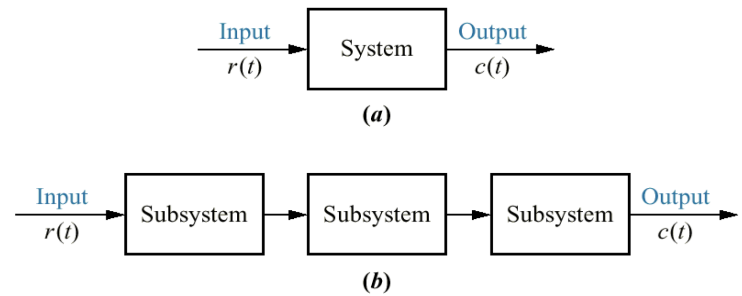
Chapter 2. Modeling in the Frequency Domain

Chapter Learning Outcomes

After completing this chapter, the student will be able to:

- Find the Laplace transform of time functions and the inverse Laplace transform (Sections 2.1–2.2)
- Find the transfer function from a differential equation and solve the differential equation using the transfer function (Section 2.3)
- Find the transfer function for linear, time-invariant electrical networks (Section 2.4)
- Find the transfer function for linear, time-invariant translational mechanical systems (Section 2.5)
- Find the transfer function for linear, time-invariant rotational mechanical systems (Section 2.6)
- Find the transfer functions for gear systems with no loss and for gear systems with loss (Section 2.7)
- Find the transfer function for linear, time-invariant electromechanical systems (Section 2.8)
- Produce analogous electrical and mechanical circuits (Section 2.9)
- Linearize a nonlinear system in order to find the transfer function (Sections 2.10–2.11)



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

Figure 2.1
a. Block diagram representation of a system;
b. Block diagram representation of an interconnection of subsystems

$$L(f(t)) = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt$$ ⟵ *Bilateral* (*two-sided*) Laplace transform.

$$L(f(t)) = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$ ⟵ *Unilateral* (*one-sided*) Laplace transform.

The bilateral and unilateral transforms are equivalent only if $x(t)=0$ for $t < 0$.

$$s = \sigma + jw, \qquad 0^- = \lim_{\varepsilon \to 0}(0-\varepsilon)$$

Complex variable

$$L^{-1}(F(s)) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\,ds$$

- Table 2.1 Laplace transform table

Item no.	f(t)	F(s)
1.	$\delta(t)$	1
2.	$u(t)$	$\boxed{\frac{1}{s}}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\boxed{\frac{1}{s+a}}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace transform

2.3 Transfer Function

2.4 Electric Network Transfer Functions

2.5 Translational Mechanical System Transfer Functions

2.6 Rotational Mechanical System Transfer Functions

2.7 Transfer Functions for Systems with Gears

2.8 Electromechanical System Transfer Functions

2.9 Electric Circuit Analogs

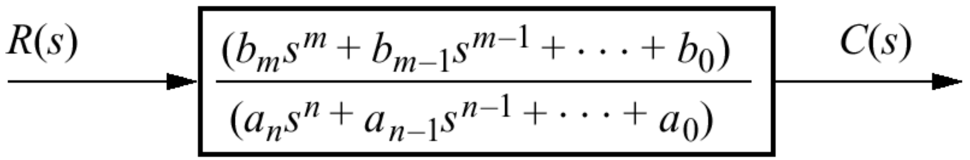
- Table 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1] For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts and no more than one can be at the origin.
[2] For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (i.e., no impulses or their derivatives at $t = 0$).

2.3 The Transfer Function

- Figure 2.2 Block diagram of a transfer function



$$a_n \frac{d^n}{dt^n} c(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} c(t) + \cdots + a_0 c(t) = b_m \frac{d^m}{dt^m} r(t) + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} r(t) + \cdots + b_0 r(t)$$

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition of } c(t)$$

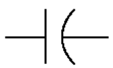
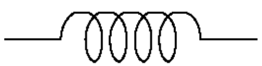
$$= b_m s^m R(s) + a_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition of } r(t)$$

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

$$C(s) = R(s)G(s)$$

2.4 Electric Network Transfer Functions

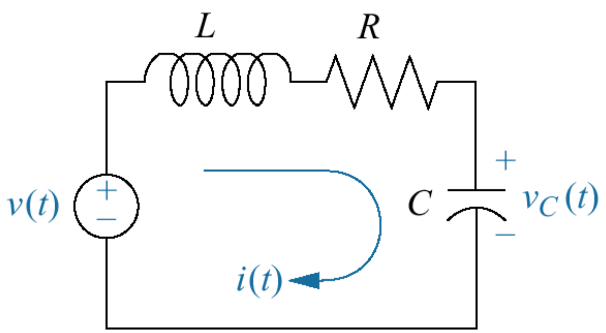
- Table 2.3
  Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Current-charge	Impedance $Z(s) = \frac{V(s)}{I(s)}$	Admittance $Y(s) = \frac{I(s)}{V(s)}$
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\boxed{\frac{1}{Cs}}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	$\boxed{R}$	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	$\boxed{Ls}$	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

Example 2.6: Transfer function – By the differential equation

$$\text{Transfer function}: \frac{V_c(s)}{V(s)} = ?$$



Summing the voltages around the loop:

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

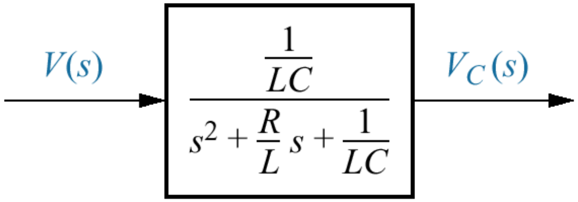
Changing variables from current to charge using

$$i(t) = \frac{d}{dt}q(t), \quad q(t) = Cv_c(t) \;\rightarrow\; \begin{cases} i(t) = \frac{d}{dt}Cv_c(t) = C\frac{d}{dt}v_c(t) \\ \frac{d}{dt}i(t) = C\frac{d^2}{dt^2}v_c(t) \\ \int i(\tau)d\tau = Cv_c(t) = C\frac{d}{dt}v_c(t) \end{cases}$$

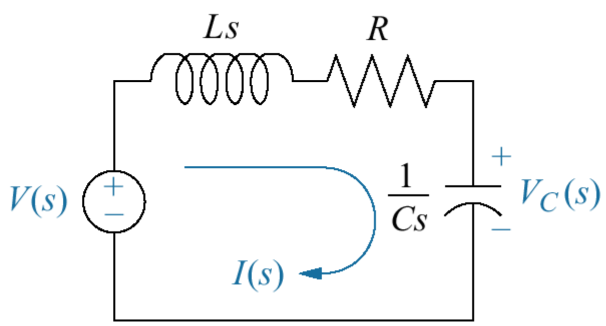
$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v(t)$$

$$\boxed{(LCs^2 + RCs + 1)}\, V_c(s) = V(s)$$

$$\therefore \quad \frac{V_c(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



Example 2.7: Transfer function – By the transform methods



← Laplace-transformed network

$$V(s)=\left(Ls+R+\frac{1}{Cs}\right)I(s)$$

$$\frac{I(s)}{V(s)}=\frac{1}{Ls+R+\frac{1}{Cs}}$$

$$V_C(s)=I(s)\frac{1}{Cs}$$

$$V_C(s)=I(s)\frac{1}{Cs}$$
$$=\frac{V(s)}{\left(Ls+R+\frac{1}{Cs}\right)}\cdot\frac{1}{Cs}$$

Example 2.9: By voltage division

$$V_c(s)=\frac{1/Cs}{Ls+R+\frac{1}{Cs}}V(s)$$

$$I(s)=CsV_C(s)$$
$$V(s)=LsI(s)+RI(s)+V_c(s)$$
$$=LsCsV_C(s)+RCsV_C(s)+V_c(s)$$
$$=\left(Ls(Cs)+RCs+1\right)\ V_c(s)$$

$$V(s)=\left(LCs^2+RCs+1\right)V_c(s)$$

$$\frac{V_c(s)}{V(s)}=\frac{1}{LCs^2+RCs+1}$$

$$\Rightarrow\quad\frac{V_c(s)}{V(s)}=\frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Example 2.10: Transfer function – multiple loops

Given the network (a), find the Transfer function, $I_2(s) / V(s)$.

$$V(s) = R_1 I_1(s) + Ls(I_1(s) - I_2(s))$$
$$= (R_1 + Ls)I_1(s) - LsI_2(s)$$

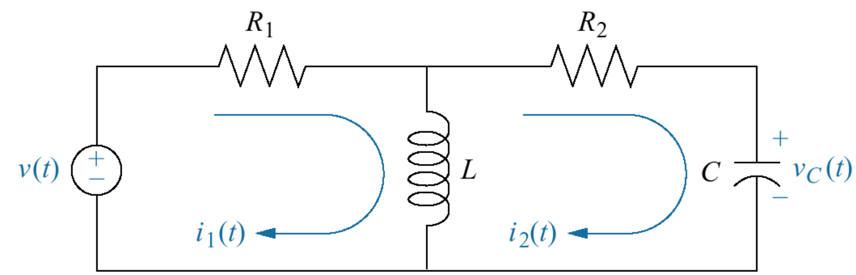
$$0 = Ls(I_2(s) - I_1(s)) + R_2 I_2(s) + \frac{1}{Cs} I_2(s)$$

$$\begin{pmatrix} R_1 + Ls & -Ls \\ -Ls & Ls + R_2 + \frac{1}{Cs} \end{pmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} V(s) \\ 0 \end{pmatrix}$$

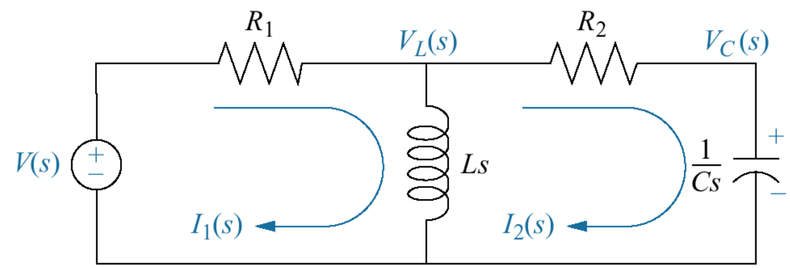
Appendix F (F.4):
Cramer's rule

$$I_2(s) = \frac{\begin{vmatrix} R_1 + Ls & V(s) \\ -Ls & \boxed{0} \end{vmatrix}}{\begin{vmatrix} R_1 + Ls & -LS \\ -Ls & Ls + R_2 + \frac{1}{Cs} \end{vmatrix}} = \frac{LsV(s)}{\Delta}$$

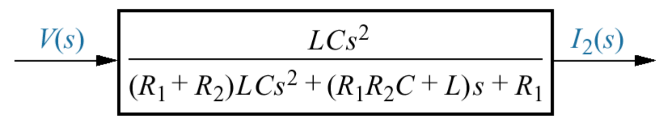
$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1 R_2 C + L)s + R_1}$$



(a)



(b)



(c)

```
'(ch2sp4) Example 2.10'            % Display label.
syms s R1 R2 L c V                 % Construct symbolic objects for frequency
                                   % variable 's', and 'R1', 'R2', 'L', 'c', and 'V'.
                                   % Note: Use lower-case "c" in declaration for
                                   % capacitor.
A2=[(R1+L*s) V;-L*s 0]             % Form Ak = A2.
A=[(R1+L*s) -L*s;-L*s (L*s+R2+(1/(c*s)))]
                                   % Form A.
I2=det(A2)/det(A);                 % Use Cramer's rule to solve for I2(s).
I2=simple(I2);                     % Reduce complexity of I2(s).
G=I2/V;                            % Form transfer function, G(s) = I2(s)/V(s).
'G(s)'                             % Display label.
pretty(G)                         % Pretty print G(s).
```

```
(ch2sp4) Example 2.10

A2 =
           [ R1 + L*s, V]
           [     -L*s, 0]

A =
           [ R1 + L*s,                -L*s]
           [     -L*s, R2 + L*s + 1/(c*s)]


G(s)
                                   2
                             L c s
             ------------------------------------------
                          2                          2
             L s + R1 (L c s  + R2 c s + 1) + L R2 c s
>>
```

Example 2.11: Transfer function – multiple nodes

Find the Transfer function, $V_C(s) / V(s)$.

$$\frac{V_L(s)-V(s)}{R_1}+\frac{V_L(s)}{Ls}+\frac{V_L(s)-V_C(s)}{R_2}=0$$

$$\frac{V_c}{\frac{1}{Cs}}+\frac{V_C(s)-V_L(s)}{R_2}=0$$

$$A\cdot V_L+B\cdot V_C=V$$

$$C\cdot V_L+D\cdot V_C=0$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} V_L \\ V_C \end{pmatrix}=\begin{pmatrix} V \\ 0 \end{pmatrix}$$

$$\frac{V_C}{V}= \ \sim$$

Use Cramer's rule … (do it yourself)

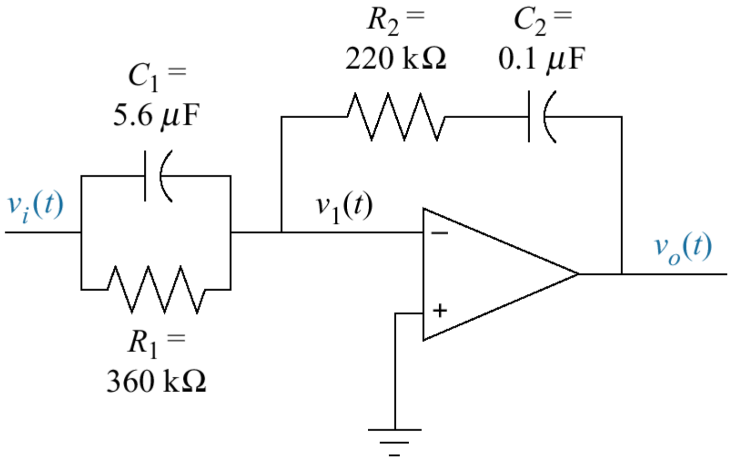
$$V(s) \longrightarrow \boxed{\dfrac{\dfrac{G_1G_2}{C}s}{(G_1+G_2)s^2+\dfrac{G_1G_2L+C}{LC}s+\dfrac{G_2}{LC}}} \longrightarrow V_C(s)$$

Example 2.14: Transfer function – Inverting operational amplifier circuit

Find the Transfer function, $V_o(s) / V_i(s)$.

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

From Eq. 2.97



$$\boxed{Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}}}$$

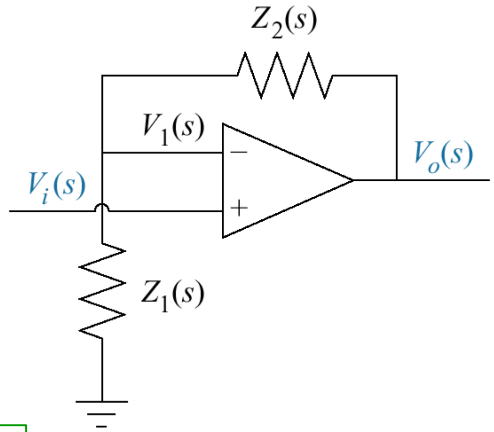
$$Z_1(s) = \frac{\frac{1}{C_1 s} R_1}{\frac{1}{C_1 s} + R_1} = \frac{R_1}{1 + R_1 C_1 s} = \frac{1}{C_1 s + \frac{1}{R_1}}$$

$$Z_2(s) = \boxed{R_2 + \frac{1}{C_2 s}}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2}{Z_1} = -\left(\frac{1 + sR_1C_1}{R_1}\right)\left(R_2 + \frac{1}{sC_2}\right) = -\frac{(1 + sR_1C_1)(sR_2C_2 + 1)}{sR_1C_2}$$

Transfer function – Noninverting operational amplifier circuit

Find the Transfer function, $V_o(s)$ / $V_i(s)$.



$$V_o(s) = A\big(V_i(s) - V_1(s)\big)$$

By using voltage division,

$$V_1(s) = \frac{Z_1(s)}{Z_1(s)+Z_2(s)} V_o(s)$$

$$V_o(s) = A\left(V_i(s) - \frac{Z_1(s)}{Z_1(s)+Z_2(s)} V_o(s)\right)$$

$$V_o(s)\left(1 + \frac{AZ_1(s)}{Z_1(s)+Z_2(s)}\right) = AV_i(s)$$

$$\frac{V_o(s)}{AV_i(s)} = \frac{1}{\left(1 + \frac{AZ_1(s)}{Z_1(s)+Z_2(s)}\right)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{A}{1 + \frac{AZ_1(s)}{Z_1(s)+Z_2(s)}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{A}{\frac{(Z_1+Z_2)+AZ_1}{Z_1+Z_2}} = \frac{A(Z_1+Z_2)}{(Z_1+Z_2)+AZ_1} = \frac{(Z_1+Z_2)}{\frac{1}{A}(Z_1+Z_2)+Z_1}$$

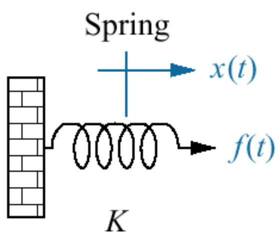
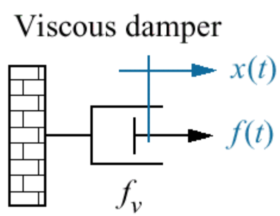
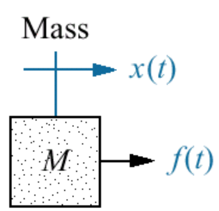
For large A,

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s)+Z_2(s)}{Z_1(s)}$$

2.5 Translational Mechanical System Transfer Functions

Component			
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$

Table 2.4: Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
	$f(t) = f_v v(t)$	$f(t) = f_v\frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

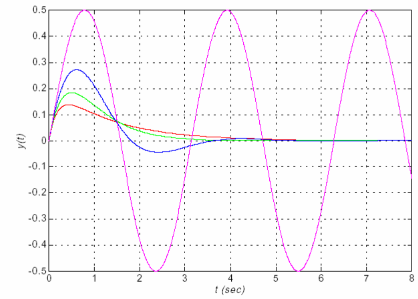
$$F = KX(s)$$

$$F = f_v sX(s)$$

$$F = Ms^2X(s)$$

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Transfer function and system responses

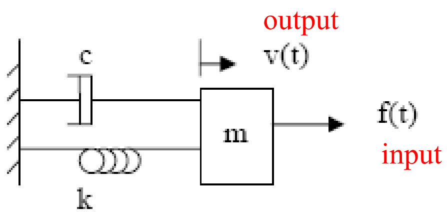


- Modeling of dynamic systems can described mathematically using physical laws.
- Based on the use of the conservation law of mass and energy, we can write the ordinary differential equations (ODEs) and algebraic equations describing system.
- Once the ordinary differential equation is obtained, *the system can be analyzed and developed to be controlled*.

Example *(Mechanical system)*

Consider a mechanical system that consists of mass-spring-damper as illustrates in the figure.

Where: *f(t) is the applied force;*
v(t) is the resulted velocity
m is the inertia
c is the damping coefficient
k is the stiffness coefficient



Mass-damper-spring System

By applying the force balance on the mass m, the ODE model is:

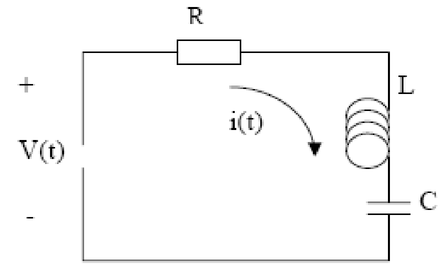
$$m\frac{dv(t)}{dt} + cv(t) + k\int v(t)dt = f(t)$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)dt = v(t)$$

***Example** (Electrical system)*

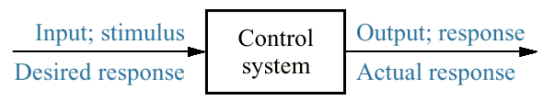
Consider an electrical system that consists of R-L-C circuit as illustrates in the figure.

Where: *V(t) is the <u>applied voltage</u>; i(t) is the <u>resulted current</u>*
R is the <u>resistance</u> in Ohm
L is the <u>inductance</u> in Henry
C is the <u>capacitance</u> in Farad



R-L-C Circuit

By applying Ohm's law on the circuit, the ODE model is:

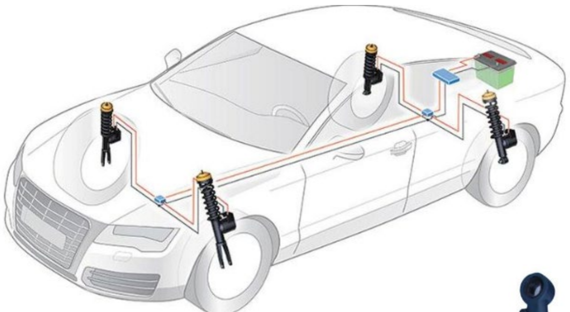


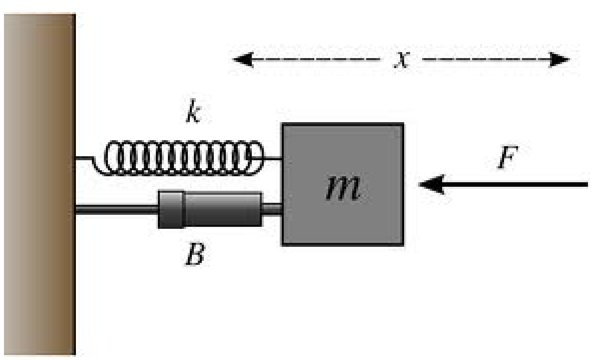
$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int i(t)dt=v(t)$$

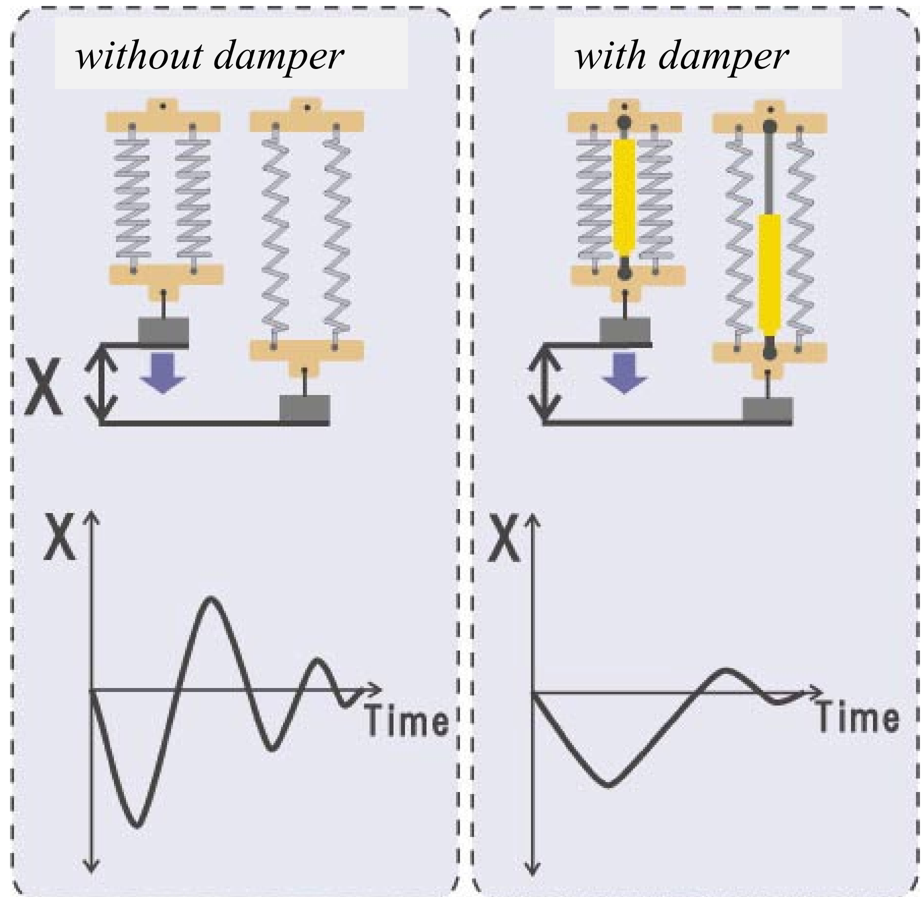
$$m\frac{dv(t)}{dt}+cv(t)+k\int v(t)dt=f(t)$$

Mechanical/Electrical systems analogy

Variable Type	Translational	R-L-C Loop Circuit
Input (forcing)	$f(t)$	$v(t)$ (voltage)
Output	$v(t)$ (velocity)	$i(t)$
Inertia	m	L
Damping	c	R
Stiffness	k	$1/C$



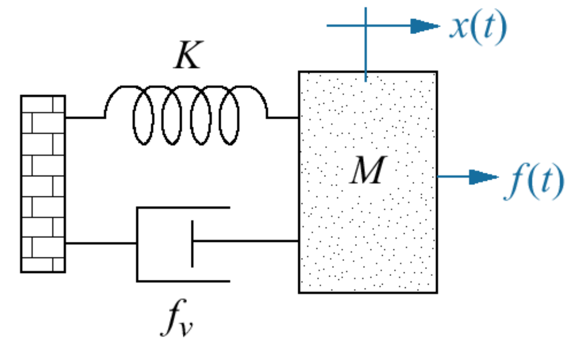

x
k
F
m
B


without damper
with damper
X
X
Time
X
Time

Example 2.16: Transfer function – One equation of motion

Find the Transfer function, X(s) / F(s).

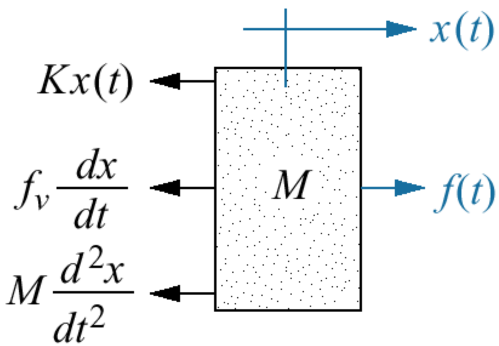
- Mass, spring, and damper system →



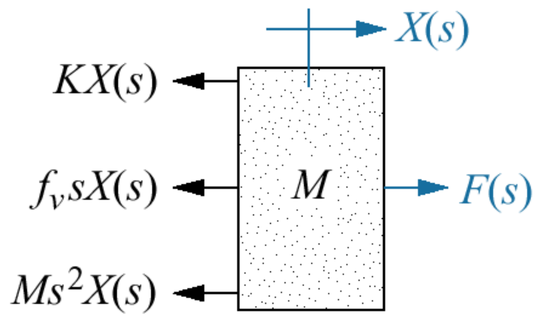
(a)

a. Free-body diagram of mass, spring, and damper system;

b. transformed free-body diagram



(a)

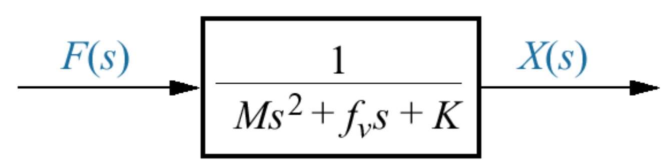


(b)

$$F(s) = (K + f_v s + Ms^2)X(s)$$

- *Spring*
- *Viscous damper*
- *Force due to acceleration point to the left*

Force-displacement
$f(t) = Kx(t)$
$f(t) = f_v \frac{dx(t)}{dt}$
$f(t) = M\frac{d^2x(t)}{dt^2}$

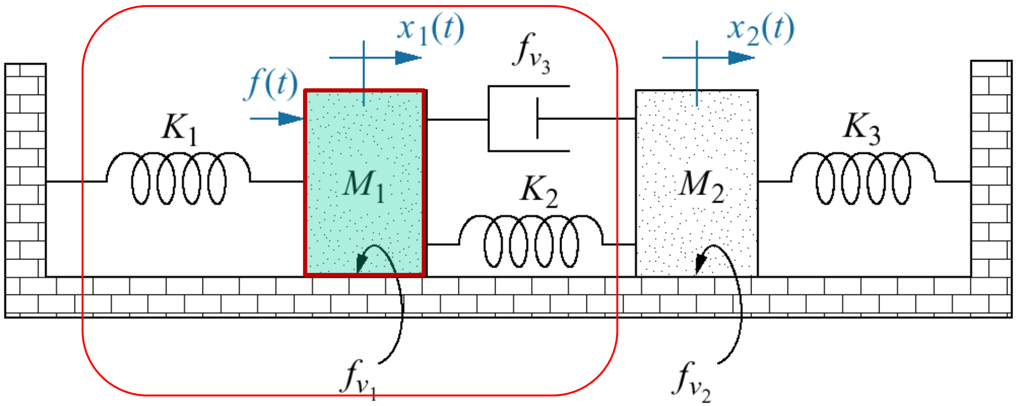


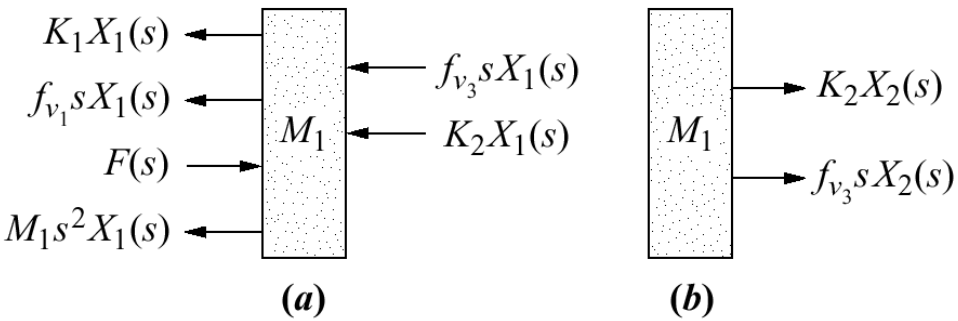
(b)

Example 2.17: Transfer function – Two degrees of freedom

Find the Transfer function, $X_2(s)$ / $F(s)$.

- Two-degrees-of-freedom translational mechanical system[8];

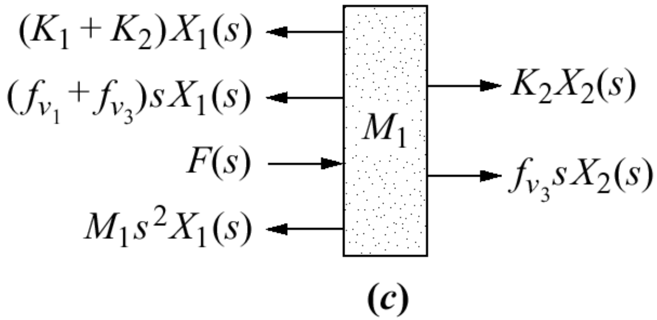




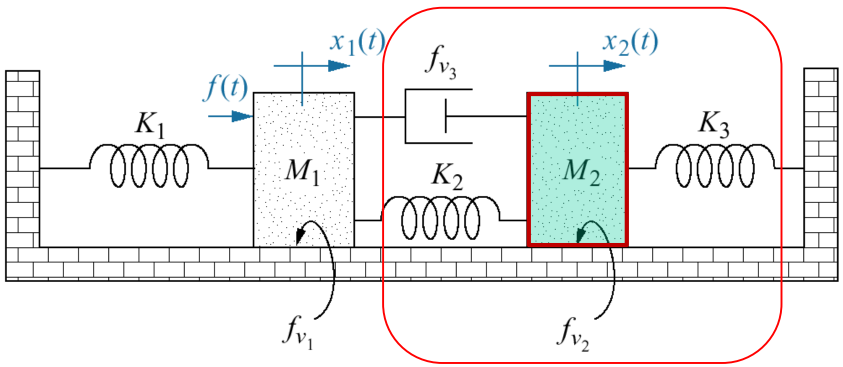
(a) forces on M_1 due only to motion of M_1

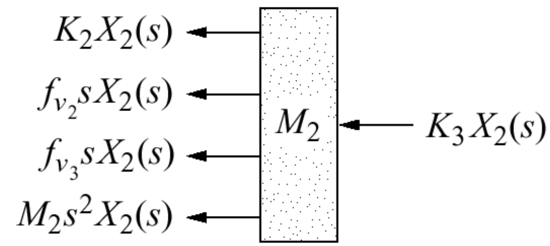
(b) forces on M_1 due only to motion of M_2

(c) all forces on M_1

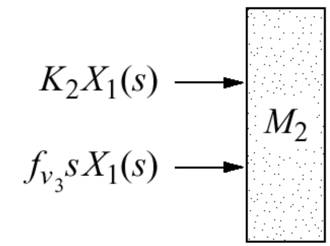


$$\boxed{\begin{aligned}&[(k_1+k_2)+(f_{v1}+f_{v3})s+M_1s^2]X_1(s)=(K_2+f_{v3}s)X_2(s)+F(s)\\&[M_1s^2+(f_{v1}+f_{v3})s+(k_1+k_2)]X_1(s)-(f_{v3}s+K_2)X_2(s)=F(s)\end{aligned}}$$





(a)

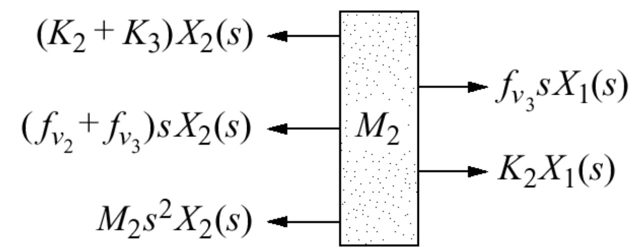


(b)

(*a*) forces on M_2 due only to motion of M_2;

(*b*) forces on M_2 due only to motion of M_1;

(*c*) all forces on M_2

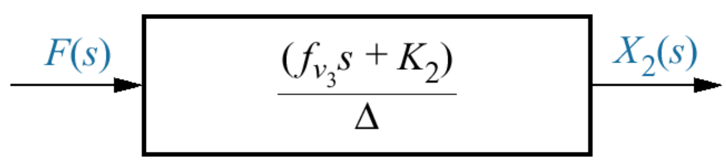


(c)

$$[M_1s^2+(f_{v1}+f_{v3})s+(k_1+k_2)]X_1(s)-(f_{v3}s+K_2)X_2(s)=F(s)$$

$$-(f_{v3}s+K_2)X_1(s)+[M_2s^2+(f_{v2}+f_{v3})s+(k_2+k_3)]X_2(s)=0$$

$$\boxed{\begin{pmatrix} \sim & \sim \\ \sim & \sim \end{pmatrix}\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix}=\begin{pmatrix} F(s) \\ 0 \end{pmatrix}}$$

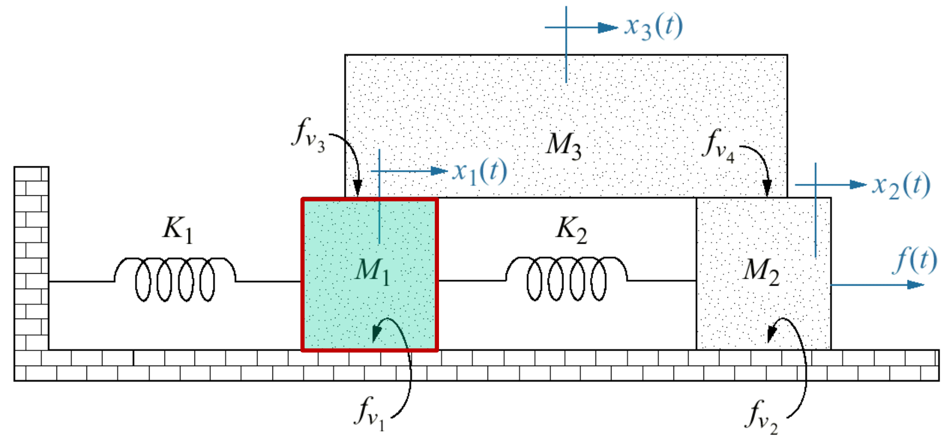


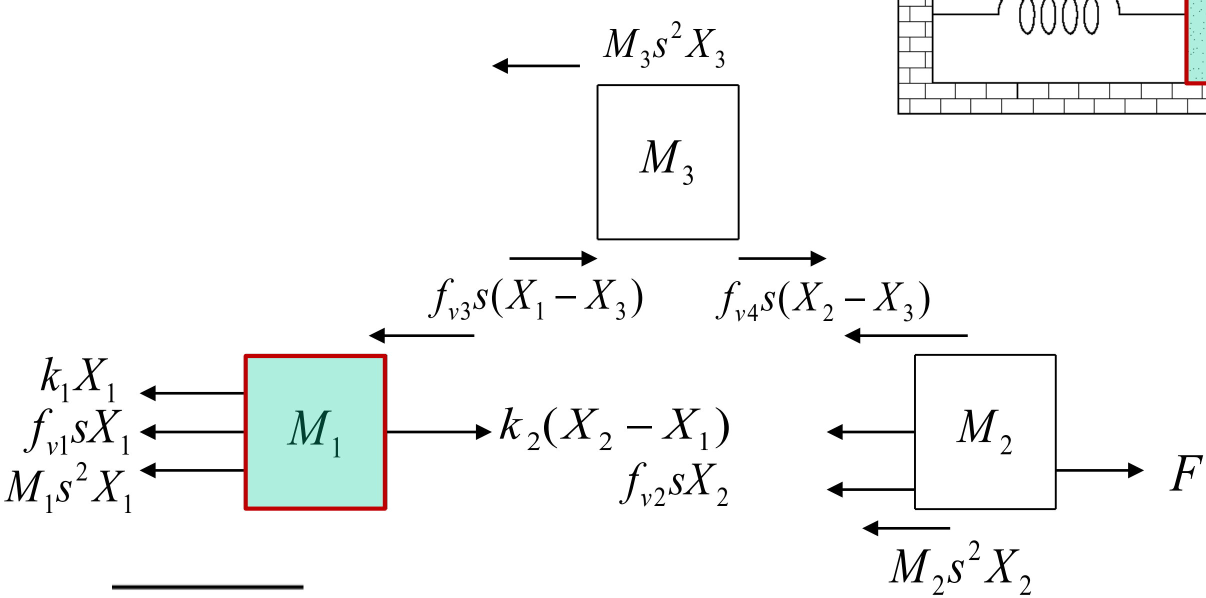
(b)

Example 2.18: Equation of motion by inspection

Find the Transfer function, $X_2(s) / F(s)$.

- Three-degrees-of-freedom translational mechanical system





$$\begin{pmatrix} \sim & \sim & \sim \\ \sim & \sim & \sim \\ \sim & \sim & \sim \end{pmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{pmatrix} = \begin{pmatrix} 0 \\ F(s) \\ 0 \end{pmatrix}$$

Force-displacement
$f(t) = Kx(t)$
$f(t) = f_v \dfrac{dx(t)}{dt}$
$f(t) = M \dfrac{d^2x(t)}{dt^2}$

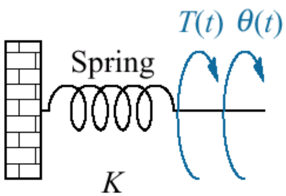
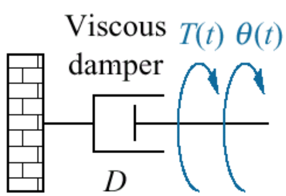
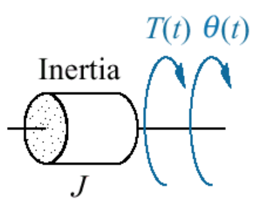
$$M_1: \quad k_1 X_1 + f_{v1} s X_1 + M_1 s^2 X_1 + f_{v3} s(X_1 - X_3) = k_2(X_2 - X_1)$$

$$M_2: \quad k_2(X_2 - X_1) + f_{v2} s X_2 + M_2 s^2 X_2 + f_{v4} s(X_2 - X_3) = F$$

$$M_3: \quad f_{v3} s(X_1 - X_3) + f_{v4} s(X_2 - X_3) = M_3 s^2 X_3$$

2.6 Rotational Mechanical System Transfer Functions

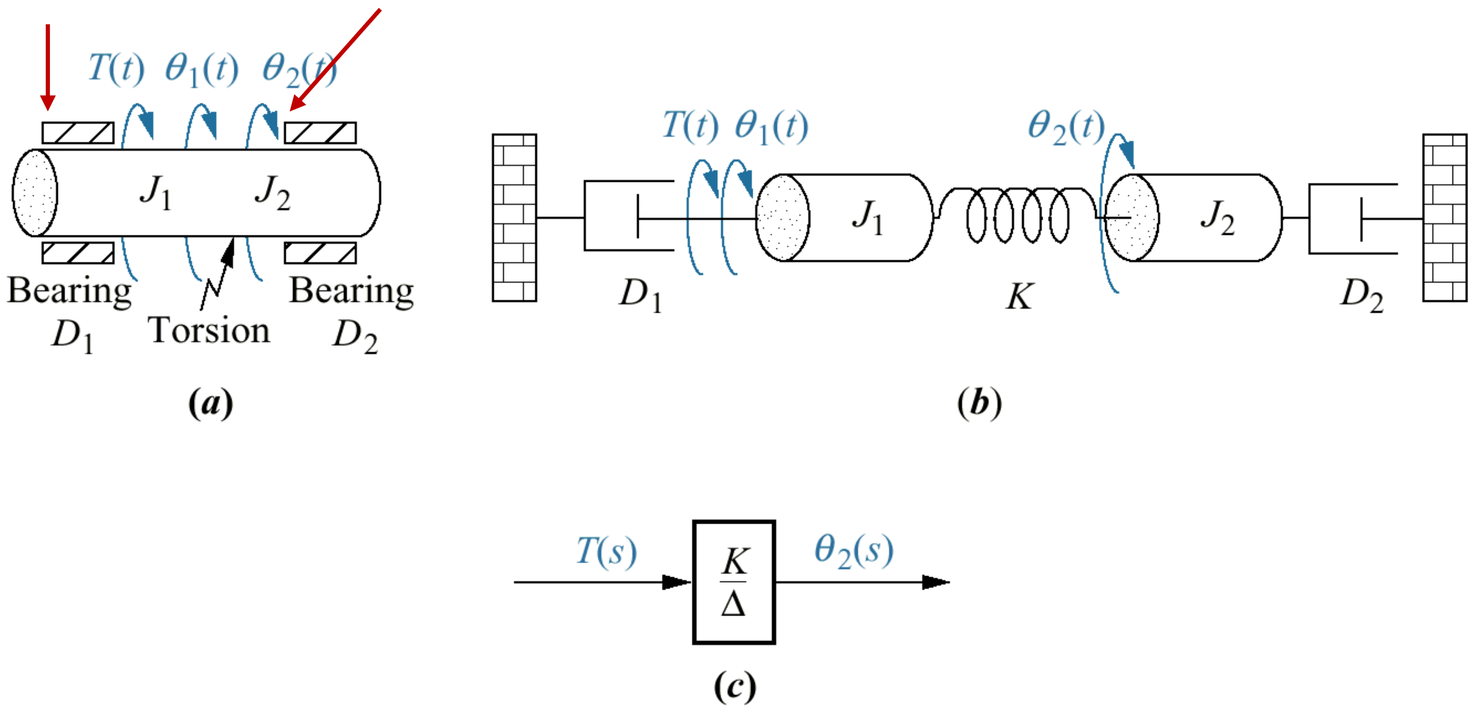
- Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$ $\boxed{T(s) = K\theta(s)}$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$ $\boxed{T(s) = Ds\theta(s)}$	$\boxed{Ds}$
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$ $\boxed{T(s) = Js^2\theta(s)}$	$\boxed{Js^2}$

Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m^2 (kilogram-meters^2 = newton-meters-seconds^2/radian).

Example 2.18: Equation of motion by inspection

Find the Transfer function, $\theta_2(s)$ / T(s).



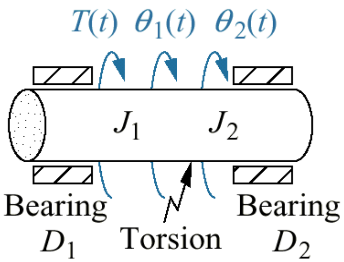
(*a*) physical system;
(*b*) schematic;
(*c*) block diagram

Torque angular displacement

Spring: $T(s) = K\theta(s)$

Viscous damper: $T(s) = Ds\theta(s)$

Inertia: $T(s) = Js^2\theta(s)$



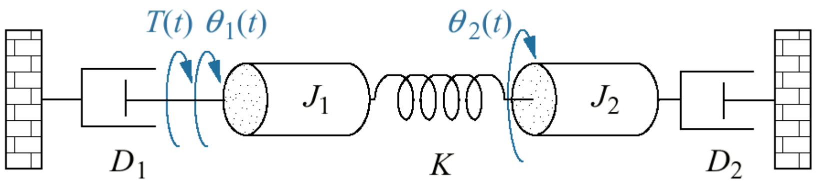
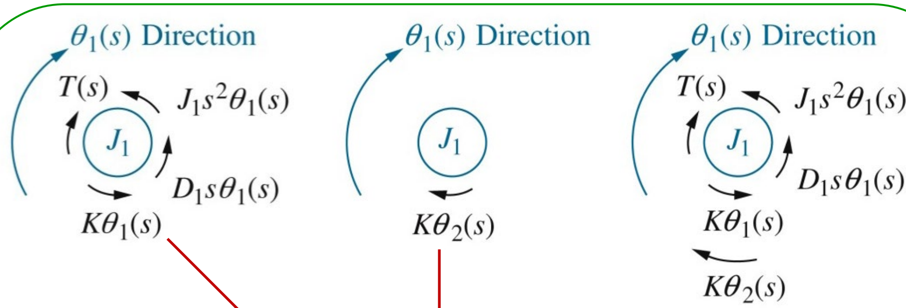


Figure 2.23 & 24



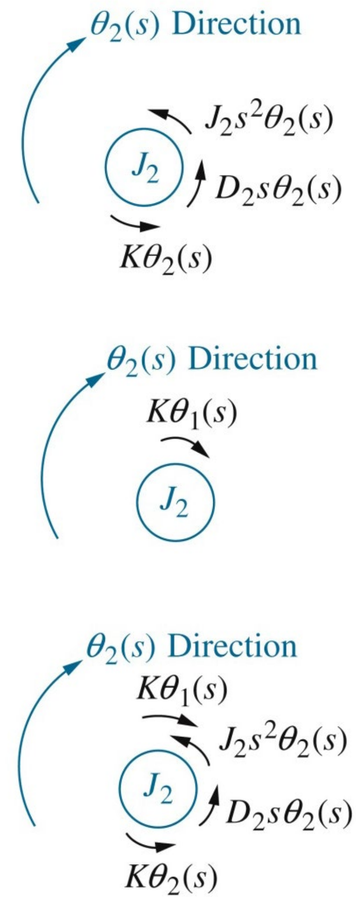
$$T$$

$$D_1 s\theta_1 \qquad K(\theta_1-\theta_2) \qquad J_1 s^2\theta_1$$

$$T = J_1 s^2\theta_1 + K(\theta_1-\theta_2) + D_1 s\theta_1$$

$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0$$



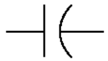
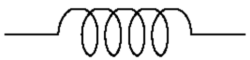
$$K(\theta_1-\theta_2)$$

$$D_2 s\theta_2 \qquad J_2 s^2\theta_2$$

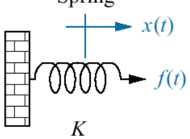
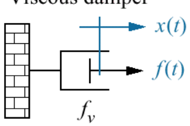
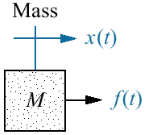
$$K(\theta_1-\theta_2) = D_2 s\theta_2 + J_2 s^2\theta_2$$

$$K\theta_1 = (K + D_2 s + J_2 s^2)\theta_2$$

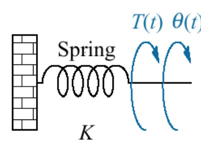
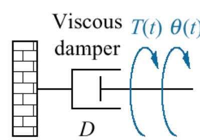
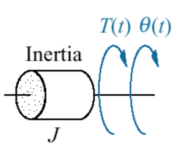
$$\boxed{\begin{pmatrix} (J_1 s^2 + D_1 s + K) & -K \\ -K & (J_2 s^2 + D_2 s + K) \end{pmatrix}\begin{pmatrix}\theta_1(s)\\ \theta_2(s)\end{pmatrix} = \begin{pmatrix} T(s) \\ 0\end{pmatrix}} \Rightarrow \frac{\theta_2(s)}{T(s)}$$

 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

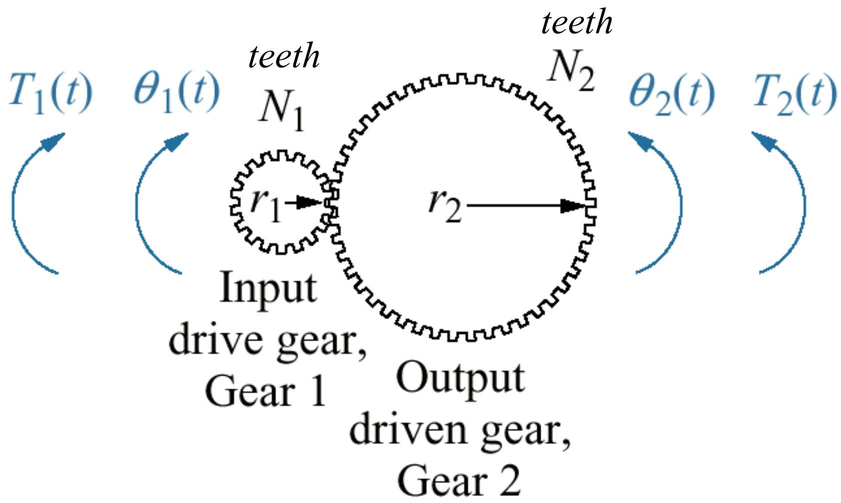
Component	**Force-velocity**	**Force-displacement**	**Impedance $Z_M(s) = F(s)/X(s)$**
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v\frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds²/meter).

Component	**Torque-angular velocity**	**Torque-angular displacement**	**Impedance $Z_M(s) = T(s)/\theta(s)$**
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m² (kilogram-meters² = newton-meters-seconds²/radian).

2.7 Transfer Functions for Systems with Gears (page 74)

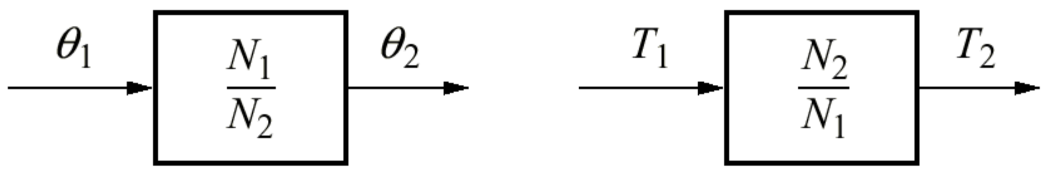


Distance traveled along each gear's circumference is the same

$$r_1\theta_1 = r_2\theta_2$$

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

Figure 2.28 Transfer functions for
(*a*) angular displacement in lossless gears
(*b*) torque in lossless gears



(*a*) **(*b*)**

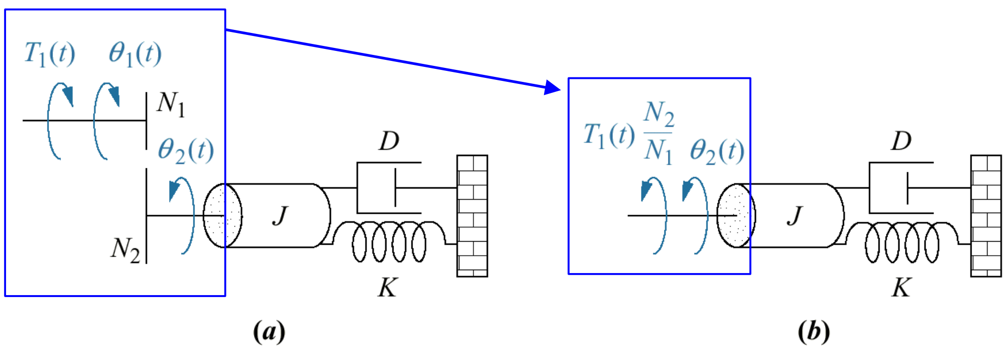
$$r_1\theta_1 = r_2\theta_2 \qquad \frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

Assume that the gears do not absorb or store energy (N: teeth)

$$T_1\theta_1 = T_2\theta_2$$

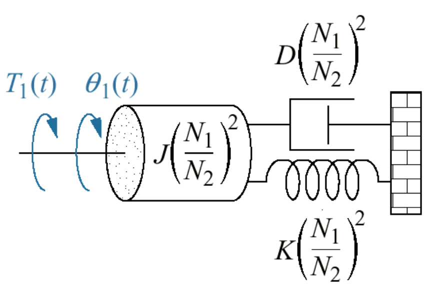
$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

$\longrightarrow$ Figure 2.28



$$(Js^2 + Ds + K)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$



(c)

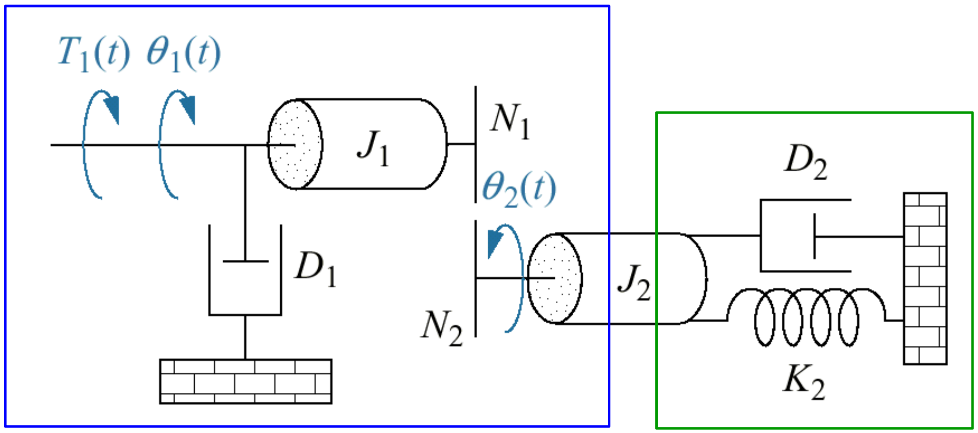
$$\left[J\left(\frac{N_1}{N_2}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s)$$

N_1: destination, N_2: Source

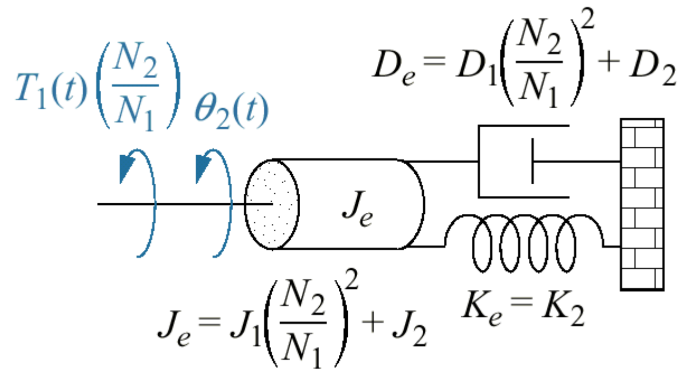
Example 2.21: Transfer function – system with lossless gears

Find the Transfer function, $\theta_2(s)$ / $T_1(s)$.

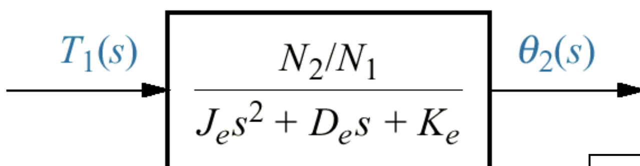
(a) rotational mechanical system with gears;
(b) system after reflection of torques and impedances to the output shaft;
(c) block diagram



(a)



(b)

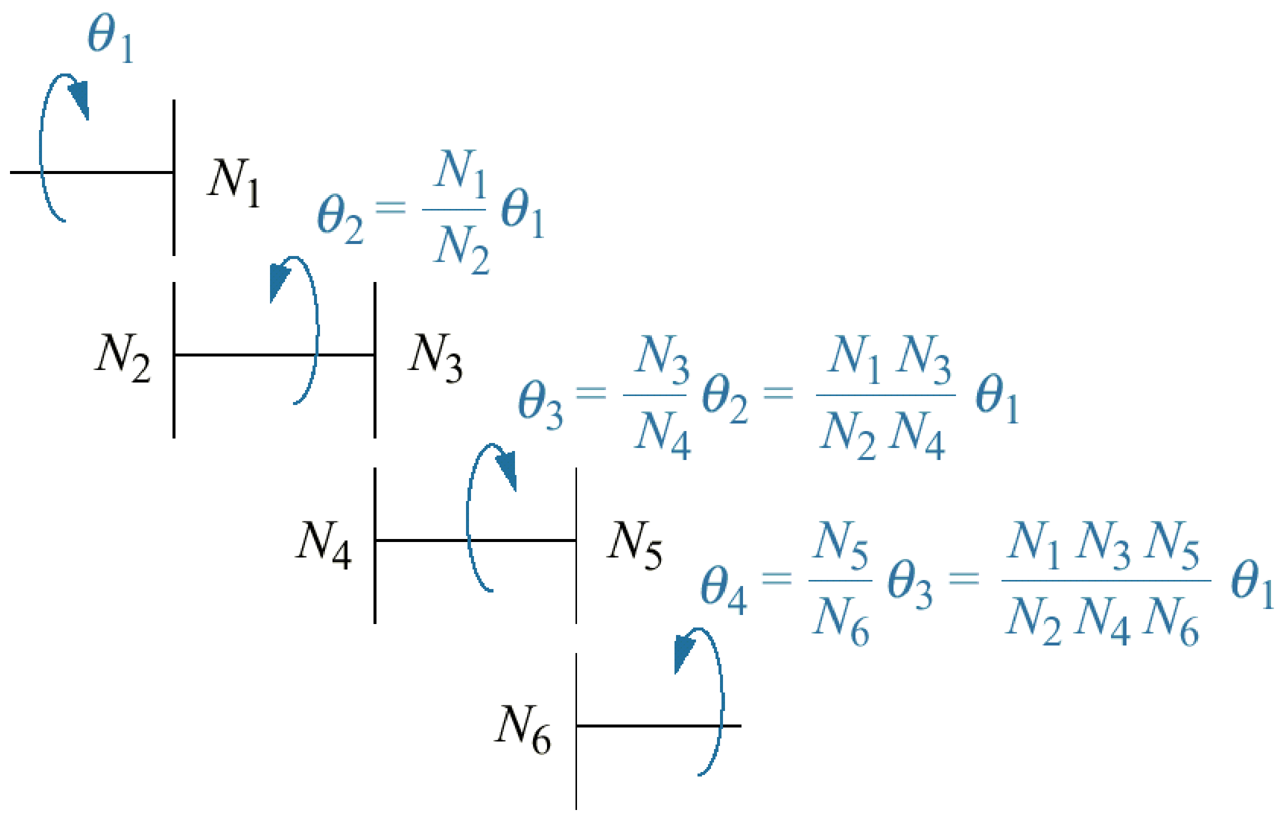


(c)

$$(J_e s^2 + D_e s + K_e)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

$$G(s) = \frac{\theta_2(s)}{T_1(s)}$$

Figure 2.31 Gear train



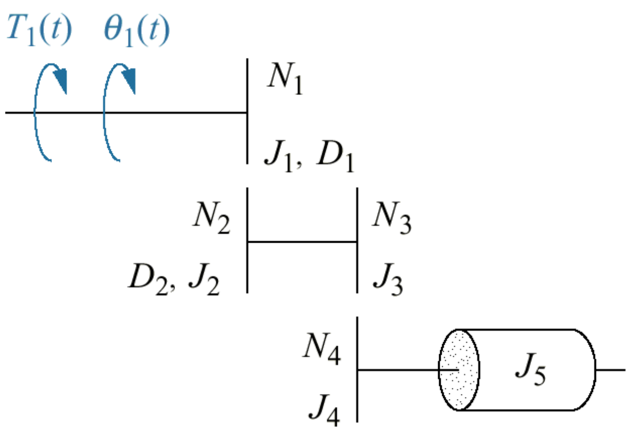
Example 2.22: Transfer function – gears with loss

Find the Transfer function, $\theta_1(s) / T_1(s)$.

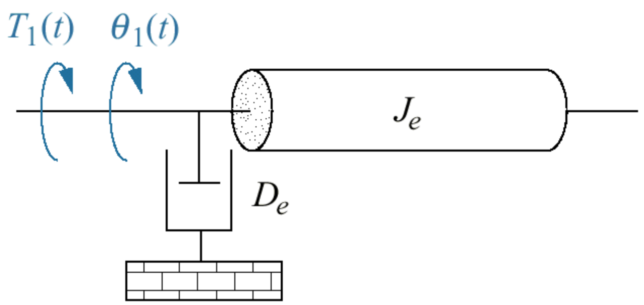
Figure 2.32
(a) System using a gear train;
(b) equivalent system at the input;
(c) block diagram

$$(J_e s^2 + D_e s)\theta_1(s) = T_1(s)$$

$$G(s) = \frac{\theta_1(s)}{T_1(s)}$$



(a)



$$J_e = J_1 + (J_2 + J_3)\left(\frac{N_1}{N_2}\right)^2 + (J_4 + J_5)\left(\frac{N_1 N_3}{N_2 N_4}\right)^2$$

$$D_e = D_1 + D_2\left(\frac{N_1}{N_2}\right)^2$$

(b)

$$T_1(s) \longrightarrow \boxed{\frac{1}{J_e s^2 + D_e s}} \longrightarrow \theta_1(s)$$

(c)

2.8 Electromechanical System Transfer Functions (page 79)

Figure 2.34
NASA flight simulator robot arm with electromechanical control system components

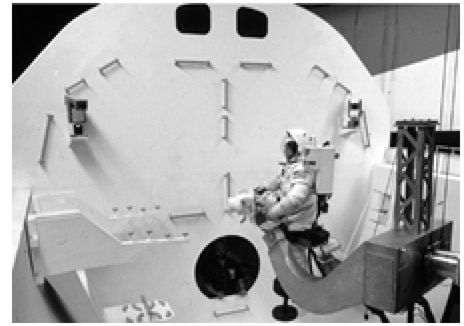


Figure 2.35 DC motor:
a. schematic;
b. block diagram
Ref: Appendix H.Derivation of a Schematic for a DC Motor.pdf

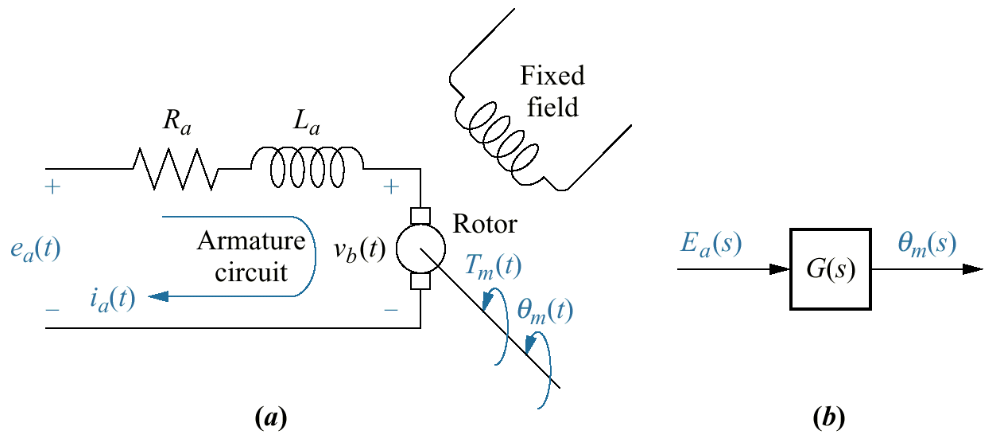
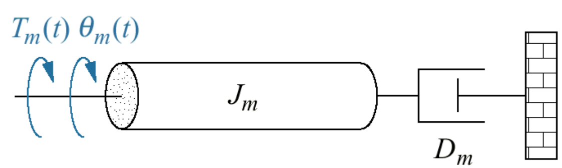


Figure 2.36 Typical equivalent mechanical loading on a motor



$$v_b(t) = K_b \frac{d}{dt}\theta_m(t) \longleftarrow$$

$$V_b(s) = K_b s\theta_m(s)$$

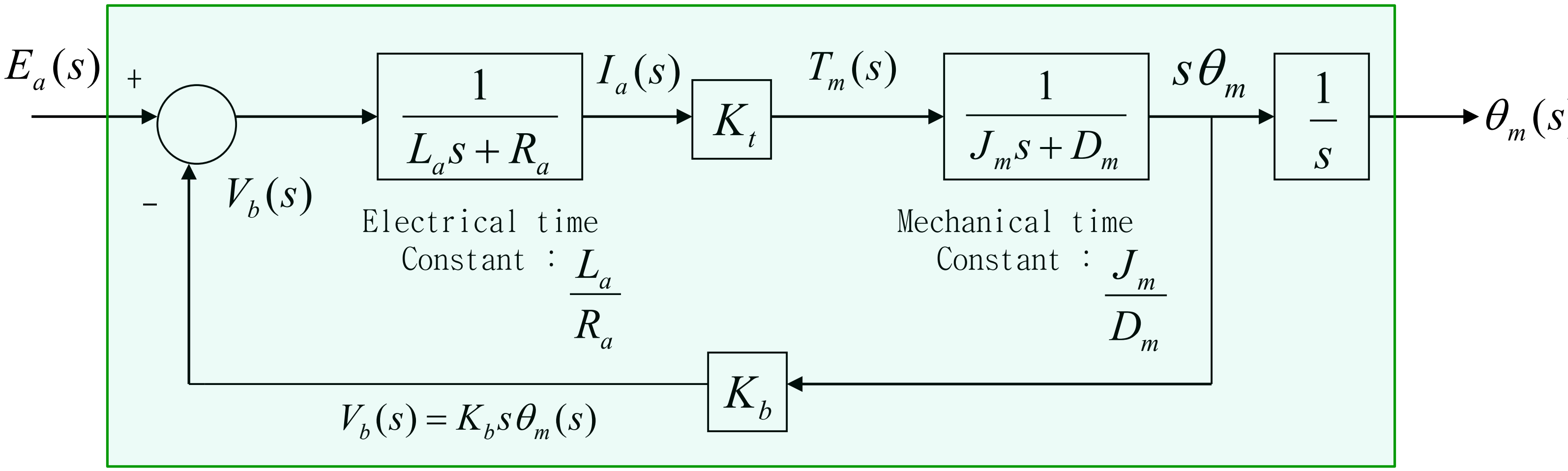
$v_b(t)$: back electromotive force (back emf)
$K_b(t)$: back emf constant

(1) $E_a(s) = R_a I_a(s) + L_a s I_a(s) + V_b(s)$ $\longrightarrow$ $E_a(s) - V_b(s) = (R_a + L_a s) I_a(s)$

(2) $T_m(s) = K_t I_a(s)$ $\longleftarrow$ Torque is proportional to the armature current

K_t: motor torque constant

(3) $T_m(s) = (J_m s^2 + D_m s)\theta_m(s)$



(1), (2) $\longrightarrow$ (4) $$\frac{(R_a + L_a s) T_m(s)}{K_t} + K_b s \theta_m(s) = E_a(s)$$

(3), (4) $\longrightarrow$ (5) $$\frac{(R_a + L_a s)(J_m s^2 + D_m s)}{K_t} + K_b s \theta_m(s) = E_a(s)$$

$$\longrightarrow \frac{\theta_m(s)}{E_a(s)} = \boxed{\frac{K}{s(s+\alpha)}}$$

2.9 Electric Circuit Analogs (page 84)

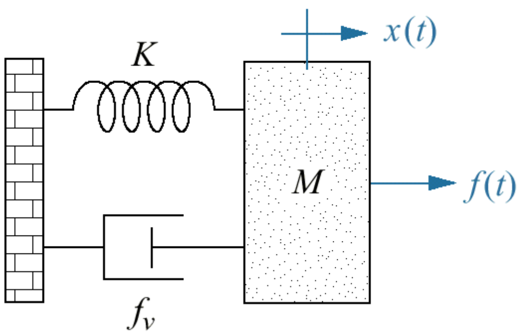
$$(Ms^2 + f_v s + K)X(s) = F(s)$$

$$\frac{Ms^2 + f_v s + K}{s} sX(s) = F(s)$$

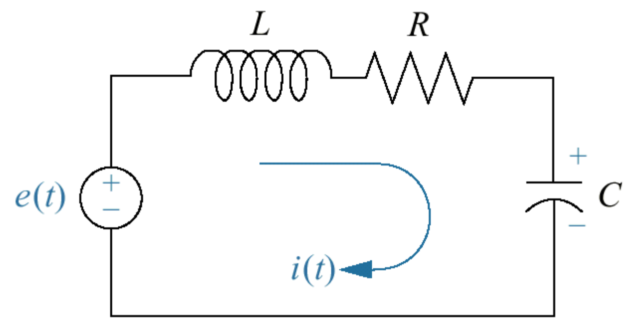
$$(Ms + f_v + \frac{K}{s})V(s) = F(s)$$

$$\boxed{(Ls + R + \frac{1}{Cs})I(s) = E(s)}$$

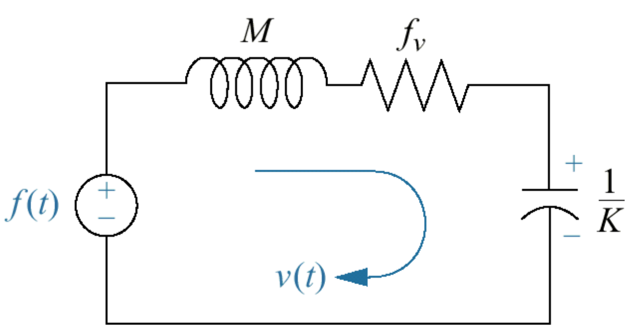
Figure 2.41 Development of series analog:
a. mechanical system;
b. desired electrical representation;
c. series analog;
d. parameters for series analog



(a)



(b)



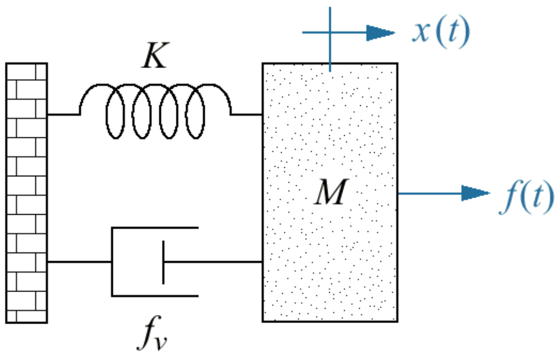
(c)

mass =	M	→	inductor =	M henries
viscous damper =	f_v	→	resistor =	f_v ohms
spring =	K	→	capacitor =	$\frac{1}{K}$ farads
applied force =	$f(t)$	→	voltage source =	$f(t)$
velocity =	$v(t)$	→	mesh current =	$v(t)$

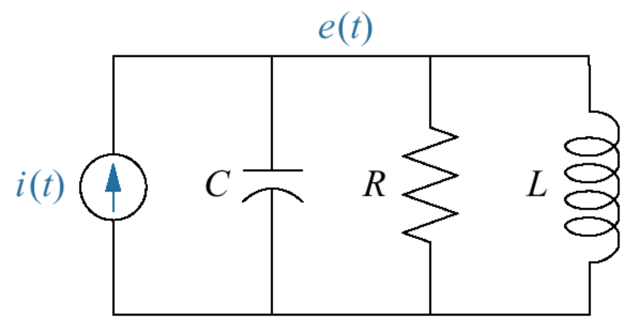
(d)

Figure 2.43 Development of parallel analog:
a. mechanical system;
b. desired electrical representation;
c. parallel analog;
d. parameters for parallel analog

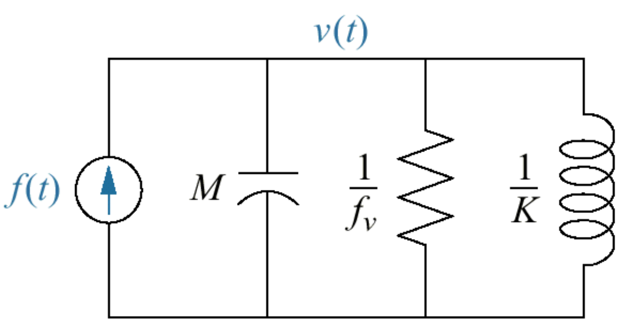
$$\boxed{(Cs+\frac{1}{R}+\frac{1}{Ls})E(s)=I(s)}$$



(*a*)



(*b*)



(*c*)

mass $= M \longrightarrow$ capacitor $= M$ farads

viscous damper $= f_v \longrightarrow$ resistor $= \frac{1}{f_v}$ ohms

spring $= K \longrightarrow$ inductor $= \frac{1}{K}$ henries

applied force $= f(t) \longrightarrow$ current source $= f(t)$

velocity $= v(t) \longrightarrow$ node voltage $= v(t)$

(*d*)

2.10 Nonlinearities (page 88)

Figure 2.45
a. Linear system;
b. nonlinear system

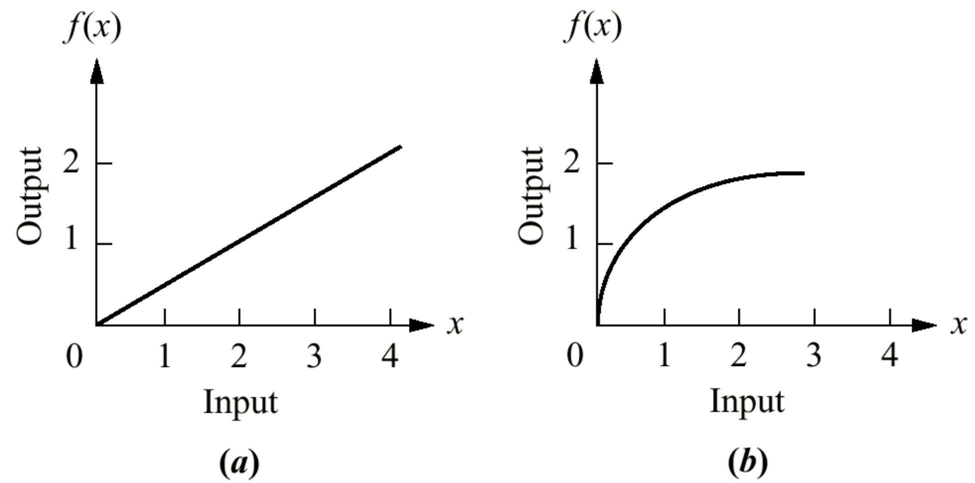
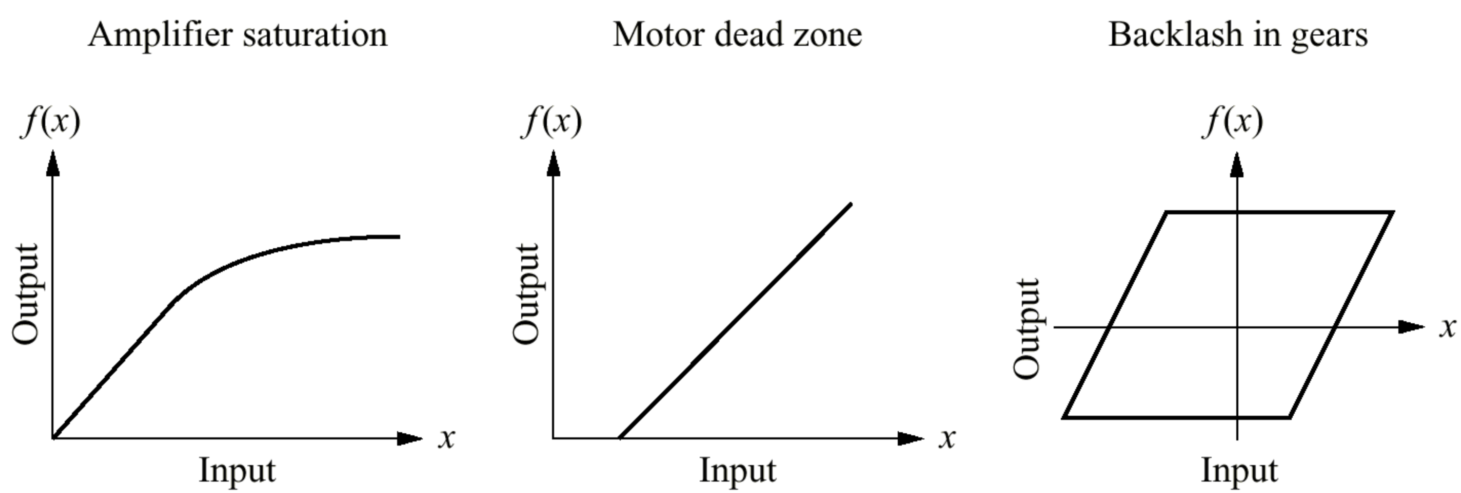


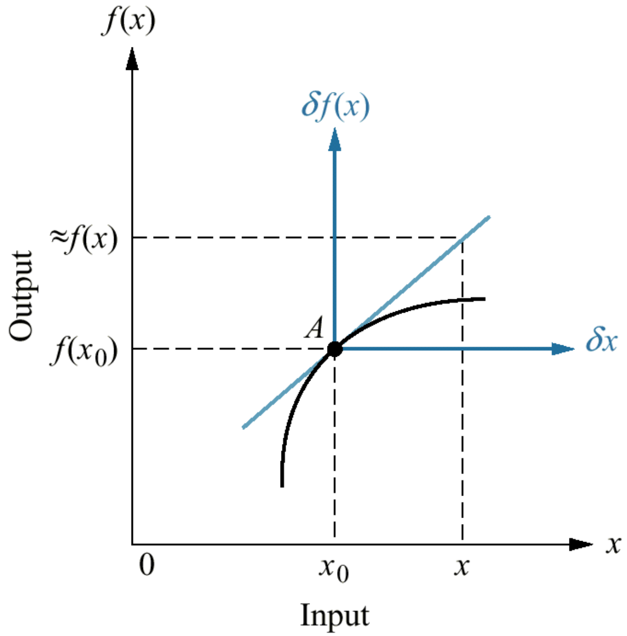
Figure 2.46
Some physical nonlinearities



2.11 Linearization (page 89)

- Linearization about a point *A*, $(x_0, f(x_0))$

If the slope of the curve at point *A* is m_a ,



$$f(x) - f(x_0) \approx \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

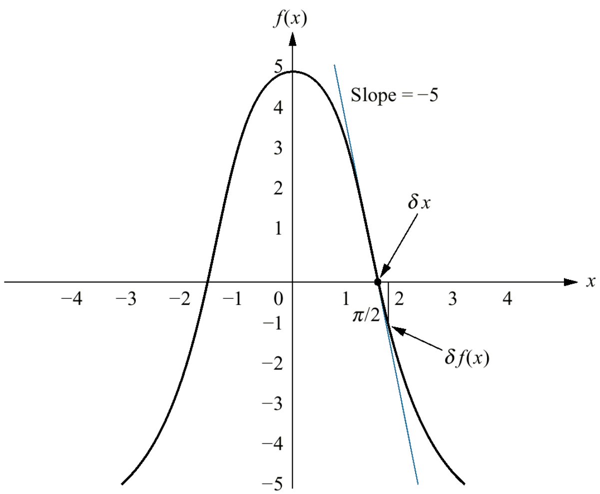
$$f(x) - f(x_0) \approx m_a (x - x_0)$$

$$\delta f(x) \approx m_a (\delta x)$$

$$f(x) \approx f(x_0) + m_a (x - x_0) \approx \boxed{f(x_0) + m_a \delta x}$$

Example 2.26: Linearizing a function

Linearize $f(x) = 5\cos x$ about $x = \pi/2$.



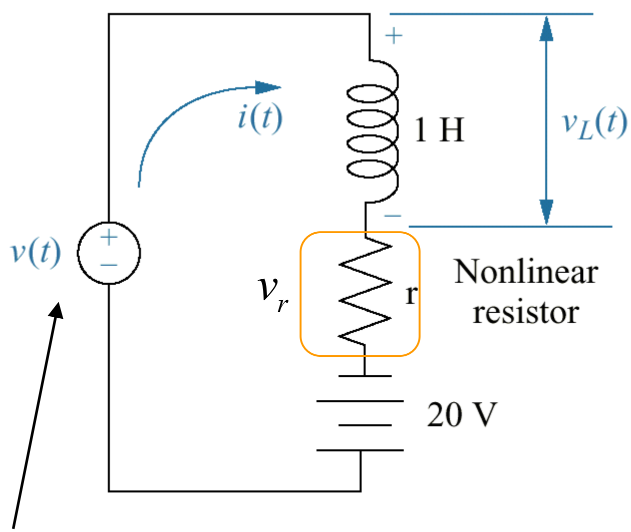
$$
\begin{aligned}
f(x) &\approx f(x_0) + m_a \delta x \\
&= 5\cos(\frac{\pi}{2}) + \left.\frac{df(x)}{dx}\right|_{x=\frac{\pi}{2}} (\delta x) \\
&= 0 - 5\sin(\frac{\pi}{2})(\delta x) \\
&= -5\delta x
\end{aligned}
$$

- Taylor series expansion: $f(x) \approx f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0} \frac{(x-x_0)}{1!} + \left.\frac{d^2 f(x)}{dx^2}\right|_{x=x_0} \frac{(x-x_0)^2}{2!} + \cdots$

$$
\begin{aligned}
&\approx 5\cos(\frac{\pi}{2}) + \left.\frac{df(x)}{dx}\right|_{x=\frac{\pi}{2}} (x - \frac{\pi}{2}) \\
&= -5\sin(\frac{\pi}{2})(x - \frac{\pi}{2}) = \boxed{-5(x - \frac{\pi}{2})}
\end{aligned}
$$

Example 2.28: Transfer function – nonlinear electrical network (page 88)

Find transfer function $V_L(s)/V(s)$.



small-signal source

- Voltage-current relationship of the nonlinear register is:

$$\frac{1}{2}i_r = e^{0.1v_r}$$

$$\ln\frac{1}{2}i_r = 0.1v_r$$

$$i_r = 2e^{0.1v_r}, \qquad v_r = 10\ln\frac{1}{2}i_r$$

$$L\frac{di}{dt} + 10\ln(\frac{1}{2}i) - 20 = v(t)$$

- Evaluate equilibrium solution: $v(t)$=0

$$0 + 10\ln(\frac{1}{2}i_0) - 20 = 0 \longrightarrow i_0 = 2\exp(2) = 14.78$$

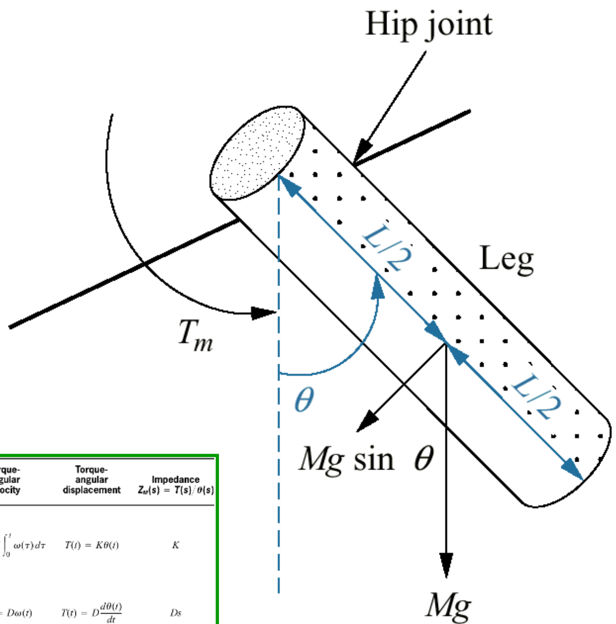
$$i = i_o + \delta i,$$

$$L\frac{d(i_0+\delta i)}{dt} + \boxed{10\ln\frac{1}{2}(i_o+\delta i)} - 20 = v(t)$$

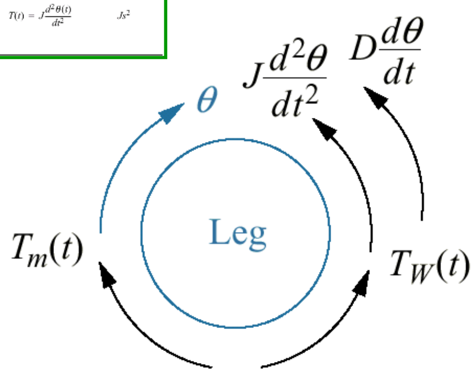
$$L\frac{d\delta i}{dt} + \boxed{10\left(\ln\frac{i_o}{2} + \frac{1}{i_o}\delta i\right)} - 20 = v(t)$$

Linearization (Do it yourself)

Case study: Find transfer function of a biological system



Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
Spring K	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
Viscous damper D	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
Inertia J	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2



Free-body diagram of leg model

- The system is human leg, which pivots from the hip joint.
- Cylinder model of a human leg
- Evaluate the nonlinear torque due to the weight

$$J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + Mg\frac{L}{2}\sin\theta = T_m(t)$$

- Need a linearization about equilibrium point, θ=0.

$$f(x) - f(x_0) \approx \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0)$$

$$\sin\theta - \sin 0 = (\cos 0)\delta\theta, \quad \sin\theta = \delta\theta$$

$$\vdots$$

$$\frac{\delta\theta(s)}{T_m(s)} = \frac{1}{Js^2 + Ds + Mg\frac{L}{2}} = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{MgL}{2J}}$$