

Chapter 6. Stability

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Chapter 6. Stability

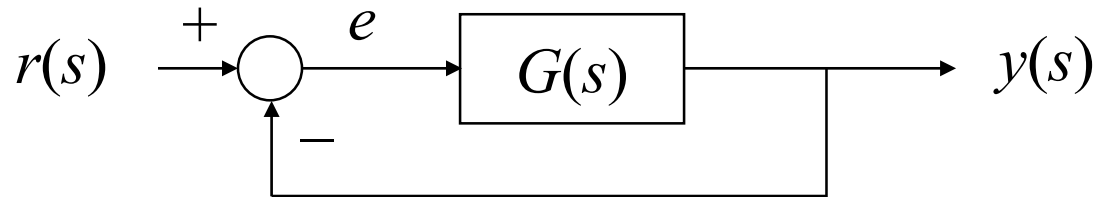
Objectives

- How to determine the stability of a system represented as a transfer function
- How to determine the stability of a system represented in state space
- How to determine system parameters to yield stability

Unity Negative Feedback

$$\frac{R(s)}{\text{Input}} \rightarrow \frac{G(s)}{1 \pm G(s)H(s)} \rightarrow \frac{C(s)}{\text{Output}}$$

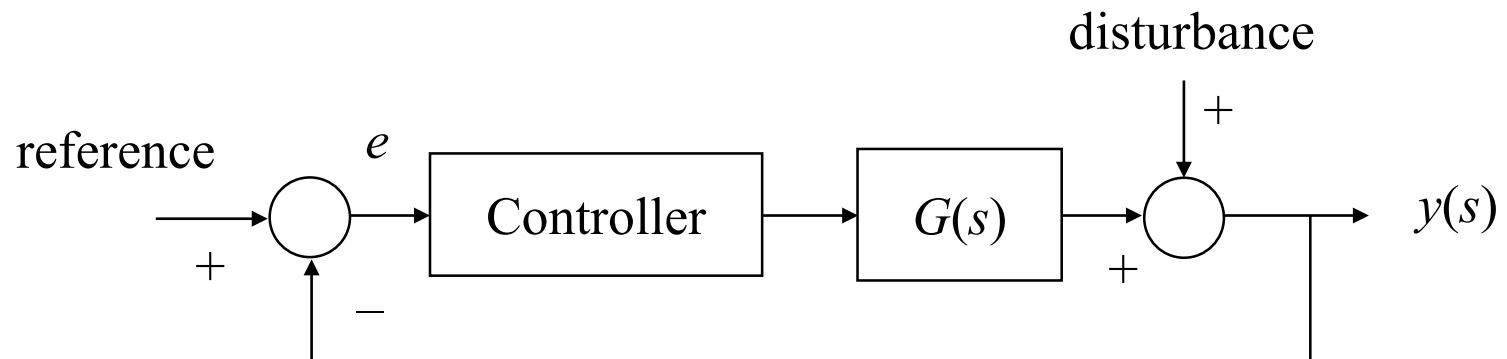
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$



- The overall transfer function of the above system is:

$$T(s) = \frac{G(s)}{1 + G(s)}$$

- System Model:

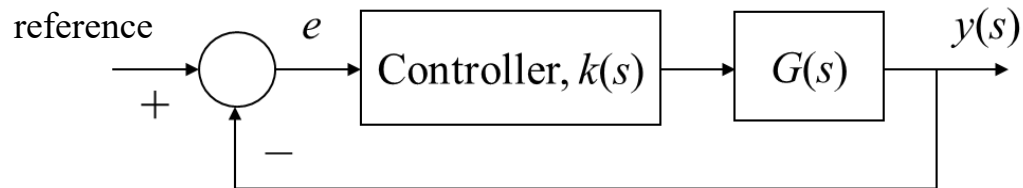


Root Locus

- The Root Locus is the graph of the closed loop poles of the system as the gain k varies from 0 to ∞ .
- For certain values of k , the system goes from being stable to being unstable.

Control Strategies

- Classical Control:



- Design:

– $k(s) = k$ (gain) \Rightarrow Choose k

– $k(s) = k \frac{s + z}{s + p} \Rightarrow$ Choose k , z and p

- Classical control
 - Objectives: Choose $k(s)$ to satisfy:
 - Stability
 - Performance
 - Robustness
- Other kinds of control
 - Nonlinear control
 - Online control
 - Adaptive control: Identifies the plant and adapts accordingly.
 - Digital sampled data control

6.1 Introduction

- 3 requirements for control systems:

- transient response (Chap. 4, 8)
- stability (Chap. 6)
- steady-state errors (Chap. 7)

- Total response of a system: $C(t) = C_{forced}(t) + C_{natural}(t)$

$$y = \frac{1}{s+1} \cdot \frac{1}{s} = \frac{1}{s} - \frac{1}{s+1} \Rightarrow y(t) = u(t) - e^{-t}u(t)$$

- stable system: $C_{natural}(t) \Rightarrow 0 \text{ as } t \rightarrow \infty$

- unstable system: $C_{natural}(t) \Rightarrow \infty \text{ as } t \rightarrow \infty$

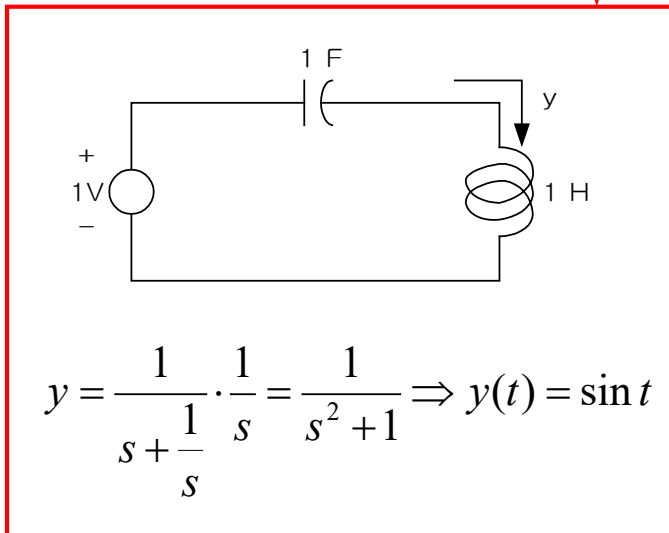
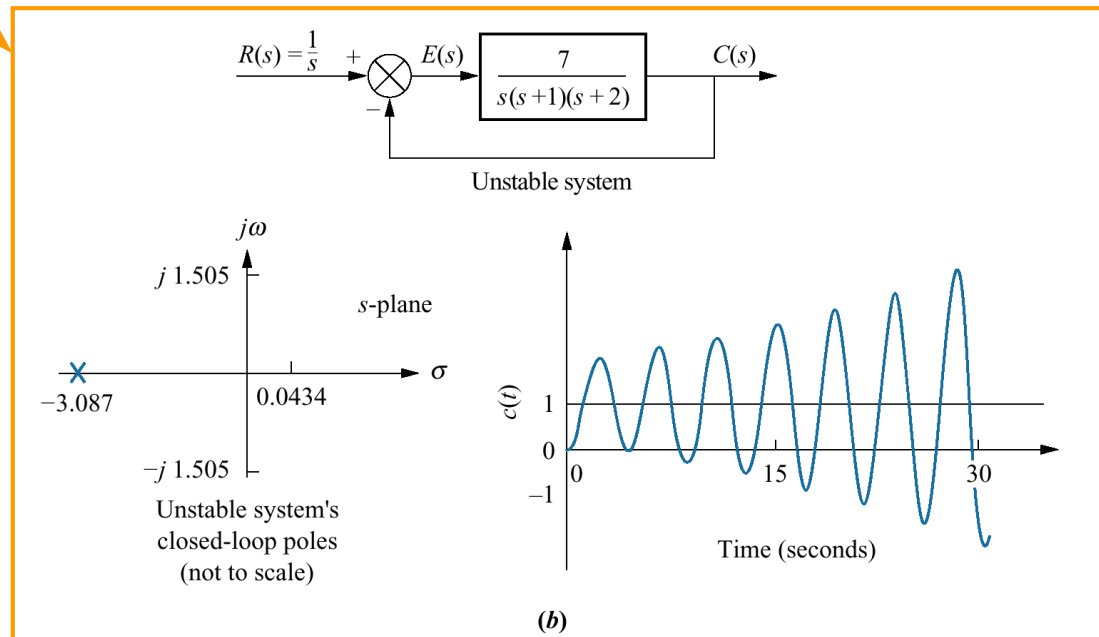
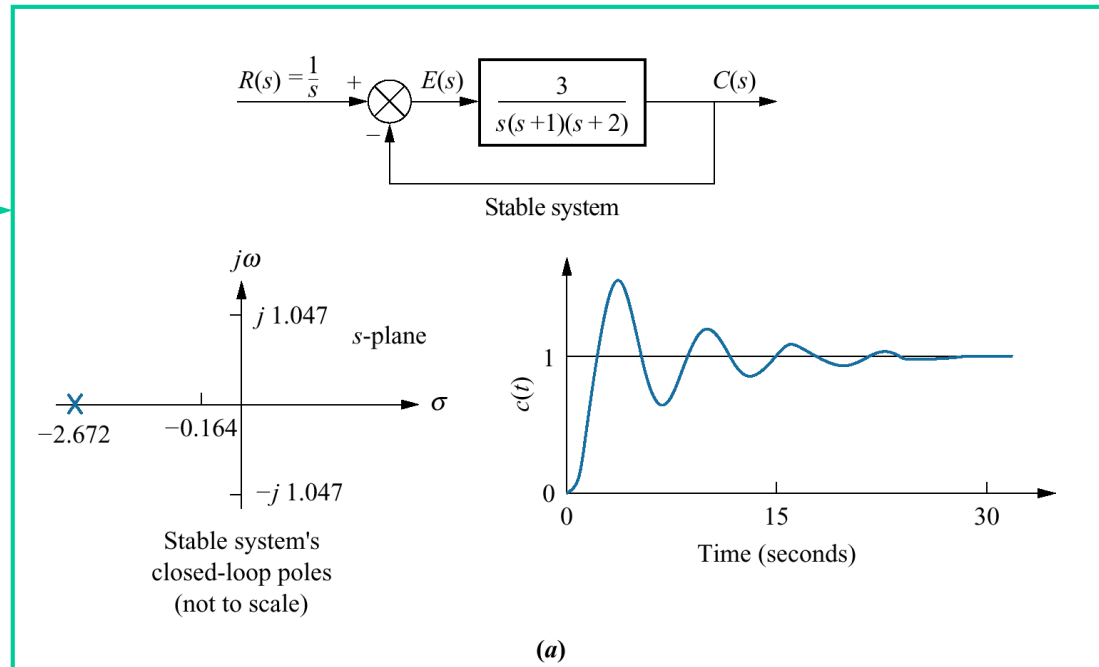
- marginally stable system: $C_{natural}(t)$ remains constant *or* oscillates

Example:

• Stable system:

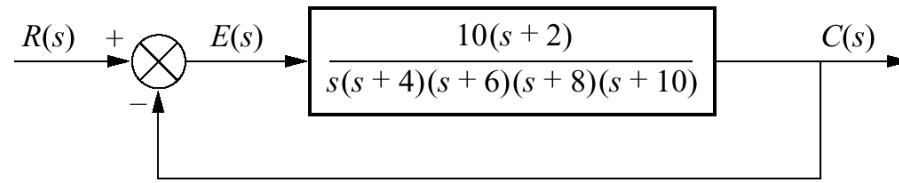
• Unstable system:

• Marginally stable system:

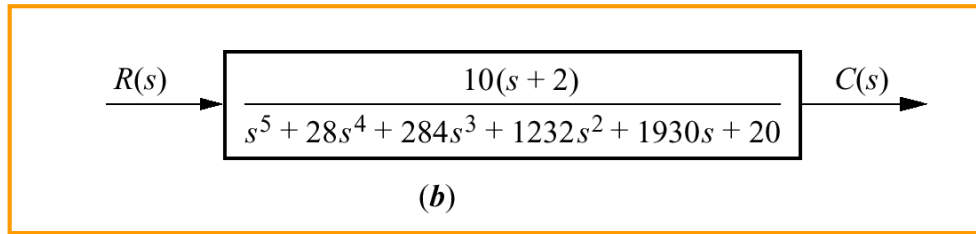


Common cause of problems in finding closed-loop poles:

- (a) original system;
- (b) equivalent system



(a)



(b)

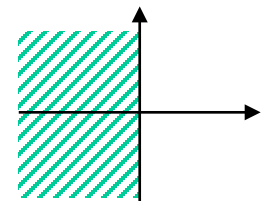
$$C(s) = \frac{Z(s)}{P(s)} R(s)$$

- Stability of a system:

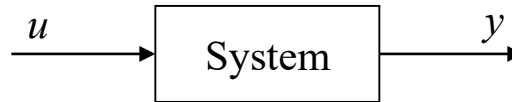
If closed-loop transfer function has only left-half-plane poles, then

$$P(s) = \prod_{i=1}^n (s + a_i)$$

a_i : real and positive, or complex with a positive real part



Stability



- A. Mode
- B. Stability
- C. Stability test: Routh criterion

$$\ddot{y} + 4\dot{y} + 3y = u$$

$$\{s^2 Y(s) + 4sY(s) + 3Y(s)\} - \{sy(0) + 4y(0) + \dot{y}(0)\} = U(s)$$

$$(s^2 + 4s + 3)Y(s) - \{(s + 4)y(0) + \dot{y}(0)\} = U(s)$$

$$\ddot{y} = -4\dot{y} - 3y + u$$

$$s[sY(s) - y(0)] - \dot{y}(0) + 4[sY(s) - y(0)] + 3Y(s) = U(s)$$

$$Y(s) = \frac{1}{s^2 + 4s + 3} U(s) + \frac{(s + 4)y(0) + \dot{y}(0)}{s^2 + 4s + 3}$$

Zero state response

Zero input response

$$\text{Let } Q(s) = s^2 + 4s + 3 = (s + 1)(s + 3) = (s - p_1)(s - p_2)$$

$$\text{where } p_1 = -1, p_2 = -3$$

$p_{1,2}$ are called poles of $G(s)$, or poles of the system.

$$Y(s) = \frac{U(s)}{s^2 + 4s + 3} - \frac{(s+4)y(0) + \dot{y}(0)}{s^2 + 4s + 3}$$

$$Y(s) = \frac{U(s)}{s^2 + 4s + 3}$$

- In the zero state response, i.e., $y(0) = \dot{y}(0) = 0$

$$Y(s) = \frac{1}{(s+1)(s+3)} U(s) = \frac{1}{(s-p_1)(s-p_2)} U(s)$$

$\Rightarrow y(t)$ has terms, e^{-t} and e^{-3t} for any input!

$\Rightarrow e^{-t}$ and e^{-3t} are called modes of the system.

- In the zero input response, i.e., $u(t)=0$

$\Rightarrow y(t)$ has the modes, e^{-t} and e^{-3t} .

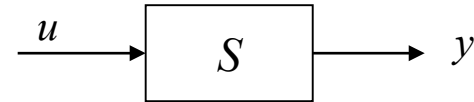
- General case: for any input u ,

$$Y(s) = G(s)U(s) = \frac{P(s)}{Q(s)} U(s) = \frac{P(s)U(s)}{Q_1(s)(s-p)}$$

- In the partial fraction expansion,

$$Y(s) \rightarrow \frac{1}{s-p}, \quad y(t) \rightarrow e^{pt} \quad : \text{a mode of the system for any input.}$$

Example: $\dot{y} = -y + u, \quad (y(0) = 0)$

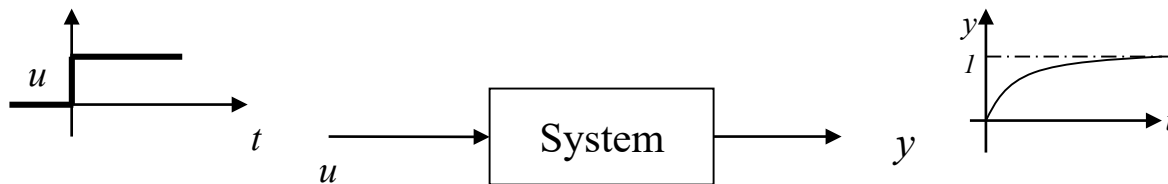


$$sY(s) = -Y(s) + U(s), \quad Y(s) = \frac{1}{s+1}U(s)$$

$p = -1$, a mode $\rightarrow e^{-t}$ in $y(t)$

$$\text{If } u(t) = \text{step} \rightarrow U(s) = \frac{1}{s} \rightarrow Y(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$y(t) = 1 - e^{-t}$$



1: **steady state** (coming from input)

$-e^{-t}$: **transient** (coming from system)

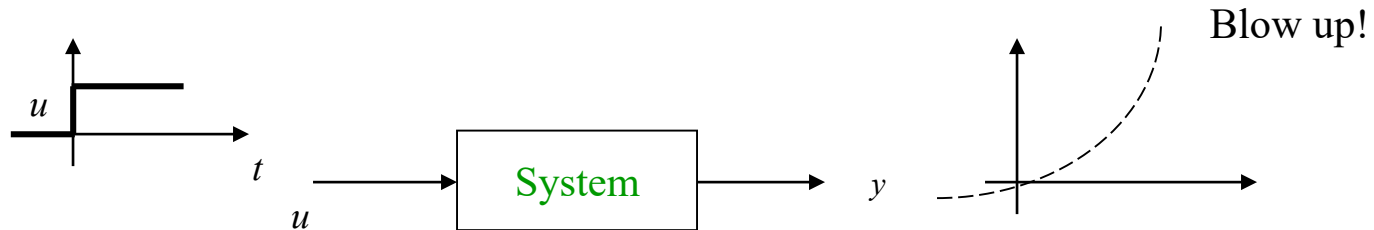
Example: $\dot{y} = y + u, \quad (y(0) = 0)$

$$sY(s) = Y(s) + U(s) \rightarrow Y(s) = \frac{1}{s-1}U(s)$$

$$p = 1 \rightarrow e^t \text{ in } y(t)$$

$$\text{If } u(t) = \text{step, then } Y(s) = \frac{1}{(s-1)s} = \frac{1}{s-1} - \frac{1}{s}$$

$$\rightarrow y(t) = e^t - 1$$



$\Rightarrow e^t$ is a system's mode: it will be the output for any input!

[Not a desirable phenomenon. Instability]

Definition: A system is bounded-input bounded-output stable (BIBO stable). If whenever the input is bounded, the output is bounded.

Complex pole: $\sigma + j\omega$

$$G(s) = \frac{P(s)}{Q(s)}, \quad Q(\sigma + j\omega) = 0 \rightarrow \text{From algebra, } Q(\sigma - j\omega) = 0$$

$$Y(s) = G(s)U(s)$$


→ In PFE, $Y(s)$ will have the terms

$$\frac{1}{s - (\sigma + j\omega)} \quad \text{and} \quad \frac{1}{s + (\sigma - j\omega)}$$

$$\rightarrow y(t) : e^{(\sigma + j\omega)t} \quad \text{and} \quad e^{(\sigma - j\omega)t}$$

$$\rightarrow e^{\sigma t} (\cos \omega t + j \sin \omega t) \quad \text{and} \quad e^{\sigma t} (\cos \omega t - j \sin \omega t)$$

$$\rightarrow e^{\sigma t} \sin(\omega t + \phi) \quad \text{or} \quad e^{\sigma t} (A \cos \omega t + B \sin \omega t)$$

Example: $G(s) = \frac{1}{s^2 + 2s + 5}$ 

$$Y(s) = G(s)U(s)$$

Let $u(t) = \text{step} \rightarrow U(s) = \frac{1}{s}$

Poles: $s^2 + 2s + 5 = 0 \rightarrow s = -1 \pm 2j$

$$Y(s) = G(s)U(s) = \frac{1}{\{s - (-1 + 2j)\} \{s - (-1 - 2j)\} s}$$

After calculation (PFE),

$$\rightarrow y(t) = \frac{1}{5} - e^{-t} \left(\frac{1}{5} \cos 2t + \frac{1}{10} \sin 2t \right)$$

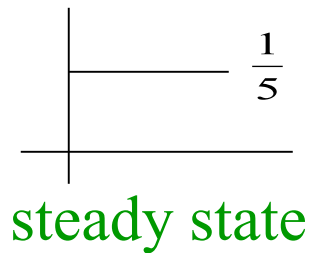
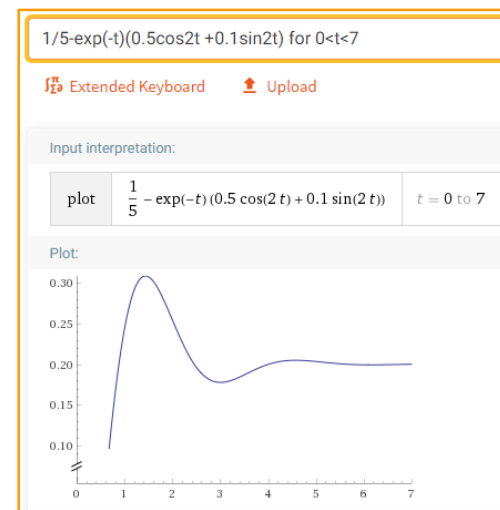
$$y(t) = \frac{1}{5} - e^{-t} C_1 \sin(2t + \phi),$$

\uparrow \uparrow

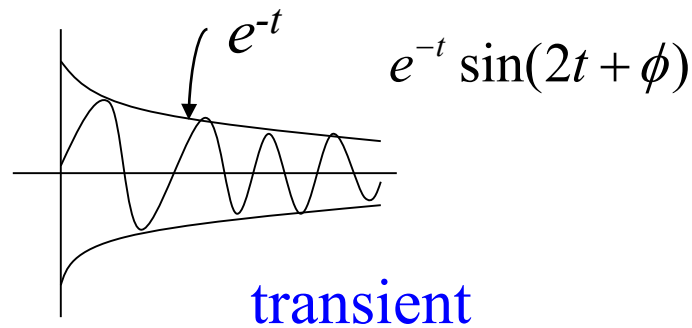
$$\phi = \arctan(2),$$

$$C_1 = \frac{1}{5 \sin \phi}$$

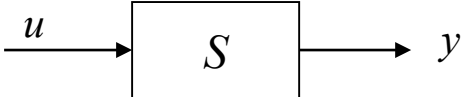
$$y(t) = \underbrace{\frac{1}{5}}_{\substack{\uparrow \\ \text{from input}}} - \underbrace{e^{-t} C_1 \sin(2t + \phi)}_{\text{from poles, } -1 \pm 2j}$$



+

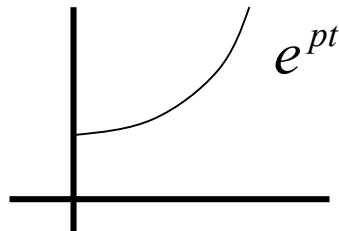


$$y(t) = \boxed{\text{steady state} + \text{transient}}$$

Summary: $Y(s) = G(s)U(s)$ 

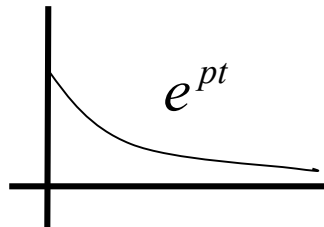
① If p is a pole of G , then the output will always have e^{pt} .

i) p is real and positive



\Rightarrow the system is **unstable**

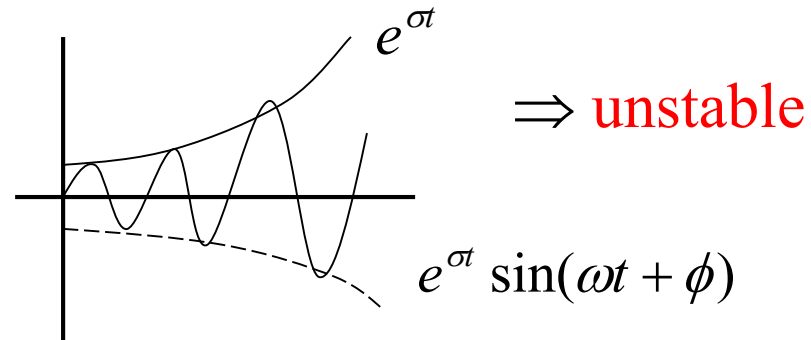
ii) p is real and negative



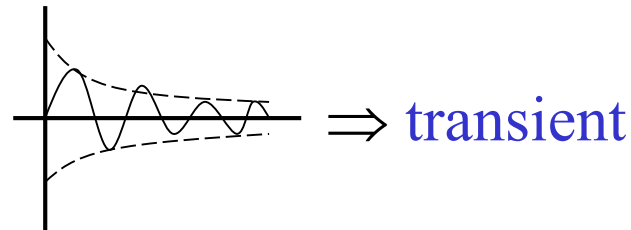
\Rightarrow transient

$$\textcircled{2} \quad p = \sigma + j\omega \rightarrow \bar{p} = \sigma - j\omega \Rightarrow e^{\sigma t} \sin(\phi + \omega t)$$

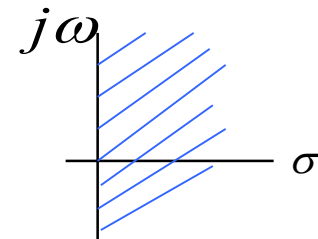
i) If $\sigma > 0$



ii) If $\sigma < 0$



- Thus, if $G(s)$ has pole in the open *RHP* (right half-plane)
 \Rightarrow the system is *unstable*.

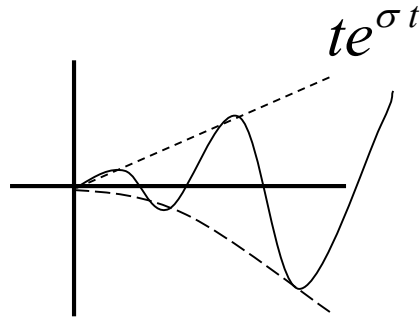


③ Double pole:

$$\sigma + j\omega, \sigma + j\omega \rightarrow \sigma - j\omega, \sigma - j\omega$$

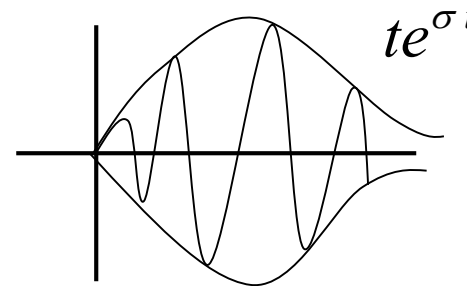
By PFE, $y(t)$ will have $te^{\sigma t} \sin(\omega t + \phi)$

i) If $\sigma > 0$



unstable

ii) If $\sigma < 0$



transient

Any pole of $G(s)$ in the open *RHP* makes the system *unstable*.
Any pole of $G(s)$ in the open *LHP* causes a *transient*.

④ Poles on the $j\omega$ -axis ?

Example: $G(s) = \frac{1}{s^2 + 4} = \frac{1}{(s + 2j)(s - 2j)}$

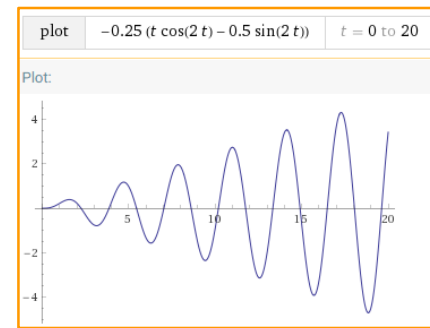
i) Let $u(t)$ =step

$$Y(s) = \frac{1}{(s + 2j)(s - 2j)s} = \frac{1}{4} \frac{1}{s} - \frac{1}{8} \frac{1}{(s + 2j)} - \frac{1}{8} \frac{1}{(s - 2j)}$$

$$y(t) = \frac{1}{4} - \frac{1}{8} \left(e^{-2jt} + e^{+2jt} \right) = \frac{1}{4} (1 - \cos 2t)$$

$$\begin{aligned} e^{-2jt} &= \cos 2t - j \sin 2t \\ e^{+2jt} &= \cos 2t + j \sin 2t \\ (e^{-2jt} + e^{+2jt}) &= 2 \cos 2t \end{aligned}$$

- y has the term $1/4$ from the input, and the term $(\cos 2t)/4$ from the system. → **Oscillation!**
- $u(t)$ =step function → y is **bounded**.
- Is the system stable? → **No!**



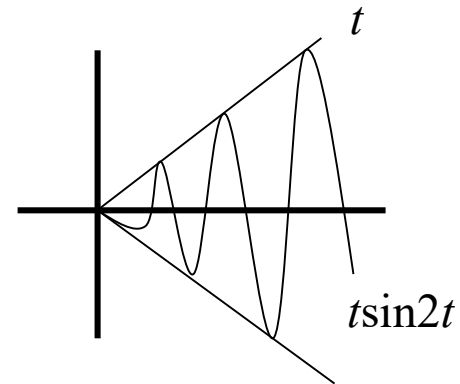
ii) Let $u(t)=\sin 2t$

$$U(s) = \frac{2}{s^2 + 4}, \quad Y(s) = \frac{2}{(s^2 + 4)} \frac{1}{(s^2 + 4)} = \frac{2}{(s - 2j)^2 (s + 2j)^2}$$

$$\rightarrow y(t) = -\frac{1}{4} \left(t \cos 2t - \frac{1}{2} \sin 2t \right)$$

↑ from double pole at $\pm 2j$

→ **unstable**



- Pole at $\pm j\omega \rightarrow \sin(\omega t + \phi)$
 Double pole at $\pm j\omega \rightarrow t \sin(\omega t + \phi)$
 \Rightarrow The system is **unstable**.

Fact: The system is stable if and only if all the poles of $G(s)$ are in the *LHP* !

Theorem: If $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s)$$

Example: $L\{e^{2t}\} = \frac{1}{s-2}$. Find $L\{te^{2t}\}$, $L\{t^2e^{2t}\}$

$$L\{te^{2t}\} = (-1) \frac{d}{ds} \left(\frac{1}{s-2} \right) = (-1) \frac{-1}{(s-2)^2} = \frac{1}{(s-2)^2}$$

$$L\{t^2e^{2t}\} = (-1)^2 \frac{d^2}{ds^2} \left(\frac{1}{s-2} \right) = \frac{d}{ds} \left(\frac{-1}{(s-2)^2} \right) = \frac{2(s-2)}{(s-2)^4} = \frac{2}{(s-2)^3}$$

$$\frac{d}{dt} \left(\frac{g}{f} \right) = \frac{g' \cdot f - g \cdot f'}{f^2}$$

$$L\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2} \rightarrow L\{t \sin \omega t\} = (-1) \frac{d}{ds} \left(\frac{\omega}{s^2 + \omega^2} \right) = \frac{2s\omega}{(s^2 + \omega^2)^2}$$

$$L\{\cos \omega t\} = \frac{s}{s^2 + \omega^2} \rightarrow L\{t \cos \omega t\} = (-1) \frac{d}{ds} \left(\frac{s}{s^2 + \omega^2} \right) = -\frac{(s^2 + \omega^2) - s(2s)}{(s^2 + \omega^2)^2} = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

$$Y(s) = \frac{2}{s^2 + 4} \frac{1}{s^2 + 4} = \left(-\frac{1}{4} \right) \left(\frac{s^2 - 4}{(s^2 + 4)^2} - \frac{1}{2} \frac{2}{s^2 + 4} \right)$$

$$\rightarrow y(t) = -\frac{1}{4} \left(t \cos 2t - \frac{1}{2} \sin 2t \right)$$

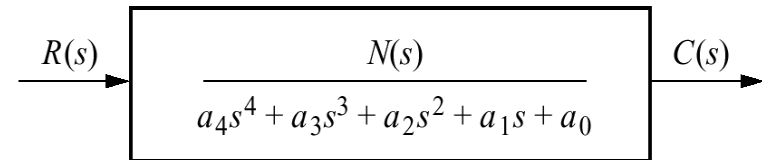
$$\begin{aligned} \leftarrow Y(s) &= \left(-\frac{1}{4} \right) \left(\frac{s^2 - 4 - (s^2 + 4)}{(s^2 + 4)^2} \right) \\ &= \left(-\frac{1}{4} \right) \left(\frac{-8}{(s^2 + 4)^2} \right) \\ &= \frac{2}{(s^2 + 4)^2} \end{aligned}$$

6.2 Routh-Hurwitz Criterion (Routh, 1905)

- In the closed-loop system,
 - How many poles are in the left half-plane, in the right half-plane, and in the $j\omega$ -axis.
 - *How many*, not where!
 - Routh Stability Test

• Generating a Basic Routh Table

- A closed-loop transfer function:
- Initial layout for Routh table



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

$$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$$

$$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$$

Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Routh Stability Test (Routh-Hurwitz Criterion)

Given $Q(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$

The Routh test tells how many zeros of Q are in the *RHP*.

$$\xrightarrow{R(s)} \left(\frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \right) \xrightarrow{C(s)}$$

Routh Table:

s^4	a_4	a_2	a_0	$b_2 = -\frac{1}{a_3} \begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}$	$b_3 = -\frac{1}{a_3} \begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}$
s^3	a_3	a_1	0		
s^2	$-\frac{1}{a_3} \begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix} = b_1$	b_2	b_3	$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}$	$c_2 = -\frac{1}{b_1} \begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}$
s^1	c_1	c_2	c_3		
s^0	d_1	d_2	d_3	$d_1 =$	

Routh Test:

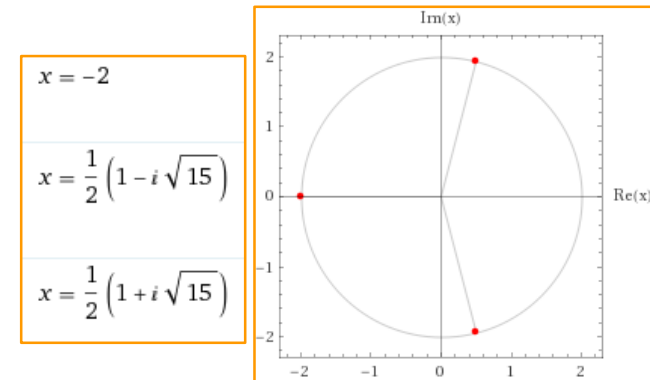
The number of *RHP* zeros of $Q(s)$
= the # of sign changes in the first column of the **Routh table**

Example: $Q(s) = s^3 + s^2 + 2s + 8$

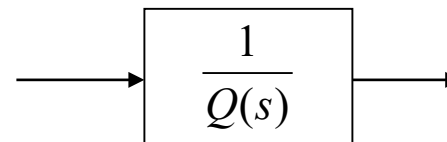
s^3	1	2
s^2	1	8
s^1	-6	
s^0	8	

2 RHP zeros →

The system is unstable since two poles exist in the RHP

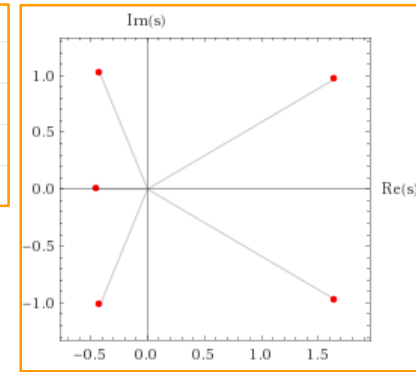


<https://www.wolframalpha.com>
solve $s^3 + s^2 + 2s + 8 = 0$

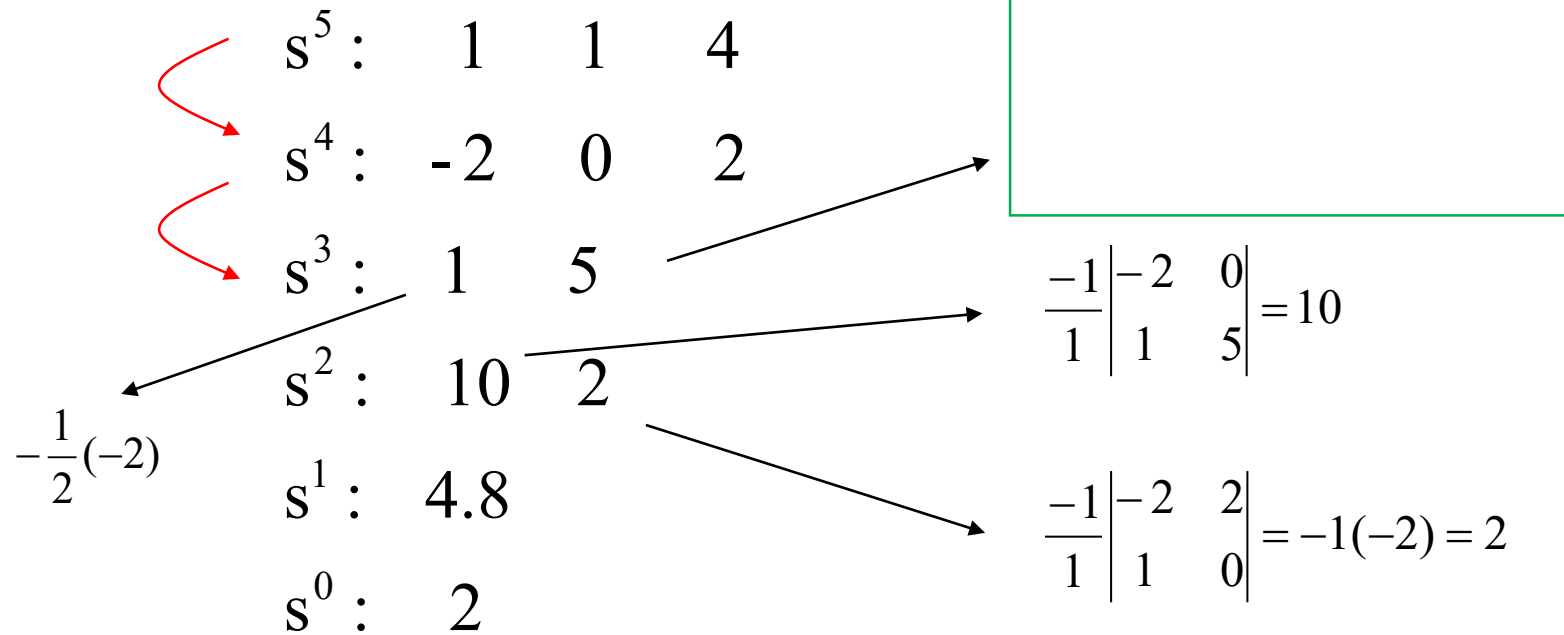


<https://www.wolframalpha.com>
 solve $s^5 - 2s^4 + s^3 + 4s + 2 = 0$

$s \approx -0.451516$
$s \approx -0.418382 - 1.020343 i$
$s \approx -0.418382 + 1.020343 i$
$s \approx 1.64414 - 0.96906 i$
$s \approx 1.64414 + 0.96906 i$

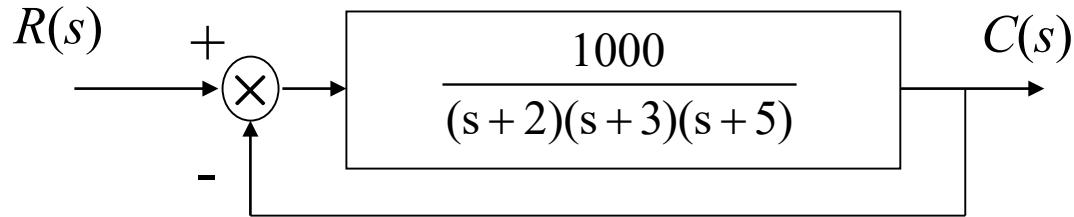


Example: $Q(s) = s^5 - 2s^4 + s^3 + 4s + 2$



$\Rightarrow 2$ RHP zeros

Example:



$$\frac{K}{1+KG}$$

$$R(s) \longrightarrow \left(\frac{1000}{s^3 + 10s^2 + 31s + 1030} \right) \longrightarrow C(s)$$

$$s^3 : 1 \quad 31$$

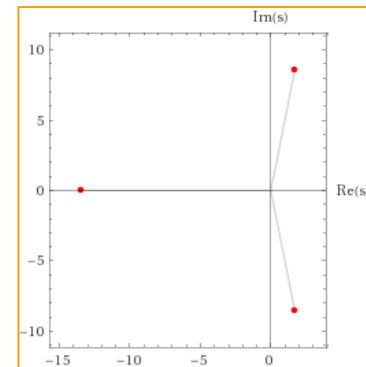
$$s^3 : \cancel{10} \quad 1 \quad \cancel{1030} \quad 103$$

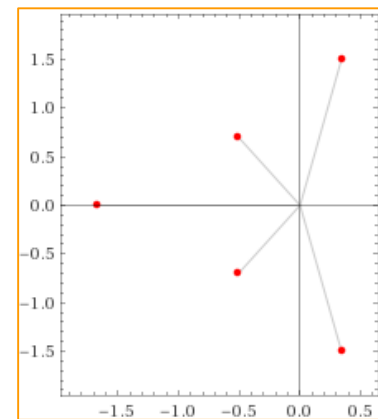
$$s^3 : \frac{-1}{1} (103 - 31) = -72, \quad \frac{-1}{1} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$s^3 : \frac{-1}{-72} \begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix} = 103, \quad \frac{-1}{-72} \begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix} = 0$$

2 RHP poles \Rightarrow unstable

<https://www.wolframalpha.com>
solve $s^3 + 10s^2 + 31s + 1030 = 0$





Example: Closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \rightarrow (1)$$

$$\begin{array}{rcl} s^5 & 1 & 3 \quad 5 \\ s^4 & 2 & 6 \quad 3 \\ s^3 & \cancel{0} \varepsilon & \frac{7}{2} \quad 0 \end{array}$$

$$s^2 \quad \frac{1}{\varepsilon}(6\varepsilon - 7) \quad \frac{1}{\varepsilon}(3\varepsilon - 0) = 3 \quad 0$$

$$s^1 \quad \frac{42\varepsilon - 49 - 6\varepsilon^2}{12\varepsilon - 14} \quad 0 \quad 0$$

$$s^0 \quad 3$$

$$s^5 \quad 1$$

$$s^4 \quad 2$$

$$s^3 \quad \varepsilon$$

$$s^2 \quad \frac{1}{\varepsilon}(6\varepsilon - 7)$$

$$s^1 \quad \frac{42\varepsilon - 49 - 6\varepsilon^2}{12\varepsilon - 14}$$

$$s^0 \quad 3$$

$$\varepsilon = + \quad \varepsilon = -$$

$$+ \quad +$$

$$+ \quad +$$

$$+ \quad -$$

$$- \quad +$$

$$+ \quad +$$

$$+ \quad +$$

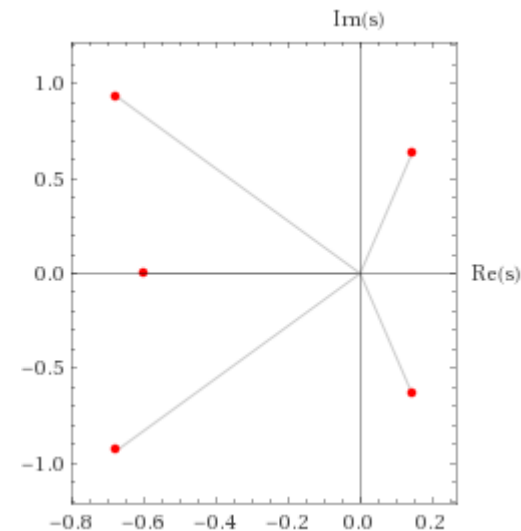
2 RHP poles \Rightarrow unstable

Example: Stability via reverse coefficient.

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

From (1) $\longrightarrow D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	-1.75		
s^0	1		

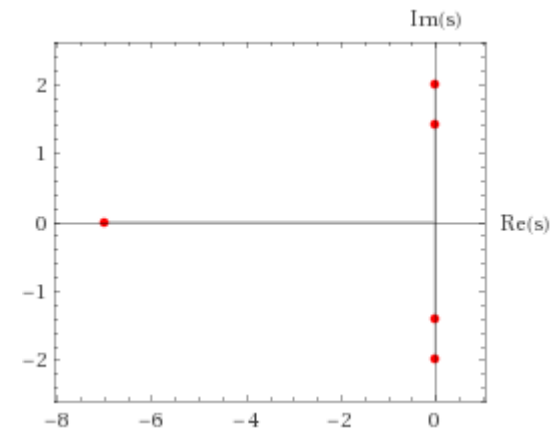


<https://www.wolframalpha.com>
 solve $3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1 = 0$

2 RHP poles \Rightarrow unstable

Example: Closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$



$$\begin{array}{c}
 \begin{array}{ccccccc}
 & s^5 & & 1 & & 6 & & 8 \\
 & \uparrow & & & & & & \\
 & s^4 & & 7 & & 1 & & 42 & & 6 & & 56 & & 8 \\
 \rightarrow & s^3 & & 0 & & 0 & & 0
 \end{array}
 \Rightarrow
 \begin{array}{l}
 (s^5 + 6s^3 + 8s) + 7(s^4 + 6s^2 + 8) \\
 = (s + 7) \underbrace{(s^4 + 6s^2 + 8)}_{P(s)}
 \end{array}
 \end{array}$$

$\downarrow \frac{d}{ds} P(s)$

$$\begin{array}{c}
 \overbrace{\frac{d}{ds} (s^4 + 6s^2 + 8)}^{P(s)} = 4s^3 + 12s + 0 \\
 = 4(s^3 + 3s)
 \end{array}$$

$$\begin{aligned}
 &(s^5 + 6s^3 + 8s) + 7(s^4 + 6s^2 + 8) \\
 &= (s + 7)(s^4 + 6s^2 + 8) \\
 &= (s + 7)(s^2 + 4)(s^2 + 2)
 \end{aligned}$$

	s^5	1	6	8	
	s^4	1	6	8	
	s^3	1	3	0	
	s^2	3	8		
	s^1	$+\frac{1}{3}$	0		
	s^0	8			

$$\begin{vmatrix} -1 & 1 & 6 \\ 1 & 1 & 3 \end{vmatrix} = 3$$

$$\begin{vmatrix} -1 & 1 & 3 \\ 3 & 3 & 8 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 1 & 8 \\ 1 & 1 & 0 \end{vmatrix} = 8$$

no sign change

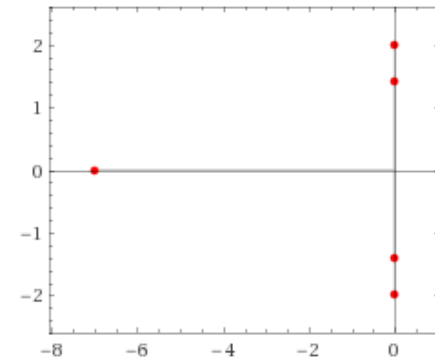
No RHP

No LHP $\Rightarrow \therefore$ roots are on the $j\omega$ - axis

\uparrow by symmetry

roots $-7, \pm 2j, \pm j1.14142$

Positive \Rightarrow No RHP poles



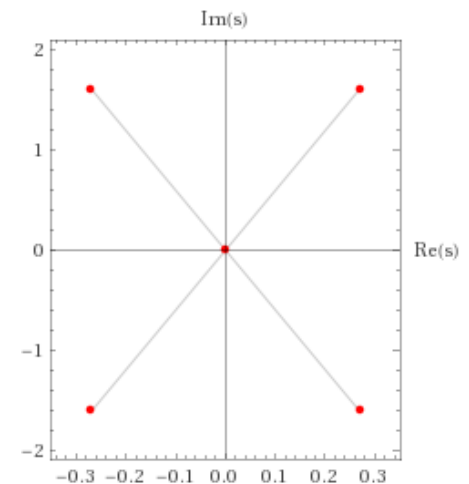
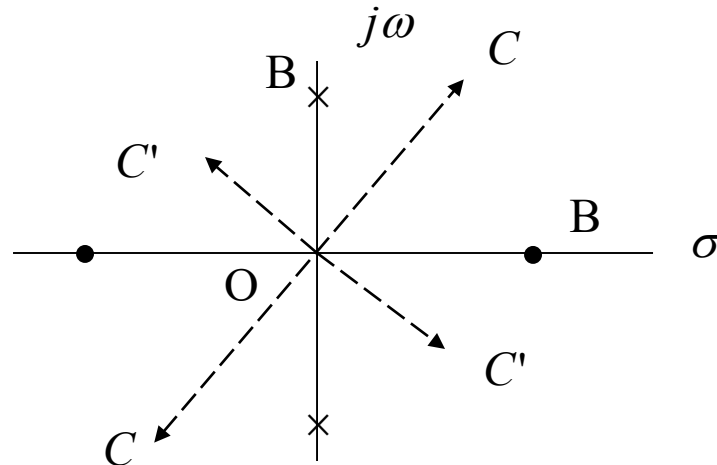
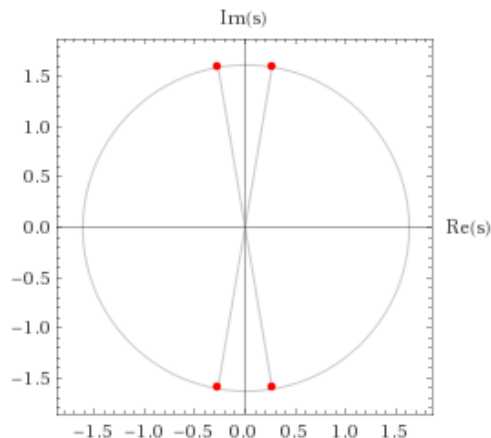
⇒ An entire row of zeros.

- even polynomial $s^4 + 5s^2 + 7 = 0 \rightarrow s = \begin{cases} -0.27 \pm j1.604 \\ +0.27 \pm j1.604 \end{cases}$

- odd polynomial $s^5 + 5s^3 + 7s = 0 \rightarrow s = \begin{cases} 0 \\ -0.27 \pm j1.604 \\ +0.27 \pm j1.604 \end{cases}$

constant term is always missing

Symmetrical about the origin



Example: $T(s) = \frac{20}{(s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20)}$

s^8	1	12	39	48	20	
s^7	1	22	59	38	0	
s^6	-10	-1	-20	-2	10	1
s^5	20	1	60	3	40	2
s^4	1	3	2	0		
s^3	0	2	0	3	0	0
s^2	3	3	2	4	0	
s^1	1	0				
s^0	4					

sign change

zeros row why?

$s^4 + 3s^2 + 2 = P(s)$

even poly.test

$\frac{d}{ds} P(s) = 4s^3 + 6s + 0$
 $= 2s(2s^2 + 3)$

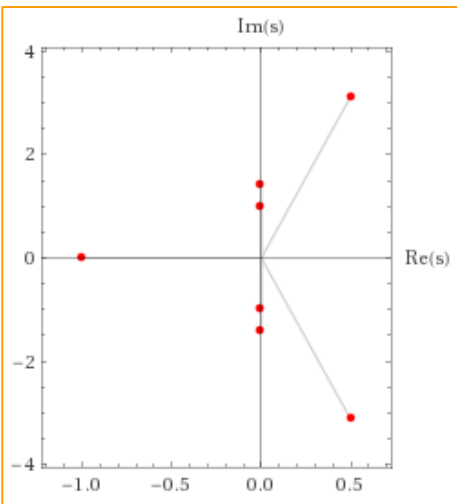
2poles. LHP
2poles. RHP

4 poles jw -axis

2 sign change \rightarrow two poles in RHP
 No sign change after $s^4 \rightarrow$ 4 poles are on the jw -axis

- Summary of pole locations for Example

Even (4th order)	Rest (4th order)	Total (8th order)
$s^4 \sim s^0$	$s^8 \sim s^5$	
0 RHP	2RHP	2RHP
0 LHP	2LHP	2LHP
$4j\omega$	$0j\omega$	$4j\omega$



```
solve s^8 + s^7 + 12 s^6 + 22 s^5 + 39 s^4 + 59 s^3 + 48 s^2 + 38 s + 20 = 0
Results:
s = -1
s = ±i
s = ±(i√2)
s = 1/2 (1 - i√39)
s = 1/2 (1 + i√39)
```

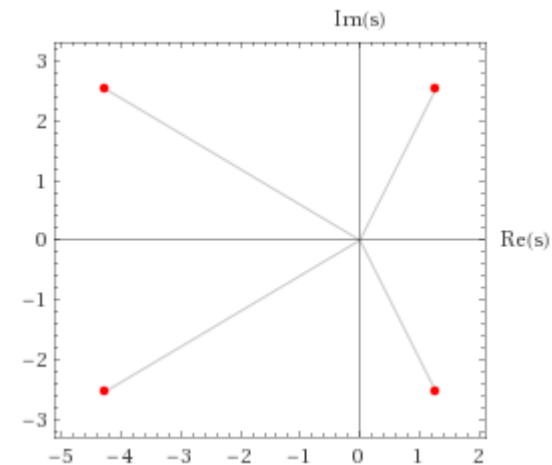
```
>> roots([1 12 22 39 59 48 38 20])
0.5000 + 3.1225i
0.5000 - 3.1225i
0.0000 + 1.4142i
0.0000 - 1.4142i
-1.0000 + 0.0000i
-1.0000 + 0.0000i
-0.0000 + 1.0000i
-0.0000 - 1.0000i
```

Example: Closed transfer function $T(s)$

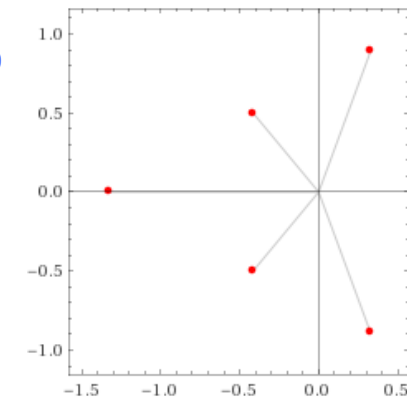
$$T(s) = \frac{1}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

s^4	1	11	200
s^3	1	1	
s^2	1	20	
s^1	-19		
s^0	20		

2 LHP poles
2 RHP poles \rightarrow unstable



<https://www.wolframalpha.com>
solve $s^4 + 6s^3 + 11s^2 + 6s + 200 = 0$



Example: $T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$

Reverse order

$$s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2$$

$$\begin{array}{r|rrr}
 s^5 & 2 & 2 & 2 \\
 s^4 & 3 & 3 & 1 \\
 + s^3 & \cancel{0} \varepsilon & \frac{4}{3} & \\
 - s^2 & \frac{3\varepsilon - 4}{\varepsilon} & 1 & \\
 + s^1 & \frac{(12\varepsilon - 16 - 3\varepsilon^2)}{(9\varepsilon - 12)} & & \\
 s^0 & 1 & &
 \end{array}$$

\Downarrow

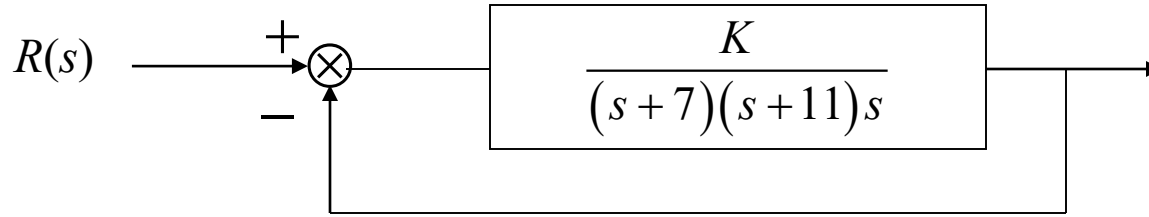
2 RHP poles
 3 LHP poles

→ unstable

⇐

$$\begin{array}{r|rrr}
 s^5 & 1 & 3 & 3 \\
 s^4 & 2 & 2 & 2 \\
 s^3 & 2 & 2 & \\
 s^2 & \cancel{0} \varepsilon & 2 & \\
 s^1 & \frac{2\varepsilon - 4}{\varepsilon} & & \\
 s^0 & 2 & &
 \end{array}$$

Stability Design via Routh- Hurwitz



$$\begin{aligned} \frac{G}{1+G} &= \frac{\frac{K}{(s+7)(s+11)s}}{1 + \frac{K}{(s+7)(s+11)s}} \\ &= \frac{K}{(s+7)(s+11)s + K} \\ &= \frac{K}{s^3 + 18s^2 + 77s + K} \end{aligned}$$

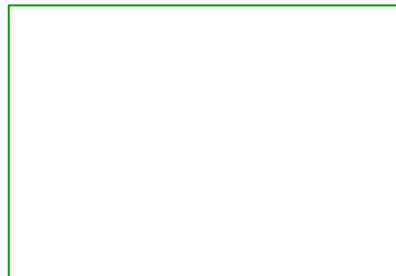
$$\Rightarrow \text{Closed loop T.F.: } T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

$$s^3 \quad 1 \quad 77$$

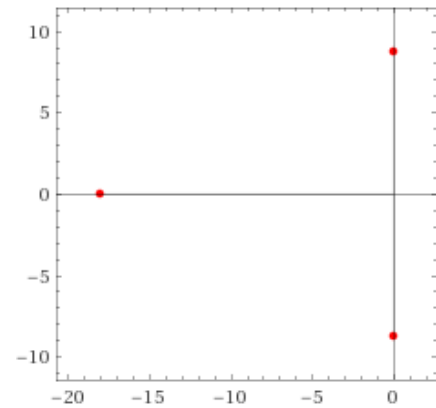
$$s^2 \quad 18 \quad K$$

$$s^1$$

$$s^0$$



$$\begin{aligned} &\left| \begin{array}{cc} 1 & 77 \\ 18 & K \end{array} \right| \\ &\quad -18 \\ &= \frac{K - 1386}{-18} \end{aligned}$$



- If $K=1386$

s^3	1	77		
s^2	18	1386	→	$p(s) = 18s^2 + 1386$
s^1	0	36	←	$\frac{d}{ds} p(s) = 36s$
s^0	1386			

2 poles on the $j\omega$ -axis → marginally stable

Chapter 6. Stability -38-