat{\beta} - \hat{\beta}_{(i)}) = \omega_i (e_i + x_i'(\hat{\beta} - \hat{\beta}_{(i)}))$$

$$\omega_i = \frac{x_i'(\hat{\beta} - \hat{\beta}_{(i)})}{e_i + x_i'(\hat{\beta} - \hat{\beta}_{(i)})} = \frac{\frac{x_i'(x'x)^{-1}x_i e_i}{1-h_{ii}}}{e_i + x_i' \frac{(x'x)^{-1}x_i e_i}{1-h_{ii}}} = \frac{\frac{h_i}{1-h_{ii}} e_i}{e_i + \frac{h_{ii}}{1-h_{ii}} e_i} = \frac{\frac{h_{ii}}{1-h_{ii}}}{1 + \frac{h_{ii}}{1-h_{ii}}} = h_{ii}$$

$$\hat{y}_{i(i)} = \frac{\hat{y}_i - h_{ii} y_i}{1-h_{ii}} = \frac{\hat{y}_i - h_{ii} e_i - h_{ii}\hat{y}_i}{1-h_{ii}} = \hat{y}_i - h_{ii} e_i$$

$$Var(\hat{y}_{i(i)}) = Var(\hat{y}_i) = h_{ii}\sigma^2$$

$$\therefore h_{ii}\sigma^2$$

4.8 $\hat{\beta} - \hat{\beta}_{(i)} = \frac{(X'X)^{-1}x_i e_i}{1-h_{ii}}$

$$\hat{\beta}_{(i)} = (X_{(i)}^t X_{(i)})^{-1} X_{(i)}^t y_{(i)} = (X'X - x_i x_i')^{-1}(X^t y - x_i y_i) \qquad \because \text{using update formula}$$

$X'X = A$. $x_i = u$. $x_i^t = V^t$.

$$\hat{\beta}_{(i)} = \left((X'X)^{-1} + \frac{(X'X)^{-1}x_i x_i'(X'X)^{-1}}{1 - x_i^t (X'X)^{-1} x_i} \right)(X'y - x_i y_i) = \hat{\beta} + \frac{(X'X)^{-1}x_i x_i'(X'X)^{-1}}{1 - x_i'(X'X)^{-1}x_i}(X'y - x_i y_i) - (X'X)^{-1}x_i y_i$$

$$= \hat{\beta} + \frac{(X'X)^{-1}x_i x_i'\hat{\beta}}{1-h_{ii}} - \frac{(X'X)^{-1}x_i h_{ii} y_i}{1-h_{ii}} - (X'X)^{-1}x_i y_i$$

$$= \hat{\beta} + \frac{(X'X)^{-1}}{1-h_{ii}}(x_i \hat{y}_i - x_i h_{ii} y_i - (1-h_{ii})x_i y_i) = \hat{\beta} - \frac{(X'X)^{-1}}{1-h_{ii}} x_i (y_i - \hat{y}_i)$$

$$= \hat{\beta} - \frac{(X'X)^{-1}x_i e_i}{1-h_{ii}}$$

$$\therefore \hat{\beta} - \hat{\beta}_{(i)} = \frac{(X'X)^{-1}x_i e_i}{1-h_{ii}}$$

5장 과제.

5.3 $n=20$, $X_1 \sim X_4$

$$C_p = \frac{SSE_p}{s^2} - (n-2p)$$

If 최대모형 $SSE_p = (n-p)s^2$

$\therefore C_p = p$ $\therefore 5$

5.4. $n=10$, $s^2=3$

$$PRESS_p = \sum_{i=1}^{10} \left(\frac{e_i}{1-h_{ii}} \right)^2 = \frac{e_1^2 + \cdots + e_{10}^2}{(1-h)^2} \quad \because h_{11} = \cdots = h_{1010} = h$$

$$= \frac{\Sigma (e_i)^2 - 0}{n(1-h)^2} = \frac{Var(e_i)}{n(1-h)^2} = \frac{(1-h)s^2}{10(1-h)^2} = \frac{3}{10(1-h)}$$

$$\therefore \frac{3}{10(1-h)}$$

5.5. $X \geq 0$, $Var(X) \Rightarrow$ finite, $E(X) = \mu$.

$\phi(x) = x^2$ 이라 가정하면.

ϕ는 열린구간 I에서. 아래로 볼록 (convex function)이고,

$P(X \in I) = 1$. $E(X) = \mu < \infty$ 이므로,

$\phi(E(X)) = \mu^2 \leq E(X^2)$ 이 항상 성립된다. $\because$ 젠센 부등식.

$\therefore \mu^2 < E(x^2)$.