Chapter 4. Frequency Analysis: The Fourier Series

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CONTENTS

Introduction

- This chapter considers
  - **Spectral representation**: The frequency representation of periodic and aperiodic signals indicates **how their power or energy is allocated to different frequencies**.
    - **The spectrum of a periodic signal is discrete**, as its power is concentrated at frequencies **multiples of fundamental frequency**.
    - **The spectrum of a aperiodic signal is a continuous function** of frequency.
  - **Eigenfunctions and Fourier analysis**: **Complex exponentials and sinusoids** are used in the Fourier representation of signals by taking advantage of the **eigenfunction property of LTI systems**.
  - **Steady-state analysis**: **Fourier analysis is in the steady state**, while Laplace analysis considers both transient and steady state.
  - **Application of Fourier analysis**: The frequency representation of signals and systems is extremely **important in signal processing and in communications**.

Eigenfunctions Revisited

If $x(t) = e^{j\Omega_0 t}, -\infty < t < \infty$ is the input to a **causal** and a **stable system** with **impulse response $h(t)$**, the output in the steady state is given by

$$\boldsymbol{y(t) = e^{j\Omega_0 t} H(j\Omega_0)}$$

where

$$\boldsymbol{H(j\Omega_0) = \int_0^{\infty} h(\tau) e^{-j\Omega_0 \tau}\, d\tau}$$

is the **frequency response** of the system at Ω_0.

- The **input signal $x(t)$** is a **linear combination of complex exponentials**, with different amplitudes, frequencies, and phases, or

$$\boldsymbol{x(t) = \sum_k X_k\, e^{j\Omega_k t}} \implies \boldsymbol{y(t) = \sum_k X_k\, e^{j\Omega_k t} H(j\Omega_k)}$$

Eigenfunctions Revisited

– The input signal is an **integral (a sum, after all)** of complex exponentials,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}\,d\Omega \implies y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}H(j\Omega)\,d\Omega$$

– The **frequency response** is $H(j\Omega) = H(s)|_{s=j\Omega}$.

- Its **magnitude** is an **even function of frequency, or** $|H(j\Omega)| = |H(-j\Omega)|$.
- Its **phase** is an **odd function of frequency, or** $\angle H(j\Omega) = -\angle H(-j\Omega)$.

– One application of LTI systems is **filtering**, where one is interested in **preserving desired frequency** and **getting rid of less-desirable components** of a signal.

Phasor Interpretation of Eigenfunction Property

For a **stable LTI system** with **transfer function $H(s)$**, if the input is

$$x(t) = A\cos(\Omega_0 t + \theta) = Re[Xe^{j\Omega_0 t}]$$

where $X = Ae^{j\theta}$ is the phasor of $x(t)$, the steady-state output of the system is

$$y(t) = Re\left[XH(j\Omega_0)e^{j\Omega_0 t}\right] = Re\left[AH(j\Omega_0)e^{j(\Omega_0 t+\theta)}\right]$$
$$= A|H(j\Omega_0)|\cos(\Omega_0 t + \theta + \angle H(j\Omega_0))$$

where the **frequency response** of the system at Ω_0 is

$$H(j\Omega_0) = H(s)\Big|_{s=j\Omega} = Y/X$$

[Ex 4.1] Consider the RC circuit with R = 1Ω and C = 1F. Let voltage source be $v_s(t) = 4\cos(t + \pi/4)$ volts, find the steady-state voltage across the capacitor.

Complex Exponential Fourier Series

- **Fourier series** is a **representation of a periodic signal $\boldsymbol{x(t)}$** in terms of complex exponentials or sinusoids of **frequency multiples of its fundamental frequency**.
  - Consider a set of complex functions $\{\psi_k(t)\}$ defined in $[a, b]$. Let's say $\psi_l(t)$ and $\psi_m(t)$, $l \neq m$, is called **orthonormal**, iff

$$\int_a^b \psi_l(t)\psi_m^*(t)\ dt = \begin{cases} 0 & l \neq m \\ 1 & l = m \end{cases}$$

  - A finite energy signal $x(t)$ defined in $[a, b]$ can be approximated by a series

$$\hat{x}(t) = \sum_k a_k \psi_k(t)$$

  - We consider a **periodic signal $\boldsymbol{x(t)}$** such that
    - It is defined for $-\infty < t < \infty$.
    - For any integer k, $x(t + kT_0) = x(t)$, where T_0 is the **fundamental period**.

Complex Exponential Fourier Series

The **Fourier series representation** of a periodic signal $x(t)$ of period T_0 is an **infinite sum of weighted complex exponentials** with **frequency multiples of the signal's fundamental frequency $\Omega_0 = 2\pi/T_0$**, or

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \qquad \Omega_0 = \frac{2\pi}{T_0}$$

where the **Fourier coefficients X_k** are found according to

$$X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jk\Omega_0 t}\, dt$$

for $k = 0, \pm 1, \pm 2, \cdots$ and any t_0.

Parseval's Theorem

- Periodic signals are **infinite-energy finite-power signals**.
  - **Fourier series** provides a way to find **how much of the signal power is in a certain band of frequencies**.

The **power P_x of a periodic signal $x(t)$ of period T_0** can be equivalently calculated in either the time or the frequency domain:

$$P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \sum_{k} |X_k|^2$$

- The power of the signal is distributed over the **harmonic frequencies $\{k\Omega_0\}$**.
  - Given the **discrete nature** of the harmonic frequencies, this plot consists of a line at each frequency, which is called the **power line spectrum**.

Symmetry of Line Spectra

For a **real-valued periodic signal $x(t)$ of period T_0**, the **Fourier coefficients $\{X_k = |X_k|e^{j\angle X_k}\}$** at **harmonic frequencies $\{k\Omega_0 = 2\pi k/T_0\}$** we have that $X_k = X^*_{-k}$. Equivalently, $|X_k| = |X_{-k}|$ (**even function**) and $\angle X_k = -\angle X_{-k}$ (**odd function**).

Trigonometric Fourier Series

The **trigonometric Fourier series** of a **real-valued periodic signal $x(t)$ of period T_0**, is given by

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \theta_k)$$

$$= X_0 + 2\sum_{k=1}^{\infty} [c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t)] \qquad \Omega_0 = \frac{2\pi}{T_0}$$

where $X_0 = c_0$ is the **DC component** and $2|X_k| \cos(k\Omega_0 t + \theta_k)$ are the **k-th harmonics**.

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) \cos(k\Omega_0 t)\, dt \quad k = 0, 1, \ldots$$

$$d_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) \sin(k\Omega_0 t)\, dt \quad k = 1, 2, \ldots$$

Example

[Ex 4.4] Find the Fourier series of a raised-cosine signal ($B \geq A$),

$$x(t) = B + A\ \cos(\Omega_0 t + \theta)$$

which is periodic with period T_0 and fundamental frequency $\Omega_0 = 2\pi/\mathrm{T}_0$.
Call

$$y(t) = B + A\ \cos(\Omega_0 t - \pi/2),$$

then find its Fourier series coefficients and compare them to those for $x(t)$.
Use symbolic MATLAB to compute the Fourier series of $y(t) = 1 + \sin(100t)$, find and plot its magnitude and phase line spectra.

Fourier Coefficients from Laplace

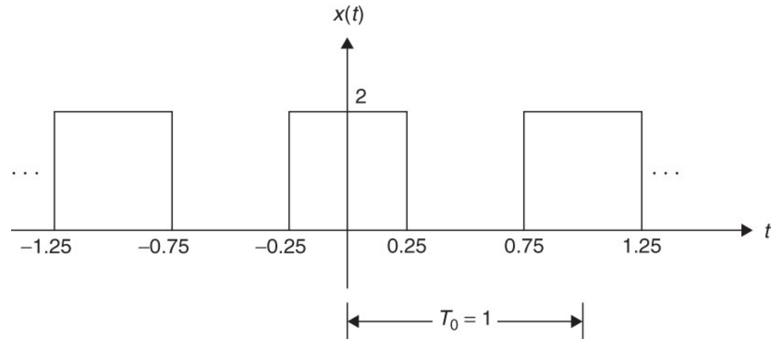
For a **periodic signal $\boldsymbol{x(t)}$ of period $\boldsymbol{T_0}$**, if we know or easily **compute the Laplace transform of a period of $\boldsymbol{x(t)}$**

$$\boldsymbol{x_1(t)} = \boldsymbol{x}(t)[u(t-t_0) - u(t-t_0-T_0)]$$

Then the **Fourier coefficients** of $\boldsymbol{x(t)}$ are given by

$$\boldsymbol{X_k = \frac{1}{T_0}\mathcal{L}[x_1(t)]_{s=jk\Omega_0} \qquad \Omega_0 = \frac{2\pi}{T_0}}$$

[Ex 4.5] Consider the periodic pulse train $x(t)$ of period $T_0 = 1$, shown below. Find its Fourier series.



Reflection and Even/Odd Periodic Signals

The **Fourier coefficients of $x(-t)$** are $\{X_{-k}\}$.

The **Fourier coefficients X_k of even periodic signal $x(t)$** are **real**, and its **trigonometric Fourier series** is

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} X_k \cos(k\Omega_0 t)$$

The **Fourier coefficients X_k of odd periodic signal $x(t)$** are **imaginary** and its **trigonometric Fourier series** is

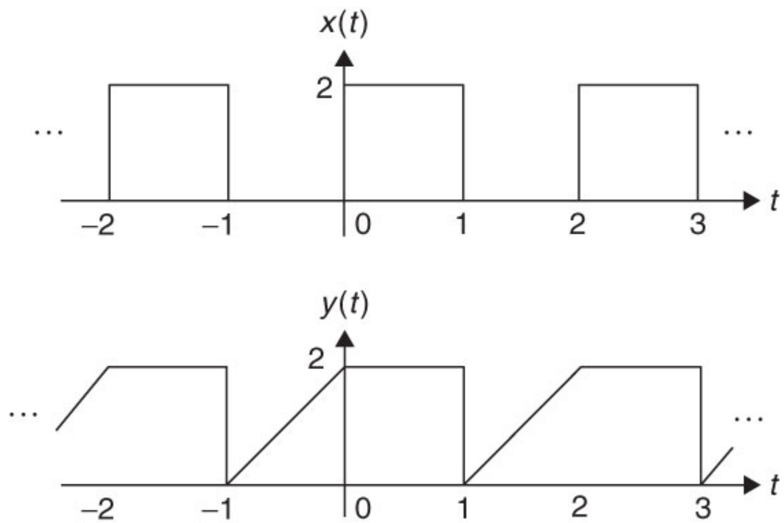
$$x(t) = 2\sum_{k=1}^{\infty} jX_k \sin(k\Omega_0 t)$$

For any periodic signal $x(t) = x_e(t) + x_o(t)$, then

$$X_k = X_{ek} + X_{ok} = 0.5[X_k + X_{-k}] + 0.5[X_k - X_{-k}]$$

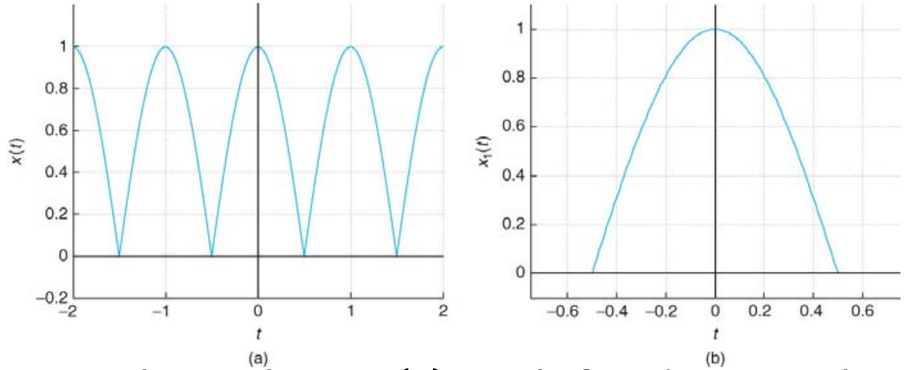
Examples

[Ex 4.6] Consider the periodic signals $x(t)$ and $y(t)$ shown in below. Determine their Fourier coefficients by using the symmetry conditions and the even-odd decomposition.

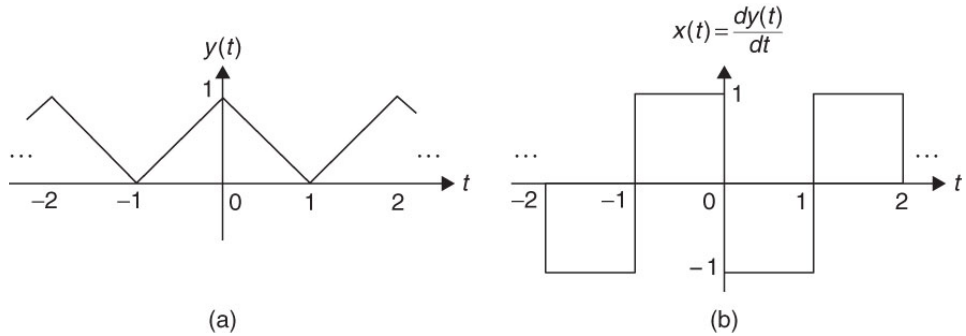


Examples

[Ex 4.7] Find the Fourier series of full-wave rectified signal $x(t) = |\cos \pi t|$ shown below.



[Ex 4.8] Consider a train of triangular pulses $y(t)$ with fundamental period $T_0 = 2$. Let $x(t) = dy(t)/dt$. Find its Fourier series and compare $|X_k|$ and $|Y_k|$ to determine which of the signals is smoother.



Convergence of Fourier Series

- A signal $x(t)$ is said to be **piecewise smooth** if it has a **finite number of discontinuities**, while a **smooth signal has a derivative** that changes continuously.

The Fourier series of a **piecewise smooth periodic signal $x(t)$** converges for all values of t. The **sufficient condition of the Fourier series convergence** is (over a one period)

– Be **absolutely integrable**

– Have a **finite number of maxima, minima, and discontinuities**.

The infinite series **equals $x(t)$ at every continuity point** and **equals the average $0.5[x(t+0^+)+x(t+0^-)]$** of the right limit $x(t+0^+)$ and the left limit $x(t+0^-)$ **at every discontinuity point**.

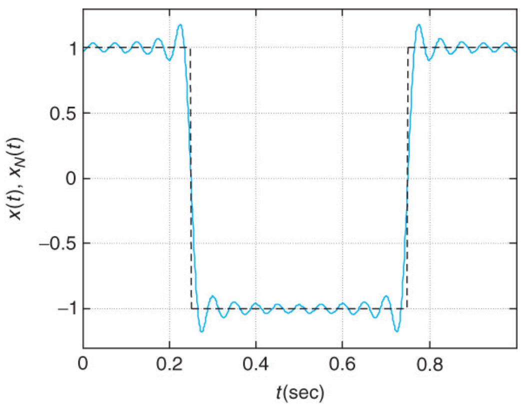
Gibb's Phenomenon

- Although the Fourier series converges to **the arithmetic average at discontinuities**, there is a **ringing** before and after the discontinuity point, called **Gibb's phenomenon**.
  - **The smoother the signal $x(t)$ is, the easier it is to approximate it** with a Fourier series with a finite number of terms.
  - The N-th order approximation of a periodic signal $x(t)$ is

$$X_N(t) = \sum_{k=-N}^{N} X_k e^{jk\Omega_0 t}$$

  - The average quadratic error over a period is

$$E_N = \frac{1}{T_0} \int_{T_0} [x(t) - x_N(t)]^2 \ dt$$

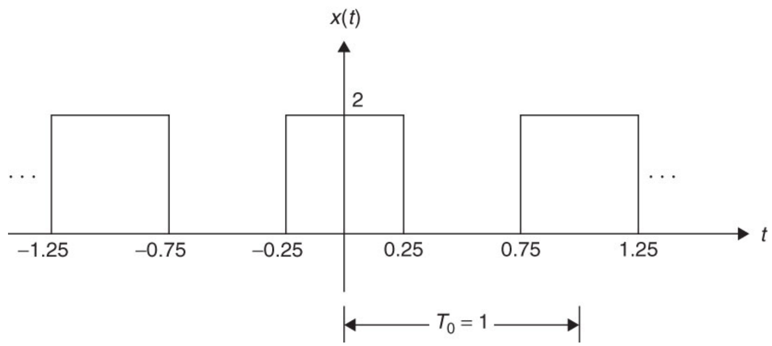


$N = 20$

Examples

[Ex 4.11] We wish to approximate $x(t)$ below by $x_2(t) = \alpha + 2\beta\cos\Omega_0 t$.
Compute α and β to minimize the mean-square error

$$E_2 = \frac{1}{T_0}\int_{T_0} [x(t) - x_2(t)]^2 \, dt$$



Time and Frequency Shifting

Time shifting causes **only a change in phase**.

- A periodic signal $x(t)$ of period T_0 remains periodic of the same period when shifted in time. If X_k are the Fourier coefficients of $x(t)$, the **Fourier coefficients for $x(t-t_0)$** are

$$\left\{ \boldsymbol{X_k e^{-jk\Omega_0 t_0} = |X_k| e^{j(\angle X_k - k\Omega_0 t_0)}} \right\}$$

When a periodic signal $x(t)$ of period T_0 **modulates a complex exponential $e^{j\Omega_1 t}$** (**frequency shifting**).

- If $\boldsymbol{\Omega_1 = M\Omega_0}$ for an integer $\boldsymbol{M \geq 1}$, the modulated signal $\boldsymbol{x(t)e^{j\Omega_1 t}}$ is **periodic with period T_0**.
- The **Fourier coefficients X_k** are shifted to frequencies $\boldsymbol{k\Omega_0 + \Omega_1}$.
- The modulated signal is real-valued by $\boldsymbol{x(t)\cos\Omega_1 t}$.

Response of LTI Systems to Periodic Signals

If the input $x(t)$ of a causal and stable LTI system, with impulse response $h(t)$, is periodic of period T_0 and has the Fourier series

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \angle X_k) \qquad \Omega_0 = \frac{2\pi}{T_0}$$

the **steady-state response** of the system is

$$y(t) = X_0 |H(j0)| \cos(\angle H(j0)) + 2\sum_{k=1}^{\infty} |X_k||H(jk\Omega_0)| \cos(k\Omega_0 t + \angle X_k + \angle H(jk\Omega_0))$$

where

$$H(jk\Omega_0) = \int_0^{\infty} h(\tau) e^{-jk\Omega_0 \tau}\, d\tau$$

is the frequency response of the system at $k\Omega_0$.

Sum of Periodic Signals

If $x(t)$ and $y(t)$ are periodic signals with the same fundamental frequency Ω_0. The **Fourier series coefficients of $\boldsymbol{z(t) = \alpha x(t) + \beta y(t)}$** are $\boldsymbol{Z_k = \alpha X_k + \beta Y_k}$.

If $x(t)$ is periodic of period T_1 and $y(t)$ is periodic of period T_2 such that $T_2/T_1 = N/M$ for non-divisible integers N and M. The **Fourier series coefficients of $\boldsymbol{z(t) = \alpha x(t) + \beta y(t)}$** are periodic of period $\boldsymbol{T_0 = MT_2 = NT_1}$, are $\boldsymbol{Z_k = \alpha X_{k/N} + \beta Y_{k/M}}$ for **integers $\boldsymbol{k}$ such that $\boldsymbol{k/N}$, and $\boldsymbol{k/M}$ are integers**.

[Ex 4.15] Consider the sum $z(t)$ of a periodic signal $x(t)$ of period $T_1 = 2$, with a periodic signal $y(t)$ with period $T_2 = 0.2$. Find the Fourier coefficients Z_k of $z(t)$ in terms of the Fourier coefficients X_k and Y_k.

Multiplication of Periodic Signals

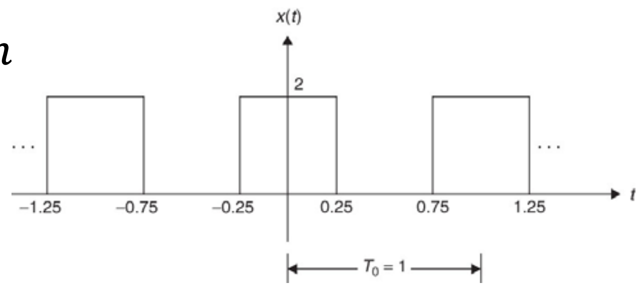
If $x(t)$ and $y(t)$ are periodic signals of the same period T_0, then their product $z(t) = x(t)y(t)$ is also periodic of period T_0, and with Fourier coefficients that are the convolution sum of the Fourier coefficients of $x(t)$ and $y(t)$:

$$\boldsymbol{Z_k = \sum_{l} X_l Y_{k-l}}$$

[Ex 4.16] Consider the train of rectangular pulses $x(t)$ shown below. Let $z(t) = 0.25\, x^2(t)$. Use the Fourier series $z(t)$ to show that

$$X_k = \alpha \sum_{m} X_m X_{k-m}$$

for some constant α. Determine α.



Derivatives and Integrals of Periodic Signals

The derivative $dx(t)/dt$ of periodic signal $x(t)$, of period T_0, is periodic of the same period T_0. If $\boldsymbol{X_k}$ are the **coefficients of the Fourier series of $\boldsymbol{x(t)}$**, the **Fourier coefficients of $\boldsymbol{dx(t)/dt}$** are $\boldsymbol{jk\Omega_0 X_k}$.

For a **zero-mean**, periodic signal $y(t)$, of period T_0, the **integral**

$$\boldsymbol{z(t) = \int_{-\infty}^{t} y(\tau)\, d\tau}$$

is periodic of the same period as $y(t)$, with **Fourier coefficients**

$$\boldsymbol{Z_k = \frac{Y_k}{jk\Omega_0} \quad k \neq 0, \qquad Z_0 = -\sum_{m \neq 0} Y_m \frac{1}{jm\Omega_0} \quad \Omega_0 = \frac{2\pi}{T_0}}$$

Examples

[Ex 4.17] Let $g(t)$ be the derivative of a triangular train of pulses $f(t)$, of period $T_0 = 1$. The period of $f(t)$, $\ 0 \leq t \leq 1$, is

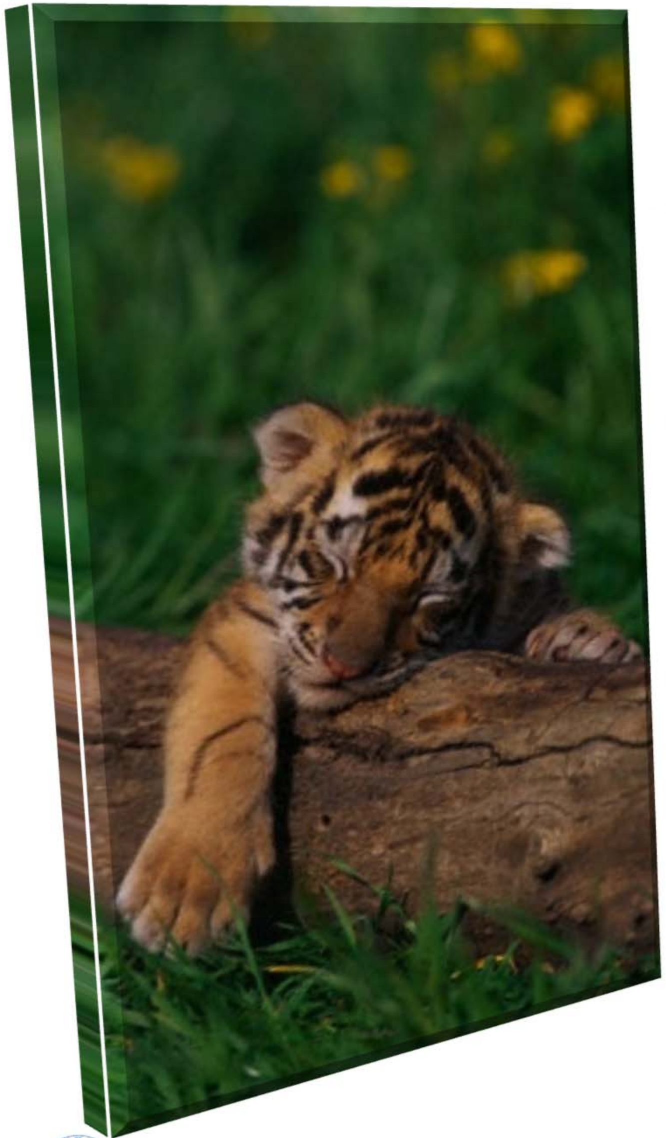
$$f_1(t) = 2r(t) - 4r(t - 0.5) + 2r(t - 1)$$

Use the Fourier series of $g(t)$ to find the Fourier series of $f(t)$.

[Ex 4.18] Find the Fourier coefficients Z_k of $z(t)$

$$z(t) = \int_{-\infty}^{t} g(\tau)\ d\tau$$

where $g(t)$ is defined in [Ex 4.17]



Thank You