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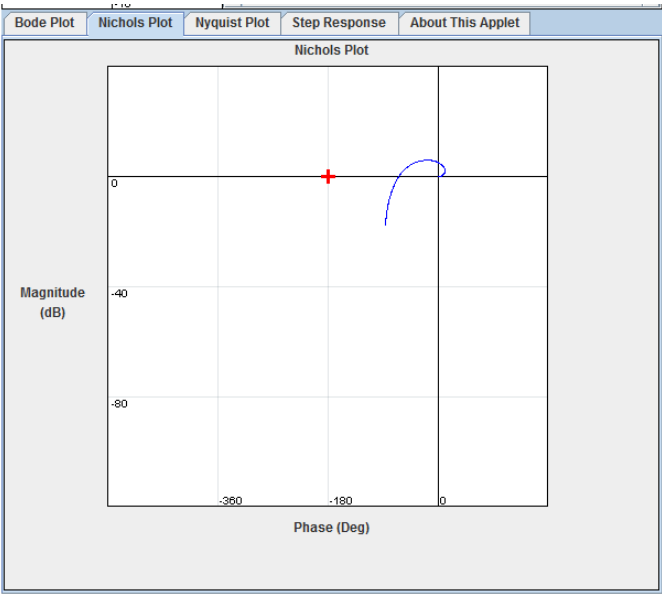
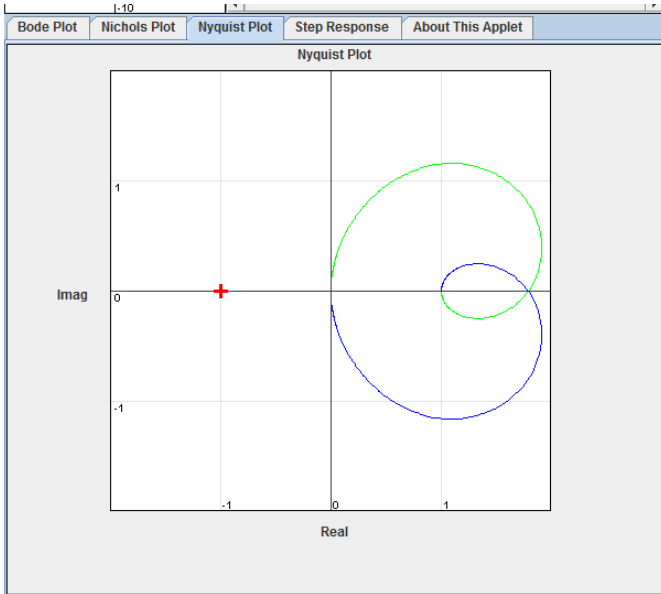
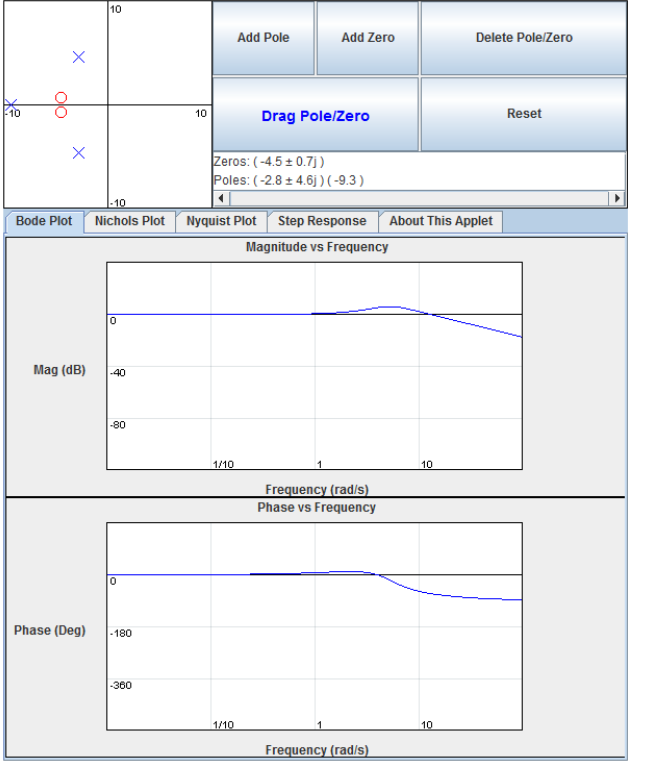
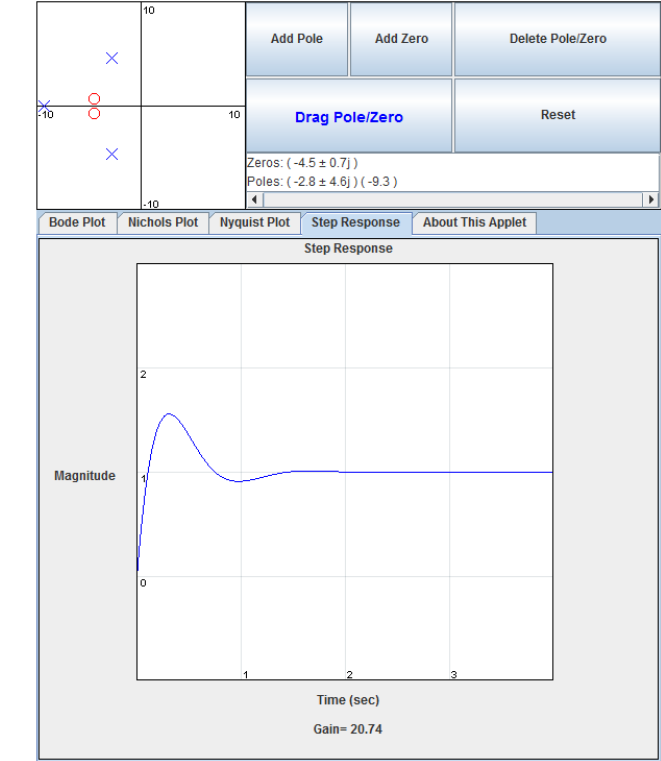
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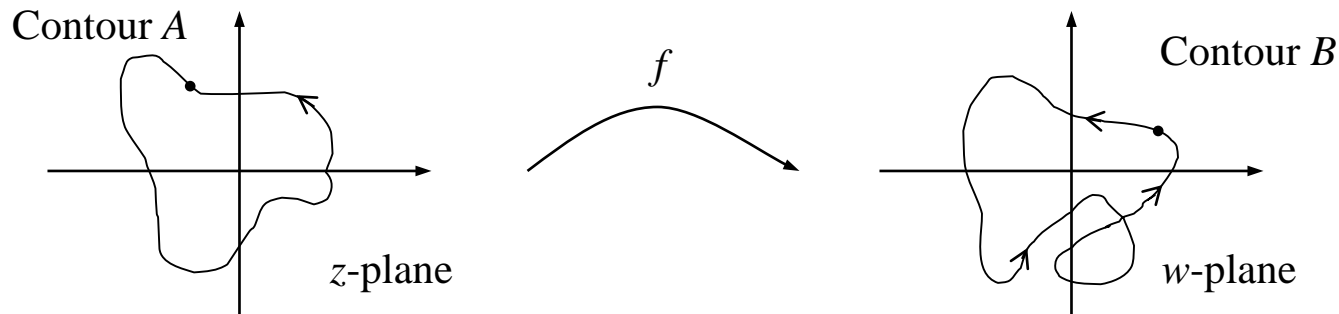


# Nyquist Stability Test

## Frequency Domain Analysis

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- Given function  $f$  from a complex plane  $z$  to a complex plane  $w$ .

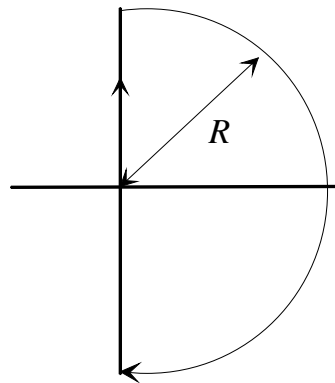


- $\Gamma \Rightarrow$  closed contour, counterclockwise

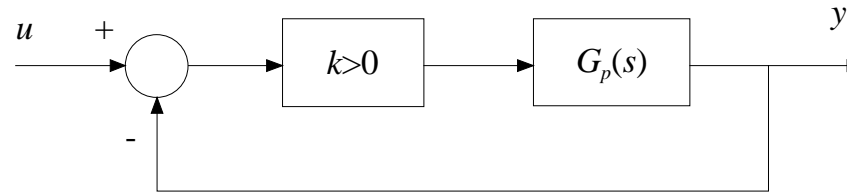
### *Theorem:*

The number of times  $f(\Gamma)$  encircles "0" (in  $w$ -plane) counterclockwise  
= number of zeros of  $f$  inside  $\Gamma$  – number of poles of  $f$  inside  $\Gamma$ .

- Now, consider stability of the following configuration:



$\Gamma$ : clockwise,  $R=\infty$ ,



$$Y(s) = \frac{k G_p(s)}{1 + k G_p(s)} U(s)$$

$$G(s) = \frac{N}{D}$$

$$1 + G(s) = 1 + \frac{N}{D} = \frac{D + N}{D}$$

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{D}{D + N}$$

pole of  $1 + G(s)$  = pole of  $G(s)$   
zero of  $1 + G(s)$  = pole of  $T(s)$

- Number of times  $1 + kG_p(s)|_{\Gamma}$  encircles 0 counterclockwise  
 = number of poles of  $1 + kG_p(s)$  inside  $\Gamma$  – number of zeros of  $1 + kG_p(s)$  inside  $\Gamma$   
 = number of RHP poles of  $1 + kG_p(s)$  – number of RHP zeros of  $1 + kG_p(s)$ .

$$N = P - Z$$

$$G(s) = \frac{N}{D}$$

$$1 + G(s) = 1 + \frac{N}{D} = \frac{D + N}{D}$$

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{D}{D + N}$$

pole of  $1 + G(s)$  = pole of  $G(s)$   
 zero of  $1 + G(s)$  = pole of  $T(s)$

- **Stability**  $\Leftrightarrow 1 + kG_p(s)$  has no zeros in RHP  $\Leftrightarrow$   
 # of times  $1 + kG_p(s)|_{\Gamma}$  encircles 0 counterclockwise  
 = # of RHP poles of  $1 + kG_p(s)$   
 = # of RHP poles of  $G_p(s)$ , defined as  $N_p$ .
- But  $1 + kG_p(s)|_{\Gamma}$  encircles 0 if and only if  $G_p(\Gamma)$  encircles  $-1/k$ .

$\Rightarrow$  **Nyquist stability test:**

The closed-loop system is stable if and only if  $G_p(\Gamma)$  encircles  $-1/k$  counterclockwise  $N_p$  times, where  $N_p$  = # of RHP poles of  $G_p(s)$ .

Example: ①  $G_P(s) = \frac{s-1}{s+1}$

$$s = j\omega: \quad G_P(j\omega) = \frac{j\omega - 1}{j\omega + 1} = \frac{(-1 + j\omega)(1 - j\omega)}{(1 + j\omega)(1 - j\omega)} = \frac{(\omega^2 - 1) + 2j\omega}{1 + \omega^2}$$

$$\omega = 0, \quad G_P = -1$$

$$\omega = \infty, \quad G_P = 1$$

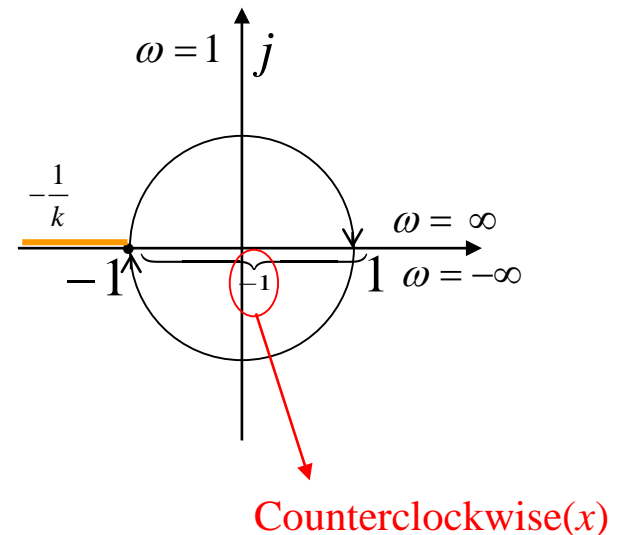
Cross Imag-axis:  $\omega^2 = 1 \rightarrow \omega = 1$

$$G_P(j \cdot 1) = \frac{2j}{2} = j$$

Cross Real-axis:  $\omega = 0$

$$G_P(0) = -1$$

$$\Rightarrow \text{stability: } -\frac{1}{k} < -1, \quad -k > -1 \quad \Rightarrow \quad k < 1$$

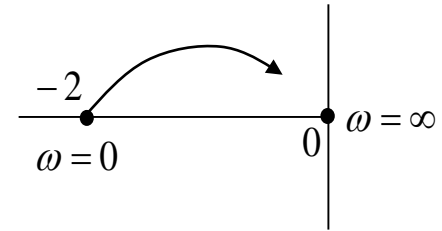


Example: ②  $G_p(s) = \frac{s+2}{(s-1)(s+1)} = \frac{s+2}{s^2-1}$

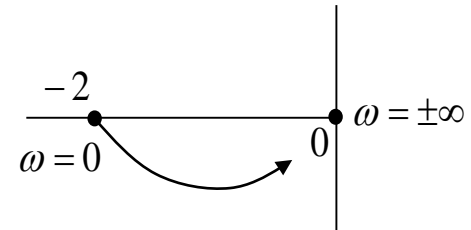
$$s = j\omega, \quad G_p = \frac{2+j\omega}{-1-\omega^2}$$

$$\omega = 0 \rightarrow G_p = -2$$

$$\omega = \infty \rightarrow G_p = 0$$



OR

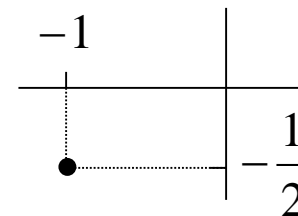


Cross Imag – axis  $\rightarrow 2 \neq 0 \rightarrow$  Never

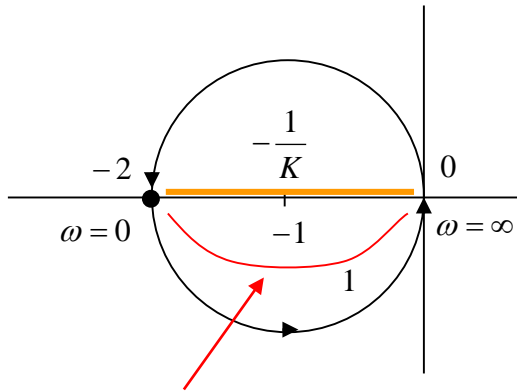
Cross Real – axis  $\rightarrow \omega = 0$

Let  $\omega=1$ , At what quadrant is  $G_p(j\omega) = ?$

$$G_p(1 \cdot j) = \frac{2+j(1)}{-2} \Rightarrow \text{III rd quadrant}$$







$$\Rightarrow -2 < -\frac{1}{K}$$

$$\Rightarrow K > \frac{1}{2}$$

⊗

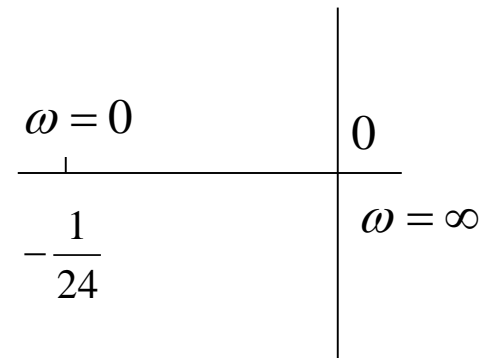
C.C.W (counterclockwise)

**Example:** ③  $G_p(s) = \frac{1}{(s+6)(s+4)(s-1)} = \frac{1}{s^3 + 9s^2 + 14s - 24}$

$$s = j\omega, \quad G_p(s) = \frac{1}{-9\omega^2 - 24 + j(14\omega - \omega^3)}$$

$$\omega = 0 \rightarrow G_p = -\frac{1}{24}$$

$$\omega = \infty \rightarrow G_p = 0$$

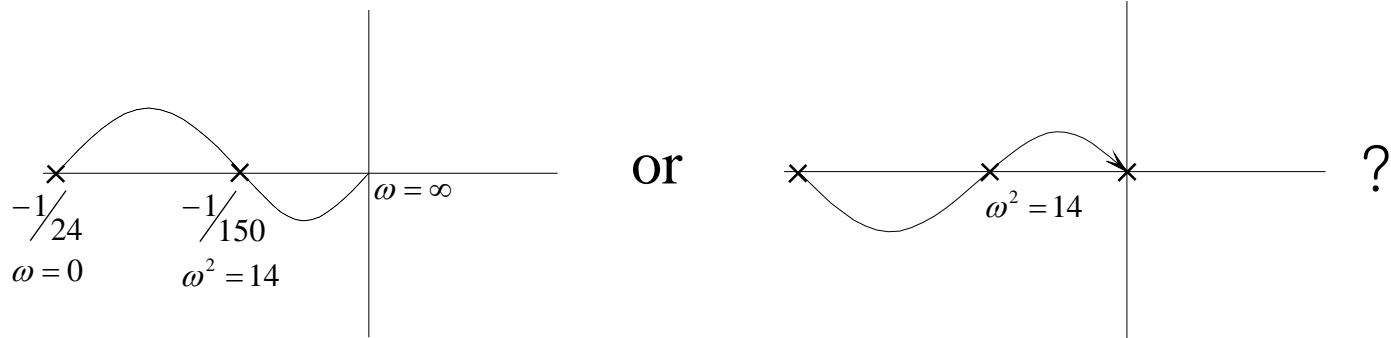


$$\frac{(-9\omega^2 - 24) - j(14\omega - \omega^3)}{\Delta}$$

Cross Imag axis:  $-9\omega^2 - 24 = 0 \rightarrow \text{never}$

Cross Real axis:  $14\omega - \omega^3 = 0 \rightarrow \omega = 0, \omega^2 = 14$

$$\rightarrow G_P = \frac{1}{-9(14) - 24} = \frac{-1}{150}$$



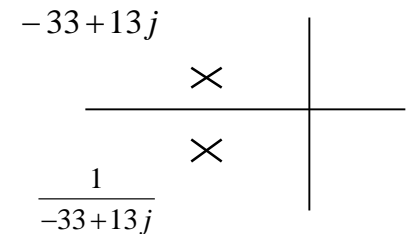
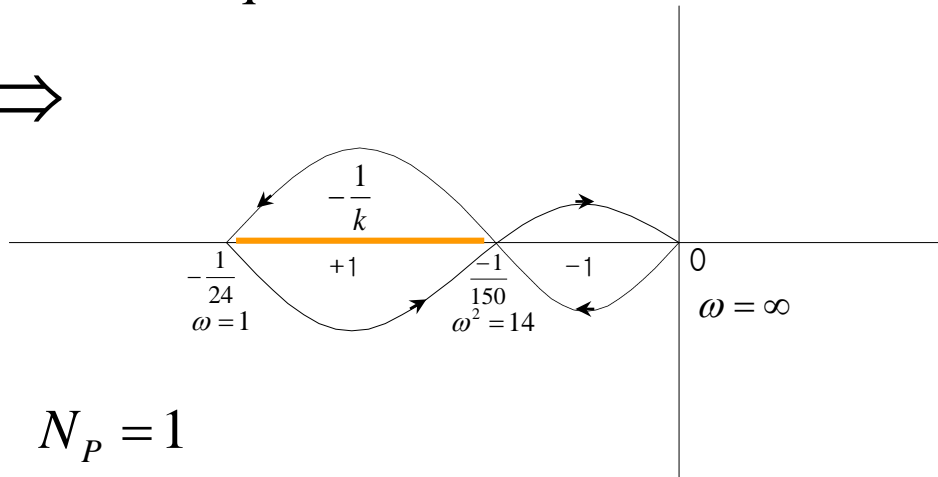
Let,  $\omega = 1, (< \sqrt{14})$  What quadrant is  $G_P(j)$  in ?

$$G_P(j\omega) = \frac{1}{-9 - 24 + j(14 - 1)} = \frac{1}{-33 + 13j} = \frac{-33 - 13j}{\Delta}$$

$\Rightarrow$  3rd quadrant

or

$\Rightarrow$



$$N_P = 1$$

$$\text{Stability : } -\frac{1}{24} < -\frac{1}{k} < -\frac{1}{150}$$

$$24 < k < 150$$



Example: ④  $G_p(s) = \frac{s+2}{(s-1)(s+4)(s^2+4s+5)}$

$N_p = \#$  of RHP poles of  $G_p(s)$

$$G_p(s) = \frac{s+2}{s^4 + 7s^3 + 13s^2 - s - 20} \quad s = j\omega$$

$$\Rightarrow G_p(j\omega) = \frac{2 + j\omega}{(\omega^4 - 7j\omega^3 - 13\omega^2 - j\omega - 20) \times (2 - j\omega)} \times (2 - j\omega)$$

$$= \frac{4 + \omega^2}{(-5\omega^4 - 27\omega^2 - 40) + j(-\omega^5 - \omega^3 + 18\omega)}$$

- $\omega = 0, \quad G_p(j \cdot 0) = -\frac{1}{10}$
- $\omega = \infty, \quad G_p(j \cdot \infty) = 0$

- Cross Real axis ;  $-\omega^5 - \omega^3 + 18\omega = 0$ ,  $\omega(\omega^4 + \omega^2 - 18) = 0$

$$\omega^2 = \frac{-1 \pm \sqrt{1 + 4 \cdot 18}}{2} = \frac{-1 \pm \sqrt{73}}{2} = 3.77$$

$$\omega = \sqrt{3.77} = 1.94$$

$$G_P(j\omega)_{\omega=1.94} = \frac{4 + \omega^2}{-5\omega^4 - 27\omega^3 - 40}_{\omega=1.94} = -0.0365$$

$$\cong -0.04$$

- Cross Imag axis ;  $5\omega^4 + 27\omega^2 + 40 = 0$

$$\omega^2 = \frac{-27 \pm \sqrt{27^2 - 800}}{10} \Rightarrow \text{never}$$

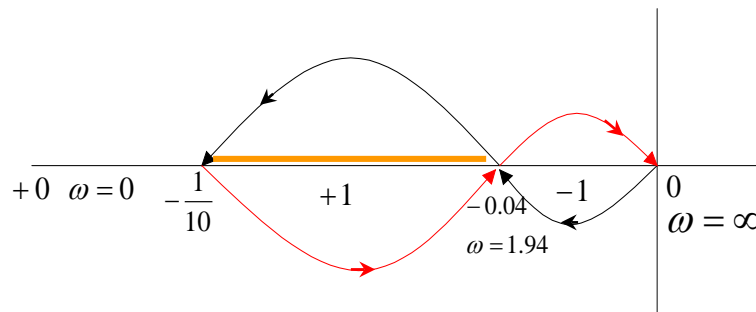
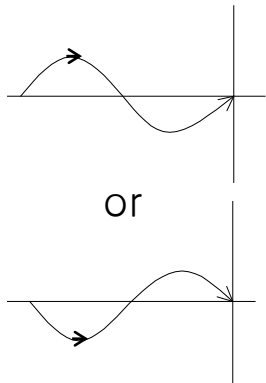
$\Rightarrow$  At  $\omega = 1$  ( $0 < 1 < 1.94$ )

$$G_P(j \cdot 1) = \frac{5}{-72 + j16} \Rightarrow \text{III}$$

$$N_P = 1$$

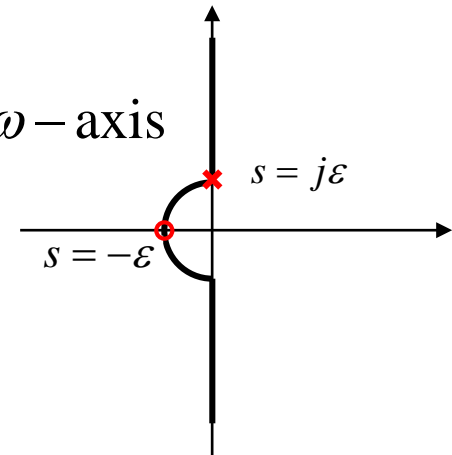
$$\text{stability: } -\frac{1}{10} < -\frac{1}{k} < -\frac{4}{100}$$

$$\therefore 10 < k < 25$$



Example: ⑤  $G_P(s) = \frac{s+1}{s(s-1)}$

Indent the  $j\omega$ -axis



$$\omega = 0 \quad G_P(0) = \infty$$

Semicircle, radius  $\varepsilon$ ,  $\varepsilon \rightarrow 0$

※ On semicircle ;  $|G_P(s)| = \left| \frac{1}{s} \cdot \frac{s+1}{s-1} \right| = \infty$

at  $s \rightarrow 0$

$$G_P(s) = \frac{1}{s} \cdot \frac{1}{-1} = -\frac{1}{s}$$

$$\angle G_P(s) = \angle -\frac{1}{s} + \angle 0$$

$$\angle G_P(s) : s = -\varepsilon : G_P = \frac{1}{\varepsilon} \Rightarrow \angle = 0$$

$$s = \varepsilon j : G_P(s) = -\frac{1}{j\varepsilon} = j\frac{1}{\varepsilon} \Rightarrow \angle = 90$$

※  $s = j\omega, \quad \omega > \varepsilon;$

$$G_P(s) = \frac{1+j\omega}{j\omega(j\omega-1)} = \frac{1+j\omega}{-\omega^2-j\omega} = j \frac{(1+j\omega)(1-j\omega)}{-(\omega^2+j\omega)(1-j\omega)} = \frac{-(1+\omega^2)}{2\omega^2+j(\omega-\omega^3)}$$

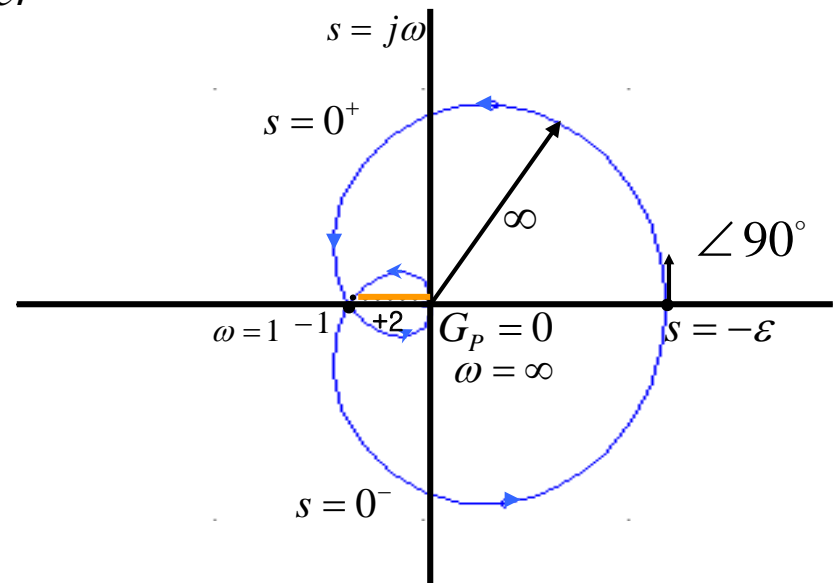
• Cross Real axis;  $\omega - \omega^3 = \omega(1 - \omega^2) = 0 \rightarrow \omega^2 = 1$

$\rightarrow \omega = 1, \quad G_P = -1$

• Cross Imag axis;  $\omega^2 = 0 \rightarrow \text{never}$

•  $\omega = \infty; \quad G_P = 0$

$N_P = 2 \rightarrow \text{stability} \quad -1 < -\frac{1}{k}$   
 $\rightarrow k > 1$



Example: ⑥  $G_P(s) = \frac{1}{s+1} e^{-sT}$

$$\frac{1}{s+1} \rightarrow \frac{1}{1+j\omega} = \frac{1-j\omega}{1+\omega^2}$$

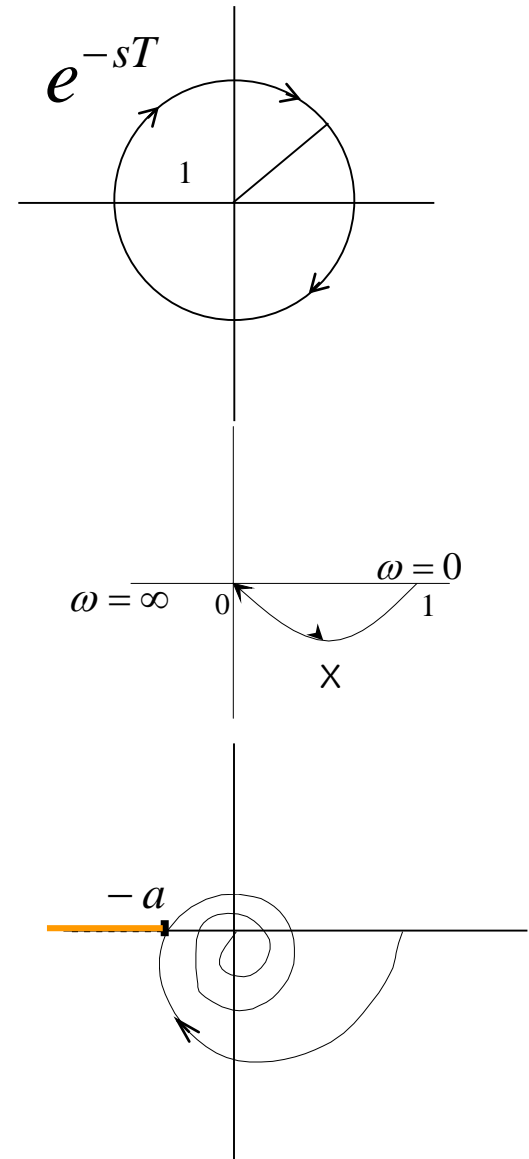
$$\omega = 0 \rightarrow G = 1$$

$$\omega = 1 \rightarrow G = (1-j)/2$$

$$\omega = \infty \rightarrow G = 0$$

$$\Rightarrow \frac{\cos(sT) - j\sin(sT)}{s+1}$$

$$N_P = 0; \quad \text{stability:} \quad -\frac{1}{k} < -a, \quad k < \frac{1}{a}$$





Find 'a':  $G_p(j\omega) = -a \leftarrow$  real number

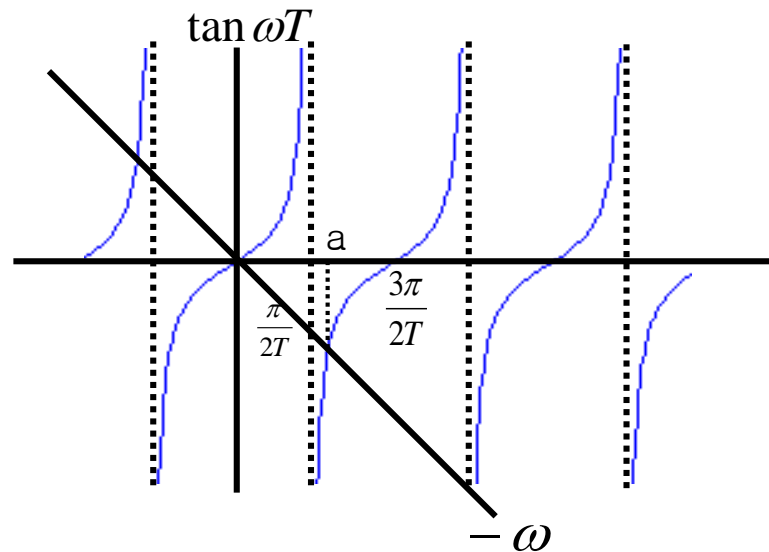
$$G_p(j\omega) = \frac{1}{1+j\omega} e^{-j\omega T} = \frac{\cos \omega T - j \sin \omega T}{1+j\omega} = -a$$

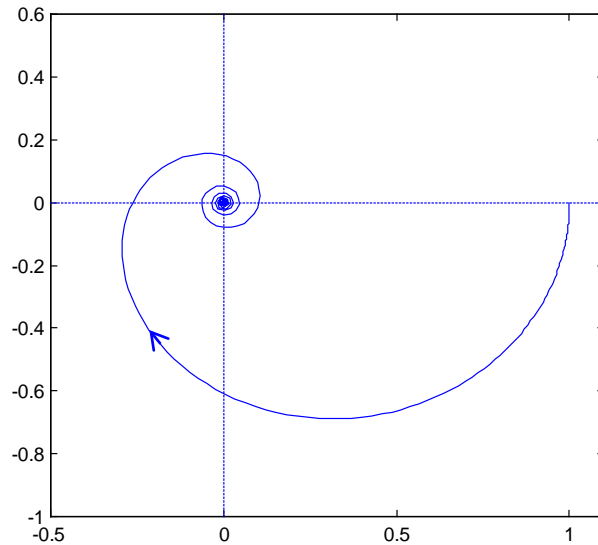
$$\cos \omega T - j \sin \omega T = -a - aj\omega$$

$$\left. \begin{array}{l} \cos \omega T = -a \\ \sin \omega T = a\omega \end{array} \right\} \rightarrow \tan \omega T = -\omega \Rightarrow \text{solve graphically}$$

$$\omega T = \frac{\pi}{2} \rightarrow \omega = \frac{\pi}{2T}$$

$$\omega T = \frac{3\pi}{2} \rightarrow \omega = \frac{3\pi}{2T}$$





```
%          1
%  Gp(s)=  -----exp(-sT)
%          s+1
function ex6
T=0.5;
w=logspace(-3, 2.2, 300);
y=gp(w,T);

plot(y)
a=-0.5; b=1.1; c=-1; d=0.6;
axis([a b c d])

hold on
plot([a b], [0 0], 'r-')
plot([0 0], [c d], 'r-')
hold off

function y=gp(w,T);
j=sqrt(-1);
temp=exp(-j.*w*T);
y=temp./((j*w+1);
```

```
%  tan(wT)=-w
```

```
T=0.1;
N=100; delta=0.2;
L=pi/2/T;
w1=linspace(-L+delta, L-delta, N);
y1=tan(w1*T);

w2=linspace(-L+delta+pi/T, L-delta+pi/T, N);
y2=tan(w2*T);

w3=linspace(w1(1), w2(N/2), N);
y3=-w3;

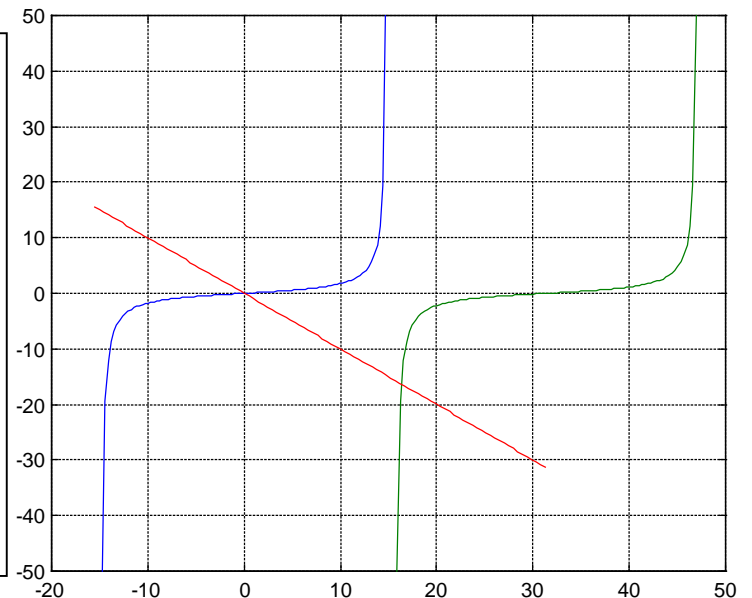
plot(w,y1, w2,y2, w3,y3, 'r-')
grid

break

w0=linspace(-L+delta+pi/T, 17, 10*N);
error=w0+tan(w0*T);
```

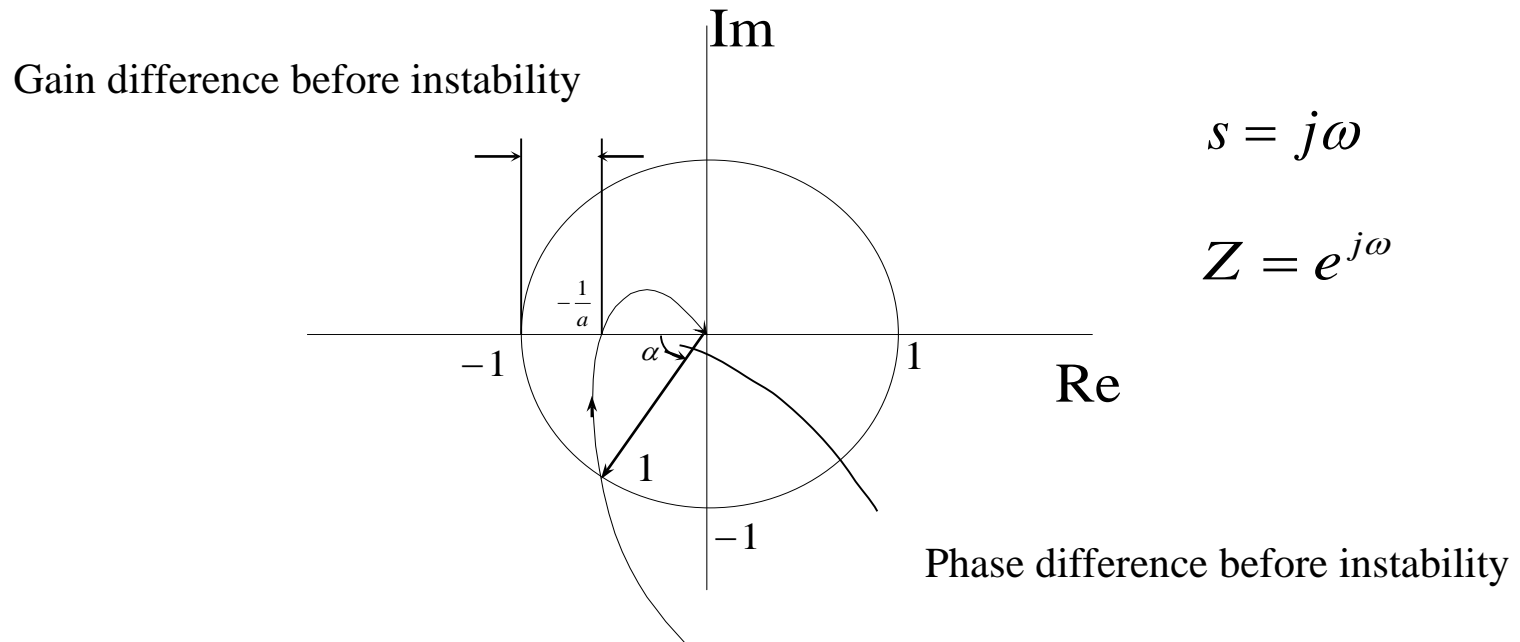
```
» [w0; error]'
```

16.3157	-0.1185
16.3168	-0.0878
16.3179	-0.0573
16.3190	-0.0268
16.3201	0.0035
16.3212	0.0338
16.3223	0.0639
16.3234	0.0940
16.3244	0.1239
16.3255	0.1538
16.3266	0.1835
16.3277	0.2132
16.3288	0.2427
16.3299	0.2721
16.3310	0.3015
16.3321	0.3307
16.3332	0.3599



# Gain margin and Phase Margin

---



$$\text{Gain margin} = G_M = 20\log a = -20\log\left(\frac{1}{a}\right)$$

$$\text{Phase margin} = \Phi_M = \alpha$$

Example:  $G(s) = \frac{k}{(s^2 + 2s + 2)(s + 2)}$

Find the gain and phase margin if  $k = 6$

$$G(j\omega) = \frac{6}{(4 - 4\omega^2) + j\omega(6 - \omega^2)}$$

– Cross Real axis:

$$\omega^2 = 6 \quad -\frac{6}{20} = -\frac{3}{10} = -0.3 = -\frac{1}{a}$$

$$G_m = 20\log a = 20\log\left(\frac{1}{0.3}\right) = 10.45 \text{ dB}$$

– Cross Imaginary axis :  $\omega^2 = 1$   $\omega = 1$

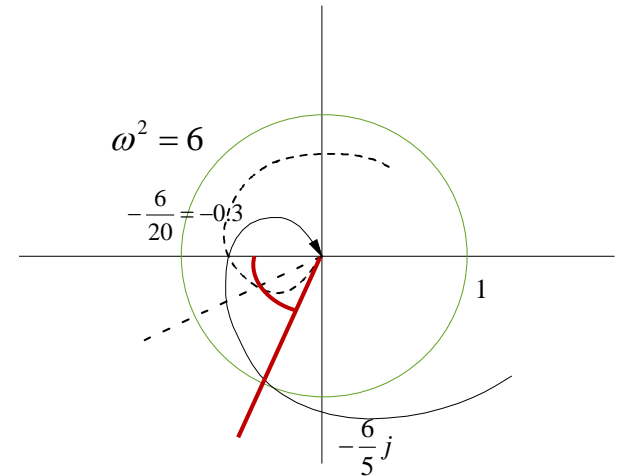
$$G = \frac{6}{j(6-1)} = -\frac{6}{5}j$$

– For the Phase Margin:  $|G(j\omega)| = 1$

$$\omega = 1.253 \text{ rad/sec}$$

$$\varphi = -112.33^\circ$$

$$\Phi_M = 180^\circ - 112.33^\circ = 67.67^\circ$$



# Bode Diagrams

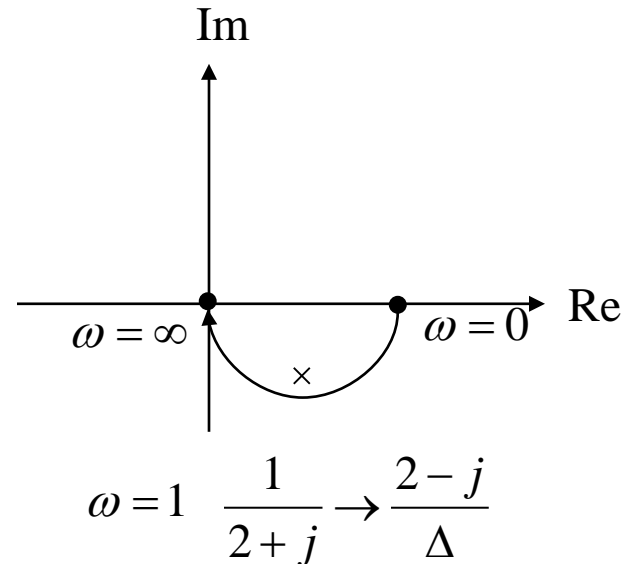
## Frequency Domain Analysis

Principle: Draw  $G_p(j\omega)$  in polar coordinates.

- Frequency response

$$G_p(s) = \frac{1}{s+2}$$

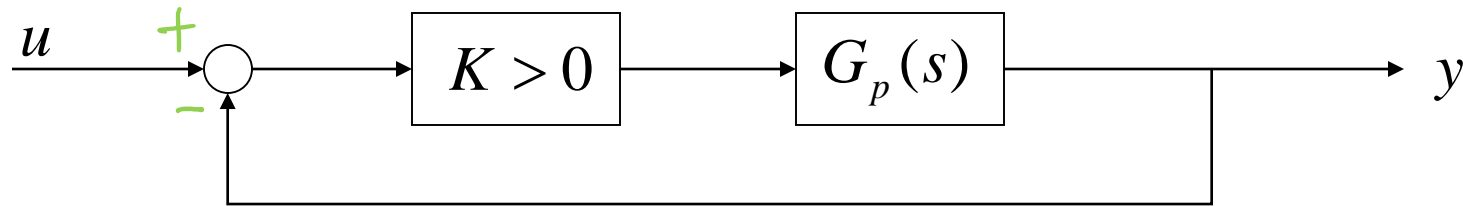
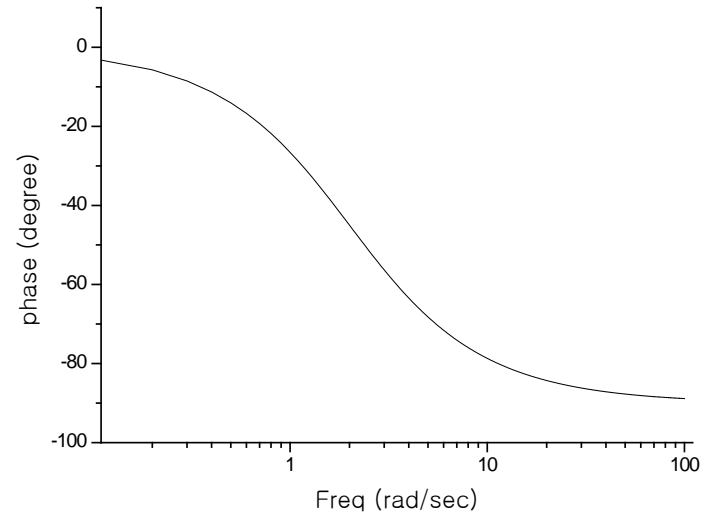
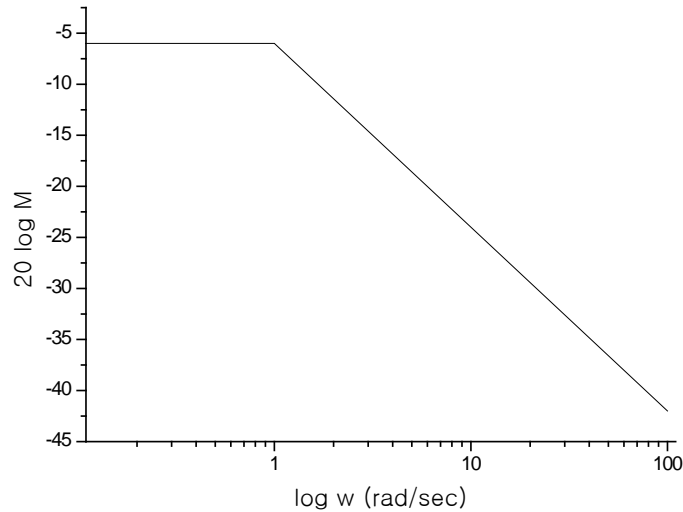
$$s = j\omega \rightarrow G = \frac{1}{j\omega + 2}$$



⇒ magnitude and phase plot

$$|G(j\omega)| = M(j\omega) = \frac{1}{\sqrt{4 + \omega^2}} \rightarrow \log(\omega) \text{ vs } 20\log M(\omega)$$

$$\angle G(j\omega) \rightarrow \phi(\omega) = -\arctan\left(\frac{\omega}{2}\right) \rightarrow \log(\omega) \text{ vs } -\arctan\left(\frac{\omega}{2}\right)$$



Can get idea about stability range?

$$G_p(s) = A \cdot \frac{\left(1 + \frac{s}{z_1}\right) \cdots \left(1 + \frac{s}{z_m}\right)}{\left(1 + \frac{s}{\omega_1}\right) \cdots \left(1 + \frac{s}{\omega_n}\right)}$$

Assume:  $z_j : j = 1, \dots, m$

$\omega : i = 1, \dots, n$  are real

$A > 0$

$$G_p(s) = A \frac{\prod_{j=1}^m \left(1 + \frac{s}{z_j}\right)}{\prod_{i=1}^n \left(1 + \frac{s}{\omega_i}\right)} \rightarrow s = j\omega :$$

$$G_p(j\omega) = A \frac{\prod_{j=1}^m \left(1 + j \frac{\omega}{z_j}\right)}{\prod_{i=1}^n \left(1 + j \frac{\omega}{\omega_i}\right)}$$

⇒ Draw:  $|G_p(j\omega)|$  in logarithmic scale, and  $\angle G_p(j\omega)$

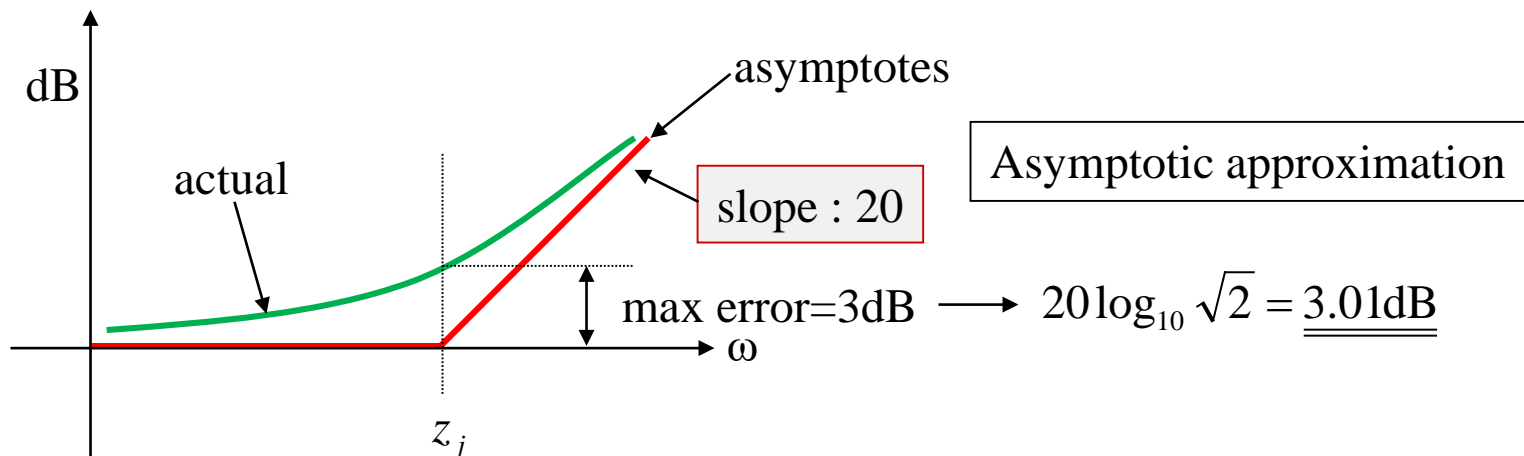


## (A) Magnitude

$$20\log_{10}|G_p(j\omega)| = 20\log \left| A \frac{\prod \left( 1 + j \frac{\omega}{z_j} \right)}{\prod \left( 1 + j \frac{\omega}{\omega_i} \right)} \right|$$

$$= 20\log A + \sum_{j=1}^m 20\log \left| 1 + j \frac{\omega}{z_j} \right| - \sum_{i=1}^n 20\log \left| 1 + j \frac{\omega}{\omega_i} \right|$$

- Consider the plot of  $20\log \left| 1 + j \frac{\omega}{z_j} \right|$



$$20\log\left|1 + j\frac{\omega}{z_j}\right| = 20\log\sqrt{1 + \frac{\omega^2}{z_j^2}} = 10\log\left(1 + \frac{\omega^2}{z_j^2}\right)$$

If  $\omega \gg |z_j|$ ,

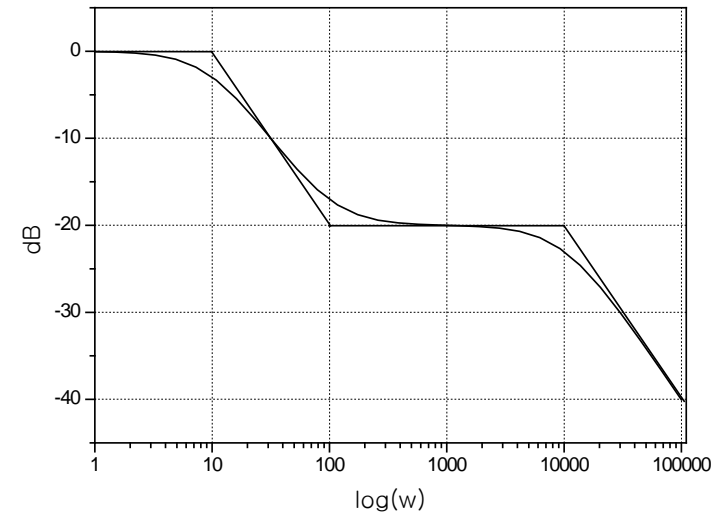
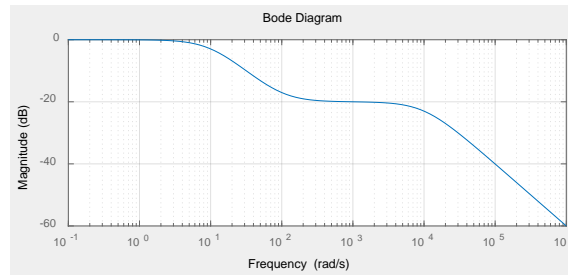
$$20\log\left|1 + j\frac{\omega}{z_j}\right| \cong 10\log\left(\frac{\omega^2}{z_j^2}\right) = 20\log\omega - 20\log|z_j|$$

If  $\omega \ll |z_j|$ ,

$$20\log\left|1 + j\frac{\omega}{z_j}\right| \cong 20\log 1 = 0$$

## Example 1

$$G_p(s) = \frac{1 + s/100}{(1 + s/10)(1 + s/1000)}$$

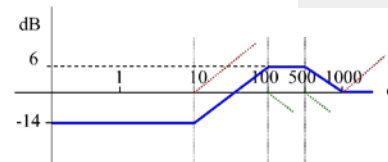
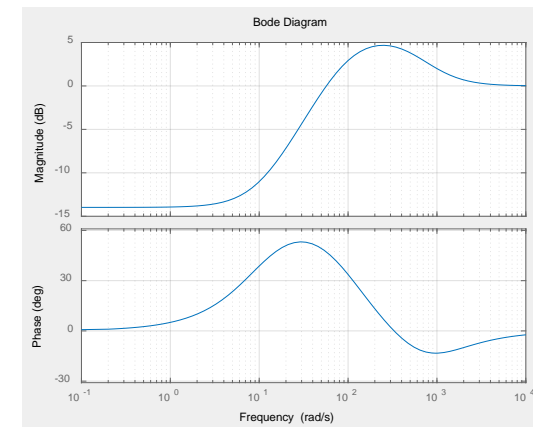
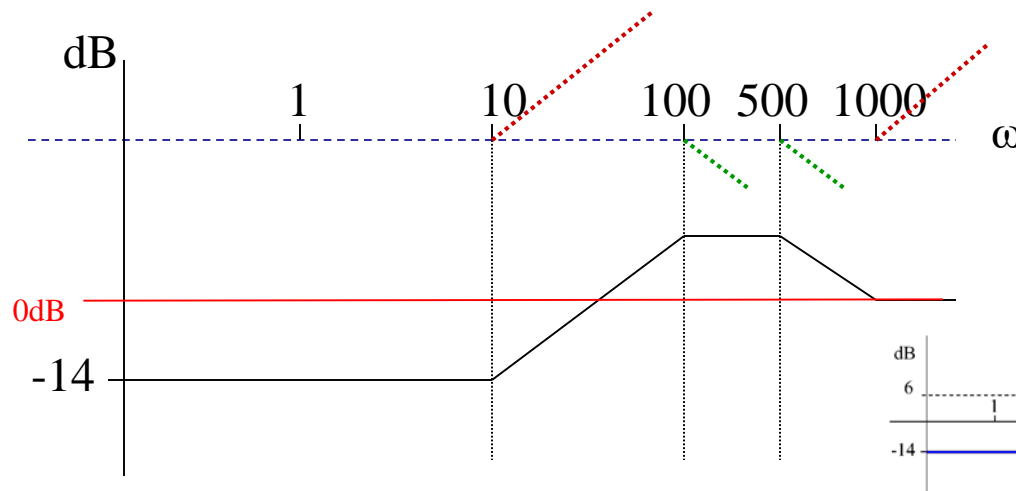


## Example 2

$$G_p(s) = \frac{(s + 1000)(s + 10)}{(s + 100)(s + 500)} = \frac{1000 \cdot 10}{100 \cdot 500} \cdot \frac{(1 + s/1000)(1 + s/10)}{(1 + s/100)(1 + s/500)}$$

```
numg=poly([-10 -1000]);  
deng=poly([-100 -500]);  
G=tf(numg,deng)  
bode(G), grid on
```

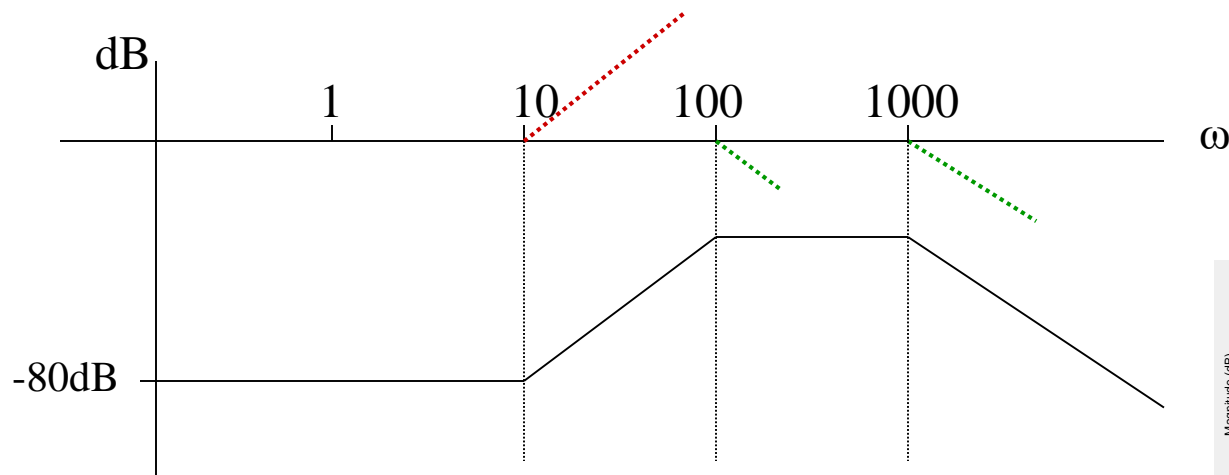
$$\rightarrow 20 \log(0.2) = -14$$



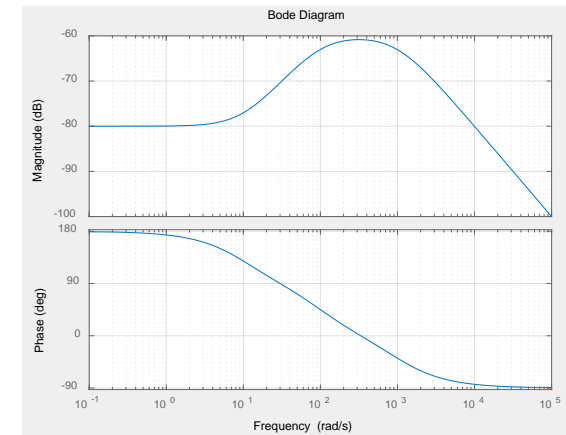
## Example 3

$$G_p(s) = \frac{(s-10)}{(s+100)(s+1000)} = -\frac{1}{10^4} \cdot \frac{1-s/10}{(1+s/100)(1+s/1000)}$$

$$\rightarrow 20\log\left(\frac{1}{10^4}\right) = -80$$



```
numg=poly([10]);  
deng=poly([-100 -1000]);  
G=tf(numg,deng)  
bode(G), grid on
```

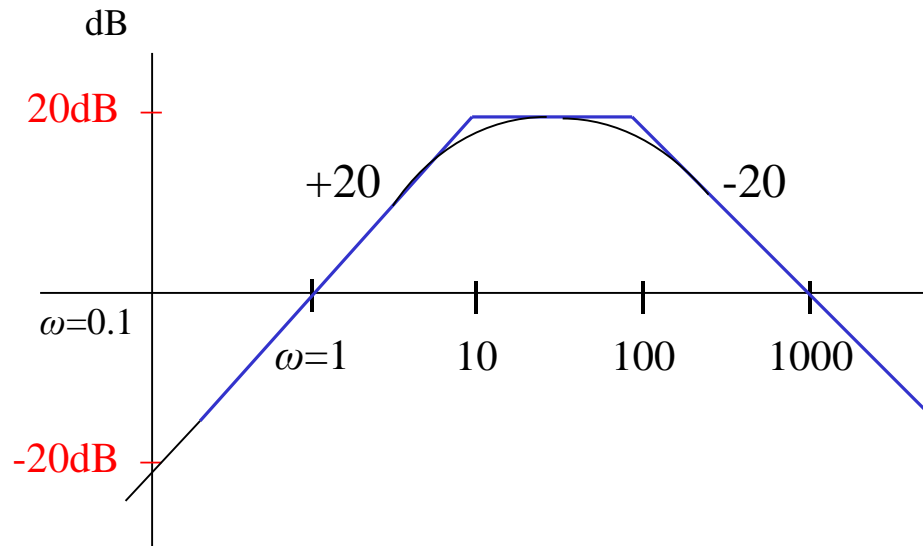


## Example 4

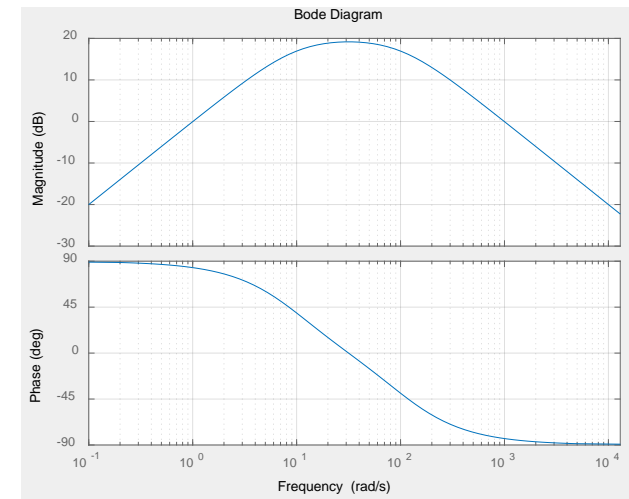
$$G_p(s) = \frac{s}{(1 + \frac{s}{10})(1 + \frac{s}{100})}$$

→ Zero at 0 !

$20\log|j\omega| = 20\log\omega \rightarrow$  add slope of 20 !



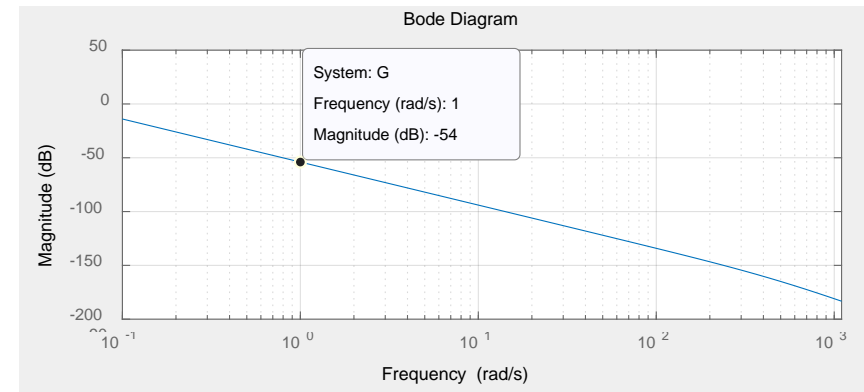
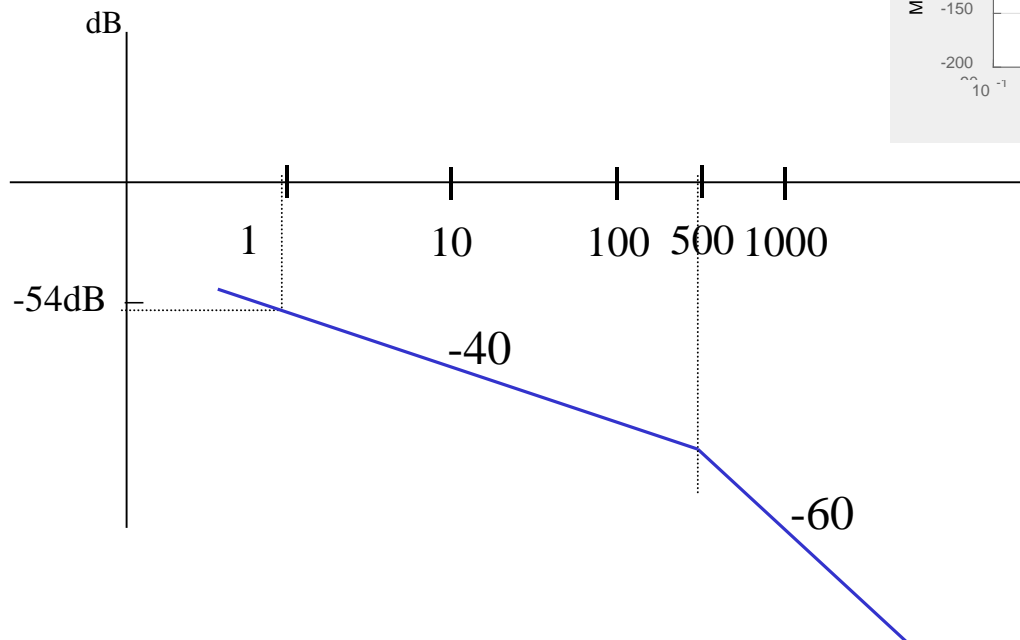
```
numg=poly([0]);  
deng=poly([-10 -100]);  
G=tf(numg,deng)*1000  
bode(G), grid on  
axis([0.1 13000 -90 90])
```



## Example 5

$$G_p(s) = \frac{1}{s^2(s+500)} = \frac{1}{500} \frac{1}{s^2 \left(1 + \frac{s}{500}\right)}$$

$$\rightarrow 20\log\left(\frac{1}{500}\right) = -54$$



```
numg=poly([]);
deng=poly([0 0 -500]);
G=tf(numg,deng)
bode(G), grid on
axis([0.1 1100 -90 90])
```

$$G = \frac{1}{s^3 + 500 s^2}$$

## (B) Phase Response

$$G_p(s) = A \frac{\prod(1 + s/z_j)}{\prod(1 + s/\omega_i)} \rightarrow \angle G_p(s) = \angle A + \sum_{j=1}^m \angle(1 + \frac{s}{z_j}) - \sum_{i=1}^n \angle(1 + \frac{s}{\omega_i})$$

- Consider:  $\angle(1 + s/z_j)$

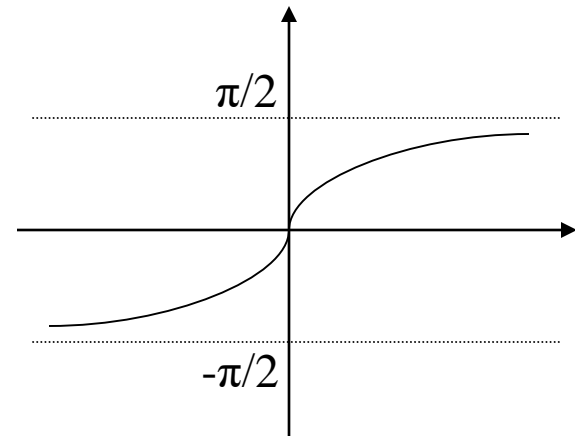
$$s = j\omega : \angle\left(1 + j\frac{\omega}{z_j}\right) = \arctan\left(\frac{\omega}{z_j}\right)$$

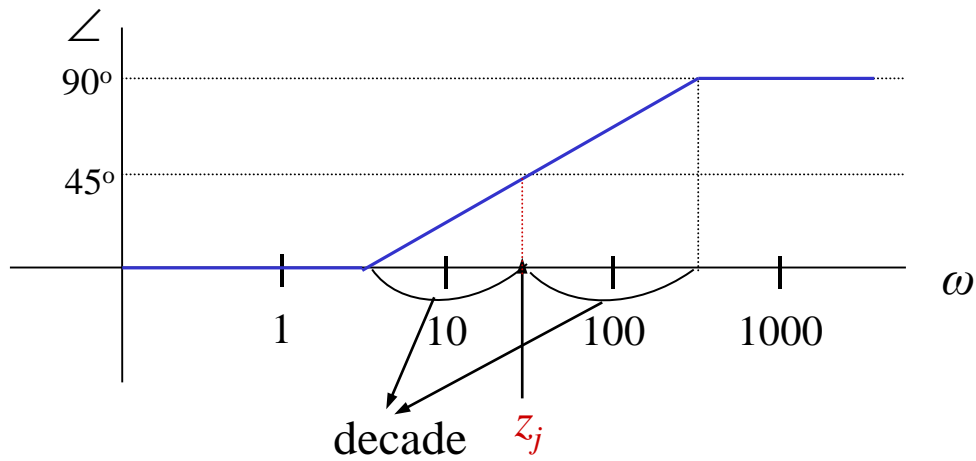
i)  $z_j > 0$

$$\omega \gg z_j : \arctan\left(\frac{\omega}{z_j}\right) \approx 90^\circ$$

$$\omega \ll z_j : \arctan\left(\frac{\omega}{z_j}\right) \approx 0^\circ$$

$$\arctan(1) = 45^\circ$$



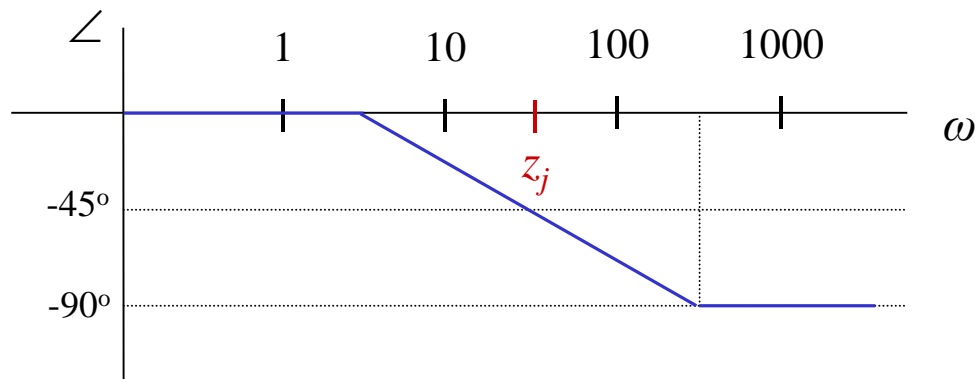


$$\text{atan}(1)=45^\circ$$

ii)  $z_j < 0$

$$\omega \gg |z_j| : \arctan\left(\frac{\omega}{z_j}\right) \approx -90^\circ$$

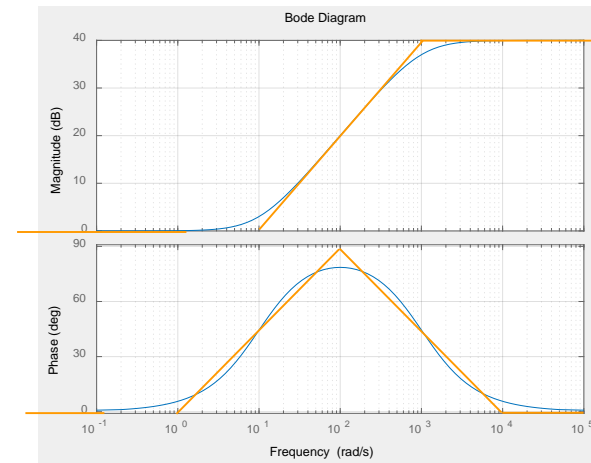
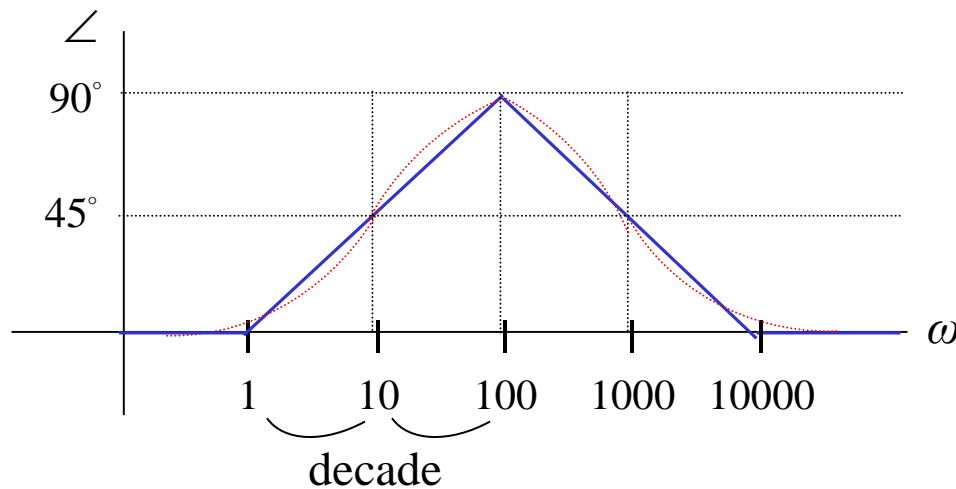
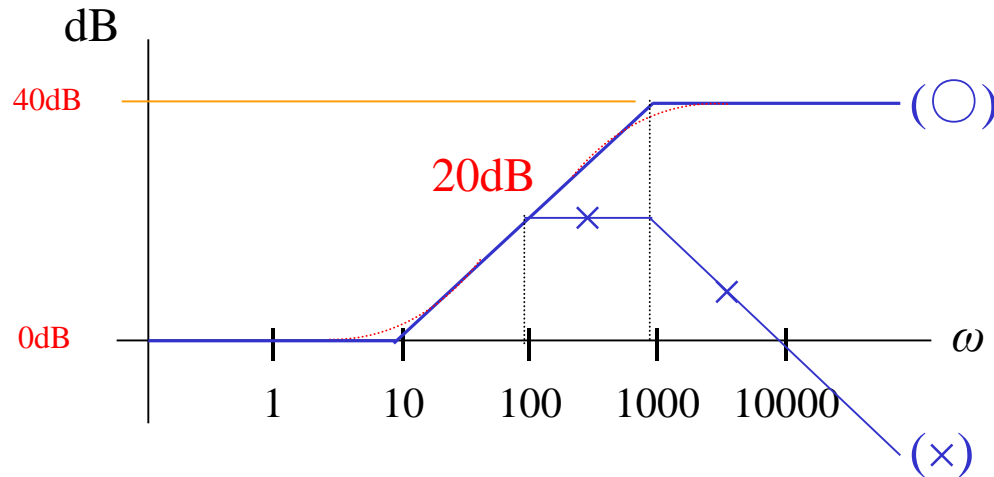
$$\omega \ll |z_j| : \arctan\left(\frac{\omega}{z_j}\right) \approx 0^\circ$$



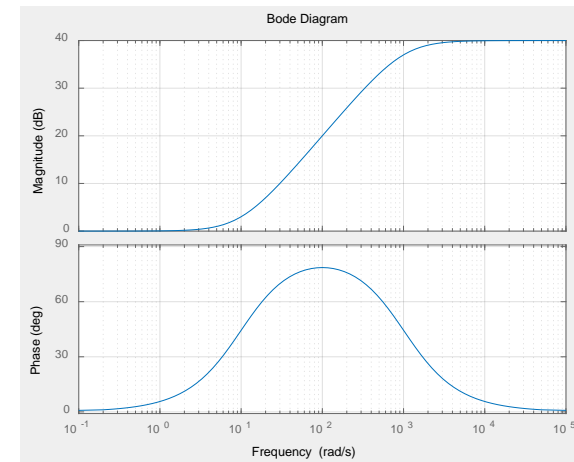


## Example 6

$$G_p(s) = \frac{1 + s/10}{1 + s/1000}$$

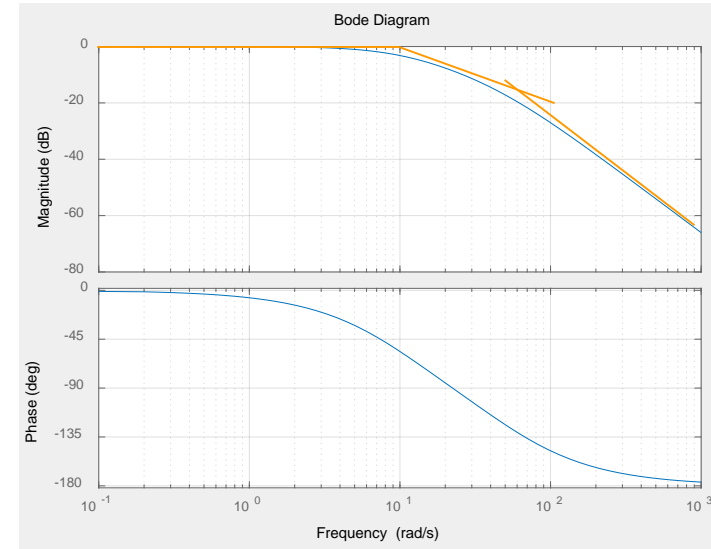
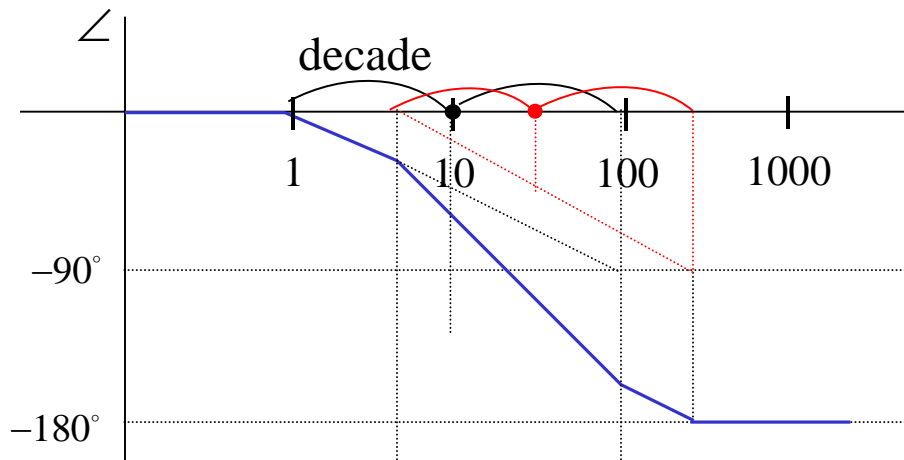
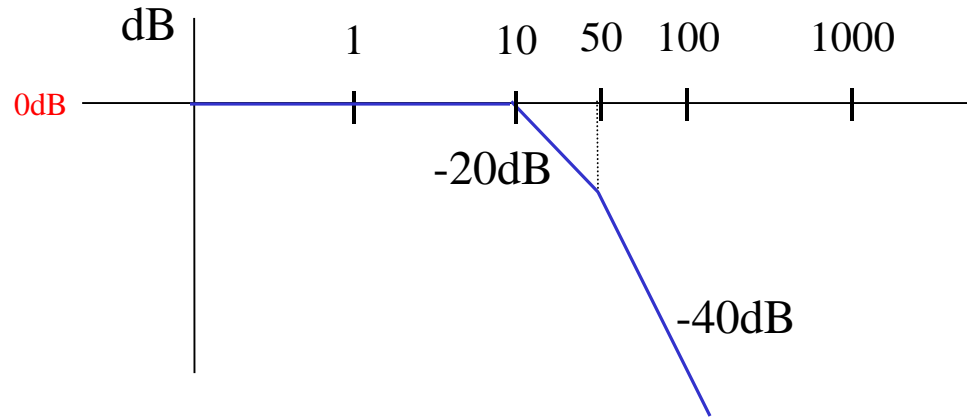


```
numg=poly([-10]);
deng=poly([-1000]);
G=tf(numg,deng)*100
bode(G), grid on
```



## Example 7

$$G_p(s) = \frac{1}{(1 + s/10)(1 + s/50)}$$

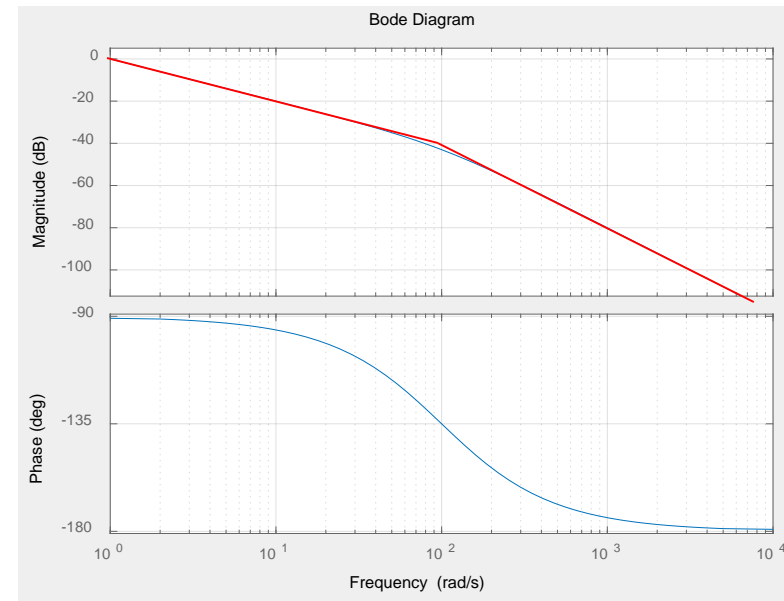
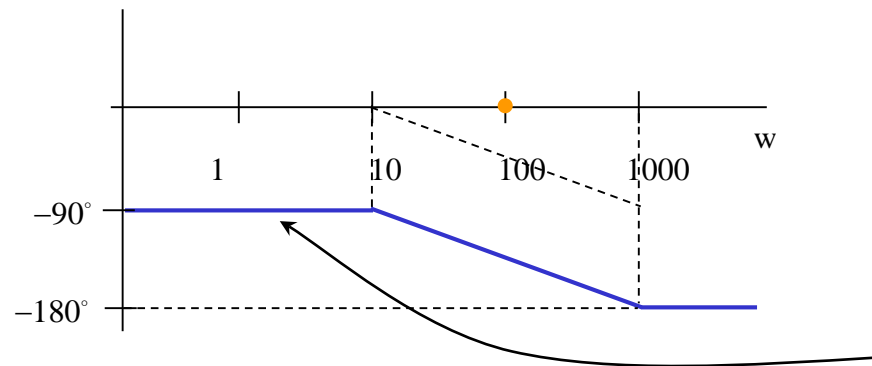
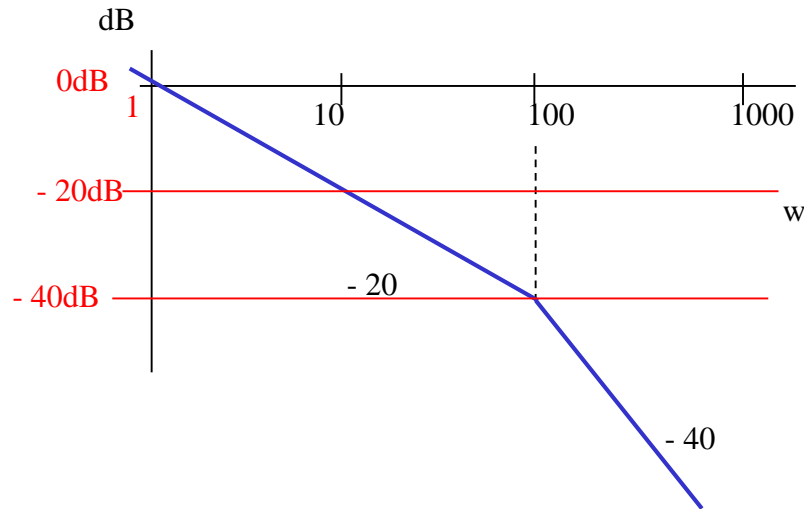


```
numg=poly([]);
deng=poly([-10 -50]);
G=tf(numg,deng)*500
bode(G), grid on
```

$$G = \frac{500}{s^2 + 60s + 500}$$

## Example 8

$$G_p(s) = \frac{1}{s(1 + \frac{s}{100})}$$



```
numg=poly([]);
deng=poly([0 -100]);
G=tf(numg,deng)*100
bode(G), grid on
```

$$G = \frac{100}{s^2 + 100s}$$

$$\angle(jw) = 90^\circ$$

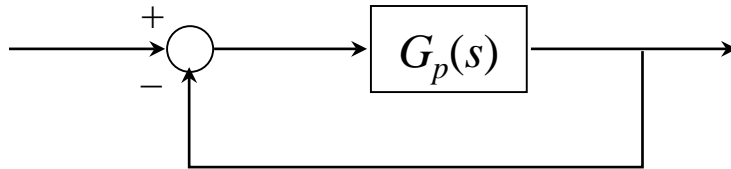


Add  $\pm 90^\circ$  to the phase

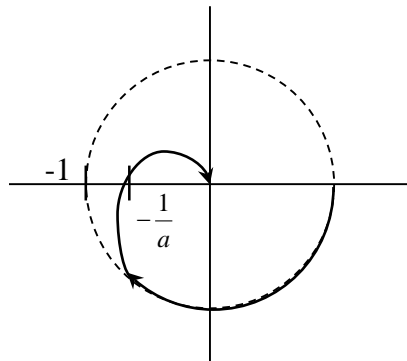
\* Pole at 0

→ Add  $-90^\circ$  to the phase

# Stability test, phase and gain margins (1)

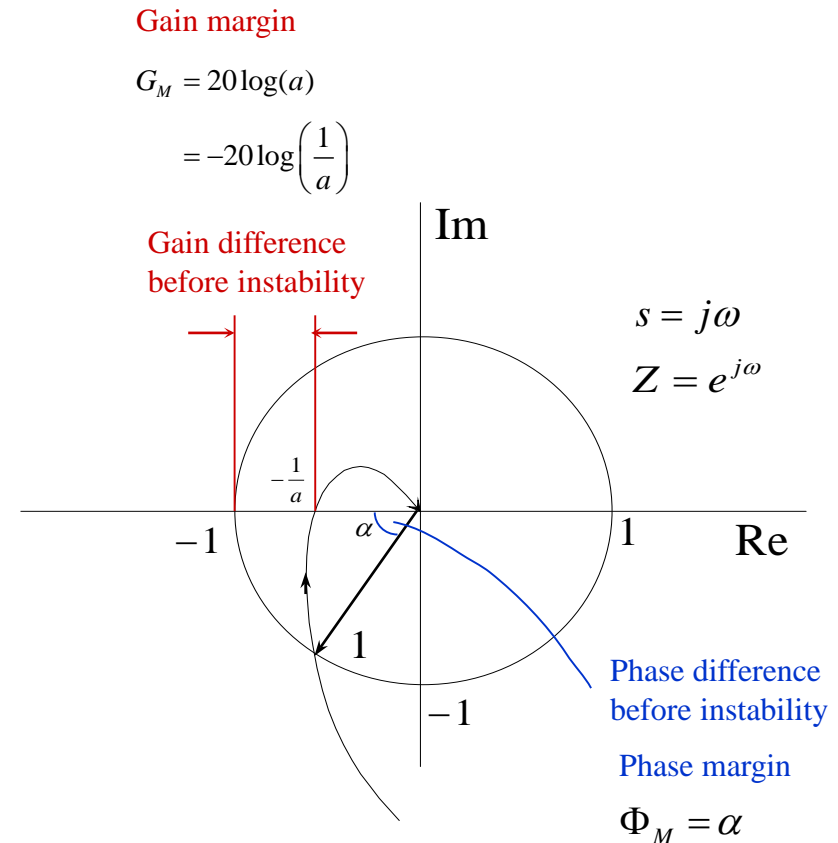


Suppose that  $G_p(s)$  has no RHP poles.  
The Nyquist plot of  $G_p(s)$  is



with  $k=1$ , the system is stable.

$$-\frac{1}{k} < -\frac{1}{a} \rightarrow k < a$$

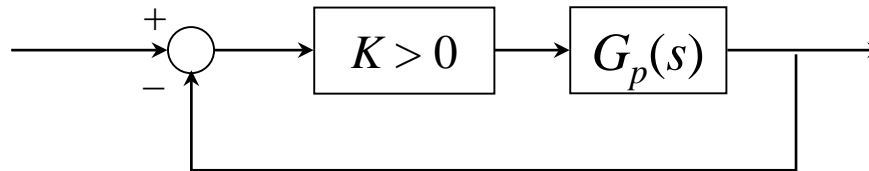


## Stability test, phase and gain margins (2)

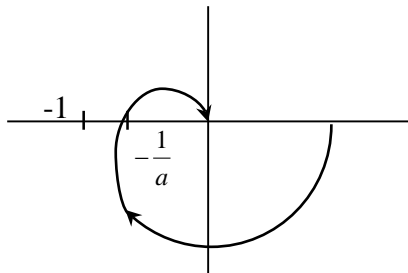
\* Two questions:

- ① By how much can you increase the gain of  $G_p$  before instability occurs?
- ② How much **delay (negative phase)** can you add to  $G_p$  before instability occurs?

① →

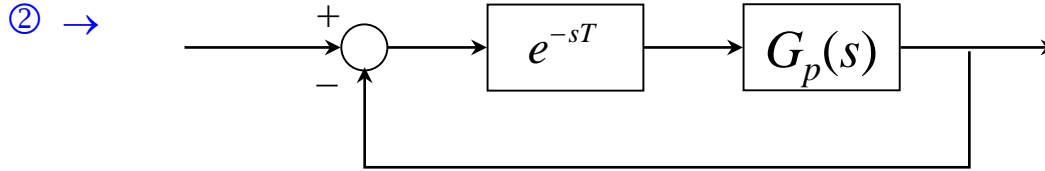


What is the maximum  $K$  with which you still have stability ? → **Gain Margin**



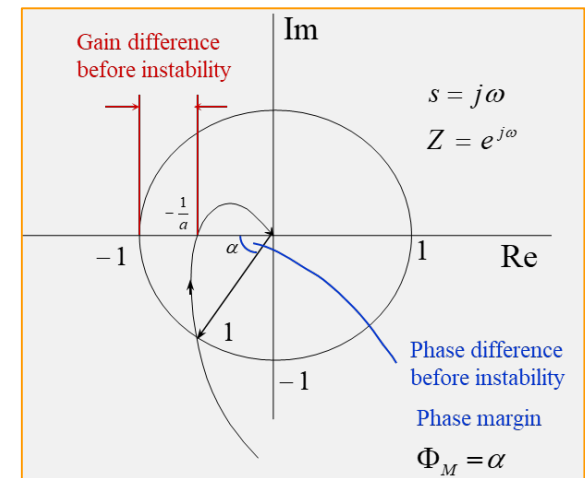
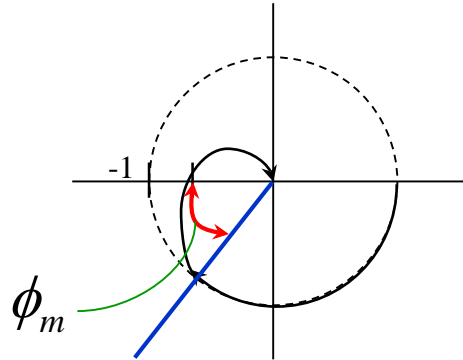
$$\text{Gain Margin} = G_M = 20\log(a) = -20\log\left(\frac{1}{a}\right) \text{ dB}$$

## Stability test, phase and gain margins (3)

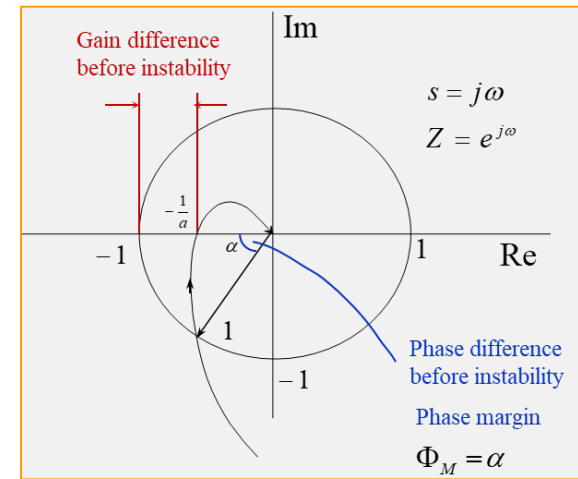
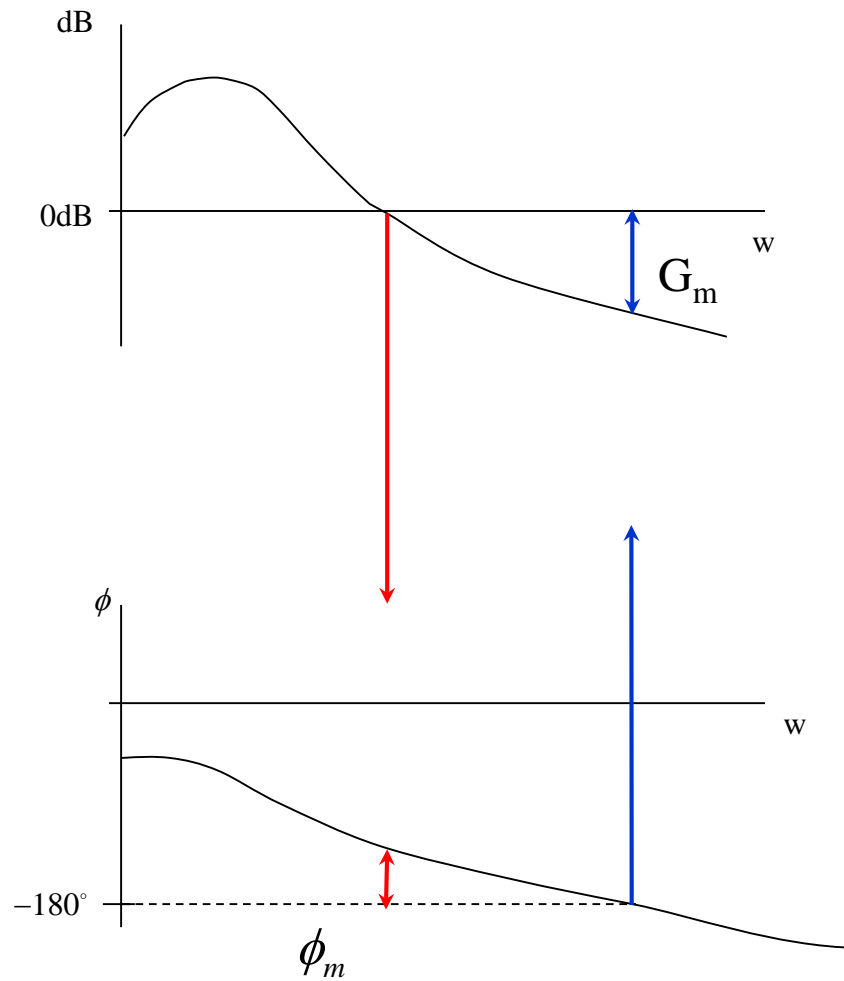


What is the maximum  $T$  with which you still have stability?

→ **Phase Margin**



⇒ From Bode Plots.



```

% Bode Diagrams
k=1.5; ng=1; dg=poly([0 -1 -2]); w=logspace(-1,1,100)';
[m,p]=bode(k*ng, dg,w);
figure(1)
%subplot(211); semilogx(w, 20*log10(m)); grid
%subplot(212); semilogx(w, p); grid
bode(k*ng, dg,w)
[gm,pm,wgc,wpc]=margin(m,p,w);
[gm,pm,wgc,wpc]    % 4.0002 41.5332  1.4142  0.6118
margin(m,p,w)

```

```

% For Nyquist
w2=linspace(0, 2*pi, 100)';
ejw=exp(j*w2); r2=real(ejw); i2=imag(ejw);
[r,i]=nyquist(k*ng, dg, w);
figure(2)
plot(r2,i2, r,i, 'r-');
axis('square'); grid
axis([-1 1 -1 1])

```

