Midterm Exam: Control Systems Eng.(I)
2019/04/16

Student Number: [ ] Name: **Solution**

1. (20 points)

(1) (15 pts)

$$C(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{K_1}{s}+\frac{K_2 s+K_3}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\begin{cases} K_1=\left.\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right|_{s=0}=1 \\ G(s)=\dfrac{(s^2+2\zeta\omega_n s+\omega_n^2)+s(K_2 s+K_3)}{s^2+2\zeta\omega_n s+\omega_n^2}=\dfrac{(K_2+1)s^2+(2\zeta\omega_n+K_3)s+\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \\ K_2=-1,\ \ K_3=-2\zeta\omega_n \end{cases}$$

$$G(s)=\frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{1}{s}-\frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2-\zeta^2\omega_n^2}=\frac{1}{s}-\frac{(s+\zeta\omega_n)+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$=\frac{1}{s}-\frac{(s+\zeta\omega_n)+\dfrac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}=\frac{1}{s}-\frac{(s+\zeta\omega_n)+\dfrac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}$$

Use Laplace transform:

$$c(t)=1-e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}t\right)=1-e^{-\zeta\omega_n t}\sqrt{1+\frac{\zeta^2}{1-\zeta^2}}\cos\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right)$$

$$\therefore\ \ \underline{c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right)},\quad \text{where }\ \underline{\tan\phi=\frac{\zeta}{\sqrt{1-\zeta^2}}}\quad\left(\cos\phi=\sqrt{1-\zeta^2},\ \ \sin\phi=\zeta\right)$$

(2) (5 pts)

At the settling time, assume that $\cos\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right)=1$.

$$c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}=0.98,\qquad \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}=0.02,\ \ e^{-\zeta\omega_n t}=0.02\sqrt{1-\zeta^2},\qquad -\zeta\omega_n t=\ln\left(0.02\sqrt{1-\zeta^2}\right)$$

$$\to T_s=\frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

2. (20 points)

(1) (15 pts)

$$L\left(\dot{c}(t)\right)=\frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}=\frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}$$

After using $\sin\omega t \leftrightarrow \dfrac{\omega}{s^2+\omega^2}$ and $L\left(e^{-at}f(t)\right) \leftrightarrow F(s+a)$,

$$\dot{c}(t)=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t$$

(2) (5 pts)

$$\dot{c}(t)=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t=0,\quad \omega_n\sqrt{1-\zeta^2}\,t=n\pi$$

$$T_p=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}},\quad (n=1)$$

3. (20 points)

$$Y(s)=G(s)U(s),\ \ U(s)=-KY(s)+V(s)\ \rightarrow\ Y(s)=G(s)\left(-KY(s)+V(s)\right)=-KG(s)Y(s)+G(s)V(s)$$

$$Y(s)=\frac{G(s)}{1+KG(s)}V(s)\quad \left(\text{or we can use } \ddot{y}+\alpha\dot{y}+5y=-ky+v\right)$$

$$\rightarrow\quad Y(s)=\frac{1}{s^2+\alpha s+(5+K)}V(s),\quad \alpha=2 \text{ or } 4$$

From the step responses, we see that the limit is well-defined and, as such, we can apply the final value theorem:

$$\lim_{t\to\infty}y(t)=\lim_{s\to 0}sY(s)=\lim_{s\to 0}s\frac{1}{s^2+\alpha s+(5+K)}\frac{1}{s}=\frac{1}{5+K}.$$

But, from the plots we see that this limit is in fact equal to 0.1. $\Rightarrow K=5$.
Now, the poles and modes associated with the two systems are:

For system 1: $Y(s)=\dfrac{1}{s^2+2s+10}V(s),\ \begin{cases} Poles: & s^2+2s+10\rightarrow s=-1\pm\sqrt{1-10}=-1\pm 3j \\ Mode: & e^{-t}\sin(3t)\end{cases}$

For system 2: $Y(s)=\dfrac{1}{s^2+4s+10}V(s),\ \begin{cases} Poles: & s^2+4s+10\rightarrow s=-2\pm\sqrt{4-10}=-2\pm\sqrt{6}j \\ Mode: & e^{-2t}\sin(\sqrt{6}\ t)\end{cases}$

As a consequence, the oscillations in systems 1 decay slower than in systems 2 (e^{-t} vs. e^{-2t}), while the oscillations in systems 1 have a higher frequency than those in systems 2 (3 vs. $\sqrt{6}$). Based on this, combined with an inspection of the two step responses, we see that step response 1 belongs to system 1, and step response 2 belongs to system 2.

$\Rightarrow K=5$
$\Rightarrow$ Step response 1 belongs to system 1
$\Rightarrow$ Step response 2 belongs to system 2

4. (20 points)

$(sI-A)X(s) = \mathrm{x}(0) + BU(s) \ \leftarrow \ sX(s) - \mathrm{x}(0) = AX(s) + BU(s)$

$Y(s) = CX(s)$

$$(sI-A) = s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ -3 & -5 \end{pmatrix} = \begin{pmatrix} s & -2 \\ 3 & s+5 \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{1}{s(s+5)+6}\begin{pmatrix} s & -2 \\ 3 & s+5 \end{pmatrix}^{-1} = \frac{1}{s^2+5s+6}\begin{pmatrix} s+5 & 2 \\ -3 & s \end{pmatrix}$$

$$BU(s) = \begin{pmatrix} 0 \\ \dfrac{1}{(s+1)} \end{pmatrix}$$

i) The state vector: $X(s) = (sI-A)^{-1}\left[\mathrm{x}(0) + BU(s)\right]$

$$= \frac{1}{s^2+5s+6}\begin{pmatrix} s+5 & 2 \\ -3 & s \end{pmatrix}\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{(s+1)} \end{pmatrix}\right] = \frac{1}{s^2+5s+6}\begin{pmatrix} s+5 & 2 \\ -3 & s \end{pmatrix}\begin{pmatrix} 2 \\ \dfrac{(s+2)}{(s+1)} \end{pmatrix}$$

$$2(s+5) + 2\frac{(s+2)}{(s+1)} = \frac{2(s^2+6s+5)+2s+4}{(s+1)} = \frac{2(s^2+7s+7)}{(s+1)}$$

$$-6 + s\frac{(s+2)}{(s+1)} = \frac{-6s-6+s^2+2s}{(s+1)} = \frac{s^2-4s-6}{(s+1)}$$

$$\therefore \ X(s) = \frac{1}{(s+1)(s+2)(s+3)}\begin{pmatrix} 2(s^2+7s+7) \\ s^2-4s-6 \end{pmatrix}$$

$$ii)\ Y(s) = \begin{pmatrix} 1 & 3 \end{pmatrix} X(s) = \frac{1}{\Delta}\begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} 2(s^2+7s+7) \\ s^2-4s-6 \end{pmatrix} = \frac{5s^2+2s-4}{(s+1)(s+2)(s+3)} = \frac{a}{(s+1)} + \frac{b}{(s+2)} + \frac{c}{(s+3)}$$

$$a = \left.\frac{5s^2+2s-4}{(s+2)(s+3)}\right|_{s=-1} = \frac{5-2-4}{(1)(2)} = \frac{-1}{2} = -0.5$$

$$b = \left.\frac{5s^2+2s-4}{(s+1)(s+3)}\right|_{s=-2} = \frac{20-4-4}{(-1)(1)} = \frac{12}{-1} = -12 \qquad \rightarrow \quad Y(s) = \frac{-0.5}{(s+1)} + \frac{-12}{(s+2)} + \frac{17.5}{(s+3)}$$

$$c = \left.\frac{5s^2+2s-4}{(s+1)(s+2)}\right|_{s=-3} = \frac{45-6-4}{(-2)(-1)} = \frac{35}{2} = 17.5$$

Inverse Laplace transform of $Y(s)$: $y(t) = -0.5e^{-t} - 12e^{-2t} + 17.5e^{-3t}$

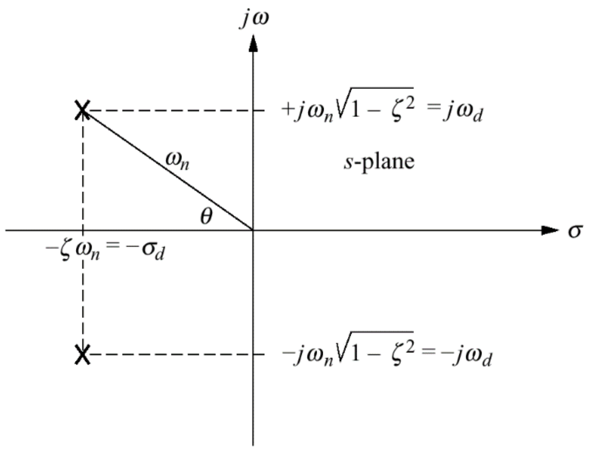
5. (10 points)

(1) $\zeta = \cos\theta = \dfrac{3}{\sqrt{3^2+4^2}} = 0.6$

(2) $\omega_n = \sqrt{3^2+4^2} = 5$

(3) $T_p = \dfrac{\pi}{\omega_d} = \dfrac{3}{4} = 0.75\,\text{sec}$

(4) $T_s = \dfrac{4}{\sigma_d} = \dfrac{4}{3} = 1.33\,\text{sec}$



$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

6. (10 points)

$$\Phi(t) = L^{-1}\left[(sI-A)^{-1}\right] = e^{At}$$

$$(sI-A) = s\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}s & -1\\ 0 & s\end{pmatrix}$$

$$(sI-A)^{-1} = \begin{pmatrix}s & -1\\ 0 & s\end{pmatrix}^{-1} = \frac{\begin{pmatrix}s & 1\\ 0 & s\end{pmatrix}}{s^2} = \begin{pmatrix}\dfrac{1}{s} & \dfrac{1}{s^2}\\ 0 & \dfrac{1}{s}\end{pmatrix}$$

Inverse Laplace transforming gives the state-transition matrix

$$\Phi(t) = e^{At} = L^{-1}\left[(sI-A)^{-1}\right] = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$$

Ref: $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$