Chapter 7. Steady-State Errors

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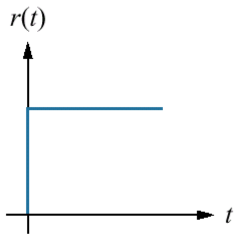
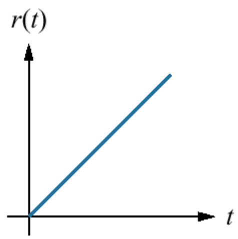
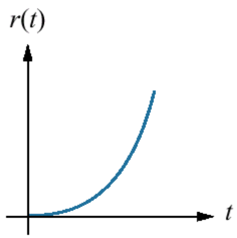
Steady-state error : difference between the input and the output for a test input as $t \to \infty$

Objectives

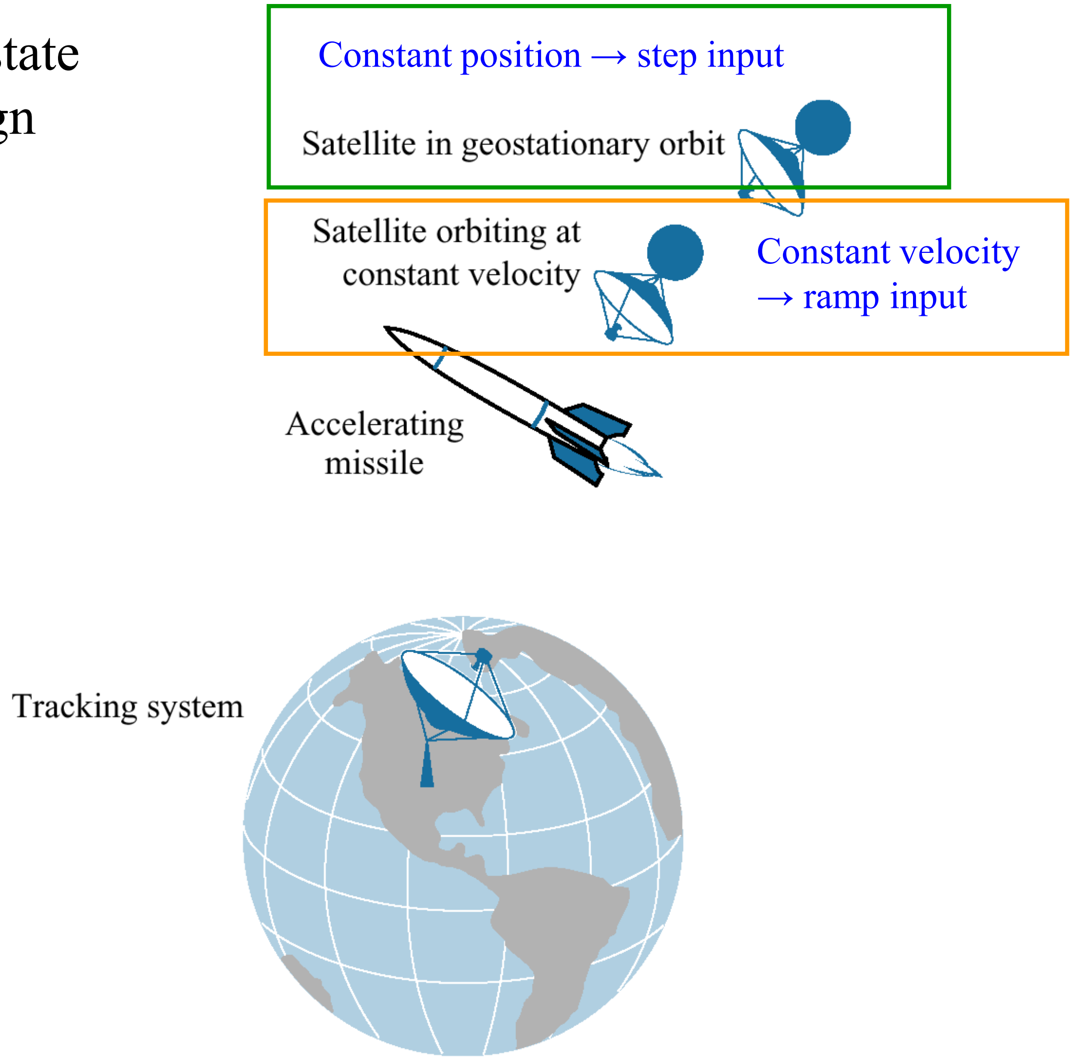
- How to find the *steady-state error* for a unity feedback system
- How to specify a system's steady-state *error performance*
- How to find the steady-state error for *disturbance inputs*
- How to find the steady-state error for *nonunity feedback systems*
- How to *design system performance* to meet steady-state error performance specifications
- How to find the steady-state error for systems represented in *state space*

7.1 Introduction

- steady-state error $\Rightarrow \quad \lim_{t\to\infty} e(t) = \lim_{t\to\infty} \{r(t) - c(t)\}$
- Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\boxed{\frac{1}{s}}$
	Ramp	Constant velocity	t	$\boxed{\frac{1}{s^2}}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

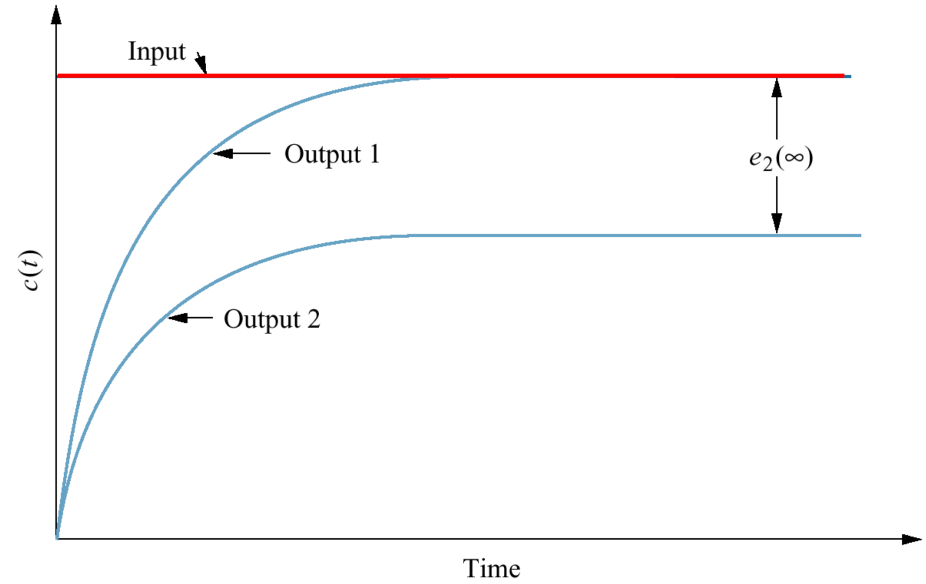
- Test inputs for steady-state error analysis and design vary with target type



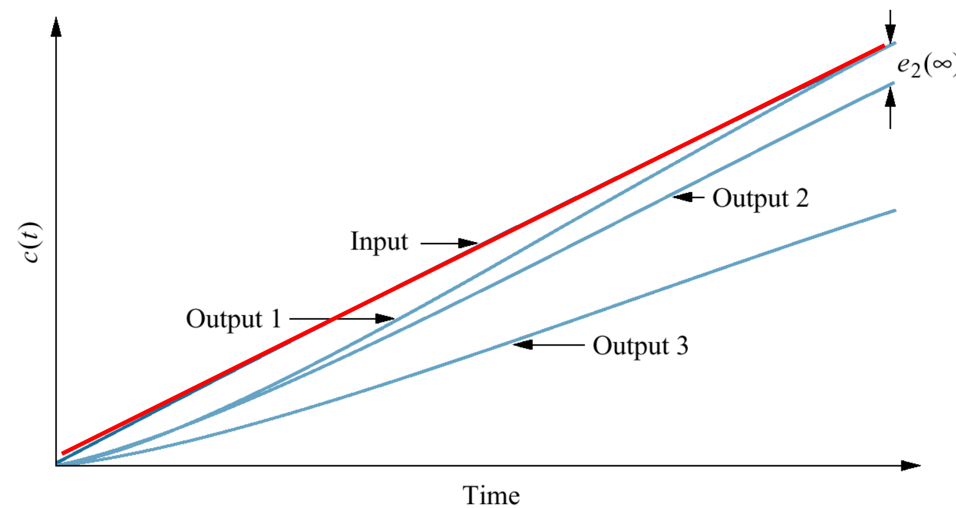
Evaluation steady-state error:

a. step input;

b. ramp input

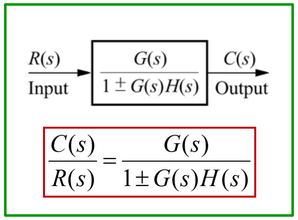


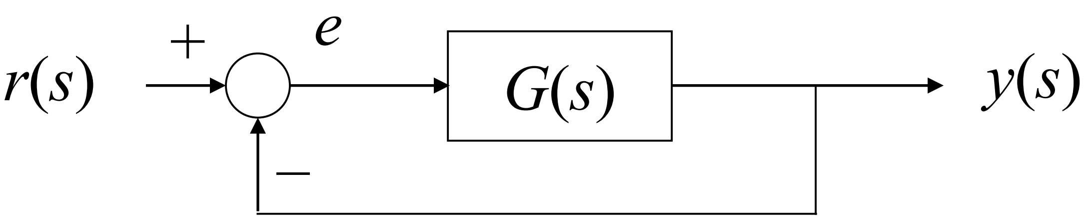
(*a*)



(*b*)

Unity Negative Feedback

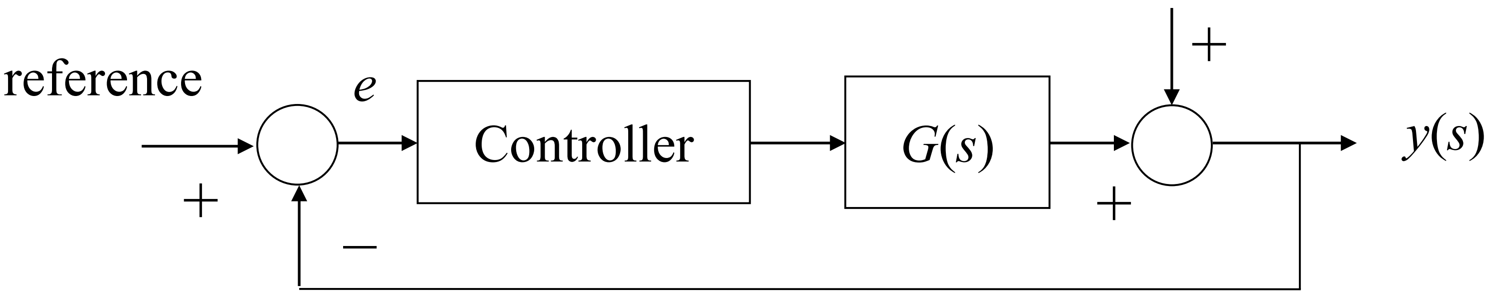




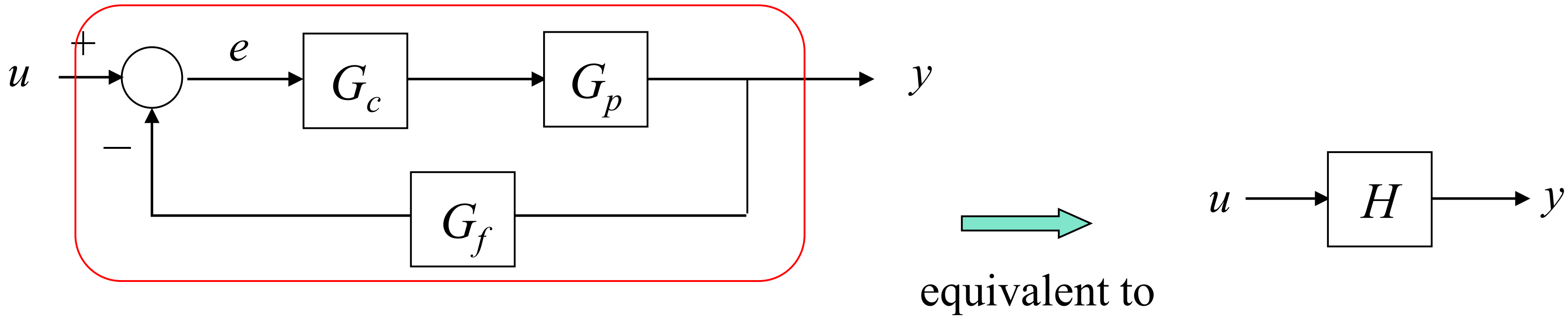
- The overall transfer function of the above system is:

$$T(s) = \frac{G(s)}{1 + G(s)}$$

- System Model:



Feedback



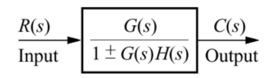
Find H ?

$$Y(s) = G_p(s) \cdot G_c(s) \cdot E(s)$$

$$E(s) = U(s) - G_f(s) \cdot Y(s)$$

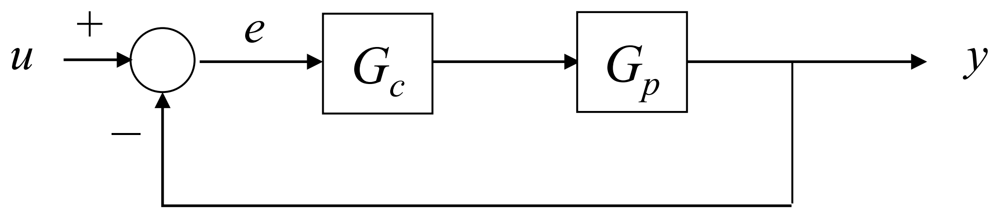
$$Y(s) = G_p(s) \cdot G_c(s)\left[U(s) - G_f(s) \cdot Y(s)\right]$$

$$Y(s) = \boxed{\frac{G_p \cdot G_c}{1 + G_p \cdot G_c \cdot G_f} U(s) = H(s) \cdot U(s)}$$



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

- Special case: unity feedback system

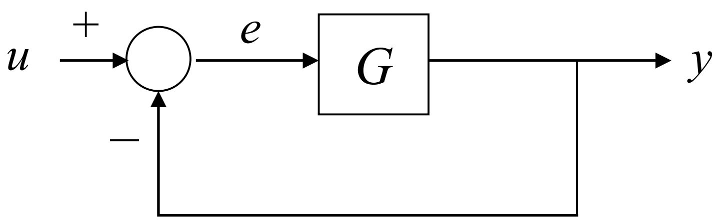


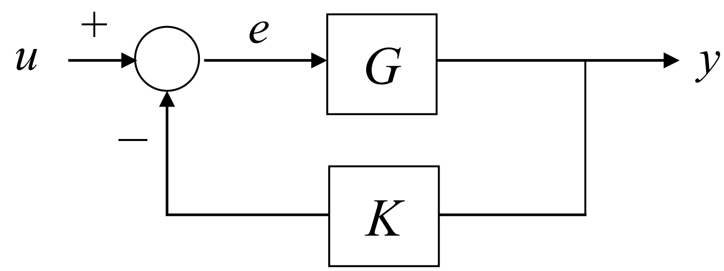
$$Y(s) = \frac{G_p(s) \cdot G_c(s)}{1 + G_p(s) \cdot G_c(s)} U(s) = H(s) \cdot U(s)$$

Purpose of feedback:

1. Change the dynamics of the system, stabilization
2. Tracking

Reference



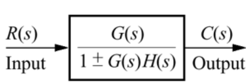


$$y = G(u - y)$$

$$Y(s) = \frac{G}{1+G}U(s)$$

$$y = G(u - Ky)$$

$$Y(s) = \boxed{\frac{G}{1+GK}U(s)}$$



$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}}$$

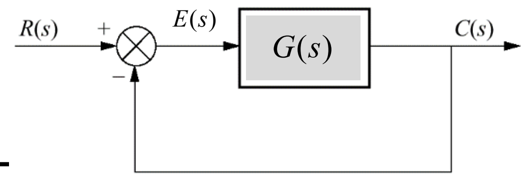
$$y = G(u - Ky)$$

$$y + GKy = Gu$$

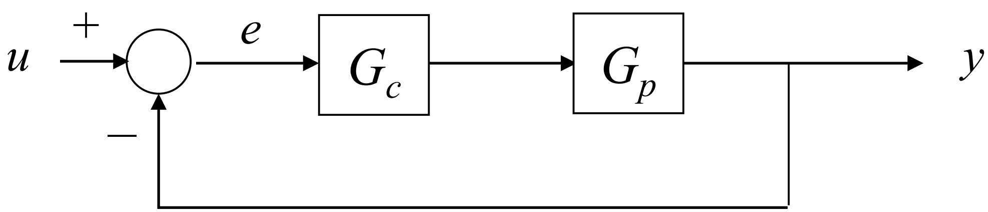
$$y(1+GK) = Gu$$

$$Y(s) = \frac{G}{1+GK}U(s)$$

Stabilization



$$E(s) = R(s) - C(s)$$
$$C(s) = E(s)G(s) = R(s)G(s) - C(s)G(s)$$
$$C(s)\left[1 + G(s)\right] = R(s)G(s)$$
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$



$$G_p(s) = \frac{1}{s-1}$$

⇨ unstable

Let $G_c(s) = K$,

$$H(s) = \frac{KG_p(s)}{1 + KG_p(s)} = \frac{\dfrac{K}{s-1}}{1 + \dfrac{K}{s-1}} = \frac{K}{s - 1 + K}$$

Pole: $s - 1 + K = 0, \quad s = 1 - K$

$$\boxed{\begin{array}{l} \rightarrow s = (1-K) < 0 \\ \rightarrow K > 1 \end{array}}$$

Tracking

- Open loop system

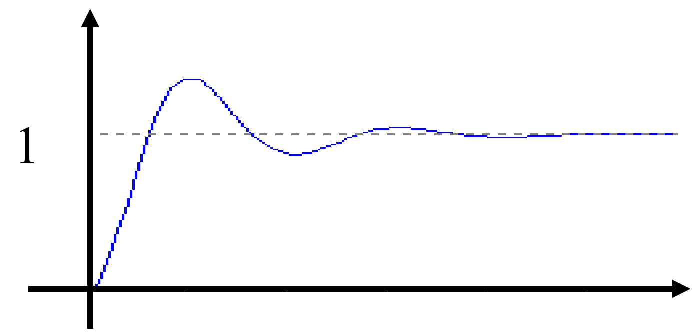
$$u \longrightarrow \boxed{G_p(s)} \longrightarrow y$$

$$G_p(s) = \frac{1}{(s+2)(s+\frac{1}{2})}$$

You want to find u such that

$$y(\infty) = \lim_{t\to\infty} y(t) = 1$$

Final value theorem



- Final value theorem : $y(\infty) = \lim_{s\to 0} sY(s)$

$$\to y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\left(\frac{1}{s}G_p(s)\right) = \frac{1}{2\cdot\frac{1}{2}} = 1$$

Try $u(t) = \text{step}(t)$

$$\rightarrow Y(s) = G_p(s) \cdot U(s) = \frac{1}{(s+2)(s+\frac{1}{2})} \cdot \frac{1}{s}$$

By the final value theorem

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{1}{(s+2)(s+\frac{1}{2})} = \boxed{1}$$

- Final value theorem:

  $x(\infty) = \lim_{s\to 0} sX(s)$

- Initial value theorem:

  $x(0) = \lim_{s\to \infty} sX(s)$

Ref: if u=step then

$$y(\infty) = \lim_{s\to 0} sG_p(s) \cdot \frac{1}{s} = G_p(0)$$ ⇨ the d_c gain of the system

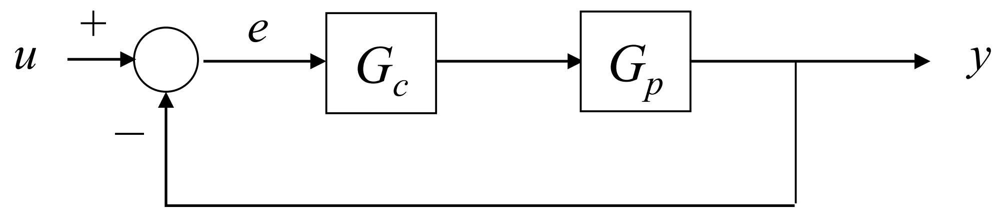
Now, suppose that $G_p(s) = \dfrac{1}{(s+2+\varepsilon)(s+\frac{1}{2})}$

for some small, ε

$$\rightarrow Y(s) = \frac{1}{(s+2+\varepsilon)(s+\frac{1}{2})} \cdot \frac{1}{s}$$

$$y(\infty) = \lim_{s\to 0} s \cdot \frac{1}{(s+2+\varepsilon)(s+\frac{1}{2})} \cdot \frac{1}{s} = \boxed{\frac{1}{1+\frac{\varepsilon}{2}} \neq 1}$$

What do you do ? $\Rightarrow$ Use feedback !



$$H(s)=\frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)}=\frac{\dfrac{1}{(s+2+\varepsilon)(s+\frac{1}{2})}G_c(s)}{1+\dfrac{1}{(s+2+\varepsilon)(s+\frac{1}{2})}G_c(s)}$$

$$Y(s)=H(s)U(s)=H(s)\frac{1}{s}\qquad (u(t)=step\ \text{ in problem})$$

$$y(\infty)=\lim_{s\to 0}sY(s)=\boxed{\lim_{s\to 0}H(s)}$$

i) Try $G_c(s) = k$

$$\rightarrow \quad H(s) = \frac{k}{(s+2+\varepsilon)(s+\frac{1}{2})+k}$$

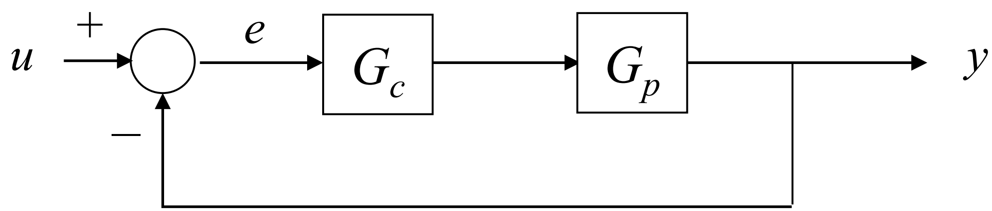
$$y(\infty) = \lim_{s\to 0} H(s) = \frac{k}{(2+\varepsilon)\frac{1}{2}+k} = \frac{k}{1+\frac{\varepsilon}{2}+k}$$

$$y(\infty) \neq 1, \quad \text{but} \quad y(\infty) \xrightarrow[k\to\infty]{} \boxed{1}$$

Need large gains

ii) Try $G_c(s) = \dfrac{1}{s}$, $\qquad H(s) = \dfrac{1}{(s+2+\varepsilon)(s+\frac{1}{2})\cdot s + 1}$

$$y(\infty) = \lim_{s\to 0} H(s) = 1$$



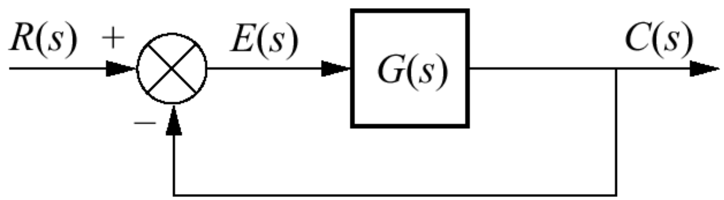
$$U(s) = \frac{1}{s} \qquad G(s) = \frac{1}{(s+2+\varepsilon)\left(s+\frac{1}{2}\right)}$$

$$G_c(s) = \frac{1}{s} \quad \Rightarrow y(\infty) = 1$$

End of tracking

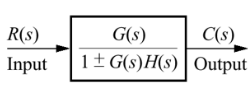
7.2 Steady-State Error for Unity Feedback Systems

- Steady-State Error for Unity Feedback Systems



$$E(s) = R(s) - C(s)$$
$$C(s) = E(s)G(s)$$
$$\Rightarrow \quad E(s) = \frac{1}{1+G(s)}R(s)$$

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$



$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}}$$

$$\boxed{e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}}$$

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}$$

- Step input: $R(s) = \dfrac{1}{s}$

$$e(\infty) = \lim_{s\to 0} \frac{s(1/s)}{1+G(s)} = \frac{1}{1+\underbrace{\lim_{s\to 0} G(s)}_{\to\infty \text{ for } n\geq 1}}$$

$$\downarrow \quad \boxed{e(\infty) = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+\infty} = 0}$$

For $e(\infty) = 0 \quad \Rightarrow \quad n \geq 1$

$(\lim_{s\to 0} G(s) = \infty)$ If $n = 0$ then $e(\infty)$ is finite.

- Ramp input: $R(s) = \dfrac{1}{s^2}$

$$\boxed{\text{For } n=1:\ e(\infty) = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{\lim_{s\to 0} s\frac{(s+z_1)}{s(s+p_1)}} = \frac{1}{\lim_{s\to 0}\frac{(s+z_1)}{(s+p_1)}} = \frac{p_1}{z_1}}$$

$$e(\infty) = \lim_{s\to 0} \frac{s(1/s^2)}{1+G(s)} = \frac{1}{\underbrace{\lim_{s\to 0} sG(s)}_{\to\infty \text{ for } n\geq 2}}$$

For $e(\infty) = 0 \quad \Rightarrow \quad n \geq 2$

If $n = 1$, $e(\infty)$ is finite.

If $n = 0$, $e(\infty)$ is infinite.

- Parabolic input: $R(s) = \dfrac{1}{s^3}$

$$e(\infty) = \lim_{s\to 0} \frac{s(1/s^3)}{1+G(s)} = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

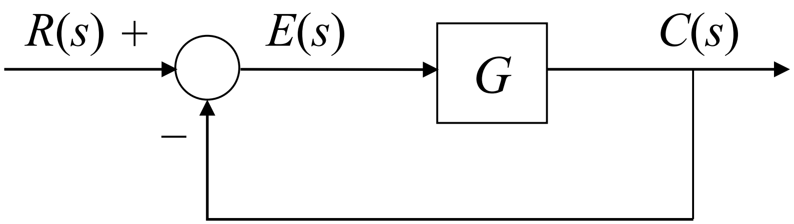
For $e(\infty) = 0 \quad \Rightarrow \quad n \geq 3$

If $n = 2$, $e(\infty)$ is finite.

If $n = 0$ or 1, $e(\infty)$ is infinite.

$$e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

Example:



$$G(s) = \frac{100(s+2)(s+6)}{s(s+3)(s+4)}$$

Find the steady-state error: i) $u(t)$, ii) $tu(t)$, iii) $\frac{1}{2}t^2u(t)$.

i) $R(s) = \frac{1}{s}$

$$e(\infty) = \lim_{s \to 0} \frac{s\left(\frac{1}{s}\right)}{1+G(s)} = \frac{1}{1+\lim_{s \to 0} G(s)} = \boxed{\frac{1}{\infty} = 0}$$

$$\boxed{e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}}$$

ii) $R(s) = \dfrac{1}{s^2}$

$$e(\infty) = \lim_{s\to 0} \frac{s\left(\dfrac{1}{s^2}\right)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \boxed{\frac{1}{\lim\limits_{s\to 0} sG(s)} = \frac{1}{100}}$$

iii) $R(s) = \dfrac{1}{s^3}$

$$e(\infty) = \lim_{s\to 0} \frac{s\left(\dfrac{1}{s^3}\right)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2+s^2G(s)} = \frac{1}{\lim\limits_{s\to 0} s^2G(s)} = \boxed{\frac{1}{0} = \infty}$$

$$e(\infty)=\lim_{s\to 0} sE(s)=\lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$

7.3 Static Error Constants and System Type

- Static error constants

i) step input $u(t)$, $e(\infty)=\dfrac{1}{1+\lim\limits_{s\to 0} G(s)}$ $=\dfrac{1}{1+K_p}$

$$\boxed{\text{position constant : } K_p=\lim_{s\to 0} G(s)}$$

ii) ramp input $tu(t)$, $e(\infty)=\dfrac{1}{\lim\limits_{s\to 0} s\,G(s)}$ $=\dfrac{1}{K_v}$

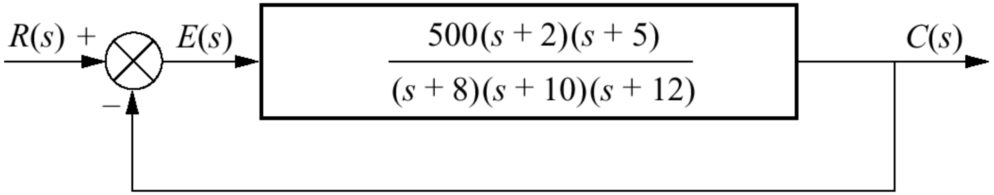
$$\boxed{\text{velocity constant : } K_v=\lim_{s\to 0} sG(s)}$$

iii) parabolic input $\dfrac{1}{2}t^2u(t)$, $e(\infty)=\dfrac{1}{\lim\limits_{s\to 0} s^2\,G(s)}$ $=\dfrac{1}{K_a}$

$$\boxed{\text{acceleration constant : } K_a=\lim_{s\to 0} s^2G(s)}$$

Example 7.4 (page 350): Steady-state error via static error constants

Evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs.



$s^n \to n = 0 \to s^0 = 1$

$$K_p = \lim_{s\to 0} G(s) = 5.208 \qquad K_v = \lim_{s\to 0} sG(s) = 0 \qquad K_a = \lim_{s\to 0} s^2 G(s) = 0$$

For a step input $\to$ $$e(\infty) = \frac{1}{1+K_p} = 0.161$$

For a ramp input $\to$ $$e(\infty) = \frac{1}{K_v} = \infty$$

For a parabolic input $\to$ $$e(\infty) = \frac{1}{K_a} = \infty$$

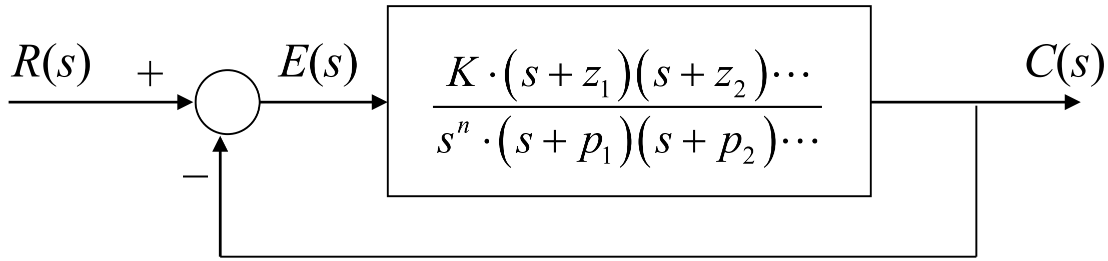
End of example

- System type

$n = 0 \rightarrow \text{Type } 0$

$n = 1 \rightarrow \text{Type } 1$

$n = 2 \rightarrow \text{Type } 2$

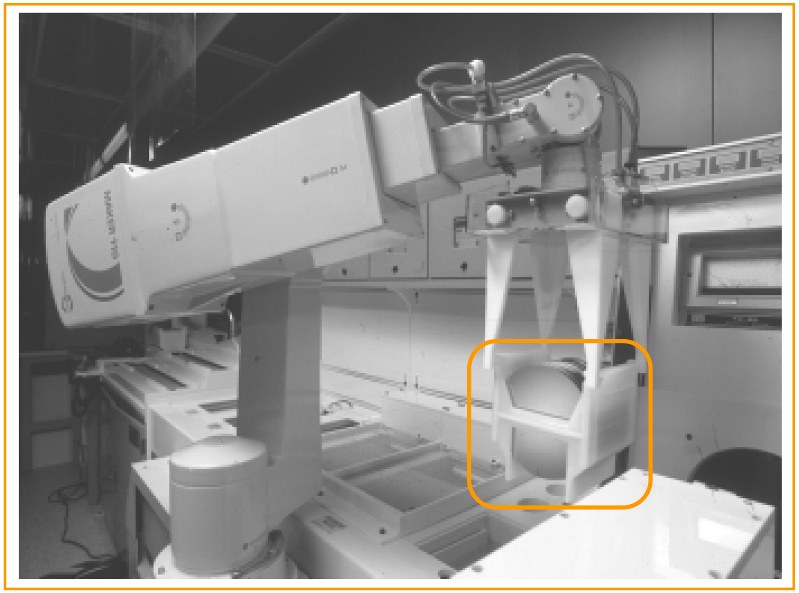


		Type 0		Type 1		Type 2	
Input	Steady-state error formula	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

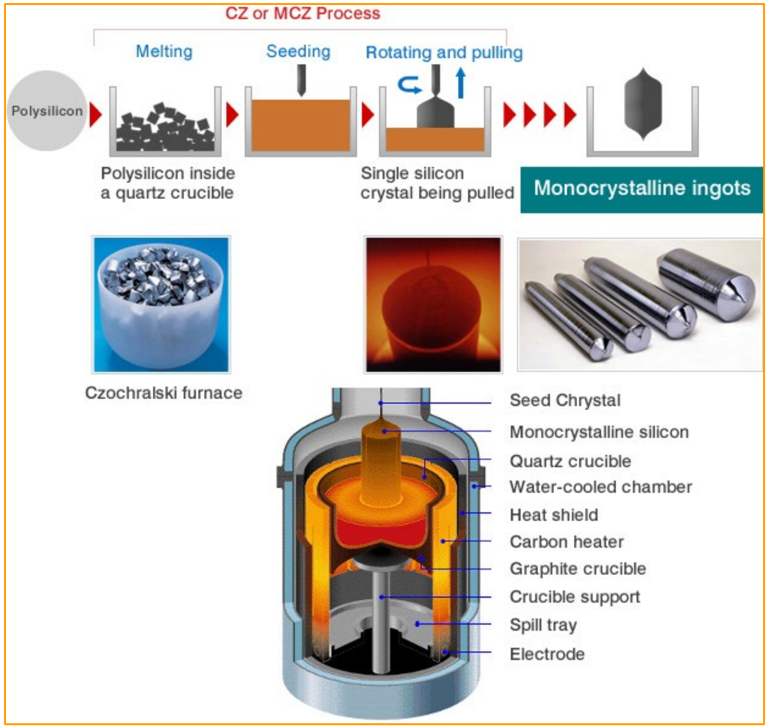
Relationships between input, system type, static error constants, and steady-state errors

7.4 Steady-State Error Specification

- A robot used in the manufacturing of semiconductor random-access memories (RAMs) similar to those in personal computers.
- Steady-state error is an important design consideration for assembly-line robots.



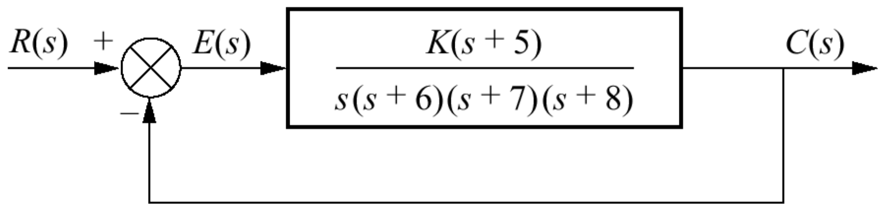
© Westlight/ Charles O'Rear.



https://www.sumcosi.com/english/products/process/step_01.html

Example 7.6 (page 355): Gain design to meet a steady-state error specification

Find the value of *K* so that there is 10% error in the steady state.



- The system is Type 1 $\Rightarrow$ a ramp input is applied to the system.

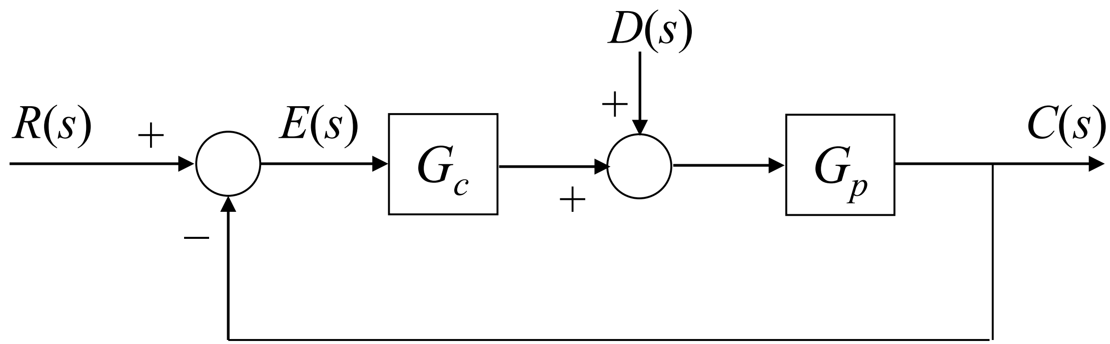
$$e(\infty) = \frac{1}{K_v} = 0.1$$

$$\rightarrow \quad K_v = 10 = \lim_{s \to 0} sG(s) = \frac{K \times 5}{6 \times 7 \times 8}$$

$$\rightarrow \quad K = 672$$

Input	Steady-state error formula
Step, $u(t)$	$\frac{1}{1+K_p}$
Ramp, $tu(t)$	$\frac{1}{K_v}$
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$

7.5 Steady-State Error for Disturbances



$$E(s) = R(s) - C(s)$$

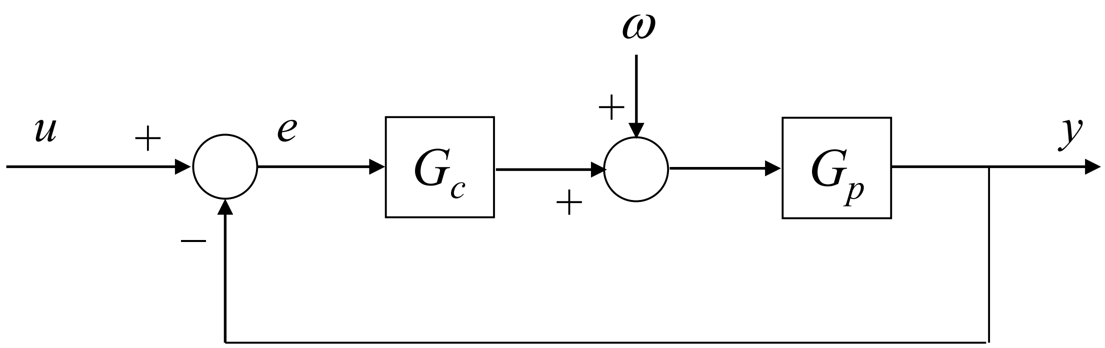
$$C(s) = \left[E(s)\cdot G_c(s) + D(s)\right]\cdot G_p(s)$$

$$E = R - E\cdot G_c G_p - D\cdot G_p \quad , \quad E\left(1 + G_c G_p\right) = R - D\cdot G_p$$

$$E(s) = \boxed{\frac{R(s)}{1+G_c(s)\cdot G_p(s)}} \boxed{- \frac{G_p(s)}{1+G_c(s)\cdot G_p(s)}\cdot D(s)}$$

$$= \boxed{e_R(\infty) + e_D(\infty)}$$

Example: Find G_c to eliminate the effect of ω on y in steady state.



$$G_p(s) = \frac{s+1}{(s+2)(s+0.51)}$$

$$\omega(t) = u(t)$$

$$\boxed{\begin{aligned} Y_\omega(s) &= \frac{G_P}{1-G_C G_P}\omega \\ &= \frac{\frac{(s+1)}{(s+2)(s+0.51)}}{1-\frac{(s+1)}{(s+2)(s+0.51)}G_C}\omega \\ &= \frac{(s+1)\omega}{(s+2)(s+0.51)-(s+1)G_C} \end{aligned}}$$

$$Y(s) = G_P(\omega - E\cdot G_C), \quad E = U - Y$$

$$= G_P(\omega - (U-Y)\cdot G_C)$$

$$= G_P\omega - UG_PG_C + G_PG_CY \Rightarrow Y(s) = \overbrace{\frac{G_P}{1-G_CG_P}\cdot\omega}^{Y_\omega(s)} - \overbrace{\frac{G_CG_P}{1-G_CG_P}\cdot U}^{Y_u(s)}$$

$$Y_\omega(s) = \frac{G_P}{1-G_CG_P}\omega \xrightarrow{\omega = \frac{1}{s}} = \frac{(s+1)\frac{1}{s}}{(s+2)(s+0.51)-(s+1)G_C}$$

$$\boxed{y_\infty(\infty) = \lim_{s\to 0} sY_\omega(s)}$$

$$Y_\omega(s) = \frac{G_P}{1 - G_C G_P}\omega$$
$$= \frac{(s+1)\frac{1}{s}}{(s+2)(s+0.51)-(s+1)G_C}$$

Let, $\quad G_C(s) = \dfrac{1}{s}$

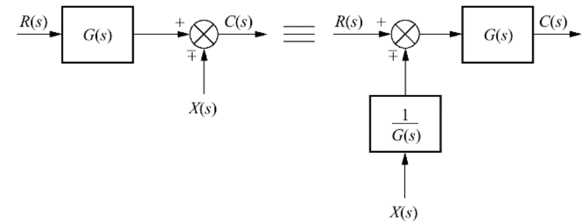
$$y_\infty(\infty) = \lim_{s\to 0} sY_\omega(s) = \lim_{s\to 0} s\cdot\frac{(s+1)\frac{1}{s}}{(s+2)(s+0.51)-(s+1)\frac{1}{s}}$$

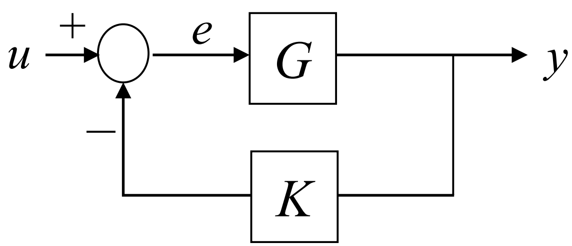
$$= \lim_{s\to 0} \frac{s(s+1)}{s(s+2)(s+0.51)-(s+1)}$$

$$=0$$

$$\therefore \quad G_C(s) = \frac{1}{s} \quad \rightarrow \quad y_\infty(\infty) = \boxed{0}$$

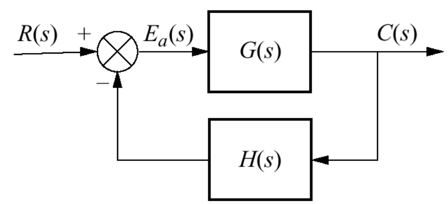
7.6 Steady-State Error for Nonunity Feedback Systems





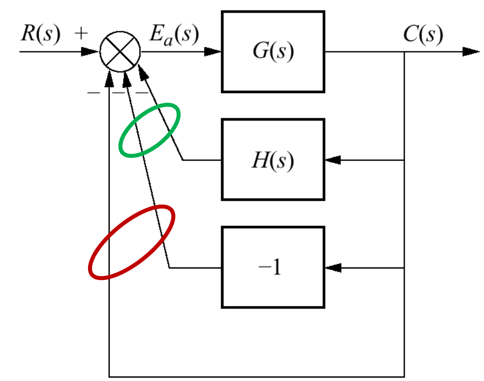
$$y = \boxed{G(u - Ky)}$$

$$Y(s) = \boxed{\frac{G}{1+GK}U(s)}$$

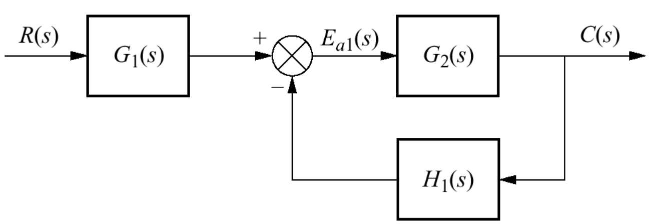


$$G = G_1G_2,\ H = \frac{H_1}{G_1}$$

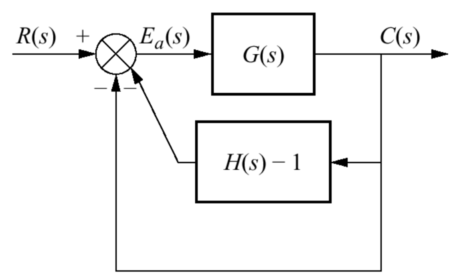
(b)



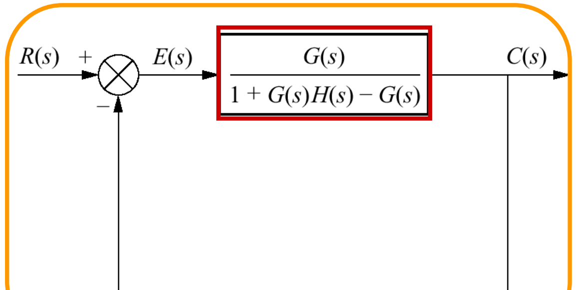
(c)



(a)



(d)



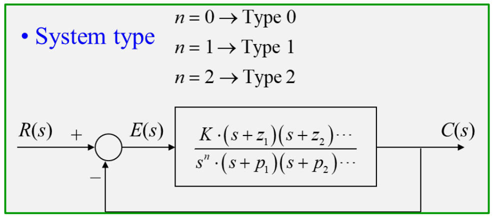
(e)

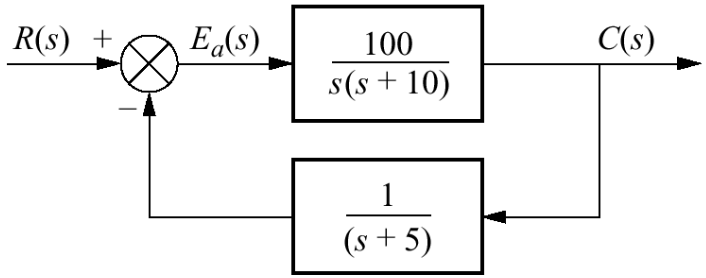
$$Y = \frac{\dfrac{G}{1+GH-G}}{1+\dfrac{G}{1+GH-G}}R = \frac{G}{(1+GH-G)+G}R = \frac{G}{1+GH}R$$

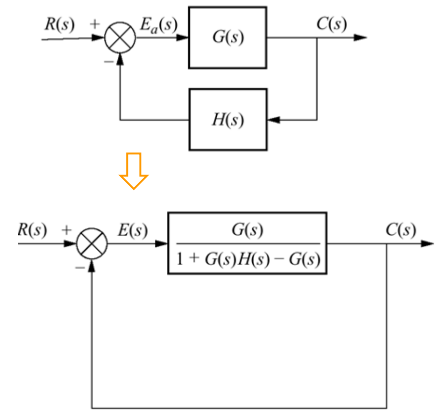
$$C(s) = \boxed{G_1\frac{G_2}{1+G_2H_1}}R(s) = \frac{G_1G_2}{1+G_1G_2\dfrac{H_1}{G_1}}R(s) = \frac{G}{1+GH}R(s)$$

Example 7.8 (page 359): Steady-state error for nonunity feedback systems

Find the system type, the appropriate static error constant, and the steady-state error for a unit step input.







$$G_e(s) = \frac{G(s)}{1+G(s)H(s)-G(s)}$$

equivalent

$$= \frac{\dfrac{100}{s(s+10)}}{1+\dfrac{100}{s(s+10)}\cdot\dfrac{1}{s+5}-\dfrac{100}{s(s+10)}}$$

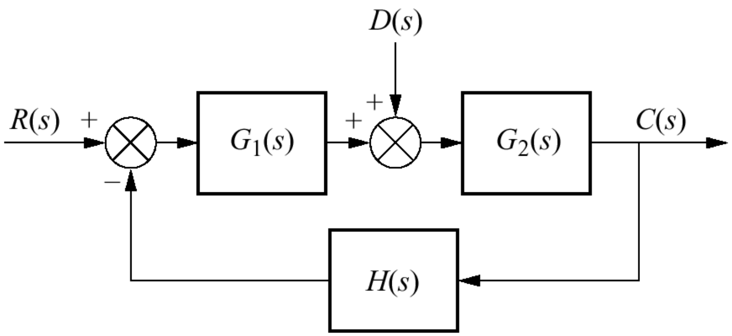
$$= \frac{100(s+5)}{s^3+15s^2-50s-400}$$

1) Type 0

2) Error Constant: $K_p = \lim_{s\to 0} G_e(s) = \boxed{\frac{500}{-400} = -\frac{5}{4}}$

3) $e(\infty) = \dfrac{1}{1+K_p} = \dfrac{1}{1-(5/4)} = -4$

• Non-Unity Feedback Control with Disturbance



input of G_1: $R - CH$

input of G_2 : $G_1 E + D = G_1(R - CH) + D$

$C = G_2\left[G_1(R - CH) + D\right] \cdots (1)$

steady-state error $E = R - C,\ C = R - E \cdots (2)$

$(2) \to (1)$;

$$\begin{aligned} R - \underline{E} &= G_1G_2R - G_1G_2(CH) + G_2D \\ &= G_1G_2R - G_1G_2(R - E)H + G_2D \\ &= G_1G_2R - G_1G_2RH + \underline{G_1G_2EH} + G_2D \end{aligned}$$

arrange for E,

$$E(1 + G_1G_2H) = R - G_1G_2R + G_1G_2RH - G_2D$$

$$\therefore E = \frac{1 - G_1G_2 + G_1G_2H}{1 + G_1G_2H}R - \frac{G_2}{1 + G_1G_2H}D$$

$$= \left(1 - \frac{G_1G_2}{1 + G_1G_2H}\right)R - \left(\frac{G_2}{1 + G_1G_2H}\right)D$$

Steady-state error : difference between the input and the output for a test input as $t \to \infty$

$$R(s) = D(s) = \frac{1}{s}$$

$$e(\infty) = \lim_{s\to 0} sE(s) = \left\{\left[1 - \frac{\lim_{s\to 0} G_1(s)G_2(s)}{\lim_{s\to 0}\left[1 + G_1(s)G_2(s)H(s)\right]}\right] - \left[\frac{\lim_{s\to 0} G_2(s)}{\lim_{s\to 0}\left[\left[1 + G_1(s)G_2(s)H(s)\right]\right]}\right]\right\}$$

For $e(\infty) = 0$, $\quad \dfrac{\lim_{s\to 0}[G_1(s)G_2(s)]}{\lim_{s\to 0}[1 + G_1(s)G_2(s)H(s)]} = 1 \quad and \quad \dfrac{\lim_{s\to 0} G_2(s)}{\lim_{s\to 0}[1 + G_1(s)G_2(s)H(s)]} = 0$

$$\Rightarrow \frac{1}{\underbrace{\frac{1}{\lim_{s\to 0} G_1(s)G_2(s)}}_{\boxed{0}} + \underbrace{\lim_{s\to 0} H(s)}_{\boxed{1}}} = 1, \quad \frac{1}{\frac{1}{\lim_{s\to 0} G_2(s)} + \underbrace{\lim_{s\to 0} G_1(s)H(s)}_{\boxed{\infty}}} = 0$$

7.7 Sensitivity

- System parameters are changed → transfer function
  ⇒ performance
  ⇒ sensibility

$$F = \frac{K}{K+a}\,, \quad K = 10\,,\ a = 100 \ \rightarrow\ F = \frac{10}{10+100} = 0.091$$

$$\boxed{\frac{300-100}{100}\times 100 = 200\%}$$

$$\boxed{\frac{0.032-0.091}{0.091}\times 100 = -65\%}$$

$$a = 300 \ \rightarrow\ \boxed{F = \frac{10}{10+300} = 0.032}$$

F has reduced sensitivity to changes in parameter a :

$$\begin{pmatrix} \text{200\% change in parameter '}a\text{'} \\ \rightarrow \quad -65\% \text{ change in } F \end{pmatrix} \Rightarrow \frac{-65\%}{200\%} = -0.325$$

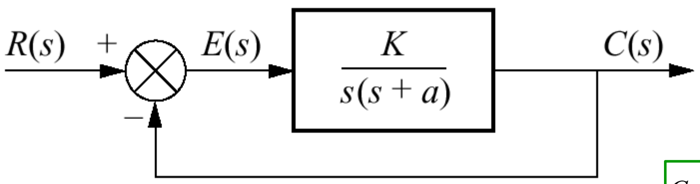
- Sensitivity:

$$S_{F:P} = \lim_{\Delta P \to 0} \frac{\dfrac{\Delta F}{F}}{\dfrac{\Delta P}{P}} = \lim_{\Delta P \to 0} \frac{P}{F} \frac{\Delta F}{\Delta P}$$

$$\boxed{S_{F:P} = \frac{P}{F} \frac{\delta F}{\delta P}}$$

Example 7.10 (page 362): Sensitivity of a closed-loop transfer function

Calculate the sensitivity to change in the parameter a.
How would you reduce the sensitivity?



$$C = \frac{G}{1+G}R$$

$$T(s) = \frac{K}{s^2 + as + K}$$

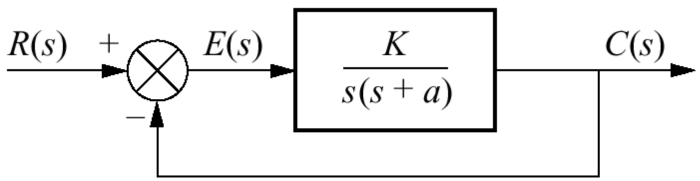
$$S_{F:P} = \lim_{\Delta P \to 0} \frac{\Delta F / F}{\Delta P / P} = \lim_{\Delta P \to 0} \frac{P}{F}\frac{\Delta F}{\Delta P} = \frac{P}{F}\frac{\delta F}{\delta P}$$

(1) $$S_{T:a} = \frac{a}{T}\frac{\delta T}{\delta a} = \frac{a}{\dfrac{K}{s^2 + as + K}} \cdot \frac{-sK}{(s^2 + as + K)^2} = \frac{-as}{(s^2 + as + K)}$$

(2) $\Rightarrow$ Increase K $\rightarrow$ $\boxed{\text{reduce sensitivity}}$

Example 7.11 (page 363): Sensitivity of steady-state error with ramp input

Find the sensitivity of the steady-state error to changes in parameters K and a with ramp input.



$$T(s) = \frac{K}{s^2 + as + K}$$

Type 1:

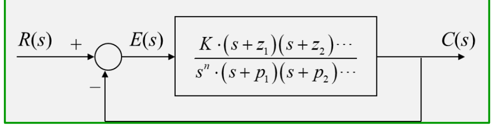
$$e = e(\infty) = \frac{1}{K_v} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{K/a} = \frac{a}{K}$$

$$S_{e:a} = \frac{a}{e}\cdot\frac{\delta e}{\delta a} = \frac{a}{a/K}\left(\frac{1}{K}\right) = 1$$

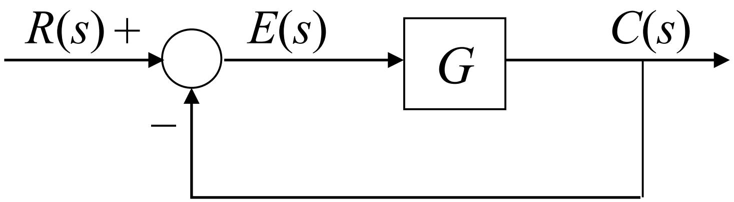
$$S_{e:K} = \frac{K}{e}\,\frac{\delta e}{\delta K} = \boxed{\frac{K}{a/K}\left[\frac{-a}{K^2}\right] = -1}$$

$$S_{F:P} = \lim_{\Delta P\to 0}\frac{\Delta F/F}{\Delta P/P} = \lim_{\Delta P\to 0}\frac{P}{F}\frac{\Delta F}{\Delta P} = \frac{P}{F}\frac{\delta F}{\delta P}$$

- System type $n = 0 \rightarrow$ Type 0, $n = 1 \rightarrow$ Type 1, $n = 2 \rightarrow$ Type 2



Example: Sensitivity of the S.S. error in K , a



$$G = \frac{K}{(s+a)(s+b)} \quad , \quad R(s) = \frac{1}{s}$$

$$E(s) = R(s) - G \cdot E(s), \quad E(s) = \frac{1}{1+G(s)} R(s)$$

$$e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s\left(\frac{1}{s}\right)}{1+G(s)} = \frac{1}{1+\lim\limits_{s \to 0} G(s)} = \boxed{\frac{1}{1+\frac{K}{ab}}}$$

$$= \frac{ab}{ab+K}$$

$$\boxed{e(\infty) = \lim_{s\to 0} sE(s) = \frac{ab}{ab+K}}$$

$$(1)\quad S_{e:a} = \frac{a}{e}\,\frac{\delta e}{\delta a} = \frac{a}{\dfrac{ab}{ab+K}} \cdot \frac{b\left(ab+K\right) - ab\left(b\right)}{\left(ab+K\right)^2} = \frac{b \cdot K}{b\left(ab+K\right)}$$

$$= \frac{K}{ab+K}$$

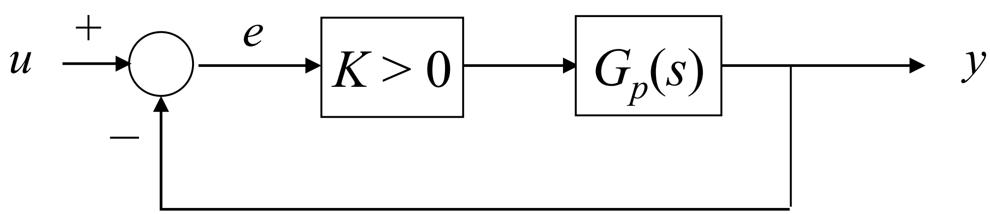
$$(2)\quad S_{e:K} = \frac{K}{e}\,\frac{\delta e}{\delta K} = \frac{K}{\dfrac{ab}{ab+K}} \cdot \frac{-ab}{\left(ab+K\right)^2} = \frac{-K}{ab+K}\,.$$

$\rightarrow$ feedback sensitivity

$$\boxed{\begin{aligned} S_{F:P} &= \lim_{\Delta P\to 0} \frac{\Delta F / F}{\Delta P / P} \\ &= \lim_{\Delta P\to 0} \frac{P}{F}\,\frac{\Delta F}{\Delta P} \\ &= \frac{P}{F}\,\frac{\delta F}{\delta P} \end{aligned}}$$

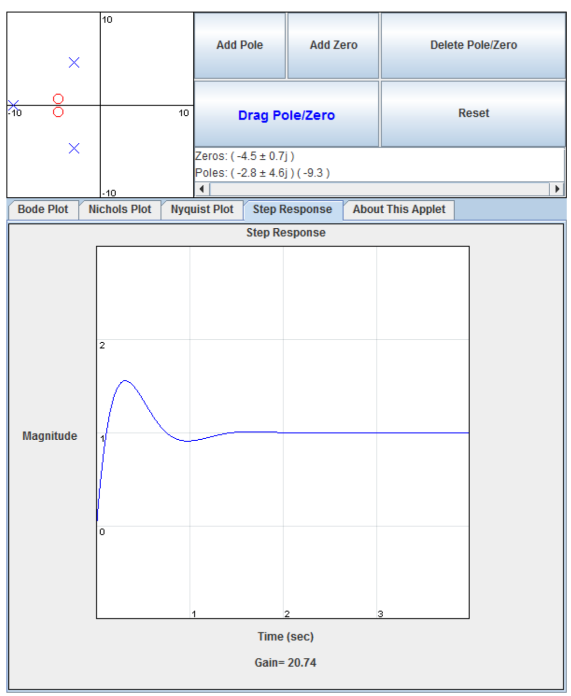
Next, Stability design.

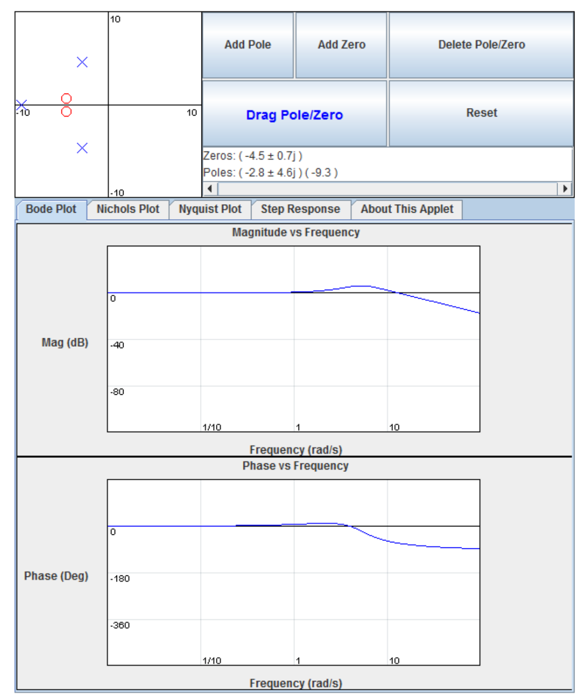
– Root locus , Nyquist Plot , Bode diagram



Question: For what value of $K > 0$ is the closed-loop system stable?

PoleZeroApplet:
http://web.mit.edu/6.302/www/pz/





7.8 Steady-State Error for Systems in State Space

- Analysis via final value theorem:

$$\begin{cases} \dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{B}r \\ y = \mathrm{Cx} \end{cases} \Rightarrow \quad E(s) = R(s) - Y(s), \quad Y(s) = R(s)T(s)$$

$$\begin{aligned} E(s) &= R(s) - R(s)T(s) \\ &= R(s)\,[1 - T(s)] \\ &= R(s)\,\boxed{[1 - \mathrm{C}(sI - \mathrm{A})^{-1}\mathrm{B}]} \end{aligned}$$

$\Rightarrow$ Applying the final value theorem:

$$\lim_{s \to 0} sE(s) = \lim_{s \to 0} sR(s)[1 - \mathrm{C}(sI - \mathrm{A})^{-1}\mathrm{B}]$$