

# Chapter 7. Steady-State Errors

## PREFACE, ix

### 1. INTRODUCTION, 1

- 1.1 Introduction, 2
- 1.2 A History of Control Systems, 4
- 1.3 System Configurations, 7
- 1.4 Analysis and Design Objectives, 10
  - Case Study, 12
- 1.5 The Design Process, 15
- 1.6 Computer-Aided Design, 20
- 1.7 The Control Systems Engineer, 21
  - Summary, 23
  - Review Questions, 23
  - Problems, 24
  - Cyber Exploration Laboratory, 30
  - Bibliography, 31

### 2. MODELING IN THE FREQUENCY DOMAIN, 33

- 2.1 Introduction, 34
- 2.2 Laplace Transform Review, 35
- 2.3 The Transfer Function, 44
- 2.4 Electrical Network Transfer Functions, 47
- 2.5 Translational Mechanical System
  - Transfer Functions, 61
- 2.6 Rotational Mechanical System
  - Transfer Functions, 69
- 2.7 Transfer Functions for Systems with Gears, 74
- 2.8 Electromechanical System
  - Transfer Functions, 79
- 2.9 Electric Circuit Analogs, 84
- 2.10 Nonlinearities, 88
- 2.11 Linearization, 89
  - Case Studies, 94
  - Summary, 97
  - Review Questions, 97

- Problems, 98
- Cyber Exploration Laboratory, 112
- Bibliography, 115

### 3. MODELING IN THE TIME DOMAIN, 117

- 3.1 Introduction, 118
- 3.2 Some Observations, 119
- 3.3 The General State-Space Representation, 123
- 3.4 Applying the State-Space Representation, 124
- 3.5 Converting a Transfer Function to State Space, 132
- 3.6 Converting from State Space to a Transfer Function, 139
- 3.7 Linearization, 141
  - Case Studies, 144
  - Summary, 148
  - Review Questions, 149
  - Problems, 149
  - Cyber Exploration Laboratory, 157
  - Bibliography, 159

### 4. TIME RESPONSE, 161

- 4.1 Introduction, 162
- 4.2 Poles, Zeros, and System Response, 162
- 4.3 First-Order Systems, 166
- 4.4 Second-Order Systems: Introduction, 168
- 4.5 The General Second-Order System, 173
- 4.6 Underdamped Second-Order Systems, 177
- 4.7 System Response with Additional Poles, 186
- 4.8 System Response With Zeros, 191
- 4.9 Effects of Nonlinearities Upon Time Response, 196

- 4.10 Laplace Transform Solution of State Equations, 199
- 4.11 Time Domain Solution of State Equations, 203
  - Case Studies, 207
  - Summary, 213
  - Review Questions, 214
  - Problems, 215
  - Cyber Exploration Laboratory, 228
  - Bibliography, 232

### 5. REDUCTION OF MULTIPLE SUBSYSTEMS, 235

- 5.1 Introduction, 236
- 5.2 Block Diagrams, 236
- 5.3 Analysis and Design of Feedback Systems, 245
- 5.4 Signal-Flow Graphs, 248
- 5.5 Mason's Rule, 251
- 5.6 Signal-Flow Graphs of State Equations, 254
- 5.7 Alternative Representations in State Space, 256
- 5.8 Similarity Transformations, 266
  - Case Studies, 272
  - Summary, 278
  - Review Questions, 279
  - Problems, 280
  - Cyber Exploration Laboratory, 297
  - Bibliography, 299

### 6. STABILITY, 301

- 6.1 Introduction, 302
- 6.2 Routh-Hurwitz Criterion, 305
- 6.3 Routh-Hurwitz Criterion: Special Cases, 308
- 6.4 Routh-Hurwitz Criterion: Additional Examples, 314
- 6.5 Stability in State Space, 320
  - Case Studies, 323
  - Summary, 325
  - Review Questions, 325
  - Problems, 326

- Cyber Exploration Laboratory, 335
- Bibliography, 336

### 7. STEADY-STATE ERRORS, 339

- 7.1 Introduction, 340
- 7.2 Steady-State Error for Unity Feedback Systems, 343
- 7.3 Static Error Constants and System Type, 349
- 7.4 Steady-State Error Specifications, 353
- 7.5 Steady-State Error for Disturbances, 356
- 7.6 Steady-State Error for Nonunity Feedback Systems, 358
- 7.7 Sensitivity, 362
- 7.8 Steady-State Error for Systems in State Space, 364
  - Case Studies, 368
  - Summary, 371
  - Review Questions, 372
  - Problems, 373
  - Cyber Exploration Laboratory, 384
  - Bibliography, 386

### 8. ROOT LOCUS TECHNIQUES, 387

- 8.1 Introduction, 388
- 8.2 Defining the Root Locus, 392
- 8.3 Properties of the Root Locus, 394
- 8.4 Sketching the Root Locus, 397
- 8.5 Refining the Sketch, 402
- 8.6 An Example, 411
- 8.7 Transient Response Design via Gain Adjustment, 415
- 8.8 Generalized Root Locus, 419
- 8.9 Root Locus for Positive-Feedback Systems, 421
- 8.10 Pole Sensitivity, 424
  - Case Studies, 426
  - Summary, 431
  - Review Questions, 432
  - Problems, 432
  - Cyber Exploration Laboratory, 450
  - Bibliography, 452

# Chapter 7. Steady-State Errors

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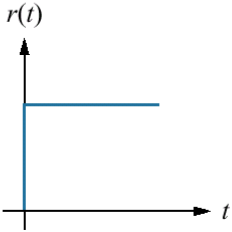
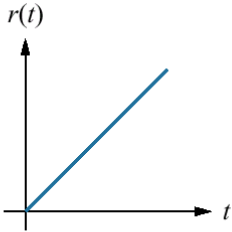
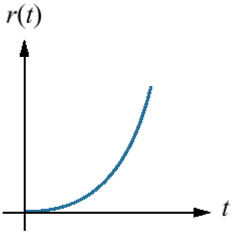
## Objectives

*Steady-state error* : difference between the input and the output for a test input as  $t \rightarrow \infty$

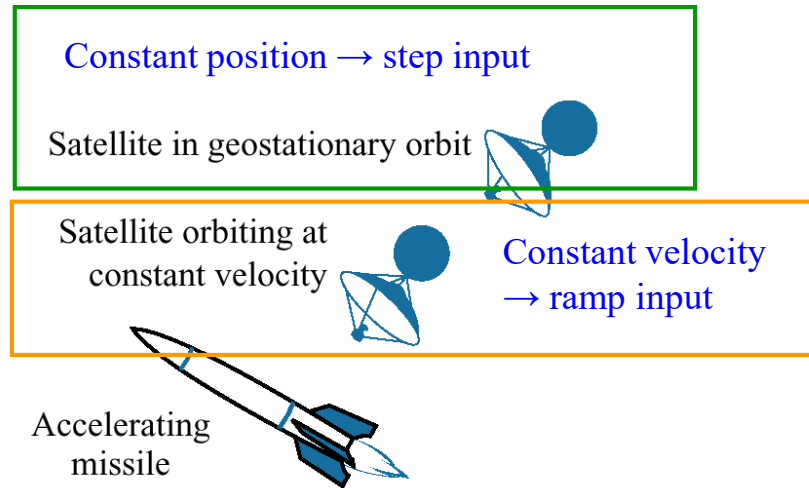
- How to find the *steady-state error* for a unity feedback system
- How to specify a system's steady-state *error performance*
- How to find the steady-state error for *disturbance inputs*
- How to find the steady-state error for *nonunity feedback systems*
- How to *design system performance* to meet steady-state error performance specifications
- How to find the steady-state error for systems represented in *state space*

## 7.1 Introduction

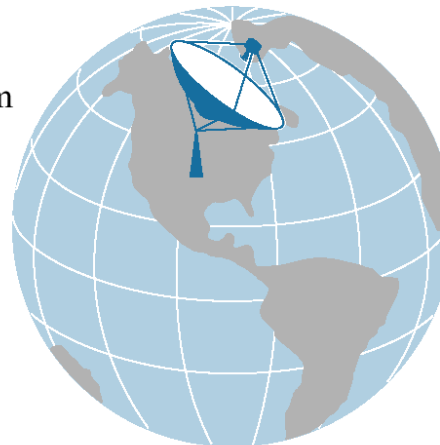
- steady-state error  $\Rightarrow \lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} \{r(t) - c(t)\}$
- Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	$t$	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

- Test inputs for steady-state error analysis and design vary with target type



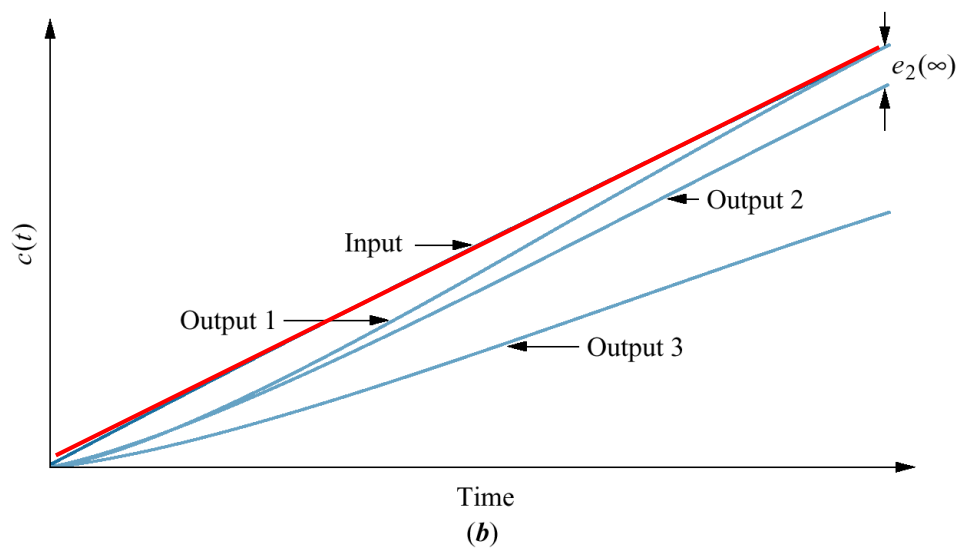
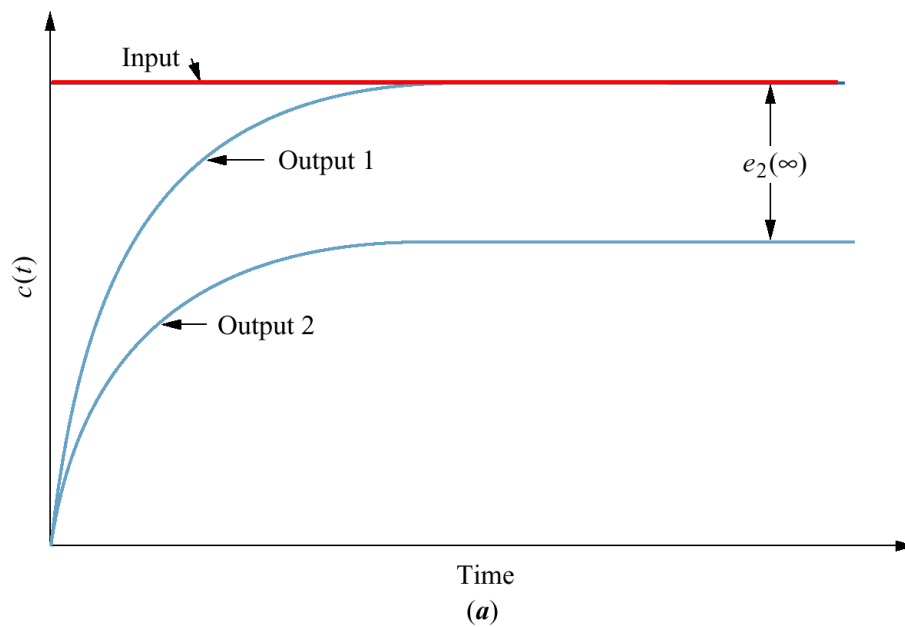
Tracking system



Evaluation steady-state error:

a. step input;

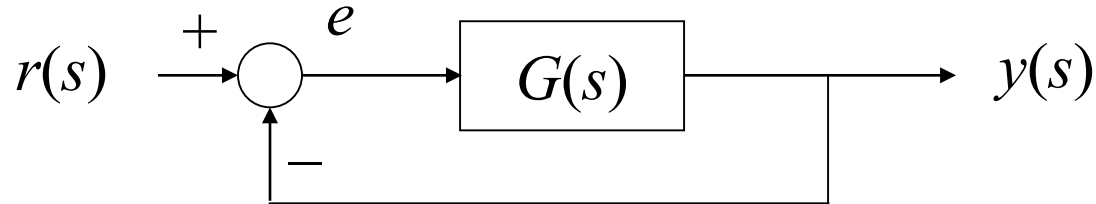
b. ramp input



# Unity Negative Feedback

$$\frac{R(s)}{\text{Input}} \rightarrow \frac{G(s)}{1 \pm G(s)H(s)} \rightarrow \frac{C(s)}{\text{Output}}$$

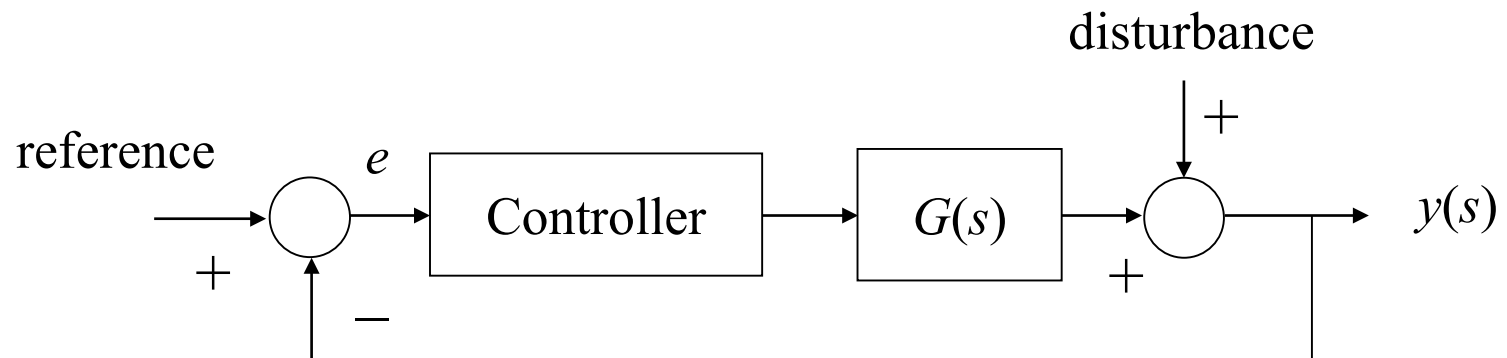
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$



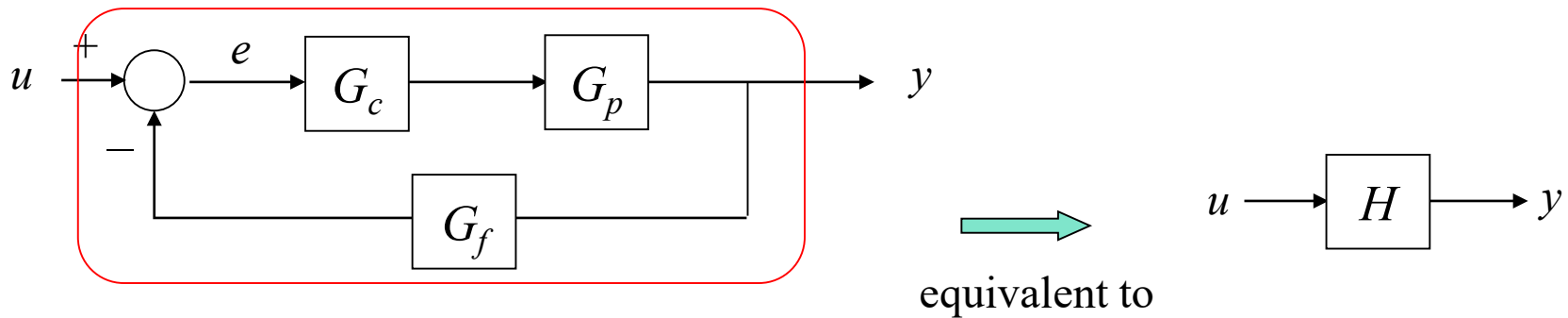
- The overall transfer function of the above system is:

$$T(s) = \frac{G(s)}{1 + G(s)}$$

- System Model:



# Feedback



Find  $H$ ?

$$Y(s) = G_p(s) \cdot G_c(s) \cdot E(s)$$

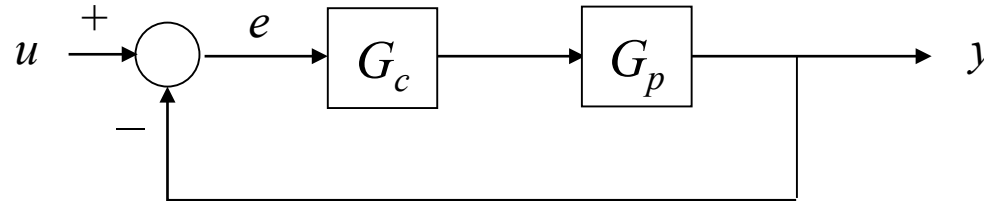
$$E(s) = U(s) - G_f(s) \cdot Y(s)$$

$$Y(s) = G_p(s) \cdot G_c(s) [U(s) - G_f(s) \cdot Y(s)]$$

$$Y(s) = \frac{G_p \cdot G_c}{1 + G_p \cdot G_c \cdot G_f} U(s) = H(s) \cdot U(s)$$

- Special case: unity feedback system

$$\frac{R(s)}{\text{Input}} \rightarrow \frac{G(s)}{1 \pm G(s)H(s)} \rightarrow \frac{C(s)}{\text{Output}}$$
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$



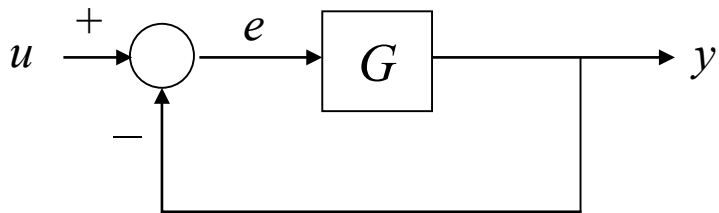
$$Y(s) = \frac{G_p(s) \cdot G_c(s)}{1 + G_p(s) \cdot G_c(s)} U(s) = H(s) \cdot U(s)$$

Purpose of feedback:

1. Change the dynamics of the system, stabilization
2. Tracking

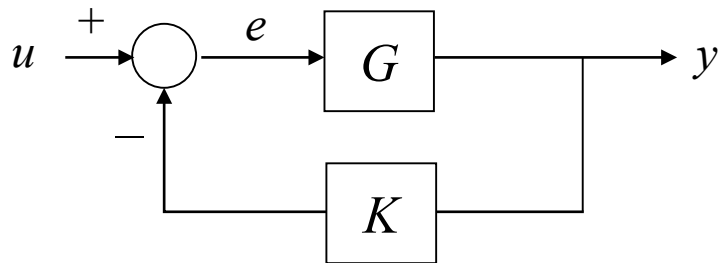


# Reference



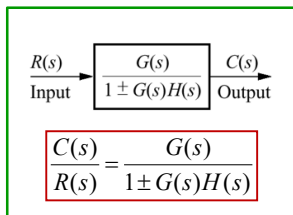
$$y = G(u - y)$$

$$Y(s) = \frac{G}{1 + G} U(s)$$



$$y = G(u - Ky)$$

$$Y(s) = \frac{G}{1 + GK} U(s)$$



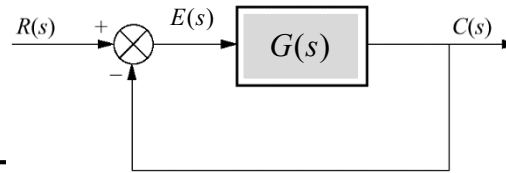
$$y = G(u - Ky)$$

$$y + GK y = Gu$$

$$y(1 + GK) = Gu$$

$$Y(s) = \frac{G}{1 + GK} U(s)$$

# Stabilization

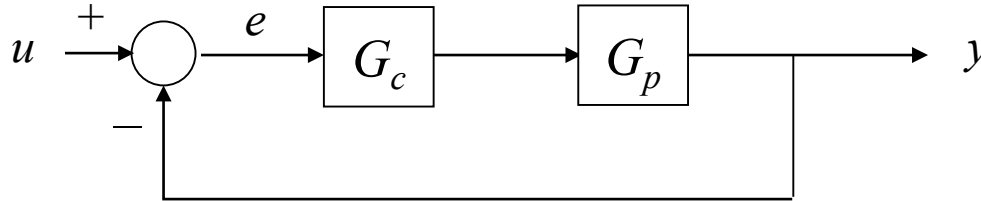


$$E(s) = R(s) - C(s)$$

$$C(s) = E(s)G(s) = R(s)G(s) - C(s)G(s)$$

$$C(s)[1 + G(s)] = R(s)G(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$



$$G_p(s) = \frac{1}{s-1}$$

$\Rightarrow$  unstable

Let  $G_c(s) = K$ ,

$$H(s) = \frac{KG_p(s)}{1 + KG_p(s)} = \frac{\frac{K}{s-1}}{1 + \frac{K}{s-1}} = \frac{K}{s-1+K}$$

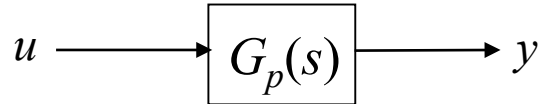
Pole:  $s-1+K=0$ ,  $s=1-K$

$$\rightarrow s = (1-K) < 0$$

$$\rightarrow K > 1$$

# Tracking

- Open loop system

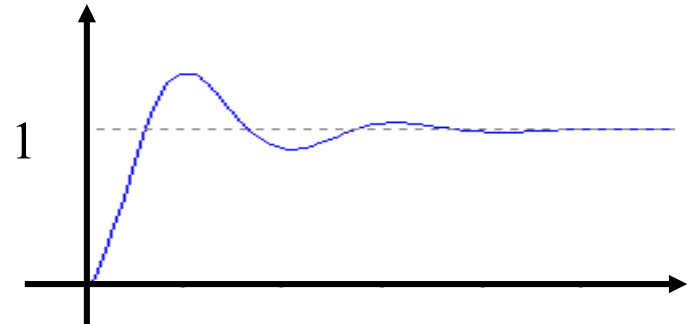


$$G_p(s) = \frac{1}{(s+2)(s+\frac{1}{2})}$$

You want to find  $u$  such that

$$y(\infty) = \lim_{t \rightarrow \infty} y(t) = 1$$

Final value theorem



- Final value theorem:  $y(\infty) = \lim_{s \rightarrow 0} sY(s)$

$$\rightarrow y(\infty) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} s \left( \frac{1}{s} G_p(s) \right) = \frac{1}{2 \cdot \frac{1}{2}} = 1$$

Try  $u(t) = \text{step}(t)$

$$\rightarrow Y(s) = G_p(s) \cdot U(s) = \frac{1}{(s+2)(s+\frac{1}{2})} \cdot \frac{1}{s}$$

By the **final value theorem**

$$y(\infty) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} \frac{1}{(s+2)(s+\frac{1}{2})} = \boxed{1}$$

• Final value theorem :

$$x(\infty) = \lim_{s \rightarrow 0} sX(s)$$

• Initial value theorem :

$$x(0) = \lim_{s \rightarrow \infty} sX(s)$$

Ref: if  $u=\text{step}$  then

$$y(\infty) = \lim_{s \rightarrow 0} sG_p(s) \cdot \frac{1}{s} = G_p(0) \quad \Rightarrow \quad \text{the } d_c \text{ gain of the system}$$

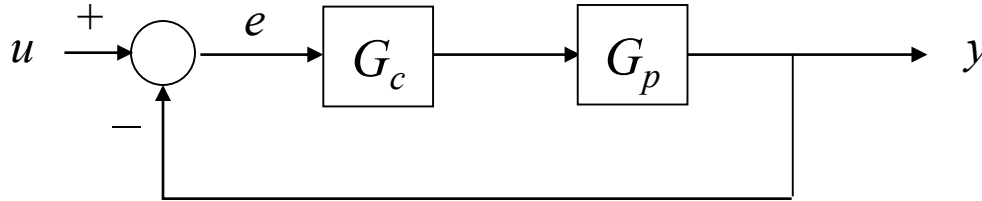
Now, suppose that  $G_p(s) = \frac{1}{(s+2+\varepsilon)(s+\frac{1}{2})}$

for some small,  $\varepsilon$

$$\rightarrow Y(s) = \frac{1}{(s+2+\varepsilon)(s+\frac{1}{2})} \cdot \frac{1}{s}$$

$$y(\infty) = \lim_{s \rightarrow 0} s \cdot \frac{1}{(s+2+\varepsilon)(s+\frac{1}{2})} \cdot \frac{1}{s} = \frac{1}{1+\frac{\varepsilon}{2}} \neq 1$$

What do you do ?  $\Rightarrow$  **Use feedback !**



$$H(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} = \frac{\frac{1}{(s+2+\varepsilon)(s+\frac{1}{2})}G_c(s)}{1 + \frac{1}{(s+2+\varepsilon)(s+\frac{1}{2})}G_c(s)}$$

$$Y(s) = H(s)U(s) = H(s)\frac{1}{s} \quad (u(t) = \text{step in problem})$$

$$y(\infty) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} H(s)$$

i) Try  $G_c(s) = k$

$$\rightarrow H(s) = \frac{k}{(s + 2 + \varepsilon)(s + \frac{1}{2}) + k}$$

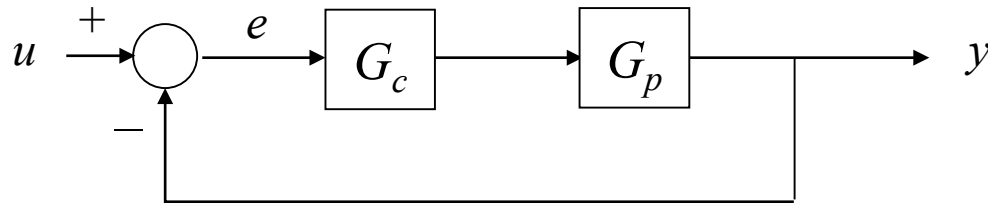
$$y(\infty) = \lim_{s \rightarrow 0} H(s) = \frac{k}{(2 + \varepsilon)\frac{1}{2} + k} = \frac{k}{1 + \frac{\varepsilon}{2} + k}$$

$$y(\infty) \neq 1, \quad \text{but} \quad y(\infty) \xrightarrow{k \rightarrow \infty} \boxed{1}$$

Need large gains

ii) Try  $G_c(s) = \frac{1}{s}$ ,  $H(s) = \frac{1}{(s + 2 + \varepsilon)(s + \frac{1}{2}) \cdot s + 1}$

$$y(\infty) = \lim_{s \rightarrow 0} H(s) = 1$$



$$U(s) = \frac{1}{s} \quad G(s) = \frac{1}{(s + 2 + \varepsilon) \left( s + \frac{1}{2} \right)}$$

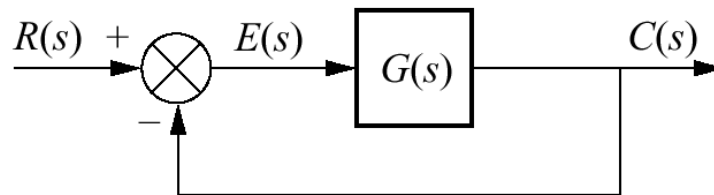
$$G_c(s) = \frac{1}{s} \Rightarrow y(\infty) = 1$$

End of tracking



## 7.2 Steady-State Error for Unity Feedback Systems

- Steady-State Error for Unity Feedback Systems



$$\begin{aligned} E(s) &= R(s) - C(s) \\ C(s) &= E(s)G(s) \end{aligned} \quad \Rightarrow \quad E(s) = \frac{1}{1 + G(s)} R(s)$$

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

• Step input:  $R(s) = \frac{1}{s}$

$$e(\infty) = \lim_{s \rightarrow 0} \frac{s(1/s)}{1 + G(s)} = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)}$$

$\rightarrow \infty$  for  $n \geq 1$

$$G(s) = \frac{(s + z_1)(s + z_2) \cdots}{s^n (s + p_1)(s + p_2) \cdots}$$

$$e(\infty) = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)} = \frac{1}{1 + \infty} = 0$$

For  $e(\infty) = 0 \Rightarrow n \geq 1$

$(\lim_{s \rightarrow 0} G(s) = \infty)$  If  $n = 0$  then  $e(\infty)$  is finite.

• Ramp input:  $R(s) = \frac{1}{s^2}$

$$e(\infty) = \lim_{s \rightarrow 0} \frac{s(1/s^2)}{1 + G(s)} = \frac{1}{\lim_{s \rightarrow 0} sG(s)}$$

$\rightarrow \infty$  for  $n \geq 2$

$$\text{For } n=1: e(\infty) = \frac{1}{\lim_{s \rightarrow 0} sG(s)} = \frac{1}{\lim_{s \rightarrow 0} sG(s)} = \frac{1}{\lim_{s \rightarrow 0} s \frac{(s + z_1)}{s(s + p_1)}} = \frac{1}{\lim_{s \rightarrow 0} \frac{(s + z_1)}{(s + p_1)}} = \frac{p_1}{z_1}$$

For  $e(\infty) = 0 \Rightarrow n \geq 2$

If  $n = 1$ ,  $e(\infty)$  is finite.

If  $n = 0$ ,  $e(\infty)$  is infinite.

• Parabolic input:  $R(s) = \frac{1}{s^3}$

$$e(\infty) = \lim_{s \rightarrow 0} \frac{s(1/s^3)}{1 + G(s)} = \frac{1}{\lim_{s \rightarrow 0} s^2 G(s)}$$

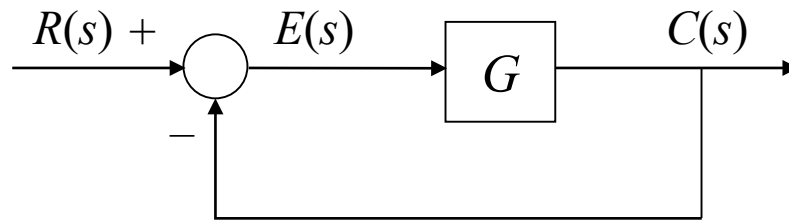
For  $e(\infty) = 0 \Rightarrow n \geq 3$

If  $n = 2$ ,  $e(\infty)$  is finite.

If  $n = 0$  or  $1$ ,  $e(\infty)$  is infinite.

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

Example:



$$G(s) = \frac{100(s+2)(s+6)}{s(s+3)(s+4)}$$

Find the steady-state error: i)  $u(t)$  , ii)  $tu(t)$  , iii)  $\frac{1}{2}t^2u(t)$  .

$$\text{i) } R(s) = \frac{1}{s}$$

$$e(\infty) = \lim_{s \rightarrow 0} \frac{s \left( \frac{1}{s} \right)}{1 + G(s)} = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)} = \frac{1}{\infty} = 0$$

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

$$\text{ii) } R(s) = \frac{1}{s^2}$$

$$e(\infty) = \lim_{s \rightarrow 0} \frac{s \left( \frac{1}{s^2} \right)}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{1}{s + sG(s)} = \frac{1}{\lim_{s \rightarrow 0} sG(s)} = \frac{1}{100}$$

$$\text{iii) } R(s) = \frac{1}{s^3}$$

$$e(\infty) = \lim_{s \rightarrow 0} \frac{s \left( \frac{1}{s^3} \right)}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim_{s \rightarrow 0} s^2 G(s)} = \frac{1}{0} = \infty$$

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

## 7.3 Static Error Constants and System Type

- Static error constants

i) step input  $u(t)$  ,  $e(\infty) = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)} = \frac{1}{1 + K_p}$

position constant :  $K_p = \lim_{s \rightarrow 0} G(s)$

ii) ramp input  $tu(t)$  ,  $e(\infty) = \frac{1}{\lim_{s \rightarrow 0} sG(s)} = \frac{1}{K_v}$

velocity constant :  $K_v = \lim_{s \rightarrow 0} sG(s)$

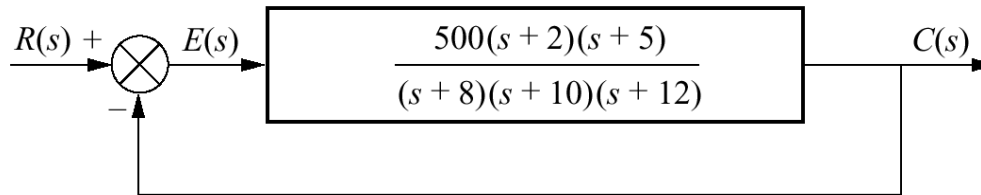
iii) parabolic input  $\frac{1}{2}t^2u(t)$  ,  $e(\infty) = \frac{1}{\lim_{s \rightarrow 0} s^2 G(s)} = \frac{1}{K_a}$

acceleration constant :  $K_a = \lim_{s \rightarrow 0} s^2 G(s)$

## Example 7.4 (page 350): Steady-state error via static error constants

Evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs.

---



$$s^n \rightarrow n=0 \rightarrow s^0=1$$

$$K_p = \lim_{s \rightarrow 0} G(s) = 5.208$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = 0$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) = 0$$

$$\text{For a step input} \quad \rightarrow \quad e(\infty) = \frac{1}{1 + K_p} = 0.161$$

$$\text{For a ramp input} \quad \rightarrow \quad e(\infty) = \frac{1}{K_v} = \infty$$

$$\text{For a parabolic input} \quad \rightarrow \quad e(\infty) = \frac{1}{K_a} = \infty$$

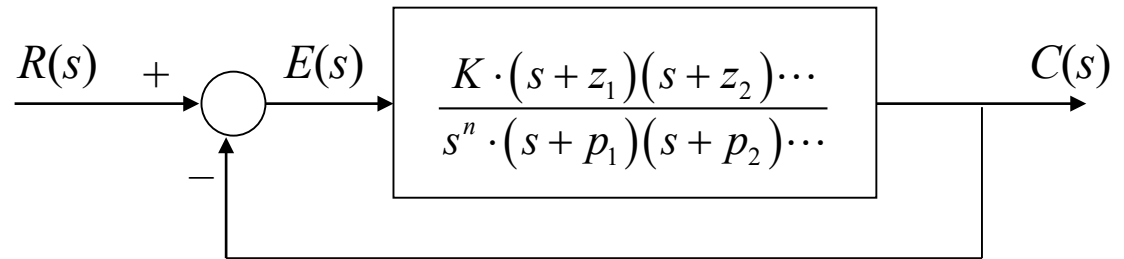
End of example

- System type

$n = 0 \rightarrow \text{Type 0}$

$n = 1 \rightarrow \text{Type 1}$

$n = 2 \rightarrow \text{Type 2}$

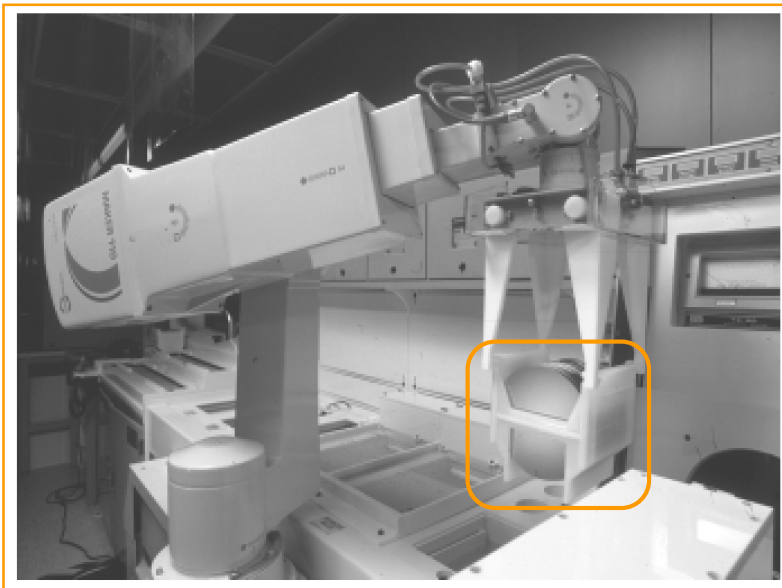


Input	Steady-state error formula	Type 0		Type 1		Type 2	
		Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1 + K_p}$	$K_p = \text{Constant}$	$\frac{1}{1 + K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	$\infty$	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	$\infty$	$K_a = 0$	$\infty$	$K_a = \text{Constant}$	$\frac{1}{K_a}$

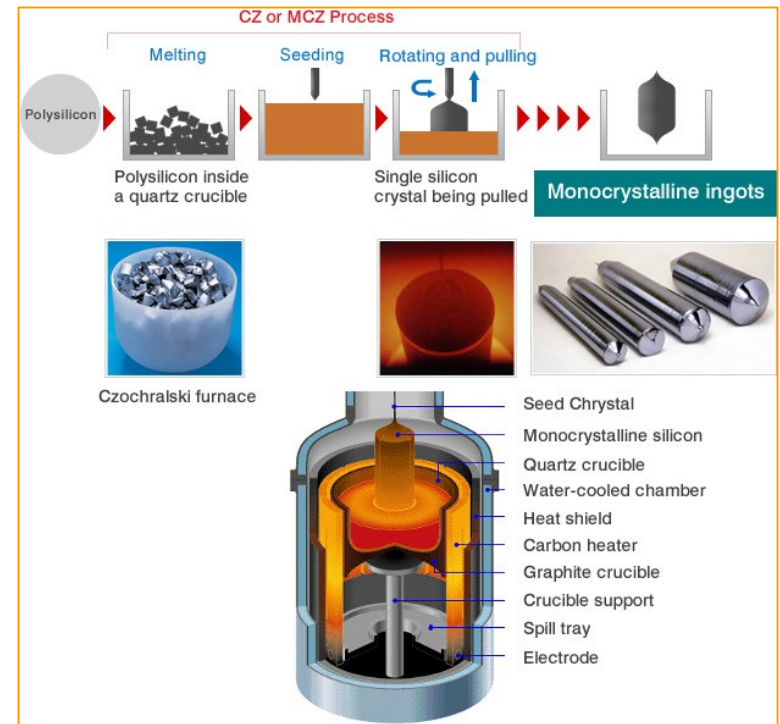
Relationships between input, system type, static error constants, and steady-state errors

## 7.4 Steady-State Error Specification

- A robot used in the manufacturing of semiconductor random-access memories (RAMs) similar to those in personal computers.
- **Steady-state error** is an important design consideration for assembly-line robots.



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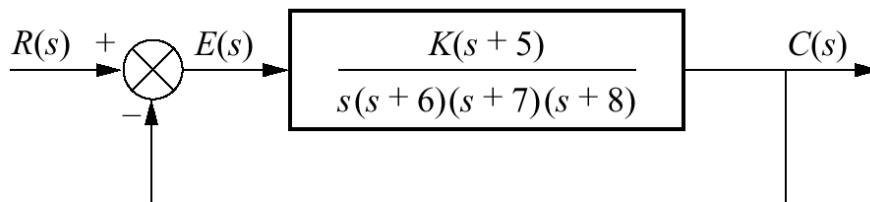
[https://www.sumcosi.com/english/products/process/step\\_01.html](https://www.sumcosi.com/english/products/process/step_01.html)



## Example 7.6 (page 355): Gain design to meet a steady-state error specification

Find the value of  $K$  so that there is **10% error** in the steady state.

---



- The system is Type 1  $\Rightarrow$  a ramp input is applied to the system.

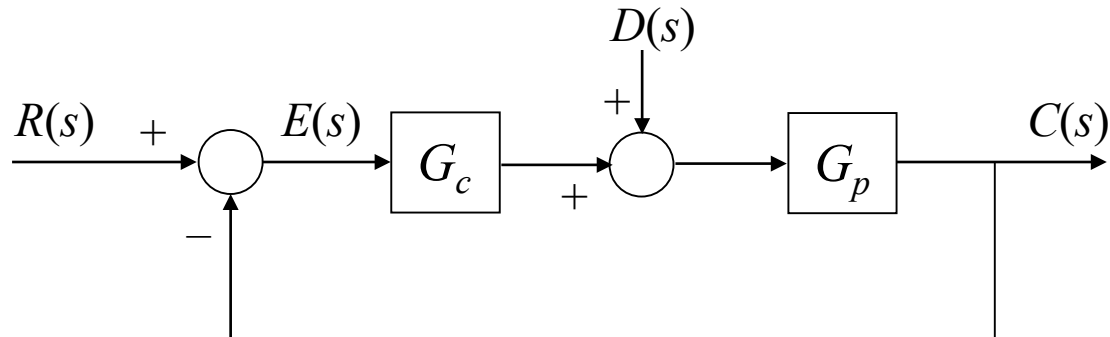
$$e(\infty) = \frac{1}{K_v} = 0.1$$

$$\rightarrow K_v = 10 = \lim_{s \rightarrow 0} sG(s) = \frac{K \times 5}{6 \times 7 \times 8}$$

$$\rightarrow K = 672$$

Input	Steady-state error formula
Step, $u(t)$	$\frac{1}{1 + K_p}$
Ramp, $tu(t)$	$\frac{1}{K_v}$
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$

## 7.5 Steady-State Error for Disturbances



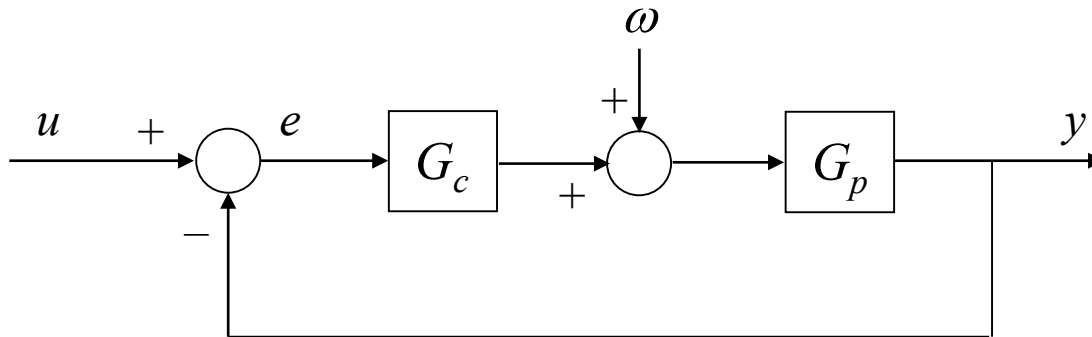
$$E(s) = R(s) - C(s)$$

$$C(s) = [E(s) \cdot G_c(s) + D(s)] \cdot G_p(s)$$

$$E = R - E \cdot G_c G_p - D \cdot G_p, \quad E(1 + G_c G_p) = R - D \cdot G_p$$

$$E(s) = \frac{R(s)}{1 + G_c(s) \cdot G_p(s)} - \frac{G_p(s)}{1 + G_c(s) \cdot G_p(s)} \cdot D(s)$$
$$= e_R(\infty) + e_D(\infty)$$

Example: Find  $G_c$  to eliminate the effect of  $\omega$  on  $y$  in steady state.



$$G_p(s) = \frac{s+1}{(s+2)(s+0.51)}$$

$$\omega(t) = u(t)$$

$$Y(s) = G_p(\omega - E \cdot G_c), \quad E = U - Y$$

$$= G_p(\omega - (U - Y) \cdot G_c)$$

$$= G_p\omega - UG_pG_c + G_pG_cY \Rightarrow Y(s) = \frac{G_p}{1 - G_cG_p} \cdot \omega - \frac{G_cG_p}{1 - G_cG_p} \cdot U$$

$$Y_\omega(s) = \frac{G_p}{1 - G_cG_p} \omega = \frac{(s+1) \frac{1}{s}}{(s+2)(s+0.51) - (s+1)G_c}$$

$$\begin{aligned} Y_\omega(s) &= \frac{G_p}{1 - G_cG_p} \omega \\ &= \frac{\frac{(s+1)}{(s+2)(s+0.51)}}{1 - \frac{(s+1)}{(s+2)(s+0.51)}G_c} \omega \\ &= \frac{(s+1)\omega}{(s+2)(s+0.51) - (s+1)G_c} \end{aligned}$$

$$y_\infty(\infty) = \lim_{s \rightarrow 0} sY_\omega(s)$$

$$Y_{\omega}(s) = \frac{G_p}{1 - G_C G_p} \omega$$

$$= \frac{(s+1) \frac{1}{s}}{(s+2)(s+0.51) - (s+1)G_C}$$

Let,  $G_C(s) = \frac{1}{s}$

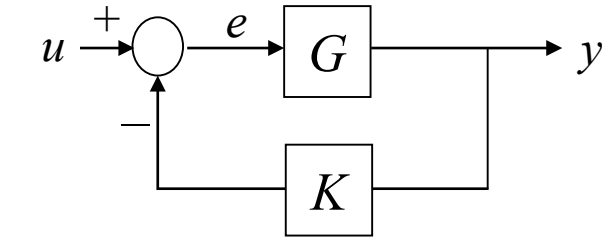
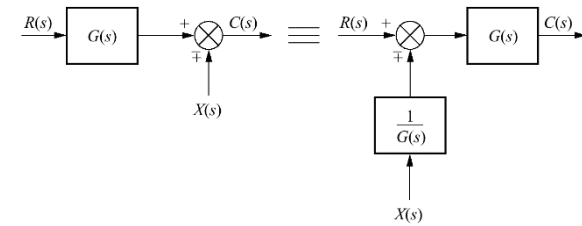
$$y_{\infty}(\infty) = \lim_{s \rightarrow 0} s Y_{\omega}(s) = \lim_{s \rightarrow 0} s \cdot \frac{(s+1) \frac{1}{s}}{(s+2)(s+0.51) - (s+1) \frac{1}{s}}$$

$$= \lim_{s \rightarrow 0} \frac{s(s+1)}{s(s+2)(s+0.51) - (s+1)}$$

$$= 0$$

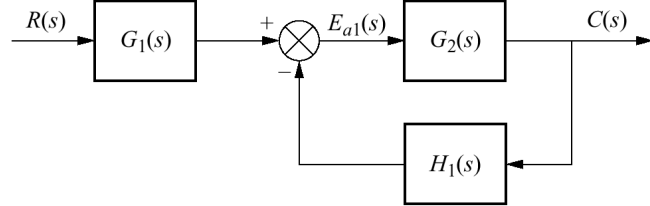
$\therefore G_C(s) = \frac{1}{s} \rightarrow y_{\infty}(\infty) = \boxed{0}$

# 7.6 Steady-State Error for Nonunity Feedback Systems



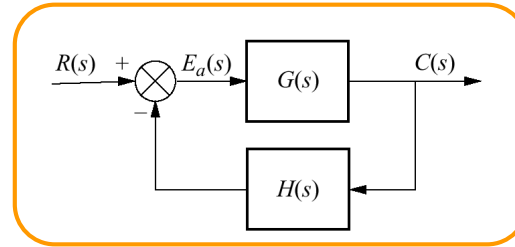
$$y = G(u - Ky)$$

$$Y(s) = \frac{G}{1 + GK} U(s)$$



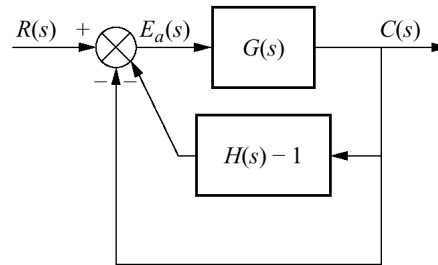
(a)

$$C(s) = G_1 \frac{G_2}{1 + G_2 H_1} R(s) = \frac{G_1 G_2}{1 + G_1 G_2 \frac{H_1}{G_1}} R(s) = \frac{G}{1 + GH} R(s)$$

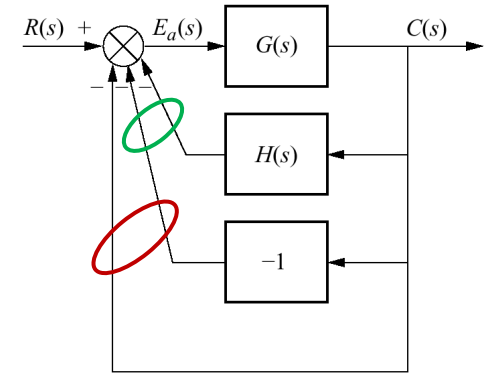


$$G = G_1 G_2, H = \frac{H_1}{G_1}$$

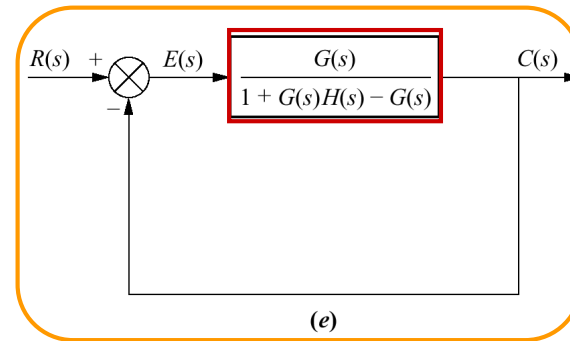
(b)



(d)



(c)



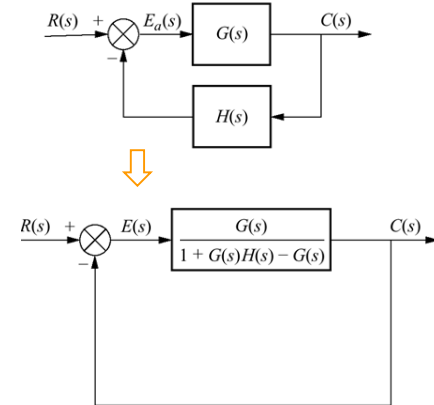
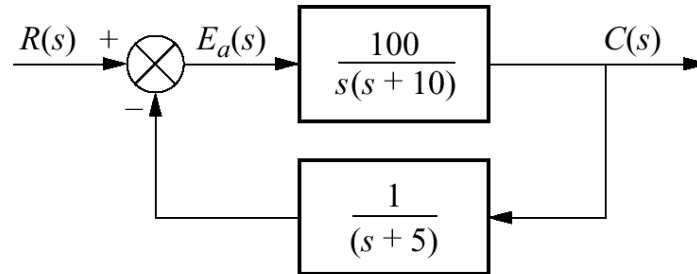
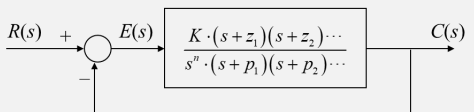
(e)

$$Y = \frac{\frac{G}{1 + GH - G}}{1 + \frac{G}{1 + GH - G}} R = \frac{G}{(1 + GH - G) + G} R = \frac{G}{1 + GH} R$$

## Example 7.8 (page 359): Steady-state error for nonunity feedback systems

Find the system type, the appropriate static error constant, and the steady-state error for a unit step input.

- System type
  - $n = 0 \rightarrow$  Type 0
  - $n = 1 \rightarrow$  Type 1
  - $n = 2 \rightarrow$  Type 2



$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)}$$

equivalent

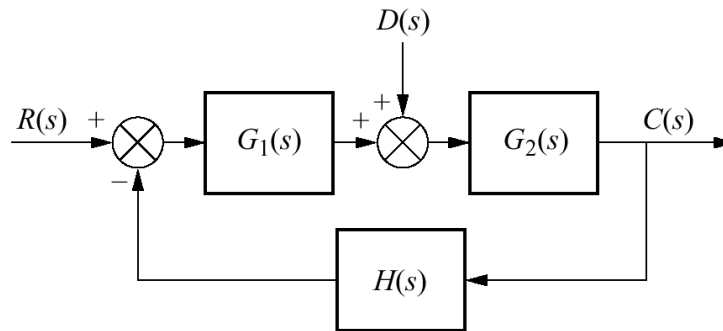
$$\begin{aligned}
 &= \frac{100}{s(s+10)} \\
 &= \frac{100}{1 + \frac{100}{s(s+10)} \cdot \frac{1}{s+5} - \frac{100}{s(s+10)}} \\
 &= \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400}
 \end{aligned}$$

1) Type 0

2) Error Constant:  $K_p = \lim_{s \rightarrow 0} G_e(s) = \frac{500}{-400} = -\frac{5}{4}$

3)  $e(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 - (5/4)} = -4$

## • Non-Unity Feedback Control with Disturbance



**Steady-state error**: difference between the input and the output for a test input as  $t \rightarrow \infty$

$$R(s) = D(s) = \frac{1}{s}$$

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \left\{ \left[ 1 - \frac{\lim_{s \rightarrow 0} G_1(s)G_2(s)}{\lim_{s \rightarrow 0} [1 + G_1(s)G_2(s)H(s)]} \right] - \left[ \frac{\lim_{s \rightarrow 0} G_2(s)}{\lim_{s \rightarrow 0} [1 + G_1(s)G_2(s)H(s)]} \right] \right\}$$

For  $e(\infty) = 0$ ,  $\frac{\lim_{s \rightarrow 0} [G_1(s)G_2(s)]}{\lim_{s \rightarrow 0} [1 + G_1(s)G_2(s)H(s)]} = 1$  and  $\frac{\lim_{s \rightarrow 0} G_2(s)}{\lim_{s \rightarrow 0} [1 + G_1(s)G_2(s)H(s)]} = 0$

$$\Rightarrow \frac{1}{\frac{1}{\lim_{s \rightarrow 0} G_1(s)G_2(s)} + \lim_{s \rightarrow 0} H(s)} = 1, \quad \frac{1}{\frac{1}{\lim_{s \rightarrow 0} G_2(s)} + \lim_{s \rightarrow 0} G_1(s)H(s)} = 0$$

$\lim_{s \rightarrow 0} G_1(s)G_2(s) \rightarrow 0$

$\lim_{s \rightarrow 0} H(s) \rightarrow 1$

$\lim_{s \rightarrow 0} G_2(s) \rightarrow \infty$

input of  $G_1$ :  $R - CH$

input of  $G_2$ :  $G_1E + D = G_1(R - CH) + D$

$$C = G_2 [G_1(R - CH) + D] \dots (1)$$

steady-state error  $E = R - C$ ,  $C = R - E \dots (2)$

(2)  $\rightarrow$  (1);

$$\begin{aligned} R - E &= G_1G_2R - G_1G_2(CH) + G_2D \\ &= G_1G_2R - G_1G_2(R - E)H + G_2D \\ &= G_1G_2R - G_1G_2RH + \underline{G_1G_2EH} + G_2D \end{aligned}$$

arrange for  $E$ ,

$$E(1 + G_1G_2H) = R - G_1G_2R + G_1G_2RH - G_2D$$

$$\begin{aligned} \therefore E &= \frac{1 - G_1G_2 + G_1G_2H}{1 + G_1G_2H} R - \frac{G_2}{1 + G_1G_2H} D \\ &= \left( 1 - \frac{G_1G_2}{1 + G_1G_2H} \right) R - \left( \frac{G_2}{1 + G_1G_2H} \right) D \end{aligned}$$

## 7.7 Sensitivity

- System parameters are changed  $\rightarrow$  transfer function  
 $\Rightarrow$  performance  
 $\Rightarrow$  sensibility

$$F = \frac{K}{K+a}, \quad K=10, a=100 \rightarrow F = \frac{10}{10+100} = 0.091$$

$$\frac{300-100}{100} \times 100 = 200\%$$

$$a=300 \rightarrow$$

$$F = \frac{10}{10+300} = 0.032$$

$$\frac{0.032-0.091}{0.091} \times 100 = -65\%$$



$F$  has reduced sensitivity to changes in parameter  $a$  :

$$\left( \begin{array}{l} 200\% \text{ change in parameter 'a'} \\ \rightarrow \quad -65\% \text{ change in } F \end{array} \right) \Rightarrow \frac{-65\%}{200\%} = -0.325$$

- Sensitivity:

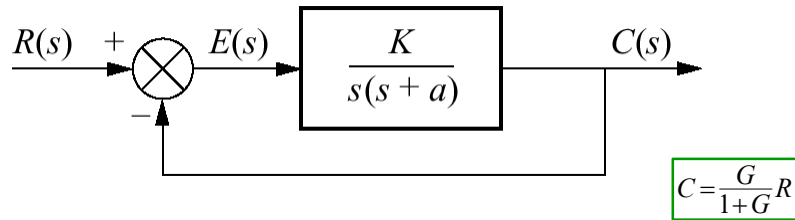
$$S_{F:P} = \lim_{\Delta P \rightarrow 0} \frac{\frac{\Delta F}{F}}{\frac{\Delta P}{P}} = \lim_{\Delta P \rightarrow 0} \frac{P}{F} \frac{\Delta F}{\Delta P}$$

$$S_{F:P} = \frac{P}{F} \frac{\delta F}{\delta P}$$

## Example 7.10 (page 362): Sensitivity of a closed-loop transfer function

Calculate the sensitivity to change in the parameter  $a$ .

How would you reduce the sensitivity?



$$T(s) = \frac{K}{s^2 + as + K}$$

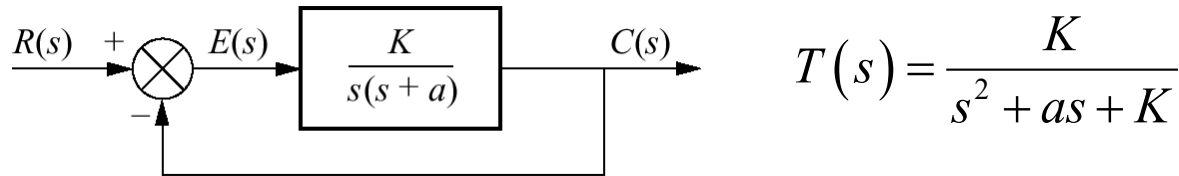
$$\begin{aligned} S_{F:P} &= \lim_{\Delta P \rightarrow 0} \frac{\Delta F / F}{\Delta P / P} \\ &= \lim_{\Delta P \rightarrow 0} \frac{P}{F} \frac{\Delta F}{\Delta P} \\ &= \frac{P}{F} \frac{\delta F}{\delta P} \end{aligned}$$

$$(1) \quad S_{T:a} = \frac{a}{T} \frac{\delta T}{\delta a} = \frac{a}{\frac{K}{s^2 + as + K}} \cdot \frac{-sK}{(s^2 + as + K)^2} = \frac{-as}{(s^2 + as + K)}$$

(2)  $\Rightarrow$  Increase  $K \rightarrow$  reduce sensitivity

## Example 7.11 (page 363): Sensitivity of steady-state error with ramp input

Find the sensitivity of the steady-state error to changes in parameters  $K$  and  $a$  with ramp input.



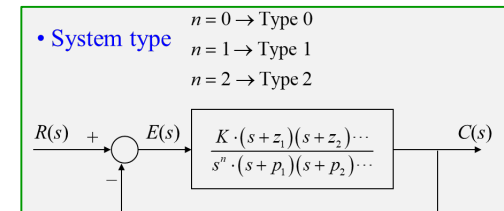
Type 1:

$$e = e(\infty) = \frac{1}{K_v} = \frac{1}{\lim_{s \rightarrow 0} sG(s)} = \frac{1}{K/a} = \frac{a}{K}$$

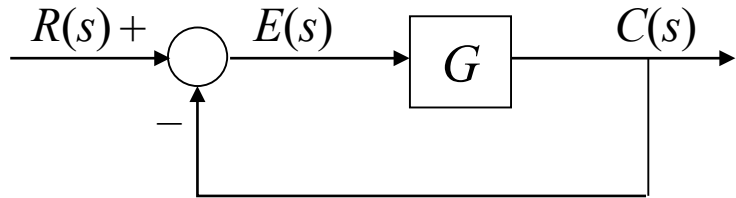
$$S_{e:a} = \frac{a}{e} \cdot \frac{\delta e}{\delta a} = \frac{a}{a/K} \left( \frac{1}{K} \right) = 1$$

$$S_{e:K} = \frac{K}{e} \frac{\delta e}{\delta K} = \frac{K}{a/K} \left[ \frac{-a}{K^2} \right] = -1$$

$$\begin{aligned} S_{F:P} &= \lim_{\Delta P \rightarrow 0} \frac{\Delta F / F}{\Delta P / P} \\ &= \lim_{\Delta P \rightarrow 0} \frac{P}{F} \frac{\Delta F}{\Delta P} \\ &= \frac{P}{F} \frac{\delta F}{\delta P} \end{aligned}$$



## Example: Sensitivity of the S.S. error in $K$ , $a$



$$G = \frac{K}{(s+a)(s+b)}, \quad R(s) = \frac{1}{s}$$

$$E(s) = R(s) - G \cdot E(s), \quad E(s) = \frac{1}{1+G(s)} R(s)$$

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{s \left( \frac{1}{s} \right)}{1+G(s)} = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)} = \boxed{\frac{1}{1 + \frac{K}{ab}}} \\ = \frac{ab}{ab + K}$$

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \frac{ab}{ab + K}$$

$$\begin{aligned} (1) S_{e:a} &= \frac{a}{e} \frac{\delta e}{\delta a} = \frac{a}{\frac{ab}{ab+K}} \cdot \frac{b(ab+K) - ab(b)}{(ab+K)^2} = \frac{b \cdot K}{b(ab+K)} \\ &= \frac{K}{ab+K} \end{aligned}$$

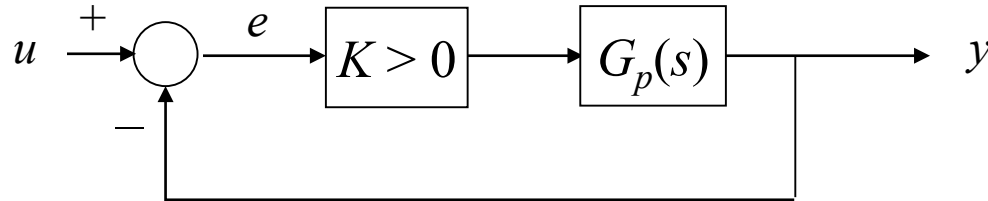
$$(2) S_{e:K} = \frac{K}{e} \frac{\delta e}{\delta K} = \frac{K}{\frac{ab}{ab+K}} \cdot \frac{-ab}{(ab+K)^2} = \frac{-K}{ab+K} \cdot$$

→ feedback sensitivity

$$\begin{aligned} S_{F:P} &= \lim_{\Delta P \rightarrow 0} \frac{\Delta F / F}{\Delta P / P} \\ &= \lim_{\Delta P \rightarrow 0} \frac{P}{F} \frac{\Delta F}{\Delta P} \\ &= \frac{P}{F} \frac{\delta F}{\delta P} \end{aligned}$$

# Next, Stability design.

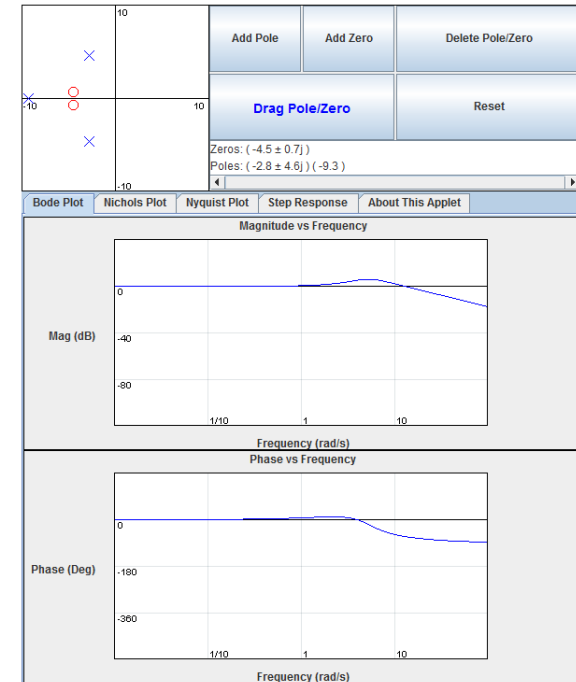
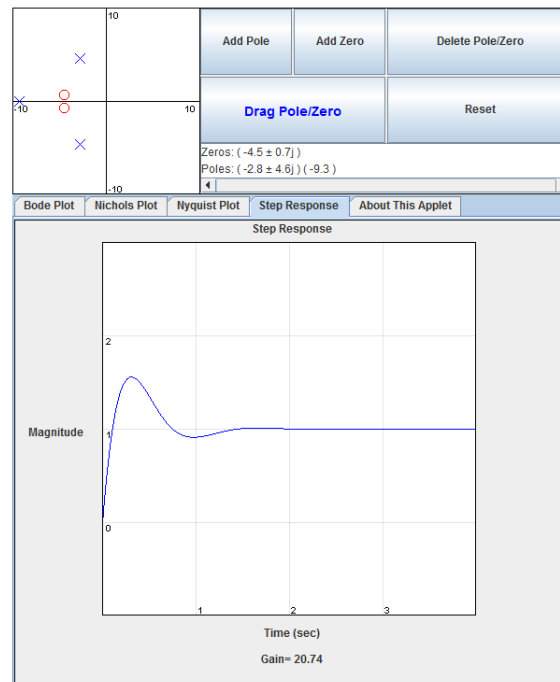
- Root locus , Nyquist Plot , Bode diagram



Question: For what value of  $K > 0$  is the closed-loop system stable?

**PoleZeroApplet:**

<http://web.mit.edu/6.302/www/pz/>



## 7.8 Steady-State Error for Systems in State Space

- Analysis via final value theorem:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}r \\ y = \mathbf{C}\mathbf{x} \end{cases} \Rightarrow \begin{aligned} E(s) &= R(s) - Y(s), \quad Y(s) = R(s)T(s) \\ E(s) &= R(s) - R(s)T(s) \\ &= R(s) [1 - T(s)] \\ &= R(s) [1 - \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}] \end{aligned}$$

$\Rightarrow$  Applying the final value theorem:

$$\lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} sR(s)[1 - \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]$$