1. $p(x,y) = \dfrac{e^{-2}}{x!(y-x)!}$

(a) Find the mgf $M(t_1,t_2)$ of $(X\ Y)'$

$$M_{XY}(t_1,t_2) = E(e^{t_1X+t_2Y}) = \sum_{y=0}^{\infty}\sum_{x=0}^{y} e^{t_1x+t_2y}\,\frac{e^{-2}}{x!(y-x)!}$$

$$= e^{-2}\sum_{y=0}^{\infty}\sum_{x=0}^{y}\frac{y!(e^{t_1})^x}{x!(y-x)!}\cdot\frac{e^{t_2y}}{y!} = e^{-2}\sum_{y=0}^{\infty}(1+e^{t_1})^y\frac{e^{t_2y}}{y!}$$

$$= e^{-2}\sum_{y=0}^{\infty}\frac{((1+e^{t_1})e^{t_2})^y}{y!} = e^{-2}e^{(1+e^{t_1})e^{t_2}} = e^{(1+e^{t_1})e^{t_2}-2}$$

(b) compute the means, the variance, the correlation coefficient of X and Y

$$E(X) = \frac{\partial}{\partial t_1}M_{XY}(t_1,t_2)\Big|_{t_1=0,t_2=0} = \frac{\partial}{\partial t_1}e^{(1+e^{t_1})e^{t_2}-2}\Big|_{t_1=0,t_2=0}$$

$$= (e^{t_1+t_2})(e^{(1+e^{t_1})e^{t_2}-2}) = 1$$

$$E(Y) = \frac{\partial}{\partial t_2}M_{XY}(t_1,t_2)\Big|_{t_1=0,t_2=0} = \frac{\partial}{\partial t_2}e^{(1+e^{t_1})e^{t_2}-2}\Big|_{t_1=0,t_2=0}$$

$$= (1+e^{t_1})e^{t_2}\cdot e^{(1+e^{t_1})e^{t_2}-2} = 2$$

$$E(X^2) = \frac{\partial^2}{\partial^2 t_1}M_{XY}(t_1,t_2)\Big|_{(0,0)} = \frac{\partial^2}{\partial^2 t_1}e^{(1+e^{t_1})e^{t_2}-2}\Big|_{(0,0)}$$

$$= \frac{\partial}{\partial t_1}(e^{t_1+t_2}\cdot e^{(1+e^{t_1})e^{t_2}-2})\Big|_{(0,0)}$$

$$= e^{t_1+t_2}e^{(1+e^{t_1})e^{t_2}-2} + e^{2t_1+2t_2}\cdot e^{(1+e^{t_1})e^{t_2}-2}\Big|_{(0,0)}$$

$$= 2$$

$$E(Y^2) = \frac{\partial^2}{\partial^2 t_2}M_{XY}(t_1,t_2)\Big|_{(0,0)} = \frac{\partial^2}{\partial^2 t_2}e^{(1+e^{t_1})e^{t_2}-2}\Big|_{(0,0)}$$

$$= \frac{\partial}{\partial t_2}(1+e^{t_1})e^{t_2}e^{(1+e^{t_1})e^{t_2}-2}\Big|_{(0,0)}$$

$$= (1+e^{t_1})e^{t_2}e^{(1+e^{t_1})e^{t_2}-2} + (1+e^{t_1})^2e^{2t_2}e^{(1+e^{t_1})e^{t_2}-2}\Big|_{(0,0)}$$

$$= 6$$

$V(X) = E(X^2) - E(X)^2 = 1$, $V(Y) = E(Y^2) - E(Y)^2 = 6-4 = 2$

$$E(XY) = \frac{\partial^2}{\partial t_1\partial t_2}M_{XY}(t_1,t_2)\Big|_{(0,0)} = \frac{\partial^2}{\partial t_1\partial t_2}e^{(1+e^{t_1})e^{t_2}-2}\Big|_{(0,0)}$$

$$= \frac{\partial}{\partial t_2}e^{t_1+t_2}e^{(1+e^{t_1})e^{t_2}-2}\Big|_{(0,0)} = e^{t_1+t_2}e^{(1+e^{t_1})e^{t_2}-2}$$

$$+ e^{t_1}(1+e^{t_1})e^{2t_2}e^{(1+e^{t_1})e^{t_2}-2}\Big|_{(0,0)} = 1+2 = 3$$

$$\rho = \mathrm{Cov}(X,Y)/\sigma_X\sigma_Y = (E(XY) - E(X)E(Y))/\sigma_X\sigma_Y = \frac{1}{\sqrt{2}} = 0.71$$

(c) Determine $E(X|Y=4)$

$$E(X|Y=4) = \sum_{x=0}^{4} \frac{P(x,4)}{P(4)}$$

$$P_Y(4) = \sum_{0}^{\infty} \frac{e^{-2}}{x!(4-x)!} = \frac{e^{-2}}{4!}\sum_{x=0}^{\infty}\frac{4!}{x!(4-x)!} = \frac{e^{-2}2^4}{4!}$$

$$= \sum_{x=0}^{4} \frac{\frac{e^{-2}}{x!(4-x)!}}{\frac{e^{-2}2^4}{4!}} = \sum_{x=0}^{4} \frac{x\,4!}{x!(4-x)!\,2^4} = \sum_{0}^{4} \frac{4!}{(x-1)!(4-x)!}2^{-4}$$

$$= \frac{4}{2^4}\sum_{x=0}^{4}\frac{(4-1)!}{(x-1)!(4-x)!} = \frac{4}{2^4}\times 2^{4-1} = \frac{4}{2} \qquad \therefore \frac{4}{2}$$

Problem 2. let $Y = X_1 + X_2$, if $Y \sim \chi^2(r)$ and $X_1 \sim \chi^2(r_1)$

then $X_2 \sim \chi^2(r - r_2)$

$$M_Y(t) = \chi^2(r) = \prod_{i=1}^{n} M_{X_i}(t) = (1-2t)^{-\frac{r_1}{2}}\cdot(1-2t)^{-\frac{r_2}{2}}\cdot\ldots$$

$$= (1-2t)^{-\frac{r}{2}}$$

$$r = r_1 + r_2 + \cdots + r_n$$

so if $Y = X_1 + X_2 \rightarrow r = r_1 + r_2$

$$r_2 = r - r_1$$

$\therefore$ If $Y \sim \chi^2(r)$, $X_1 \sim \chi^2(r_1)$, $X_2 \sim \chi^2(r - r_1)$

Problem 3. (a) show that $H(x) = f(x)/(1-F(x))$

$$H(x) = \lim_{\Delta\to 0}\frac{P(x \le X \le x+\Delta \mid X \ge x)}{\Delta} = \lim_{\Delta\to 0}\frac{P(x\le X\le x+\Delta)}{P(X\ge x)\,\Delta} = \frac{1}{1-F(x)}\lim_{\Delta\to 0}\frac{P(x\le X\le x+\Delta)}{\Delta}$$

$$= \frac{F'(x)}{1-F(x)} = \frac{f(x)}{1-F(x)}$$

(b) when $H(x) = Cx^b$, find the pdf of X.

$$H(x) = \frac{f(x)}{1-F(x)} \rightarrow 1-F(x) = e^{-\int H(x)dx + c} \quad , F(0) = 0.$$

$$1-F(x) = e^{-\frac{c}{b+1}x^{b+1} + C_k} \qquad C_k = 0 \; (\because F(0)=0)$$

$$F(x) = 1 - e^{-\frac{c}{b+1}x^{b+1}}$$

$$F'(x) = f(x) = cx^b e^{-\frac{c}{b+1}x^{b+1}}$$

Problem 4) $X_1 \sim N(6,1)$, $X_2 \sim N(7,1)$ Find $P(X_1 > X_2)$

$P(X_1 - X_2 > 0)$ $\quad E(X_1 - X_2) = E(X_1) - E(X_2) = -1$

$Var(X_1 - X_2) = Var(X_1) + Var(X_2) = 2$

$$P(X_1 - X_2 > 0) = P\left(\frac{X_1 - X_2 - (-1)}{\sqrt{2}} > \frac{0-(-1)}{\sqrt{2}}\right) = P\left(Z > \frac{1}{\sqrt{2}}\right)$$

$$= 1 - \Phi\left(\frac{1}{\sqrt{2}}\right) = 1 - \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 1 - 0.7603$$

$\therefore 0.2397$

Problem 5. $X, Y \sim N(0,1)$, find the mgf of the random variable.

$$M_W(t) = E(e^{XYt}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{xyt} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dxdy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(xy)t} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}} dxdy = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x-yt)^2}{2}} e^{+\frac{y^2t^2}{2}} e^{-\frac{y^2}{2}} dxdy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2\frac{(1-t^2)}{2}} dy = \frac{1}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{\sqrt{1-t^2}} = \frac{1}{\sqrt{1-t^2}}$$

$$\therefore \frac{1}{\sqrt{1-t^2}}$$

Problem 6. $X = (X_1, X_2)' \sim N_2(\mu, \Sigma)$. let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$.

(1) Find distribution of Y

If $Y = aX \Rightarrow \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

mean $= A\mu$, Variance $= A\Sigma A'$

$\therefore$ mean $= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\mu$, variance $= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\Sigma\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}'$

(2) Find condition that Y_1 and Y_2 are independent.

$Var(X_1) = \sigma_1^2$, $Var(X_2) = \sigma_2^2$, $COV(X_1, X_2) = \rho\sigma_1\sigma_2$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \quad A\Sigma A' = \begin{bmatrix} \sigma_1^2+\rho\sigma_1\sigma_2 & \sigma_2^2+\rho\sigma_1\sigma_2 \\ \sigma_1^2-\rho\sigma_1\sigma_2 & -\sigma_2^2+\rho\sigma_1\sigma_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2 & \sigma_1^2-\sigma_2^2 \\ \sigma_1^2-\sigma_2^2 & \sigma_1^2+\sigma_2^2 - 2\rho\sigma_1\sigma_2 \end{bmatrix}$$

If $Var(X_1) = Var(X_2) = \sigma_1 = \sigma_2$, $A\Sigma A'_{12} = 0$

then, Y_1, Y_2 are independent

$\therefore Var(X_1) = Var(X_2)$

Problem 7. $\mu_x = 1,\ \mu_y = 4.\ \sigma_x^2 = 4.\ \sigma_y^2 = 6.\ \rho_{xy} = 1/2$

find the mean and variance of $Z = 3X - 2Y$

$$Z = (3\ \ -2)\begin{pmatrix} X \\ Y \end{pmatrix} \qquad A = (3\ \ -2)$$

$$\text{mean}: \ (3\ \ -2)\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} = 3\mu_x - 2\mu_y = -5$$

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} 4 & \sqrt{6} \\ \sqrt{6} & 6 \end{pmatrix}$$

$$\mathrm{Var}(Z) = A\Sigma A' = (3\ \ -2)\begin{pmatrix} 4 & \sqrt{6} \\ \sqrt{6} & 6 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = (12-2\sqrt{6}\ \ 3\sqrt{6}-6)\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{matrix} 36-6\sqrt{6} \\ -6\sqrt{6}+12 \end{matrix}$$

$$= 48 - 12\sqrt{6}$$

$$\therefore \ \text{mean}(Z) = -5$$

$$\mathrm{Var}(Z) = 48 - 12\sqrt{6}$$

Problem 8. Is S be an unbiased estimator of sigma?

$$E(X_i) = \mu,\ \ \mathrm{var}(X_i) = \sigma^2,\ \ E(X_i^2) = \mu^2 + \sigma^2,\ \ \ E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2$$

$$E(\Sigma(X - \bar{X})^2) = E(\Sigma X_i^2 - 2\bar{X}\Sigma X_i + n\bar{X}^2) = \Sigma E(X_i^2) - E(n\bar{X})^2$$

$$= n\mu^2 + n\sigma^2 - \sigma^2 - n\mu^2 = (n-1)\sigma^2.$$

$$E(S^2) = E\left(\frac{\Sigma(X_i - \bar{X})^2}{n-1}\right) = \frac{1}{n-1}E(\Sigma(X_i - \bar{X}))^2 = \frac{n-1}{n-1}\sigma^2 = \sigma^2.$$

$\therefore$ S be an unbiased estimator of σ.