Problem 1.

$$f(x) = \begin{cases} e^{-(x-\theta)} & x > \theta \\ 0 & \text{oth} \end{cases}$$

If $x < \theta$, $F_{X_1}(x) = 0$

$x > \theta$, $F_{X_1}(x) = \int_\theta^x e^{-(t-\theta)} dt = e^\theta \int_\theta^x e^{-t} dt$

$= e^\theta (e^{-\theta} - e^{-x})$

$= 1 - e^{\theta - x}$

$F_{Y_n}(x) = P(Y_n \le x) = P(\min(X_1, \cdots X_n) \le x)$

$= 1 - P(\min(X_1, \cdots, X_n) > x) = 1 - P(X_1 > x) \cdot P(X_2 > x) \cdots P(X_n > x)$

$= 1 - (1 - F_{X_1}(x))^n = 1 - (1 - e^{\theta - x})^n \quad (x > \theta)$

using Chebishev's inequality,

$P(|Y_n - \theta| \le \varepsilon) \overset{Y_n > \theta}{=\!=} P(Y_n - \theta \le \varepsilon) = P(Y_n \le \theta + \varepsilon)$

$= 1 - e^{\theta - \varepsilon - \theta} = 1 - e^{-\varepsilon} \to 1$ for $n \to \infty$

$\therefore Y_n \xrightarrow{P} \theta$, $n \to \infty$

Problem 2. $Y_n = \max\{X_1, \cdots X_n\}$, $Z_n = n(1-F(Y_n))$

$$F_{Y_n}(x) = P(Y_n < x) = P(\max\{X_1, \cdots, X_n\} < x)$$

$$= P(X_1 < x, \cdots, X_n < x) = P(X_1 < x) \cdots P(X_n < x)$$

$$= F^n(x)$$

$$Z_n(x) = P(Z_n < x) = P(n(1-F(Y_n)) < x)$$

$$= P(F_{Y_n} \geq 1 - \frac{x}{n}) = 1 - P(F_{Y_n} < 1 - \frac{x}{n})$$

$$= 1 - P(Y_n < F_{Y_n}^{-1}(1 - \frac{x}{n}))$$

$$= 1 - F_{Y_n}(F_{Y_n}^{-1}(1 - \frac{x}{n}))^n$$

$$= 1 - (1 - \frac{x}{n})^n$$

$$\lim_{n \to \infty} F_{Z_n}(x) = \lim_{n \to \infty} (1 - (1 - \frac{x}{n})^n) = 1 - e^{-x} \quad x > 0$$

$$\therefore F_Z(x) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

problem 3 $\bar{X} \sim \text{poisson}(\lambda = 1)$, $Y_n = \sqrt{n}(\bar{X}_n - 1)$

(a) $M_{Y_n}(t) = E(e^{tY_n}) = E(e^{t\sqrt{n}(\bar{X}_n - 1)}) = E(e^{-t\sqrt{n}} e^{t\sqrt{n}\bar{X}_n})$

$= e^{-t\sqrt{n}} E(e^{t\sqrt{n}\bar{X}_n}) = e^{-t\sqrt{n}} M_{\bar{X}_n}(t\sqrt{n})$

$M_{\bar{X}_n}(t) = E(e^{t\bar{X}_n}) = E(e^{\frac{t}{n}(x_1 + \cdots + x_n)}) = E(e^{\frac{t}{n}x_1}) \cdots E(e^{\frac{t}{n}x_n})$

$= M_{x_1}(\frac{t}{n}) \cdot \cdots \cdot M_{x_n}(\frac{t}{n})$

$= (e^{e^{\frac{t}{n}} - 1})^n = e^{n(e^{\frac{t}{n}} - 1)}$

$\therefore e^{-t\sqrt{n}} M_{\bar{X}_n}(t\sqrt{n}) = e^{-t\sqrt{n}} e^{n(e^{\frac{t\sqrt{n}}{n}} - 1)}$

(b) $\lim_{n\to\infty} M_{Y_n}(t) = \lim_{n\to\infty} e^{-t\sqrt{n} + n(e^{\frac{t\sqrt{n}}{n}} - 1)}$

$= \lim_{n\to\infty} e^{-t\sqrt{n} + n(1 + \frac{t}{\sqrt{n}} + \frac{t^2}{2!n} + \frac{t^3}{3!\sqrt{n}} + \cdots - 1)}$

$= \lim_{n\to\infty} e^{\frac{t^2}{2!} + \frac{t^3}{3!\sqrt{n}} + \cdots} = e^{\frac{t^2}{2}} = M_Y(t)$ for standard normal

$N(0,1)$

(c) since $\sqrt{n}(\bar{X}_n - 1) \xrightarrow{d} N(0,1)$.

let $g(t) = \sqrt{t}$, $\theta = 1$, $g(1) = 1$.

$g'(t) = \frac{1}{2\sqrt{t}}$, $g'(1) = \frac{1}{2}$

$\sqrt{n}(g(X_n) - g(\theta)) \xrightarrow{d} N(0, \sigma^2 g'(\theta)^2)$

using Δ-method.

$\sqrt{n}(\sqrt{\bar{X}_n} - 1) \xrightarrow{d} N(0, (\frac{1}{2})^2)$

$\sqrt{n}(\sqrt{\bar{X}_n} - 1) \xrightarrow{d} N(0, \frac{1}{4})$

$\therefore N(0, \frac{1}{4})$

Problem 4.

(a) $E(\overline{x}) = \mu$. $Var(\overline{x}) = \frac{\mu}{n}$. Poission Distribution. mean $= \lambda = \mu$.

" Variance $= \lambda = \sigma^2$

$\therefore \sim N(\mu, \frac{\mu}{n})$

(b)

$g(\mu) = \sqrt{\mu}$

$E(\sqrt{n}(\overline{x}-\mu)) = \sqrt{n}(E(\overline{x})-\mu) = \sqrt{n}(\mu-\mu) = 0$

$Var(\sqrt{n}(\overline{x}-\mu)) = n\,var(\overline{x}-\mu) = n \cdot \frac{\mu}{n} = \mu$

$\sqrt{n}(\overline{x}-\mu) \simeq N(0, \mu)$

$g(t) = \sqrt{t}$ $\quad g'(t) = \frac{1}{2\sqrt{t}}$ $\quad g'(\mu) = \frac{1}{2\sqrt{\mu}}$

$\sqrt{n}(\sqrt{\overline{x}} - \sqrt{\mu}) \sim N(0, \mu(\frac{1}{2\sqrt{\mu}})^2) = N(\sqrt{\mu}, \frac{1}{4n})$

$\therefore$ Variance of $\sqrt{x}$ does not depends on μ.

Problem 5.

If $X_n \xrightarrow{D} N_k(\mu, \Sigma)$, mgf of X_n $\quad M_{a'X_n}(t) = E(e^{ta'X_n})$

$$= E(e^{(ta')X_n})$$

$$= M_{X_n}(ta')$$

$$= e^{ta'\mu + \frac{1}{2} ta'\Sigma ta}$$

$$= e^{ta'\mu + \frac{1}{2}t^2(a'\Sigma a)}$$

$$\rightarrow N(a'\mu, a'\Sigma a)$$

If $a'X_n \xrightarrow{D} N_1(a'\mu, a'\Sigma a)$ $\quad M_{X_n}(t) = M_{a'^{-1}a'X_n}(t)$

$$= E(e^{ta'^{-1}a'X_n})$$

$$= M_{a'X_n}(t(a')^{-1})$$

$$= e^{t(a')^{-1}(a'\mu) + \frac{1}{2}t^2(a')^{-1}a'\Sigma a a^{-1}}$$

$$= e^{t\mu + \frac{1}{2}t^2\Sigma}$$

$\therefore\ X_n \xrightarrow{D} N_k(\mu, \Sigma) \iff a'X_n \xrightarrow{D} N_1(a'\mu, a'\Sigma a),\ \forall a \in R^k$