Chapter 1. Continuous-Time Signals

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- Representation using Basic Signals

Introduction

- We focus on the **representation** and **processing** of **continuous-time signals**.
  - Almost all signals vary **randomly** and **continuously** with time.
  - To input signals into a computer, they must be in **binary form**.
  - The processing of signals requires us to consider **systems**.

- List of issues to be addressed
  - **Mathematical representation** of (time-dependent) signals
  - **Classification of signals** indicating the way it is stored, processed or both.
  - **Signal manipulation**
    - Delay/advance, reflect, and find odd/even components
  - **Basic signal representation**
    - Fourier representation

Classification of Time-Dependent Signals

- **Predictability**
  - **Random or deterministic**
- **Variation of time and amplitude**
  - **Continuous-time, discrete-time, digital**
- **Energy**
  - **Finite or infinite energy**
- **Repetitive behavior**
  - **Periodic or aperiodic**
- **Symmetry w.r.t. the time origin**
  - **Even or odd**
- **Support**
  - **Finite or infinite support**

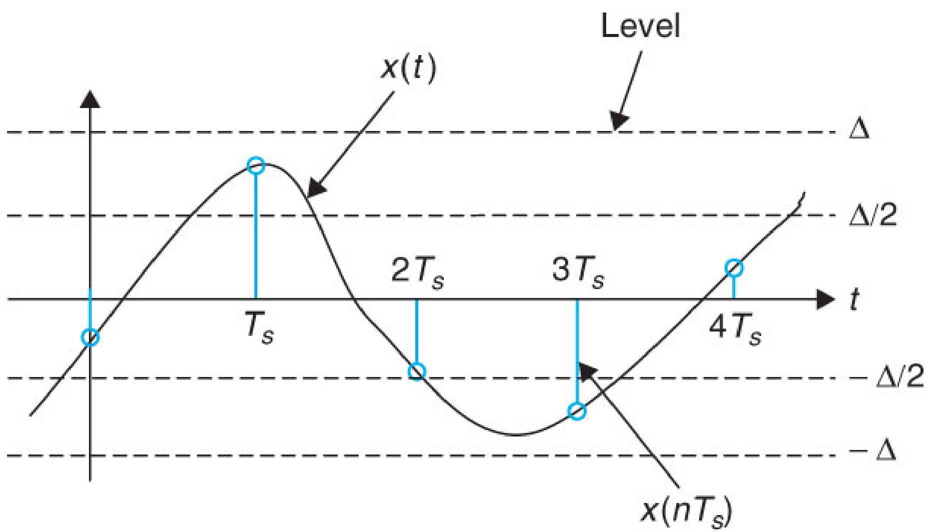
Analog, Discrete-Time, and Digital Signals

- **Time-dependent signals**
  - An **analog signal** is a continuous-amplitude, continuous-time signal.
    - Speech signal: $v(t)$, $t_b \leq t \leq t_f$
  - A **discrete-time signal** is a continuous-amplitude, discrete-time signal, usually obtained from **sampling** of an analog signal.
    - Discrete-time speech signal: $v(nT_s) = v(t)|_{t=nT_s}$, $0 \leq n \leq N-1$
  - A **digital signal** has discrete time and discrete amplitude, usually obtained from **quantization and coding** of a discrete-time signal.

Analog-to-Digital and Digital-to-Analog

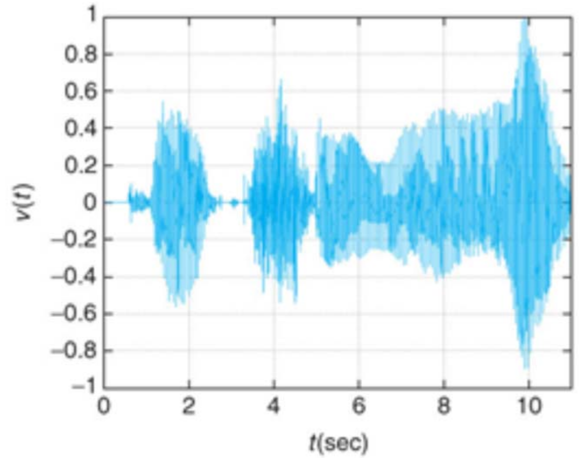
- **Analog-to-digital converter (ADC)** converts an analog signal into a digital signal.
  - **Sampling + Quantization**.

- **Digital-to-analog converter (DAC)** converts a digital signal into an analog signal.



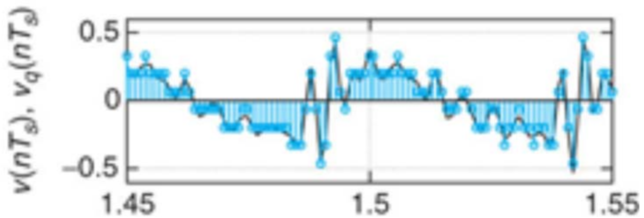
Example of Analog-to-Digital Converter

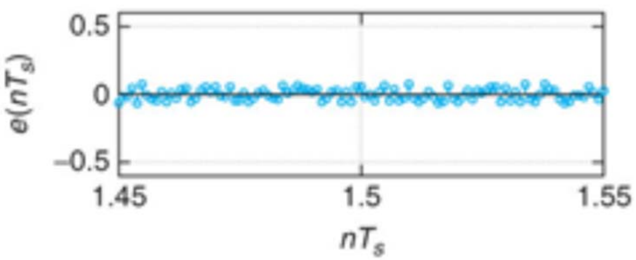
- ADC for speech signals



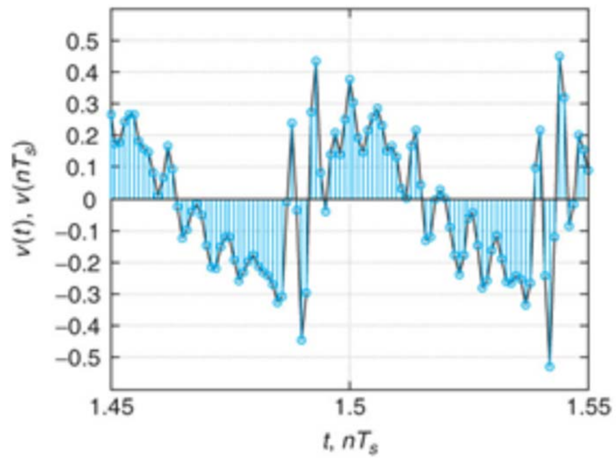
Segment of Speech Signal (Analog)

Digital Signal (Sampling + Quantization)





Quantization Error



Sampled Signal (Discrete-Time)

Continuous-Time Signal

A **continuous-time signal** can be thought of as a **real-(or complex-) valued function of time**:

$$x(\cdot):\ \Re \rightarrow \Re\ (C)$$
$$t \qquad x(t)$$

[Ex 1.1] Characterize the sinusoidal signal

$$x(t) = \sqrt{2}\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right), \qquad -\infty < t < \infty$$

[Ex 1.2] A complex signal $y(t)$ is defined as

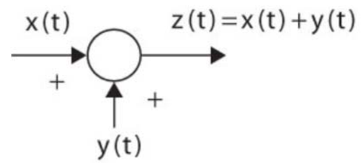
$$y(t) = (1+j)e^{\frac{j\pi t}{2}}, \qquad 0 \le t \le 10$$

and zero otherwise. Express $y(t)$ in terms of the signal $x(t)$ in [Ex 1.1].

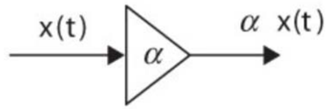
[Ex 1.3] Consider the pulse signal $p(t) = 1,\ \ 0 \le t\ \le 10$ and zero elsewhere. Characterize this signal and use it along with $x(t)$ in [Ex 1.1] to represent $y(t)$ in [Ex 1.2].

Basic Signal Operations

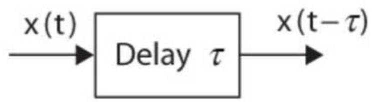
- Basic Signal Operations



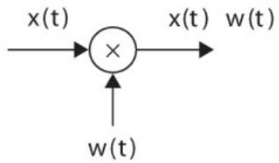
  - **Signal addition**: $z(t) = x(t) + y\ (t)$ using adder
  - **Constant multiplication**: $z(t) = \alpha x(t)$ using constant multiplier
  - **Time and frequency shifting**:



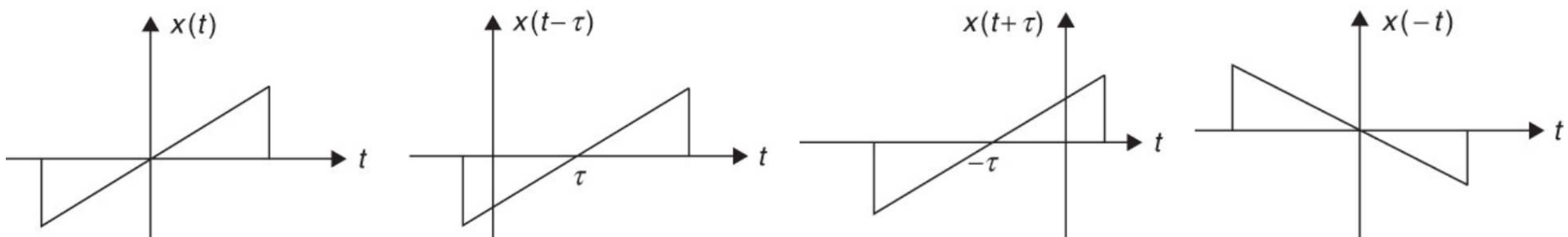
    - $x(t)$ **delayed/advanced** by τ : $x(t-\tau)$ / $x(t+\tau)$
    - $x(t)$ shifted by frequency or **frequency modulated** : $x(t)e^{j\Omega_0 t}$



  - **Time scaling**: $x(\alpha t)$ with constant α
    - $\alpha = -1, x(-t)$, reversed in time or **reflected**
    - $\alpha \neq 1$, signal **compressed/expanded**



  - **Time windowing**: $z(t) = x(t)w(t)$, window signal $w(t)$



Not implemented in real time

Examples

[Ex 1.4] Consider an analog pulse

$$x(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & otherwise \end{cases}$$

Find mathematical expression for $x(t)$ delayed by 2, advanced by 2, and the reflected signal $x(-t)$.

[Ex 1.5] When the shifting and reflecting operations are considered together the best approach to visualize the operation is to make a table computing several values of the new signal and comparing these with those from the original signal. Consider the pulse below and plot the signal $x(-t+2)$.

$$x(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 0 & otherwise \end{cases}$$

Even and Odd Signals

Even and **odd signals** are defined as follows:

$x(t)$ **even**: $x(t) = x(-t)$

$x(t)$ **odd**: $x(t) = -x(-t)$

Even and odd decomposition: Any signal $y(t)$ is representable as a sum of an even component $y_e(t)$ and an odd component $y_o(t)$.

$$y(t) = y_e(t) + y_o(t)$$
$$y_e(t) = 0.5[y(t) + y(-t)]$$
$$y_o(t) = 0.5[y(t) - y(-t)]$$

[Ex 1.6] Consider the analog signal

$$x(t) = \cos(2\pi t + \theta) \qquad -\infty < t < \infty$$

Determine the value of θ for which $x(t)$ is even and odd. If $\theta = \frac{\pi}{4}$, is $\cos(2\pi t + \pi/4), -\infty < t < \infty$, even or odd?

Periodic and Aperiodic Signals

[Ex 1.7] Consider the signal

$$x(t) = \begin{cases} 2\cos 4t & t > 0 \\ 0 & otherwise \end{cases}$$

Find its even and odd decomposition. What would happen if $x(0) = 2$ instead of 0?

An analog signal $x(t)$ is **periodic** if

- it is defined for all possible values of t
- there is a positive real value T_0, the period with $x(t)$

$$x(t + kT_0) = x(t), \qquad \forall\, integer\ k$$

Examples

[Ex 1.8] Consider the analog sinusoid

$$x(t) = A\cos(\Omega_0 t + \theta), \qquad -\infty < t < \infty$$

Determine the period of this signal, and indicate for what frequency Ω_0 the period of $x(t)$ is not clearly defined.

[Ex 1.10] Let $x(t) = e^{j2t}$ and $x(t) = e^{j\pi t}$, and consider their sum $z(t) = x(t) + y(t)$, and their product $\mathrm{z(t)} = x(t)y(t)$. Determine $\mathrm{z(t)}$ and $\mathrm{w(t)}$ are periodic, and if so, find their periods. Is $p(t) = (1 + x(t))(1 + y(t))$ periodic?

Finite-Energy and Finite Power Signals

The **energy** and the **power** of an analog signal $x(t)$ are defined for either finite or infinite-support signals as:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \; dt \,, \qquad P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \; dt$$

The signal $x(t)$ is then said to be **finite energy**, or **square integrable**, whenever $\boldsymbol{E_x} < \infty$.

The signal is said to have **finite power** if $\boldsymbol{P_x} < \infty$.

[Ex 1.12] Find the energy and the power of the following:

(a) Complex signal $y(t) = (1 + j)e^{j\pi t/2}$, for $0 \leq t \leq 10$, and zero otherwise.

(b) Pulse $\mathrm{z}(\mathrm{t}) = 1$, for $0 \leq t \leq 10$ and zero otherwise.

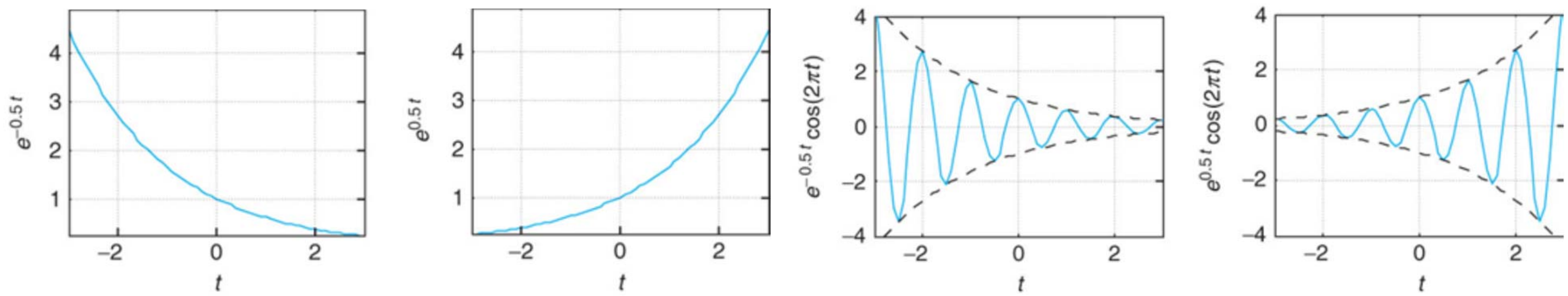
Complex Exponentials

[Ex 1.14] Consider the signal $x(t) = \cos(2\pi t) + \cos(4\pi t)$ and $y(t) = \cos(2\pi t) + \cos(2t)$, $-\infty < t < \infty$. Determine if these signals are periodic, and if so, find their periods. Compute the power of these signals.

A **complex exponential** is a signal of the form

$$x(t) = Ae^{at} \qquad -\infty < t < \infty$$
$$= |A|e^{rt}[\cos(\Omega_0 t + \theta) + j\sin(\Omega_0 t + \theta)]$$

where $A = |A|e^{j\theta}$ and $a = r + j\Omega_0$ are **complex numbers**.



Sinusoids

Sinusoids are of the general form

$$A\cos(\Omega_0 t + \theta) = A\sin(\Omega_0 t + \theta + \pi/2) \quad -\infty < t < \infty$$

where A is the **amplitude** of the sinusoid Ω_0 is the **frequency**, and θ is the **phase shift**

$$\Omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

- Modulation in communication systems

$$A(t)\cos[\Omega(t)t + \theta(t)]$$

  - **Amplitude modulation (AM)** changes $A(t)$, $\Omega(t)$ and $\theta(t)$ const.
  - **Frequency modulation (FM)** changes $\Omega(t)$, $A(t)$ and $\theta(t)$ const.
  - **Phase modulation (PM)** changes $\theta(t)$, $A(t)$ and $\Omega(t)$ const.

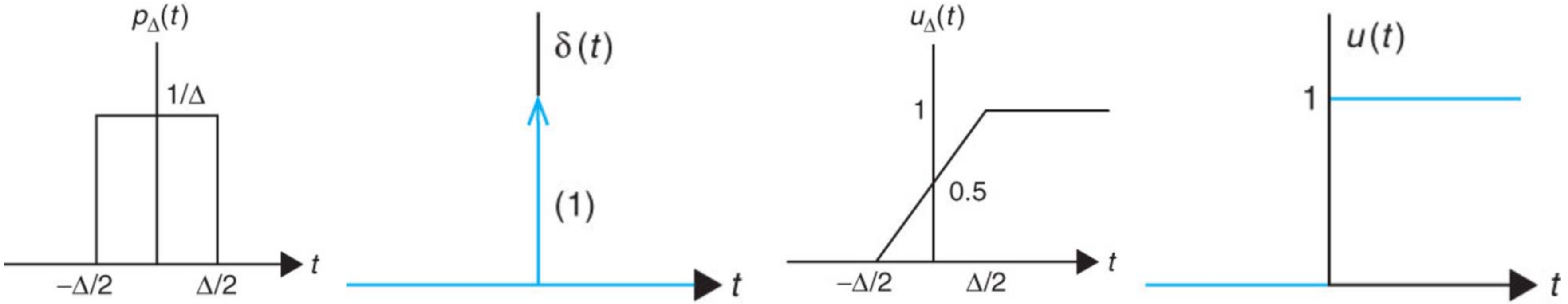
Unit-Impulse and Unit-Step Signals

- Consider a rectangular pulse of **duration Δ** and **unit area**

$$p_\Delta(t) = \begin{cases} \frac{1}{\Delta} & -\frac{\Delta}{2} \leq t \leq \frac{\Delta}{2} \\ 0 & otherwise \end{cases} \qquad \boldsymbol{\delta(t) = \lim_{\Delta \to 0} p_\Delta(t)}$$

- Its integral is

$$u_\Delta(t) = \int_{-\infty}^{t} p_\Delta(\tau)\, d\tau = \begin{cases} 1 & t > \Delta/2 \\ \frac{1}{\Delta}(t + \frac{\Delta}{2}) & -\frac{\Delta}{2} \leq t \leq \frac{\Delta}{2} \\ 0 & t < -\Delta/2 \end{cases} \qquad \boldsymbol{u(t) = \lim_{\Delta \to 0} u_\Delta(t)}$$



Ramp Signals

- The **impulse signal $\boldsymbol{\delta(t)}$** and the **unit-step signal $\boldsymbol{u(t)}$** are related as follows:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \qquad \delta(t) = \frac{du(t)}{dt}$$

The **ramp signal** is defined as

$$r(t) = t\, u(t)$$

Its relation to the unit-step and unit-impulse signals is

$$\frac{dr(t)}{dt} = u(t) \qquad \frac{d^2 r(t)}{dt^2} = \delta(t)$$

Examples

[Ex 1.15] Consider the discontinuous signals

$x_1(t) = \cos(2\pi t)[u(t) - u(t-1)], \qquad x_2(t) = u(t) - 2u(t-1) + u(t-2)$

Represent each signal as the sum of continuous signal and unit-step signals, and find their derivatives.

[Ex 1.16$^{\text{MATLAB}}$] Write a script and the necessary functions to generate a signal

$$y(t) = 3r(t+3) - 6r(t+1) + 3r(t) - 3u(t-3)$$

plot it and verify analytically that the obtained figure is correct.

[Ex 1.18$^{\text{MATLAB}}$] Use symbolic MATLAB to generate the following analog signals.

(a) The damped sinusoid signal $y(t) = e^{-t}\cos(2\pi t)$, and its envelope.

(b) The signal $x(t) = 1 + 1.5\cos(2\Omega_0 t) - 0.6\cos(4\Omega_0 t)$ which is a rough approximation of a periodic pulse generated by adding 3 cosines of frequencies multiples of $\Omega_0 = \pi/10$.

Examples

[Ex 1.19] Consider the generation of a triangular signal,

$$\Lambda(t) = \begin{cases} t & 0 < t \leq 1 \\ -t + 2 & 1 < t \leq 2 \\ 0 & otherwise \end{cases}$$

using ramp signal r(t). Use unit-step signal to represent its derivative.

[Ex 1.20] Consider a full-wave rectified signal,

$$x(t) = |\cos(2\pi t)| \quad -\infty < t < \infty$$

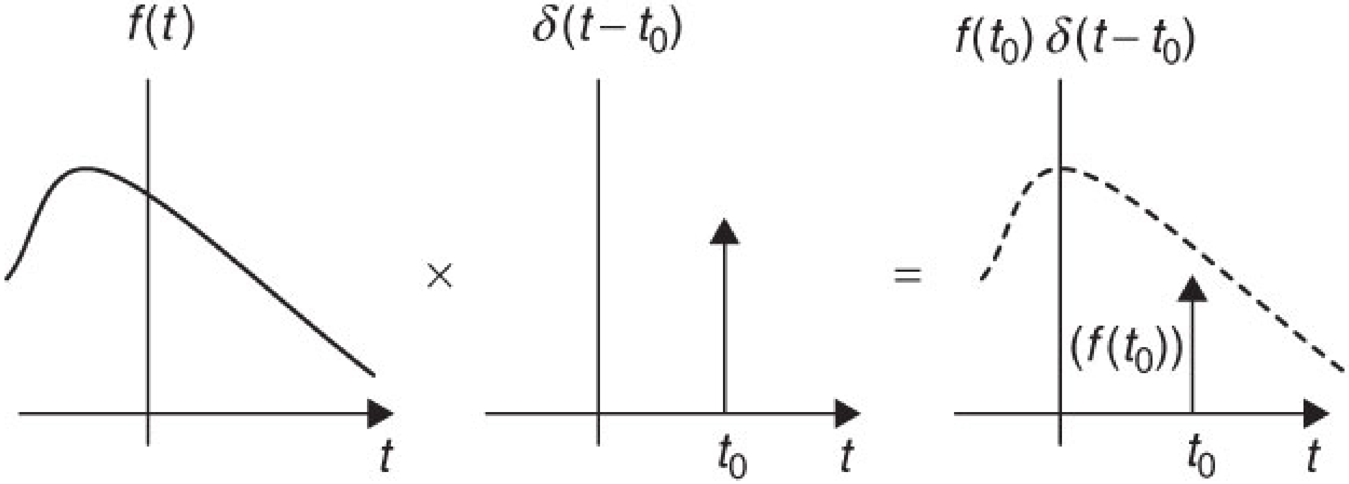
of period $T_0 = 0.5$. Obtain a representation for a period between 0 and 0.5, and represent $x(t)$ in terms of shifted versions of it.

Generic Representation of Signals

By the **sifting property** of the impulse function $\delta(t)$, any signal $x(t)$ can be represented by

$$\boldsymbol{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\, dt = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\, dt = f(\tau)\int_{-\infty}^{\infty} \delta(t-\tau)\, dt = f(\tau)$$



Sampling Signal and Sinc

- **Sampling** of a continuous-time signal consists in **taking samples** of the signal **at uniform times**

$$x_s(t) = x(t)\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

- The **sinc function** is defined as

$$\boldsymbol{S(t) = \frac{\sin \pi t}{\pi t}} \qquad -\infty < t < \infty$$

  - The **time support** of sinc is **infinite**
  - Sinc is an **even function** of t
  - At $t = 0$ both numerator and denominator are zero, but $\boldsymbol{S(0) = 1}$
  - $S(t)$ is bounded, $-1/\pi t \leq S(t) \leq 1/\pi t$
  - The zero-crossing time of $S(t)$ are found at $\boldsymbol{t = k}$ **for nonzero integer** $\boldsymbol{k}$

Time Scaling, Frequency Shifting, and Windowing

Given a signal $x(t)$ and real values $\alpha \neq 0\ or\ 1$, and $\phi > 0$,

1. $x(\alpha t)$ is said to be **contracted**, if $|\boldsymbol{\alpha}| > \mathbf{1}$, and if $\boldsymbol{\alpha} < \mathbf{0}$ it is also **reflected**,

2. $x(\alpha t)$ is said to be **expanded**, if $|\boldsymbol{\alpha}| < \mathbf{1}$, and if $\boldsymbol{\alpha} < \mathbf{0}$ it is also **reflected**,

3. $\boldsymbol{x(t)e^{j\phi t}}$ is said to be **shifted in frequency** by ϕ radians.

4. For a **window signal $\boldsymbol{w(t)}$**, $\boldsymbol{x(t)w(t)}$ displays $x(t)$ within the support of $w(t)$.

- The process of **multiplying the signal by a complex exponential or** equivalently **by sines and cosines** is called the **modulation**.

Examples

[Ex 1.22] Let $x_1(t), 0 \leq t \leq T_0$, be a period of a periodic signal $x(t)$ of fundamental period T_0. Represent $x(t)$ in terms of advanced and delayed versions of $x_1(t)$. What would be $x(2t)$?

[Ex 1.23] An acoustic signal $x(t)$ has a duration of 3.3 minutes and a radio station would like to use the signal for a 3-minute segment. Indicate how to make it possible.

[Ex 1.24$^{\text{MATLAB}}$] One way of transmitting a message over the airwaves is to multiply it by a sinusoid of frequency higher than those in the message, thus changing the frequency content of the signal. To recover the message from the transmitted signal, one can relate the envelope of the modulated signal $y(t) = 2r(t+2) - 4r(t) + r(t-2) + 3r(t-3) - u(t-3)$ to modulate so-called carrier signal $x(t) = \sin(5\pi t)$ to give the modulated signal $z(t) = y(t)x(t)$. Obtain a script to generate the signal, and to plot it. Indicate whether the envelope of the modulation is connected with the message signal $y(t)$.

Thank You