Chapter 2. Continuous-Time Systems

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Introduction

- **Systems and their classification**
  - The concept of **system** is useful in dealing with actual devices or processes for purposes of analysis and synthesis.

- **Linear time-invariant (LTI) systems**
  - We propose the **LTI model** as a mathematical idealization of the behavior of systems.

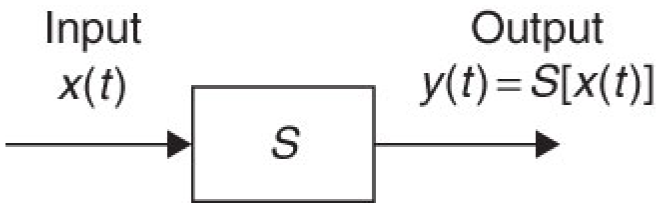
- **Convolution integral, causality, and stability**
  - **Causality**, or **non-anticipatory behavior of the system**, relates to the **cause-effect relationship** between the input and the output.

System Concept

- We view a **system** as **a mathematical transformation of an input signal into an output signal**.
  - A system can be considered as a connection of subsystems.
- System classification
  - A **dynamic system** has the capability of **storing energy or remembering its state**, while a **static system does not**.
  - In the **lumped-parameter systems**, the behavior of spatially distributed systems is modeled as a **lumped element**.
  - A system is **passive** if it is **not able to deliver energy to the outside** world.
- We consider only **dynamic systems** with **lumped parameters** possibly changing with time, with **a single input and a single output**.

LTI Continuous-Time Systems

A **continuous-time system** is a **system in which the signals at its input and output are continuous-time signals.** Mathematically, we represent it as a transformation S that converts an input signal $x(t)$ into an output signal $\boldsymbol{y(t) = S[x(t)]}$.



- Characteristics of continuous-time systems
  - **Linearity**, **time invariance**, **causality**, and **stability**

Linearity

A system represented by S is said to be **linear** if for inputs $x(t)$ and $v(t)$, and any constants α and β, superposition holds, i.e.,

$$\boldsymbol{S[\alpha x(t)+\beta v(t)]} = S[\alpha x(t)] + S[\beta v(t)]$$
$$= \boldsymbol{\alpha S[x(t)] + \beta S[v(t)]}$$

– **Linearity = Additivity + Scaling**

[Ex 2.1 – **Biased Averager**] For an input $x(t)$, the output $y(t)$ of such a system is given by

$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\, d\tau \;+ B.$$

Is this system linear? If not, is there a way to make it linear? Explain.

Example: Op-Amp

[Ex 2.2] Consider the following input-output relations that show the corresponding systems are nonlinear:

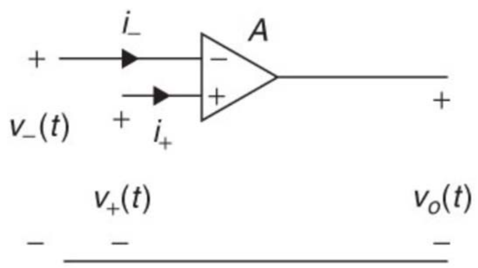
$$y(t) = |x(t)|$$

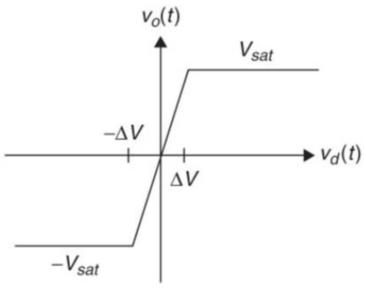
$$z(t) = \cos(x(t)) \text{ assuming } |x(t)| \leq 1$$

$$v(t) = x^2(t)$$

[Ex 2.3] Consider each of the components of an RLC circuit and determine under what conditions they are linear.

- **Operational Amplifier (Op-Amp)**
  - Input: $v_-(t)$ inverting terminal and $v_+(t)$: noninverting terminal
  - Output: $\boldsymbol{v_o(t) = f[v_+(t) - v_-(t)] = f\left(v_d(t)\right) = Av_d(t)}$





Time Invariance

- Ideal Op-Amp
  - Assuming that $\boldsymbol{A} \to \infty$ **and** $\boldsymbol{R} \to \infty$

$$i_- = i_+ = 0$$
$$v_d(t) = v_+(t) - v_-(t) = 0$$
$$-V_{sat} \leq v_o(t) \leq V_{sat}$$

A continuous-time system S is **time invariant** if whenever for an input $x(t)$ with a corresponding output $S[x(t)]$, the output corresponding to **a shifted input $\boldsymbol{x(t \pm \tau)}$** is the **original output shifted in time $\boldsymbol{S}[\boldsymbol{x(t \pm \tau)}]$**, $i.e.$,

$$x(t) \quad \Longrightarrow \quad y(t) = S[x(t)]$$
$$\boldsymbol{x(t \pm \tau)} \quad \Longrightarrow \quad \boldsymbol{y(t \pm \tau) = S[x(t \pm \tau)]}$$

The system **does not age –** its parameters are **constant!**

Example: AM Communication Systems

- Remarks
  - Linearity and time invariance are **independent of each other**.
  - Most actual systems are **nonlinear** and **time varying**.
    - **Linear models** are used to approximate **around an operating point of the nonlinear behavior**, while **time-invariant models** are used to approximate in **short segments of the system's time-varying behavior**.

- A direct transmission of our voice $m(t)$ requires **huge antenna**.
  - **AM communication systems** modulates this signal into

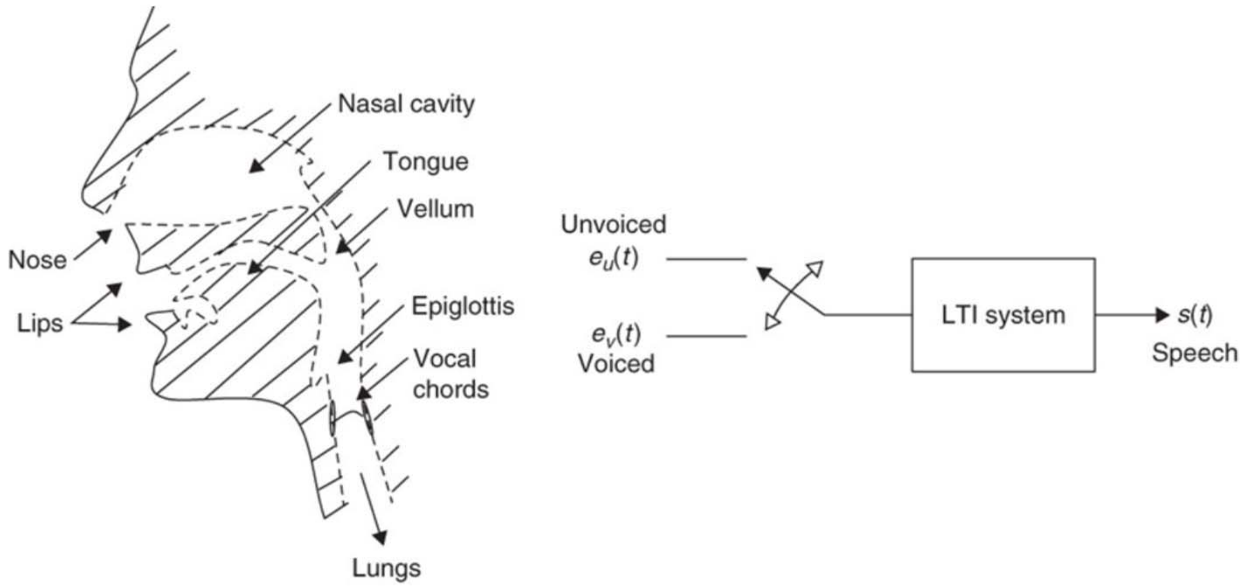
$$y(t) = m(t)\cos\Omega_0 t$$: **linear** and **time varying**

  - **FM communication systems** modulates this signal into

$$z(t) = \cos(\Omega_c t + \int_{-\infty}^{t} m(\tau)\ d\tau)$$: **nonlinear**

Example: Vocal System

- A **typical vocal system** is modeled as a distributed system and represented by **partial differential equations**.
  - A typical LTI model for speech production considers segments of speech of about 20 msec, and **for each develops a low-order LTI system.**



Examples

[Ex 2.4] Characterize time-varying resistors, capacitors and inductors. Assume zero initial conditions in the capacitors and inductors.

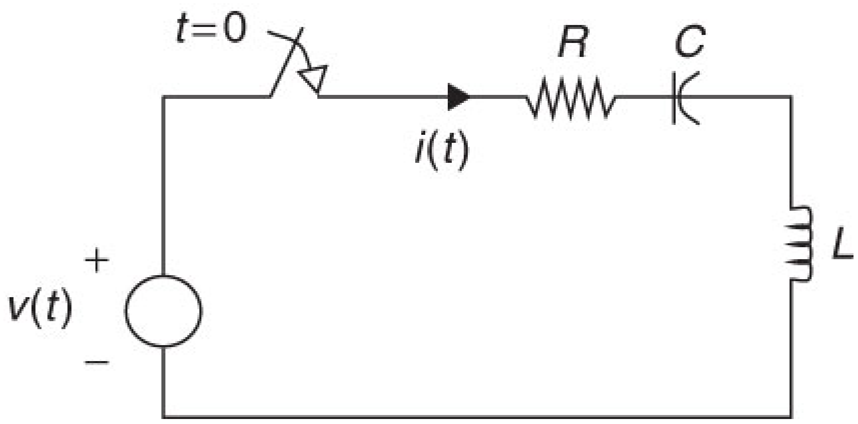
[Ex 2.5] Consider constant linear capacitors and inductors, represented by differential equations

$$\frac{dv_c(t)}{dt} = \frac{1}{C} i(t), \qquad \frac{di_L(t)}{dt} = \frac{1}{L} v(t)$$

with initial conditions $v_c(0) = 0$ and $i_L(0) = 0$. Under what conditions are these time-invariant systems?

Example: RLC Circuits

- If the **initial conditions of the RLC circuit are zero**, and **the input is zero for $t < 0$**, then the system represented by the linear differential equation with constant coefficients is **LTI**.



$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(\tau)\ d\tau$$

$$\frac{dv(t)}{dt} = R\frac{di(t)}{dt} + L\frac{d^2 i(t)}{dt^2} + \frac{1}{C}i(t)$$

Systems in forms of Differential Equations

Given a **dynamic system** represented by a **linear differential equations** with constant coefficients,

$$\boldsymbol{a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + \frac{d^N y(t)}{dt^N}}$$
$$\boldsymbol{= b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_M \frac{d^M x(t)}{dt^M}}, \qquad t \geq 0$$

with **N conditions $y(0), d^k y(t)/dt^k|_{t=0}$** for $k = 0, \cdots, N-1$ and **input $x(t) = 0$ for $t < 0$**, its **complete response $y(t)$ for $t \geq 0$** has two components:

- The **zero-state response $y_{zs}(t)$** due exclusively to the input, because the initial conditions are zero.
- The **zero-input response $y_{zi}(t)$** due exclusively to the initial conditions, because the input is zero.

Systems in forms of Differential Equations

- Most **continuous-time dynamic systems** with lumped parameters are represented by **linear ordinary differential equations with constant coefficients**.
  - Defining derivative operator as $D^n[y(t)] = \frac{d^n y(t)}{dt^n}$ $\quad n > 0$, integer, the differential equation of the system is written by

$$(a_0 + a_1 D + \cdots + D^N)[y(t)] = (b_0 + b_1 D + \cdots + b_M D^M)[x(t)], \qquad t \geq 0$$

  - This system is **LTI** if **the initial conditions as well as the input are zero for $t < 0$** – that is, the system is not energized for $t < 0$.

$$(a_0 + a_1 D + \cdots + D^N)[y_{zi}(t)] = 0, \qquad D^k[y_{zi}(t)]_{t=0} \; k = 0, \cdots, N-1$$

$$(a_0 + a_1 D + \cdots + D^N)[y_{zs}(t)] = (b_0 + b_1 D + \cdots + b_M D^M)[x(t)]$$

  - The **characteristic polynomial** is

$$\boldsymbol{a_0 + a_1 s + \cdots + s^N = \prod_k (s - p_k)}$$

eigenvalues

Analog Mechanical Systems

[Ex 2.7] Consider a circuit that is a series of a resistor $R = 1\ \Omega$ and an inductor $L = 1\ H$, with a voltage source $v(t) = Bu(t)$, and I_o amps is the initial current in the inductor. Find and solve the differential equation for $B = 1$ and $B = 2$ for initial condition $I_o = 1$ and $I_o = 0$, respectively. Determine the zero-input and the zero-output responses. Under what conditions is the system linear and time invariant?

- Analog mechanical systems
  - The applied force $f(t)$ equals the sum of the forces generated by the mass M and the damper D

$$f(t) = M\frac{dw(t)}{dt} + Dw(t)$$

  - If **the initial velocity and the external force are zero for $t < 0$**, the above differential equation represents a **LTI mechanical system**.

Superposition and Time Invariance

If S is the transformation corresponding to an LTI system

$\boldsymbol{y(t) = S[x(t)]}$ for an input $x(t)$

Then we have

$$\boldsymbol{S\left[\sum_k A_k x(t-\tau_k)\right] = \sum_k A_k S[x(t-\tau_k)] = \sum_k A_k y(t-\tau_k)}$$

$$\boldsymbol{S\left[\int g(\tau)x(t-\tau)d\tau\right] = \int g(\tau)S[x(t-\tau)]d\tau = \int g(\tau)y(t-\tau)d\tau}$$

This property allows us to **find the response of an LTI system due to any signal**, if we know the response of the system to an **impulse signal**.

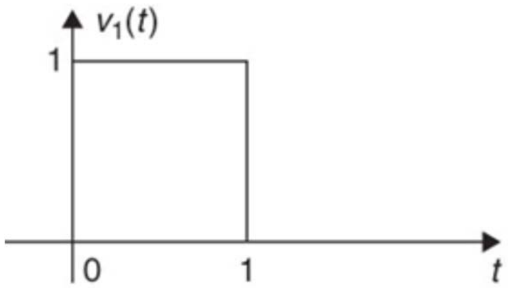
Examples

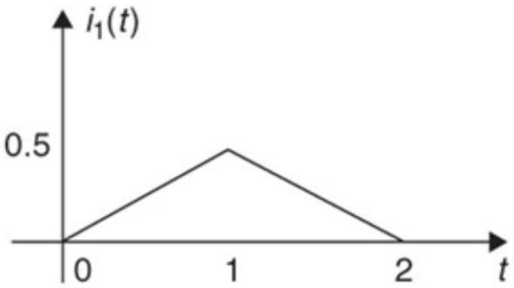
[Ex 2.8] The response of RL circuit to a unit-step function $u(t)$ is

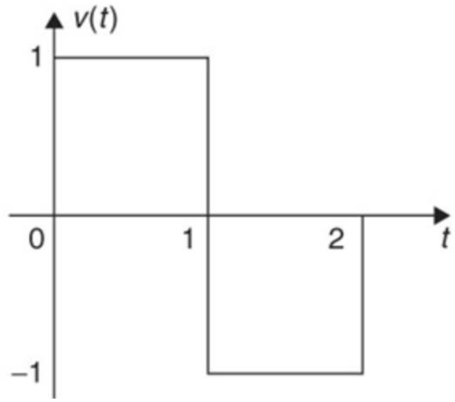
$$i(t) = (1 - e^{-t})u(t)$$

Find the response to a source $v(t) = u(t) - u(t-2)$.

[Ex 2.9] Suppose we know that the response to a rectangular pulse $v_1(t)$ is the current $i_1(t)$ shown in the figure below. If the input voltage is a train of two pulses, $v(t)$, find the corresponding current $i(t)$.







Convolution Integral

The **impulse response** of an analog LTI system, $h(t)$, is the **output of the system** corresponding to **an impulse $\boldsymbol{\delta(t)}$ as input** and **initial conditions equal to zero**.

The **response of an LTI system $\boldsymbol{S}$** with $\boldsymbol{h(t) = S[\delta(t)]}$ to any signal $x(t)$ is the **convolution integral**

$$\begin{aligned} \boldsymbol{y(t)} &= \boldsymbol{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau} \\ &= \boldsymbol{[x*h](t) = [h*x](t)} \end{aligned}$$

assuming that no energy is initially stored in the system.

- The impulse response is fundamental in the characterization of LTI systems.
- A system characterized by the convolution integral is **linear and time invariant** by the above construction.

Examples

[Ex 2.10] Obtain the impulse response of a capacitor and use it to find its unit-step response by means of the convolution integral. Let $C = 1\ F$.

[Ex 2.11] The output of an analog average is given by

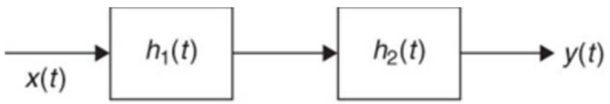
$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\,d\tau$$

which corresponds to the accumulation of values of $x(t)$ in a segment $[t-T, t]$ divided by its length T, or the average of $x(t)$ in $[t-T, t]$. Use the convolution integral to find the response of the average to a ramp.

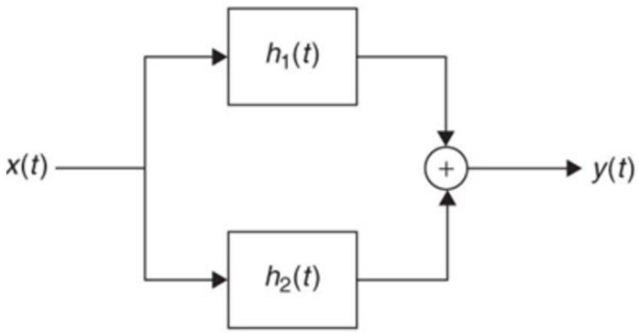
[Ex 2.12] The impulse response of a LTI system is $h(t) = u(t) - u(t-1)$. Consider inputs $x_2(t) = 0.5[\delta(\mathrm{t}) + \delta(t-0.5)]$, $x_4(t) = 0.25[\delta(t) + \delta(t-0.25) + \delta(t-0.5) + \delta(t-0.75)]$ find the corresponding outputs of the system.

Interconnection of Systems - Block Diagrams

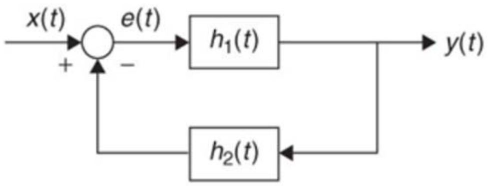
- The **flow of signals** is indicated by **arrows**, and the **addition of signals or multiplication of a signal by a constant** is indicated by means of **circles**.
  - The **cascade** and the **parallel** connections.
  - The **feedback** connection is found in control systems.



cascade



parallel



negative feedback

Cascade and Parallel Connection

Two LTI systems with **impulse responses $h_1(t)$ and $h_2(t)$** connected in **cascade** have an overall impulse response

$$h(t) = [h_1 * h_2](t) = [h_2 * h_1](t)$$

where $h_1(t)$ and $h_2(t)$ **commute**.

- When dealing with **time-varying systems**, the **order in which we connect the systems in cascade is important**.

If we **connect in parallel** two LTI systems with **impulse responses** $h_1(t)$ and $h_2(t)$, the impulse response of the overall system is

$$h(t) = h_1(t) + h_2(t)$$

Feedback Connection

- The feedback output is either **added to the input** giving a **positive feedback** system or **subtracted from the input** giving a **negative feedback** system.

Given two LTI systems with **impulse responses $h_1(t)$ and $h_2(t)$**, a **negative feedback** connection is such that the output is

$$y(t) = [h_1 * e](t)$$

where the **error signal**

$$e(t) = x(t) - [y * h_2](t)$$

The **impulse response $h(t)$** is given by

$$h(t) = [h_1 - h * h_1 * h_2](t)$$

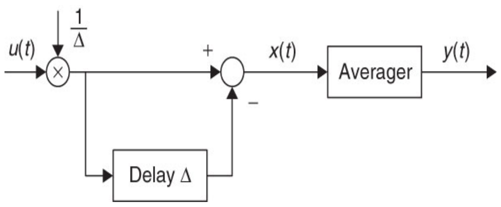
Examples

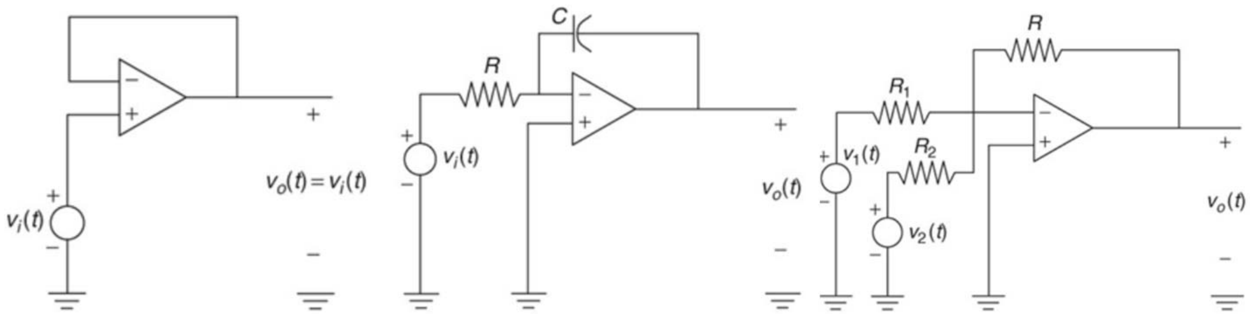
[Ex 2.13] Consider a block diagram with input a the unit-step signal $u(t)$. The average is such that for an input $x(t)$ its output is

$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\ d\tau$$

Determine what the system is doing as we let the delay $\Delta \to 0$. Consider that the average and the system with input $u(t)$ and output $y(t)$ are LTI.

[Ex 2.14] Consider the circuits obtained with an operational amplifier when we feed back its output with a wire, a resistor, and a capacitor. Assume the linear model for the op-amp. The circuits are called a voltage follower, an integrator, and an adder.





Causality

A **continuous-time system S** is called **causal** if

- Whenever the **input $x(t) = 0$** and there are **no initial conditions**, the **output $y(t) = 0$**.
- The output $y(t)$ **does not depend on future inputs**.

An **LTI system** represented by its **impulse response $h(t)$** is **causal** if

$$h(t) = 0, \quad \text{for } t < 0$$

The output of a causal LTI system with a **causal input $x(t)$** (i.e. $x(t) = 0$ **for** $t < 0$) is

$$y(t) = \int_0^t x(\tau)h(t-\tau)\ d\tau$$

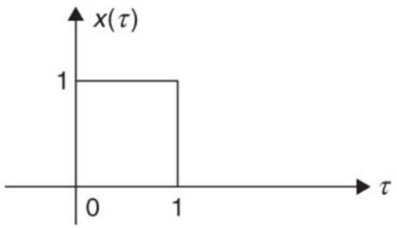
Graphical Computation of Convolution

[Ex 2.15] Graphically find the unit-step $y(t)$ response of an average, with $T = 1$ sec, which has an impulse response

$$h(t) = u(t) - u(t-1)$$

[Ex 2.16] Consider the graphical computation of the convolution integral of two pulses of the same duration.

The length of the support $\boldsymbol{y(t) = [x * h](t)}$ is equal to the **sum of the lengths of the supports of $\boldsymbol{x(t)}$ and $\boldsymbol{h(t)}$**.



[Ex 2.17$^{\text{MATLAB}}$] Compute the corresponding output using convolution integral of the following inputs and impulse responses of LTI systems:

(a) $x_1(t) = u(t) - u(t-1), \quad h_1(t) = u(t) - u(t-2)$,

(b) $x_2(t) = h_2(t) = r(t) - 2r(t-1) + r(t-2)$,

(c) $x_3(t) = e^{-t}u(t), \quad h_3(t) = e^{-10t}u(t)$.

Bounded-Input Bounded-Output Stability

- A **stable system** is such that **well-behaved outputs are obtained for well-behaved inputs**.

Bounded-input bounded-output (BIBO) stability establishes that **for a bounded input $x(t)$ the output of a BIBO stable system $y(t)$ is also bounded**.

An **LTI system with an absolutely integrable impulse response** – that is

$$\int_{-\infty}^{\infty} |h(t)| \; dt < \infty$$

is **BIBO stable**.

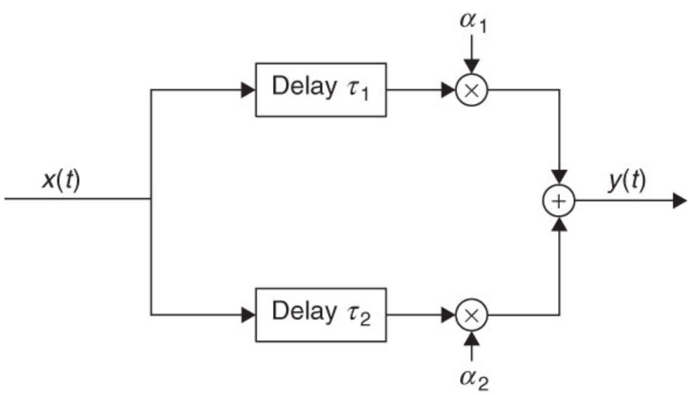
Examples

[Ex 2.18] Consider the BIBO stability and causality of RLC circuits. Consider, for instance, a series RL circuit where $R = 1\ \Omega$ and $L = 1\ H$, and a voltage source $v_s(t)$, which is bounded. Discuss why such a system would be causal and stable.

[Ex 2.19] Consider the causality and BIBO stability of an echo system (or a multipath system). Let the output $y(t)$ be given by

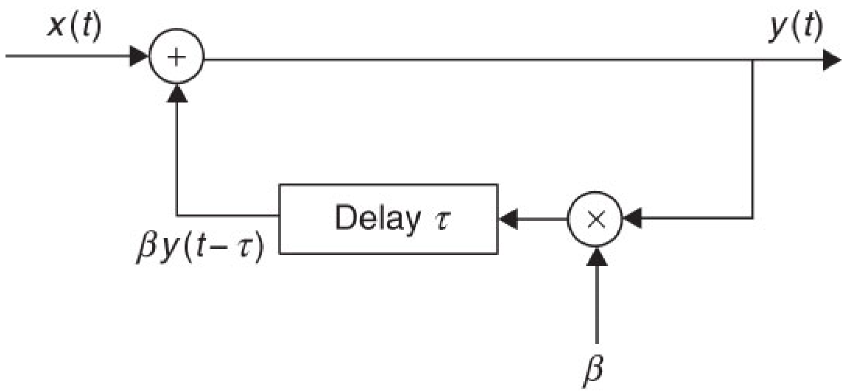
$$y(t) = \alpha_1 x(t - \tau_1) + \alpha_2 x(t - \tau_2)$$

where $x(t)$ is the input, and $|\alpha_i| < 1, \tau_i > 0$, for $i = 1$ and 2, are attenuation factors and delays. Is this system causal and BIBO stable?



Example

[Ex 2.20] Consider a positive feedback system created by a microphone close to a set of speakers that are putting out an amplified acoustic signal. The microphone picks up the input signal $x(t)$ as well as the amplified and delayed signal $\beta y(t-\tau)$, $|\beta| \geq 1$. Find the equation that connects the input $x(t)$ and output $y(t)$ and recursively from it obtain an expression for $y(t)$ in terms of past values of the input. Determine if the system is BIBO stable or not – use $x(t) = u(t)$, $\beta = 2$, and $\tau = 1$ in doing so.



Thank You