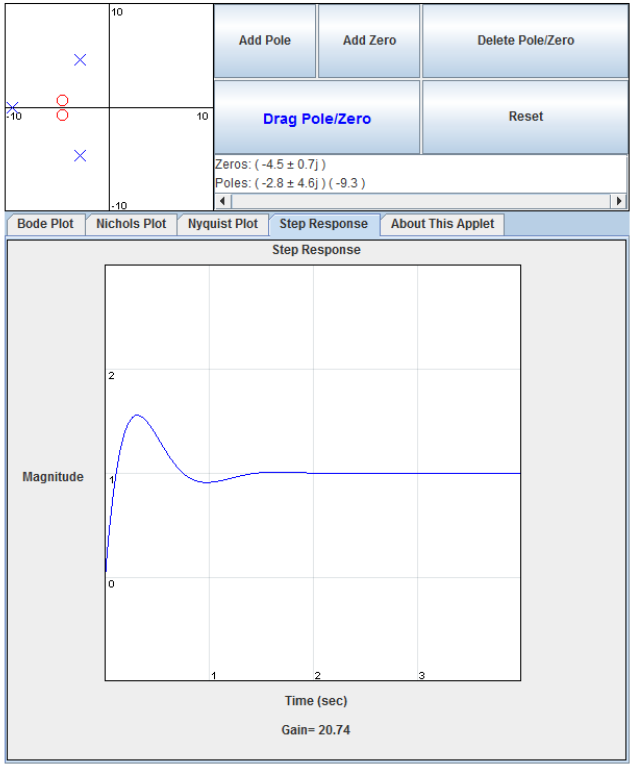
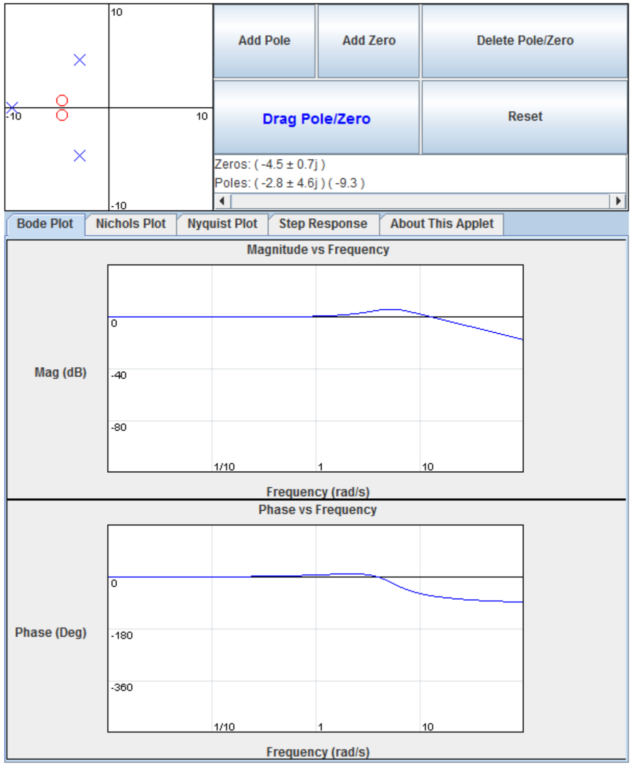
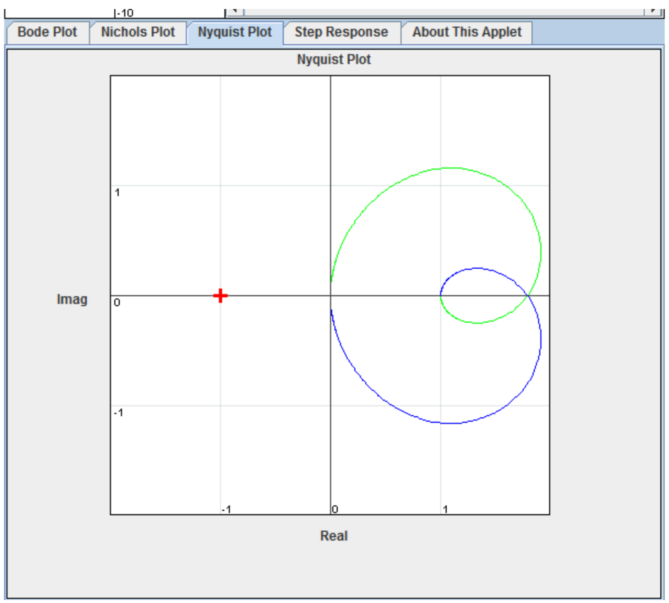
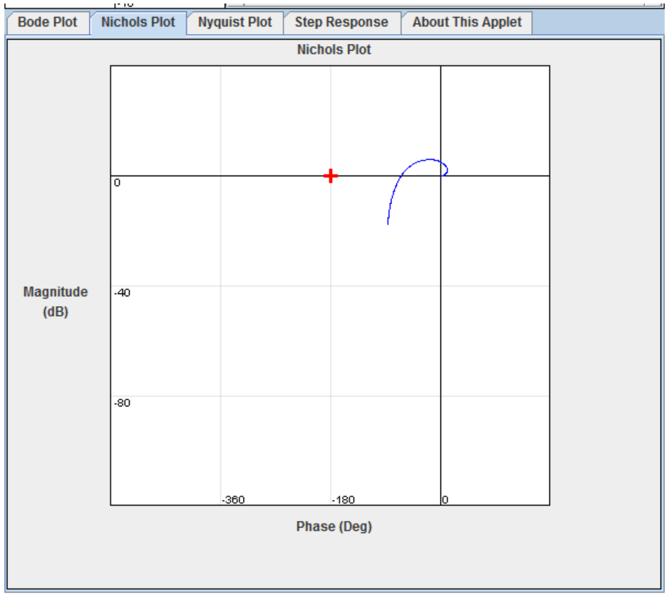
CH.10 Frequency Response Techniques

http://web.mit.edu/6.302/www/pz/





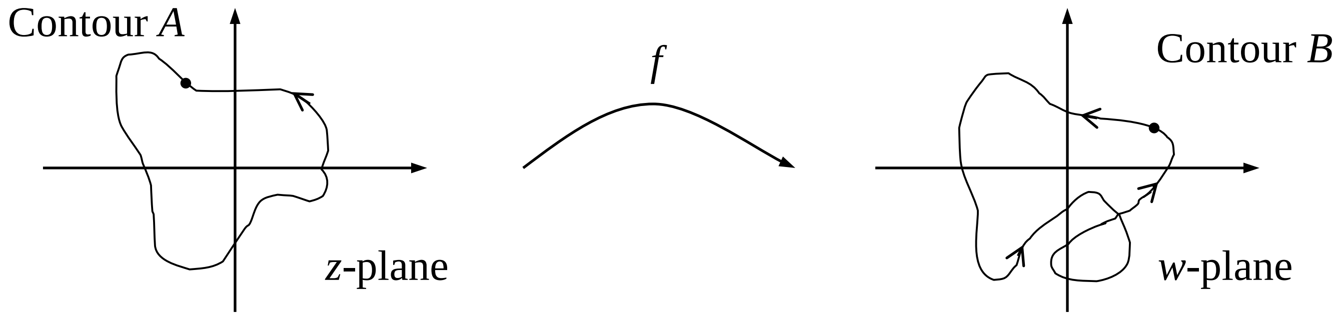




Nyquist Stability Test

Frequency Domain Analysis

- Given function f from a complex plane z to a complex plane w.

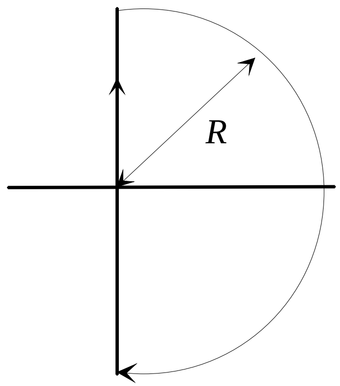


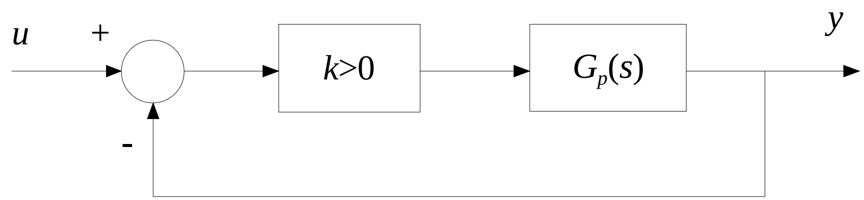
- $\Gamma \Rightarrow$ closed contour, counterclockwise

Theorem:

The number of time $f(\Gamma)$ encircles "0" (in w-plane) counterclockwise = number of zeros of inside Γ – number of poles of f inside Γ.

- Now, consider stability of the following configuration:





Γ: clockwise, $R=\infty$,

$$Y(s)=\frac{k\,G_p(s)}{1+k\,G_p(s)}U(s)$$

$$G(s)=\frac{N}{D}$$

$$1+G(s)=1+\frac{N}{D}=\frac{D+N}{D}$$

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{D}{D+N}$$

pole of $1+G(s)=$ pole of $G(s)$

zero of $1+G(s)=$ pole of $T(s)$

- Number of times $1+kG_p(s)|_\Gamma$ encircles 0 counterclockwise

  = number of poles of $1+kG_p(s)$ inside Γ – number of zeros of $1+kG_p(s)$ inside Γ

  = number of RHP poles of $1+kG_p(s)$ – number of RHP zeros of $1+kG_p(s)$.

$$\boxed{N=P-Z}$$

$$G(s)=\frac{N}{D}$$

$$1+G(s)=1+\frac{N}{D}=\frac{D+N}{D}$$

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{D}{D+N}$$

pole of $1+G(s)$ = pole of $G(s)$

zero of $1+G(s)$ = pole of $T(s)$

- Stability $\Leftrightarrow$ $1+kG_p(s)$ has no zeros in RHP $\leftrightarrow$
  \# of times $1+kG_p(s)|_\Gamma$ encircles 0 counterclockwise
  = # of RHP poles of $1+kG_p(s)$
  = # of RHP poles of $G_p(s)$, defined as N_p.
- But $1+kG_p(s)|_\Gamma$ encircles 0 if and only if $G_p(\Gamma)$ encircles $-1/k$.

$\Rightarrow$ Nyquist stability test:

The closed-loop system is stable if and only if $G_p(\Gamma)$ encircles $-1/k$ counterclockwise N_p times, where N_p = # of RHP poles of $G_p(s)$.

Example: ① $G_P(s)=\dfrac{s-1}{s+1}$

$$s=j\omega: \quad G_P(j\omega)=\frac{j\omega-1}{j\omega+1}=\frac{(-1+j\omega)(1-j\omega)}{(1+j\omega)(1-j\omega)}=\frac{(\omega^2-1)+2j\omega}{1+\omega^2}$$

$\omega=0, \quad G_p=-1$

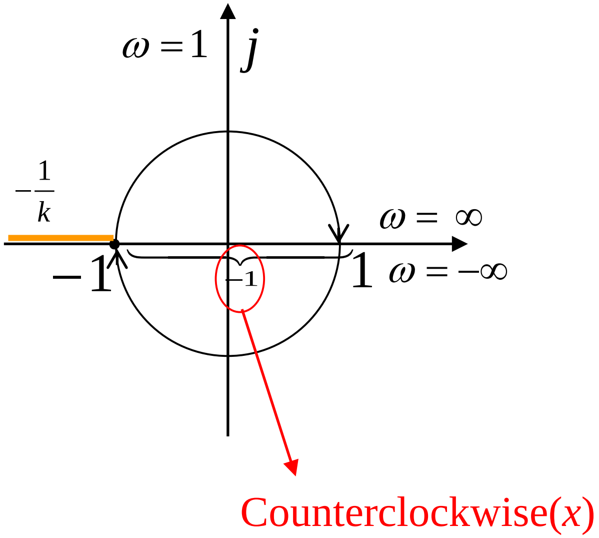
$\omega=\infty, \quad G_P=\ 1$

Cross Imag-axis : $\omega^2=1\rightarrow\omega=1$

$$G_P(j\cdot 1)=\frac{2j}{2}=j$$

Cross Real-axis: $\omega=0$

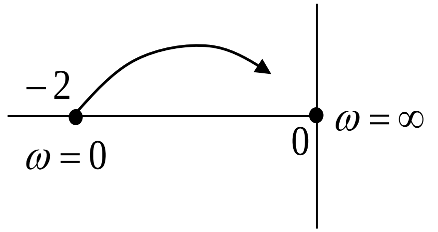
$$G_P(0)=-1$$



$\Rightarrow$ stability: $-\dfrac{1}{k}<-1, \qquad -k>-1 \quad \Rightarrow \quad k<1$

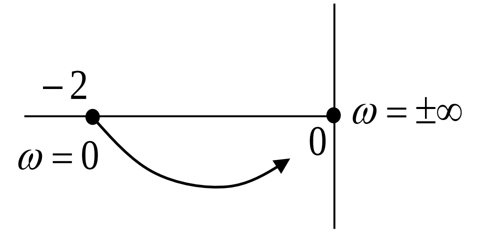
☒

Example: ② $G_p(s) = \dfrac{s+2}{(s-1)(s+1)} = \dfrac{s+2}{s^2-1}$



$$s = j\omega, \quad G_p = \frac{2 + j\omega}{-1-\omega^2}$$

OR



$$\omega = 0 \rightarrow G_p = -2$$

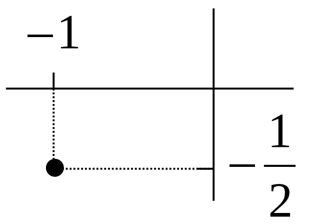
$$\omega = \infty \rightarrow G_p = 0$$

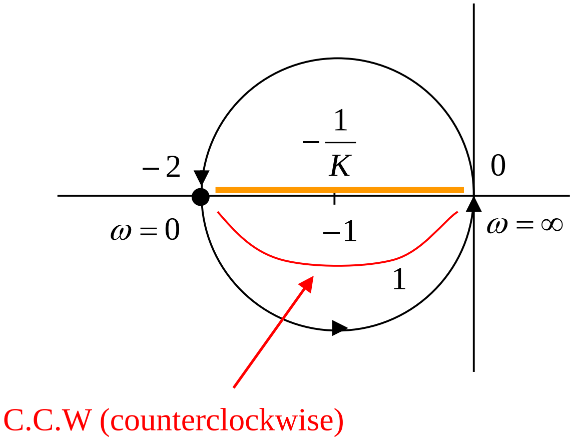
$\text{Cross Imag} - \text{axis} \rightarrow 2 \neq 0 \rightarrow \text{Never}$

$\text{Cross Real} - \text{axis} \rightarrow \omega = 0$

$$\boxed{\text{Let } \omega\text{=1, At what quadrant is } G_p(j\omega) = ?}$$

$G_p(1 \cdot j) = \dfrac{2 + j(1)}{-2}$ ⇨ III rd quadrant





$$\Rightarrow \quad -2<-\frac{1}{K}$$

$$\Rightarrow \quad K>\frac{1}{2}$$

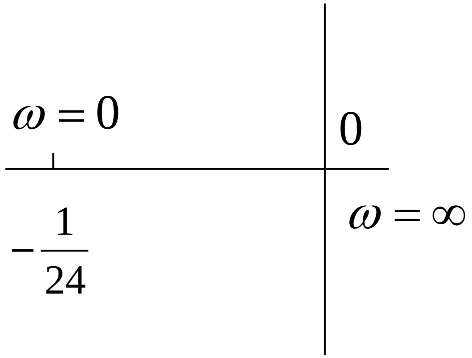
$\otimes$

Example: ③ $$G_p(s)=\frac{1}{(s+6)(s+4)(s-1)}=\frac{1}{s^3+9s^2+14s-24}$$

$$s=j\omega,\quad G_p(s)=\frac{1}{-9\omega^2-24+j(14\omega-\omega^3)}$$

$$\omega=0\rightarrow G_p=-\frac{1}{24}$$

$$\omega=\infty\rightarrow G_p=0$$

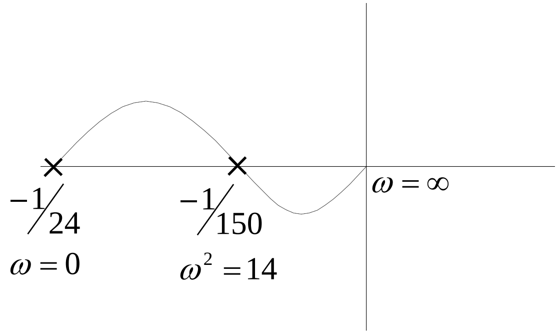


$$\frac{(-9\omega^2 - 24) - j(14\omega - \omega^3)}{\Delta}$$

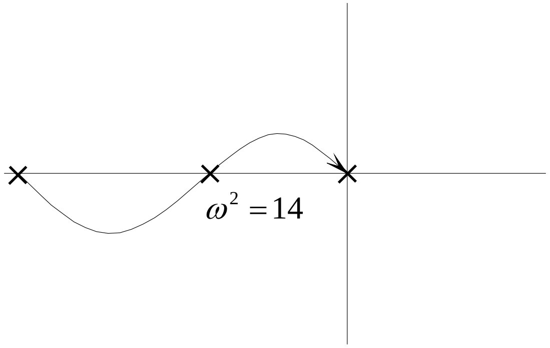
Cross Imag axis: $\quad -9\omega^2 - 24 = 0 \rightarrow \text{never}$

Cross Real axis: $\quad 14\omega - \omega^3 = 0 \rightarrow \omega = 0, \omega^2 = 14$

$$\rightarrow G_P = \frac{1}{-9(14) - 24} = \frac{-1}{150}$$



or



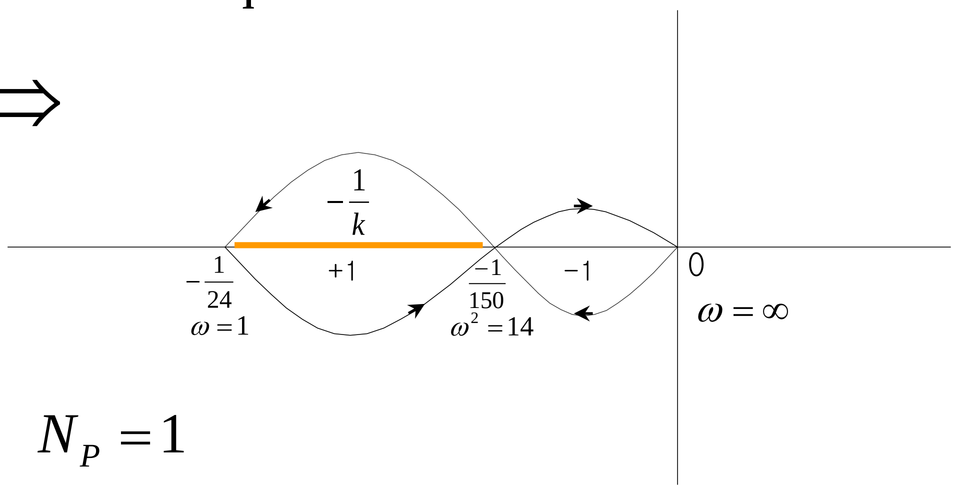
?

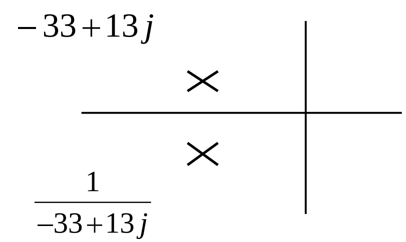
Let, $\omega = 1, \ (< \sqrt{14})$ What quadrant is $G_P(j)$ in ?

$$G_P(j\omega) = \frac{1}{-9-24+j(14-1)} = \frac{1}{-33+13j} = \frac{-33-13j}{\Delta}$$

$\Rightarrow$ 3rd quadrant or

$\Rightarrow$





$N_P = 1$

Stability : $-\frac{1}{24} < -\frac{1}{k} < -\frac{1}{150}$

$$24 < k < 150$$

Example: ④ $G_p(s) = \dfrac{s+2}{(s-1)(s+4)(s^2+4s+5)}$

N_P=# of RHP poles of $G_P(s)$

$$G_P(s) = \frac{s+2}{s^4+7s^3+13s^2-s-20} \qquad\qquad s = j\omega$$

$$\Rightarrow G_P(j\omega) = \frac{2+j\omega \qquad\qquad\qquad \times(2-j\omega)}{(\omega^4-7j\omega^3-13\omega^2-j\omega-20)\times(2-j\omega)}$$

$$= \frac{4+\omega^2}{(-5\omega^4-27\omega^2-40)+j(-\omega^5-\omega^3+18\omega)}$$

- $\omega = 0, \quad G_P(j\cdot 0) = -\dfrac{1}{10}$
- $\omega = \infty, \quad G_P(j\cdot\infty) = \quad 0$

- Cross Real axis ; $-\omega^5-\omega^3+18\omega=0, \quad \omega(\omega^4+\omega^2-18)=0$

$$\omega^2=\frac{-1\pm\sqrt{1+4\cdot 18}}{2}=\frac{-1\pm\sqrt{73}}{2}=3.77$$

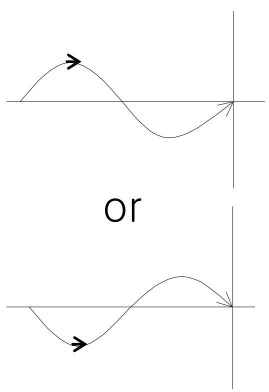
$$\omega=\sqrt{3.77}=1.94$$

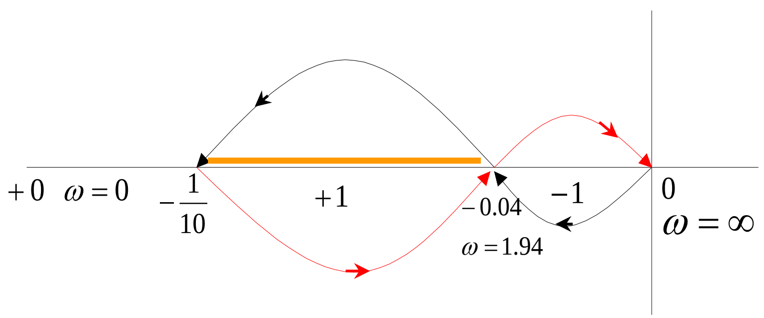
$$G_P(j\omega)\Big|_{\omega=1.94}=\frac{4+\omega^2}{-5\omega^4-27\omega^3-40}\Bigg|_{\omega=1.94}=-0.0365$$

$$\cong\ -0.04$$

- Cross Imag axis ; $5\omega^4+27\omega^2+40=0$

$$\omega^2=\frac{-27\pm\sqrt{27^2-800}}{10} \quad \Rightarrow \quad \text{never}$$





$\Rightarrow At \quad \omega=1 \ \ (0<1<1.94)$

$$G_P(j\cdot 1)=\frac{5}{-72+j16} \quad \Rightarrow \text{III}$$

$$N_P=1$$

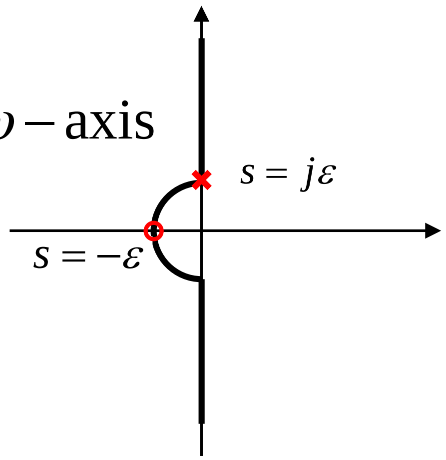
stability: $-\frac{1}{10}<-\frac{1}{k}<-\frac{4}{100}$

$\therefore\ 10<k<25$

☒

Example: ⑤ $G_P(s) = \dfrac{s+1}{s(s-1)}$

Indent the $j\omega -$ axis



$\omega = 0 \quad G_P(0) = \infty$

Semicircle, radius $\varepsilon, \quad \varepsilon \to 0$

※ On semicircle ; $\quad |G_P(s)| = \left|\dfrac{1}{s}\cdot\dfrac{s+1}{s-1}\right| = \infty$

at $s \to 0$

$$G_P(s) = \frac{1}{s}\cdot\frac{1}{-1} = -\frac{1}{s} \qquad \angle G_P(s) = \angle -\frac{1}{s} + \angle 0$$

$$\angle G_P(s): \quad s = -\varepsilon: \quad G_P = \frac{1}{\varepsilon} \quad \Rightarrow \quad \angle = 0$$

$$s = \varepsilon j \quad : \quad G_P(s) = -\frac{1}{j\varepsilon} = j\frac{1}{\varepsilon} \Rightarrow \angle = 90$$

※ $s = j\omega, \quad \omega > \varepsilon;$

$$G_P(s) = \frac{1+j\omega}{j\omega(j\omega-1)} = \frac{1+j\omega}{-\omega^2 - j\omega} = j\frac{(1+j\omega)(1-j\omega)}{-(\omega^2+j\omega)(1-j\omega)} = \frac{-(1+\omega^2)}{2\omega^2 + j(\omega-\omega^3)}$$

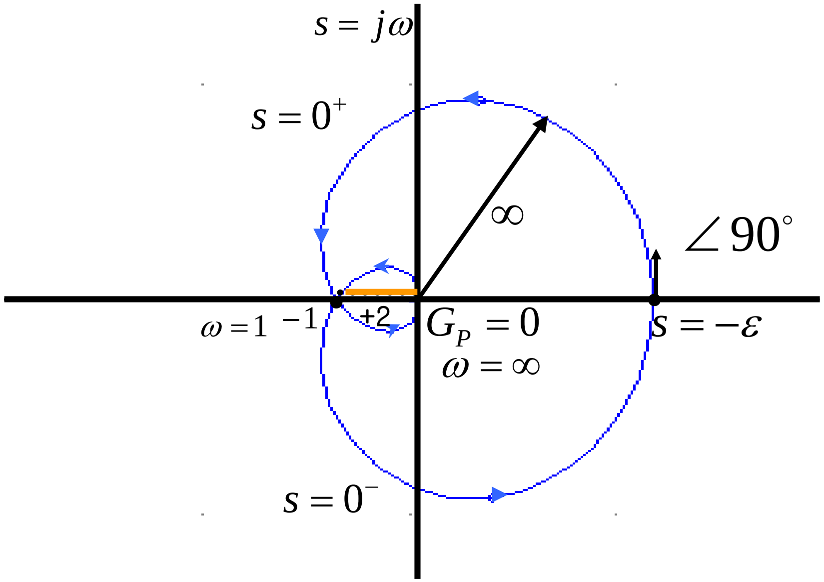
- Cross Real axis; $\quad \omega - \omega^3 = \omega(1-\omega^2) = 0 \rightarrow \omega^2 = 1$

$$\rightarrow \omega = 1, \quad G_P = -1$$

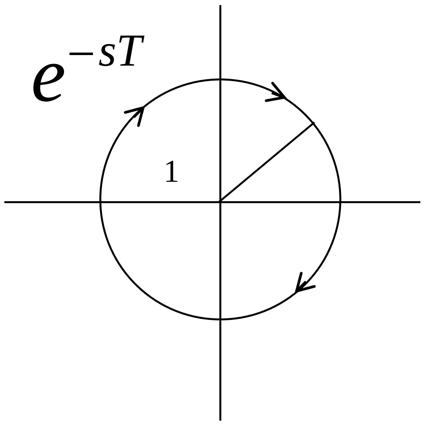
- Cross Imag axis; $\quad \omega^2 = 0 \rightarrow never$
- $\omega = \infty; \quad G_P = 0$

$$N_P = 2 \rightarrow \text{ stability } \quad -1 < -\frac{1}{k}$$

$$\rightarrow \quad k > 1$$



Example: ⑥ $G_P(s) = \dfrac{1}{s+1} e^{-sT}$

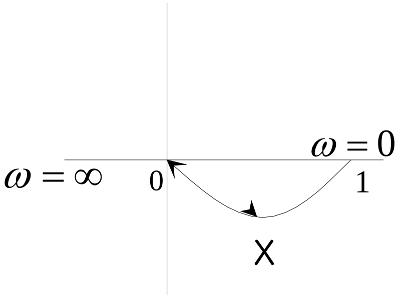


$$\frac{1}{s+1} \rightarrow \frac{1}{1+j\omega} = \frac{1-j\omega}{1+\omega^2}$$

$$\omega = 0 \rightarrow G = 1$$

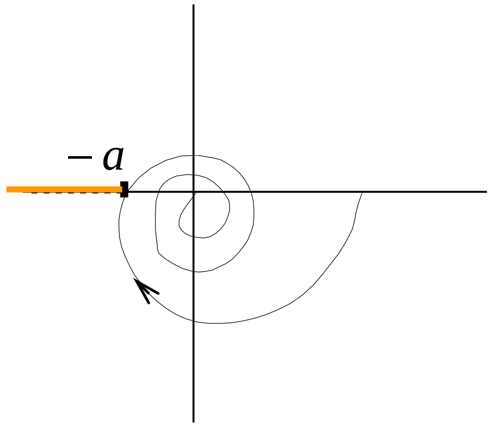
$$\omega = 1 \rightarrow G = (1-j)/2$$

$$\omega = \infty \rightarrow G = 0$$



$$\Rightarrow \quad \frac{\cos(sT) - j\sin(sT)}{s+1}$$

$$N_P = 0; \quad \text{stability:} \quad -\frac{1}{\mathrm{k}} < -a, \quad k < \frac{1}{a}$$



Find 'a': $G_P(j\omega) = -a \quad \leftarrow$ real number

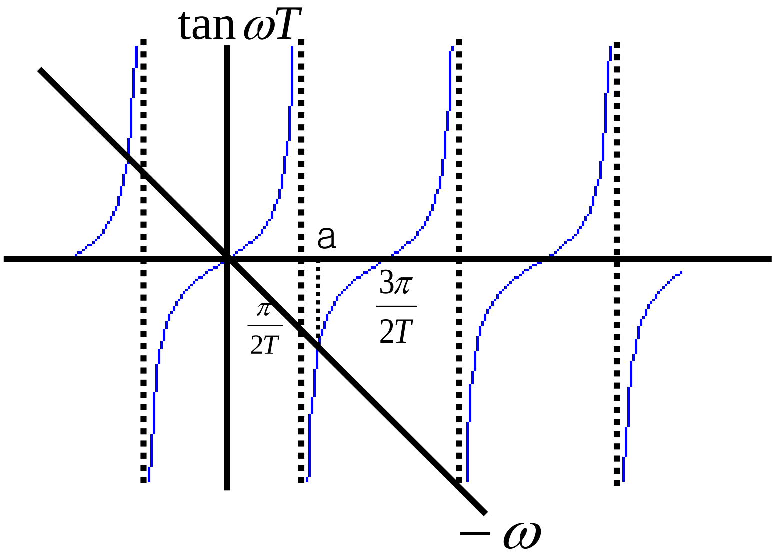
$$G_p(j\omega) = \frac{1}{1+j\omega} e^{-j\omega T} = \frac{\cos\omega T - j\sin\omega T}{1+j\omega} = -a$$

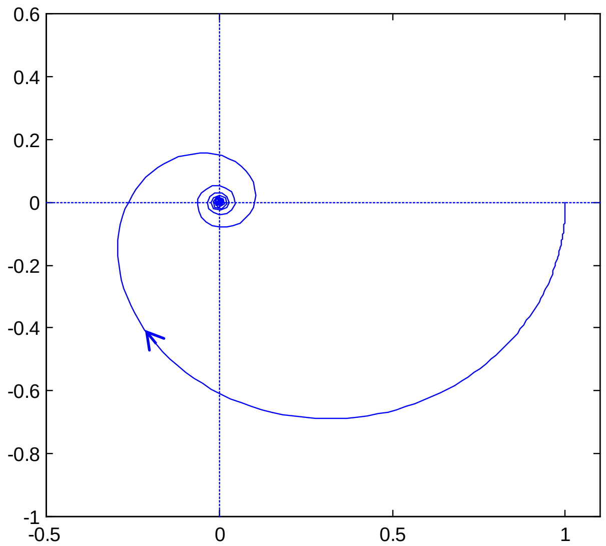
$$\cos\omega T - j\sin\omega T = -a - aj\omega$$

$$\left.\begin{aligned}\cos\omega T &= -a \\ \sin\omega T &= a\omega\end{aligned}\right\} \rightarrow \tan\omega T = -\omega \quad \Rightarrow \text{ solve graphically}$$

$$\omega T = \frac{\pi}{2} \rightarrow \omega = \frac{\pi}{2T}$$

$$\omega T = \frac{3\pi}{2} \rightarrow \omega = \frac{3\pi}{2T}$$

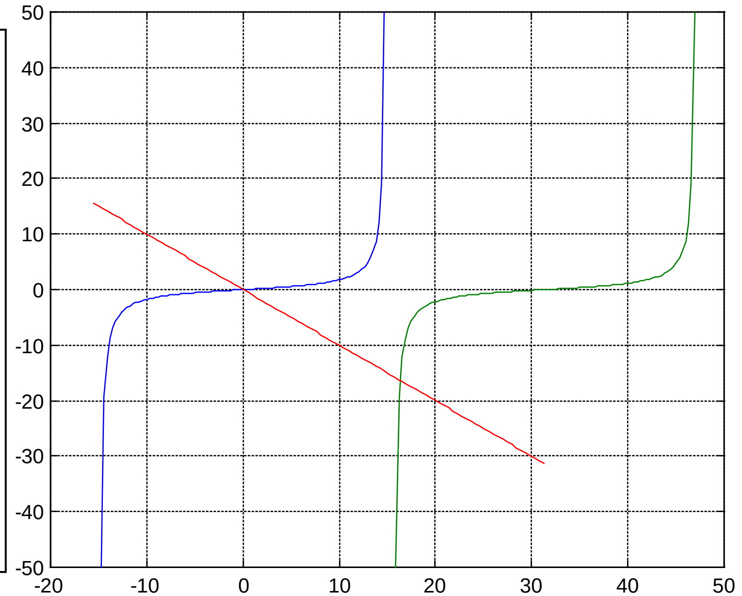





```
%             1
%  Gp(s)= ------exp(-sT)
%            s+1
function ex6
T=0.5;
w=logspace(-3, 2.2, 300);
y=gp(w,T);

plot(y)
a=-0.5; b=1.1; c=-1; d=0.6;
axis([a b c d])

hold on
plot([a b], [0 0], ':')
plot([0 0], [c d], ':')
hold off

function y=gp(w,T);
j=sqrt(-1);
temp=exp(-j.*w*T);
y=temp./(j*w+1);
```

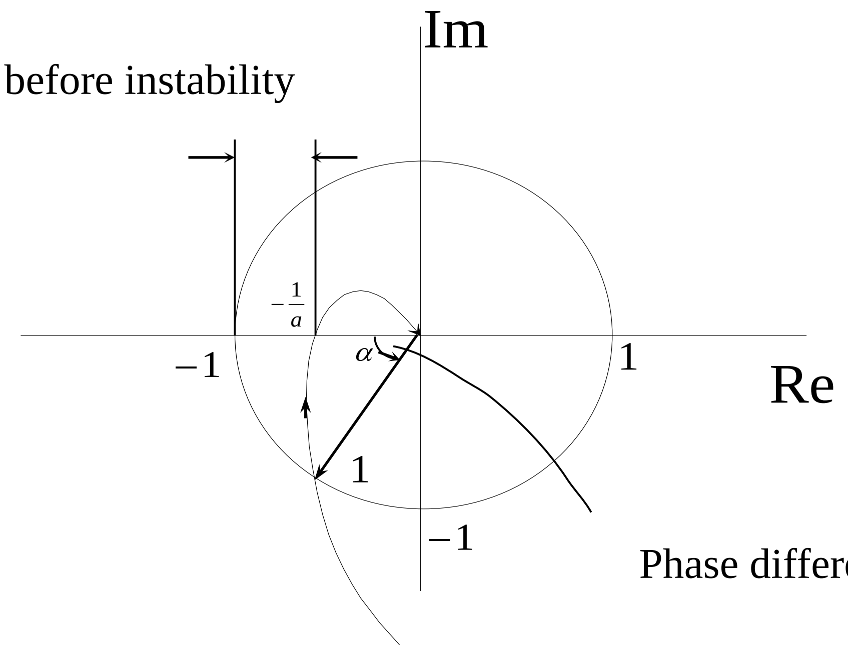
```
%  tan(wT)=-w

T=0.1;
N=100; delta=0.2;
L=pi/2/T;
w1=linspace(-L+delta, L-delta, N);
y1=tan(w1*T);

w2=linspace(-L+delta+pi/T, L-delta+pi/T, N);
y2=tan(w2*T);

w3=linspace(w1(1), w2(N/2), N);
y3=-w3;

plot(w,y1, w2,y2, w3,y3, 'r-')
grid

break
w0=linspace(-L+delta+pi/T, 17, 10*N);
error=w0+tan(w0*T);
```

```
» [w0; error]'
   16.3157   -0.1185
   16.3168   -0.0878
   16.3179   -0.0573
   16.3190   -0.0268
   16.3201    0.0035
   16.3212    0.0338
   16.3223    0.0639
   16.3234    0.0940
   16.3244    0.1239
   16.3255    0.1538
   16.3266    0.1835
   16.3277    0.2132
   16.3288    0.2427
   16.3299    0.2721
   16.3310    0.3015
   16.3321    0.3307
   16.3332    0.3599
```



Gain margin and Phase Margin

Gain difference before instability



$$s = j\omega$$

$$Z = e^{j\omega}$$

Phase difference before instability

$$\text{Gain margin} = G_M = 20\log a = -20\log\left(\frac{1}{a}\right)$$

$$\text{Phase margin} = \Phi_M = \alpha$$

Example: $$G(s)=\frac{k}{(s^2+2s+2)(s+2)}$$

Find the gain and phase margin if $k=6$

$$G(j\omega)=\frac{6}{(4-4\omega^2)+j\omega(6-\omega^2)}$$

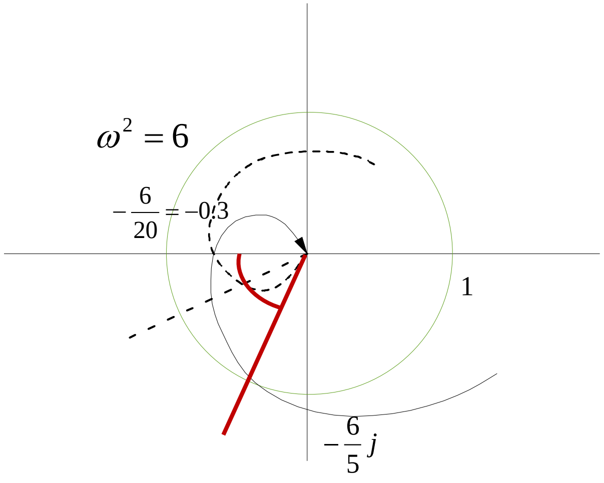
– Cross Real axis:

$$\omega^2=6 \qquad -\frac{6}{20}=-\frac{3}{10}=-0.3=-\frac{1}{a}$$

$$G_m=20\log a=20\log\left(\frac{1}{0.3}\right)=10.45 \quad dB$$

– Cross Imaginary axis : $\omega^2 = 1 \quad \omega = 1$

$$G = \frac{6}{j(6-1)} = -\frac{6}{5} j$$



– For the Phase Margin: $|G(j\omega)| = 1$

$$\omega = 1.253 \quad \text{rad/sec}$$

$$\varphi = -112.33^\circ$$

$$\Phi_M = 180^\circ - 112.33^\circ = 67.67^\circ$$

Bode Diagrams

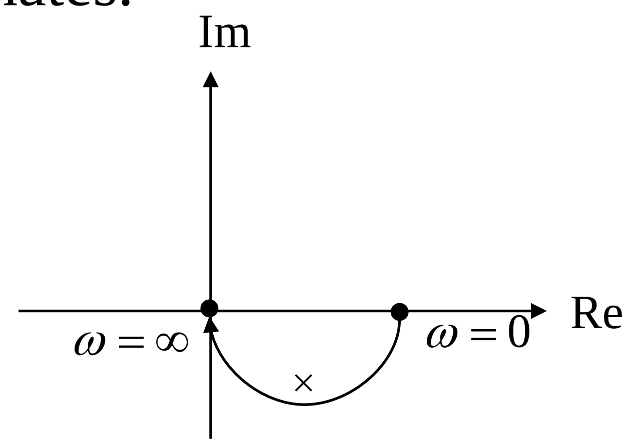
Frequency Domain Analysis

Principle: Draw $G_p(j\omega)$ in polar coordinates.

- Frequency response

$$G_p(s) = \frac{1}{s+2}$$

$$s = j\omega \rightarrow G = \frac{1}{j\omega + 2}$$

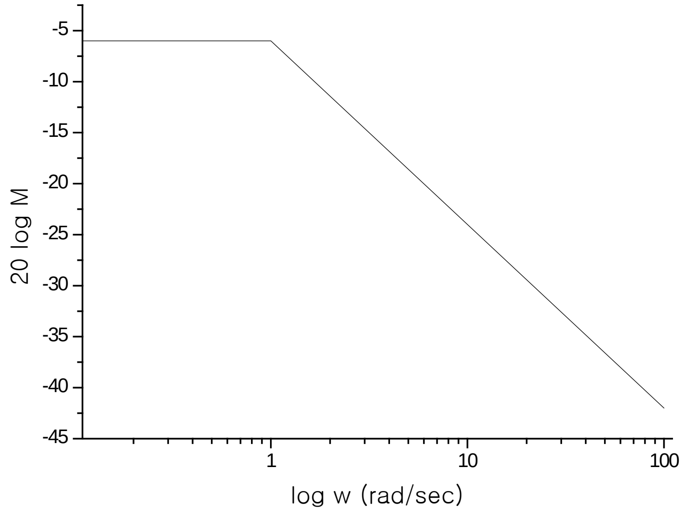


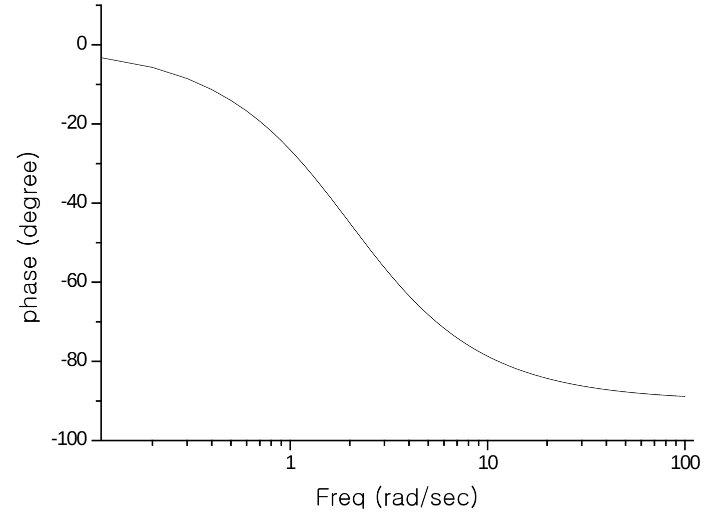
$$\omega = 1 \quad \frac{1}{2+j} \rightarrow \frac{2-j}{\Delta}$$

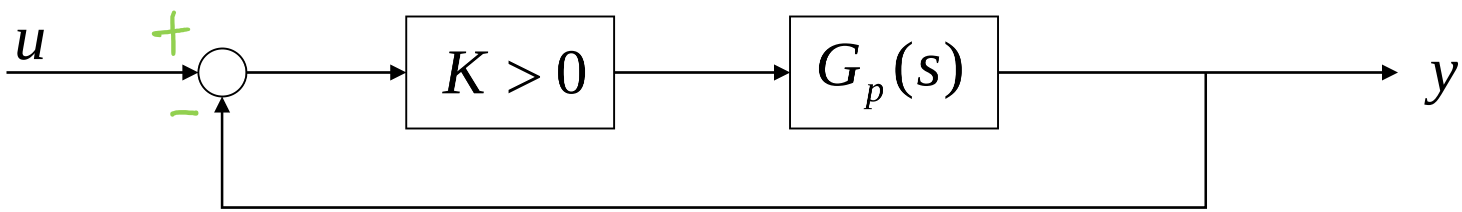
⇨ magnitude and phase plot

$$|G(j\omega)| = M(j\omega) = \frac{1}{\sqrt{4+\omega^2}} \rightarrow \log(\omega) \text{ vs } \boxed{20\log M(\omega)}$$

$$\angle G(j\omega) \rightarrow \phi(\omega) = -\arctan\left(\frac{\omega}{2}\right) \rightarrow \log(\omega) \text{ vs } -\arctan\left(\frac{\omega}{2}\right)$$







Can get idea about stability range?

$$G_p(s) = A \cdot \frac{\left(1+\dfrac{s}{z_1}\right)\cdots\left(1+\dfrac{s}{z_m}\right)}{\left(1+\dfrac{s}{\omega_1}\right)\cdots\left(1+\dfrac{s}{\omega_n}\right)}$$

Assume: $z_j : j = 1,\cdots,m$

$\omega : i = 1,\cdots,n$ are real

$A > 0$

$$G_p(s) = A\frac{\prod_{j=1}^{m}\left(1+\frac{s}{z_j}\right)}{\prod_{i=1}^{n}\left(1+\frac{s}{\omega_i}\right)} \rightarrow s = j\omega :$$

$$G_p(j\omega) = A\frac{\prod_{j=1}^{m}\left(1+j\frac{\omega}{z_j}\right)}{\prod_{i=1}^{n}\left(1+j\frac{\omega}{\omega_i}\right)}$$

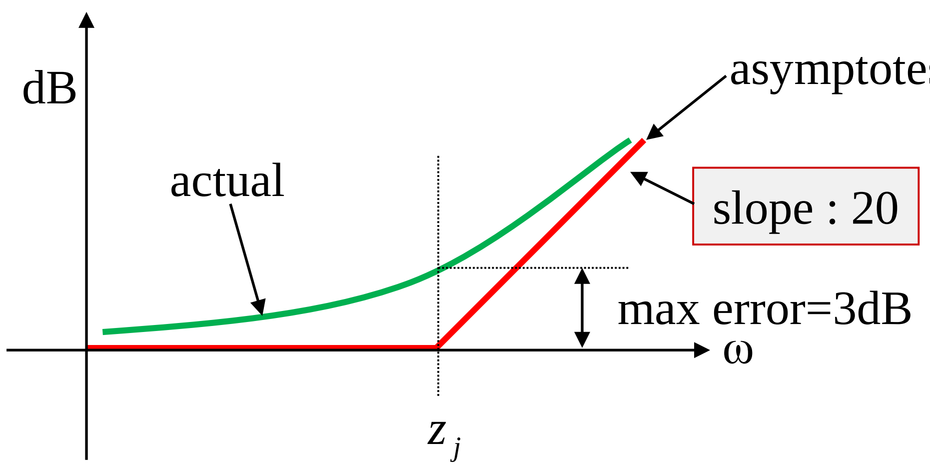
⇨ Draw: $\left|G_p(j\omega)\right|$ in logarithmic scale, and $\angle G_p(j\omega)$

(A) Magnitude

$$20\log_{10}\left|G_p(j\omega)\right| = 20\log\left|A\frac{\Pi\left(1+j\dfrac{\omega}{z_j}\right)}{\Pi\left(1+j\dfrac{\omega}{\omega_i}\right)}\right|$$

$$= 20\log A + \sum_{j=1}^{m} 20\log\left|1+j\frac{\omega}{z_j}\right| - \sum_{i=1}^{n} 20\log\left|1+j\frac{\omega}{\omega_i}\right|$$

- Consider the plot of $20\log\left|1+j\dfrac{\omega}{z_j}\right|$



Asymptotic approximation

$20\log_{10}\sqrt{2} = \underline{\underline{3.01\text{dB}}}$

$$20\log\left|1+j\frac{\omega}{z_j}\right| = 20\log\sqrt{1+\frac{\omega^2}{z_j^2}} = 10\log\left(1+\frac{\omega^2}{z_j^2}\right)$$

If $\omega >> |z_j|$,

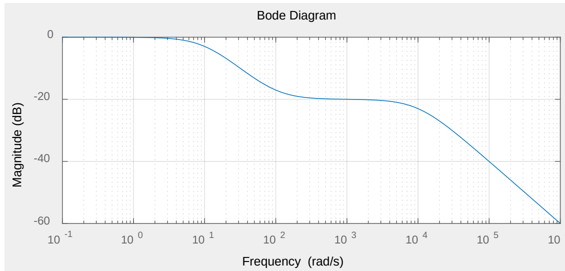
$$20\log\left|1+j\frac{\omega}{z_j}\right| \cong 10\log\left(\frac{\omega^2}{z_j^2}\right) = 20\log\omega - 20\log|z_j|$$

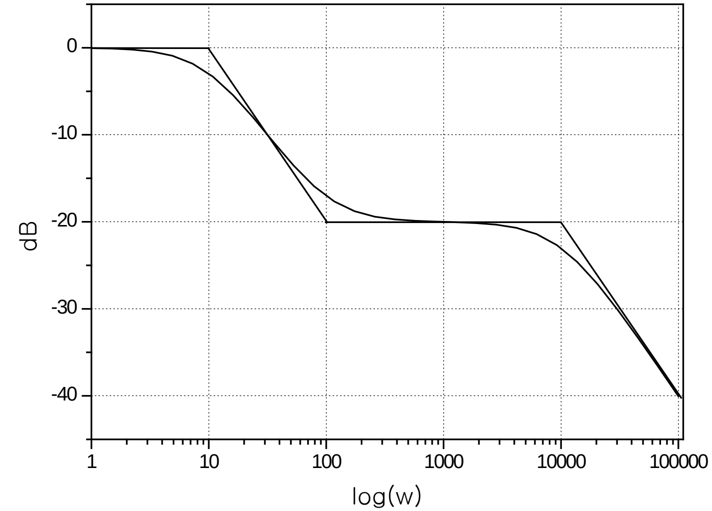
If $\omega << |z_j|$,

$$20\log\left|1+j\frac{\omega}{z_j}\right| \cong 20\log 1 = 0$$

Example 1

$$G_p(s)=\frac{1+s/100}{(1+s/10)(1+s/1000)}$$



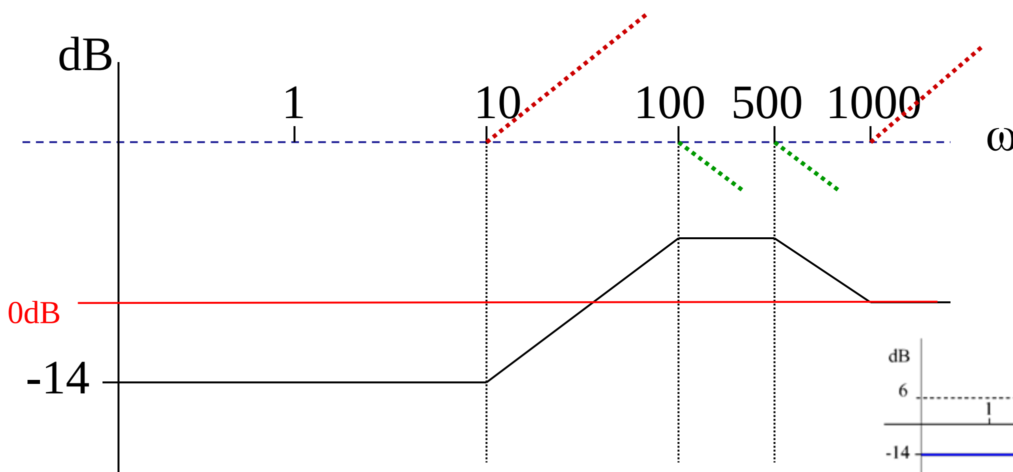


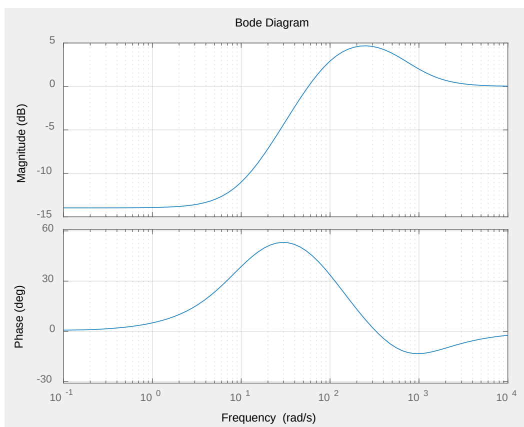
Example 2

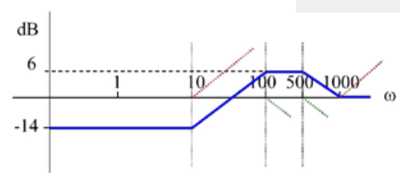
$$G_p(s)=\frac{(s+1000)(s+10)}{(s+100)(s+500)}=\frac{1000\cdot 10}{100\cdot 500}\cdot\frac{(1+s/1000)(1+s/10)}{(1+s/100)(1+s/500)}$$

```
numg=poly([-10 -1000]);
deng=poly([-100 -500]);
G=tf(numg,deng)
bode(G), grid on
```

$$\rightarrow 20\log(0.2)=-14$$



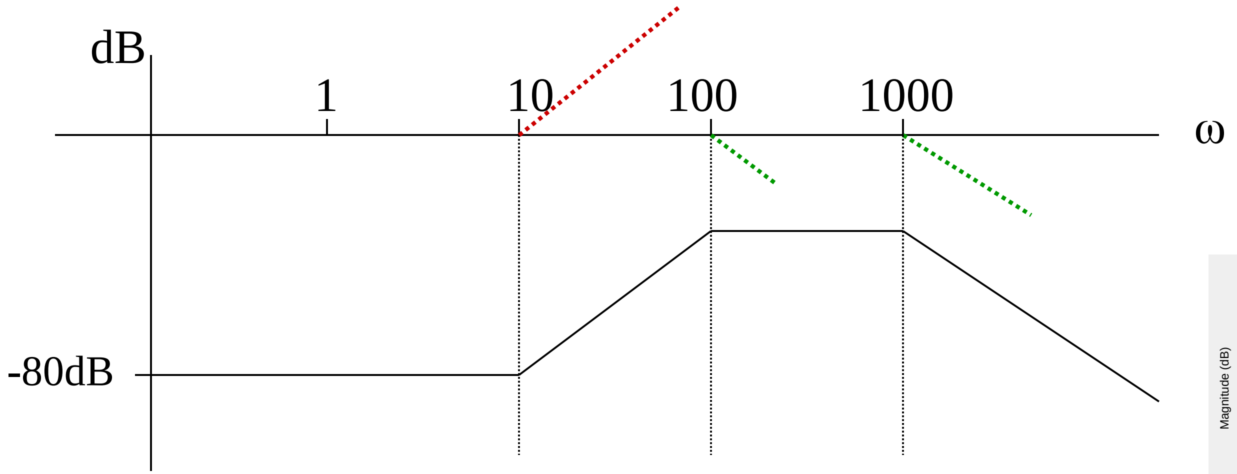




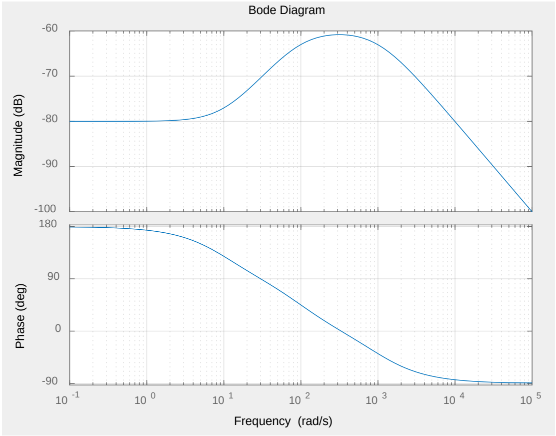
Example 3

$$G_p(s) = \frac{(s-10)}{(s+100)(s+1000)} = -\frac{1}{10^4} \cdot \frac{1 - s/10}{(1+s/100)(1+s/1000)}$$

$$\rightarrow 20\log\left(\frac{1}{10^4}\right) = -80$$




```
numg=poly([10]);
deng=poly([-100 -1000]);
G=tf(numg,deng)
bode(G), grid on
```



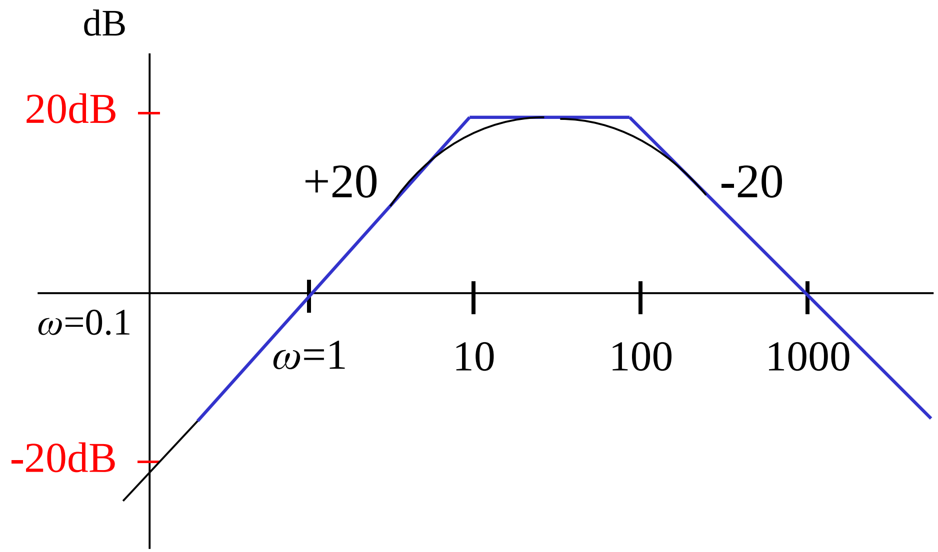
Example 4

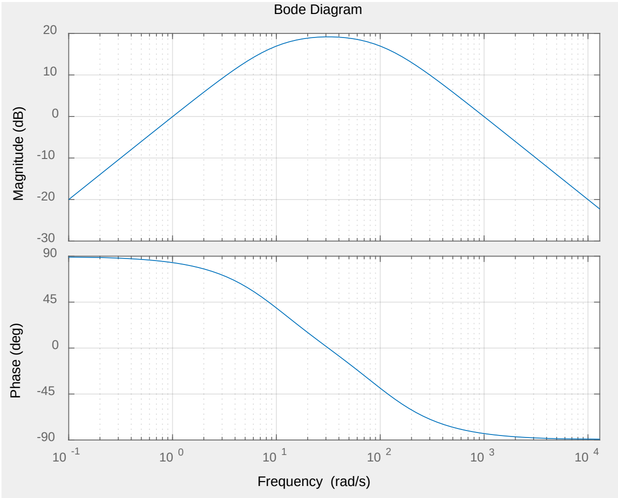
$$G_p(s) = \frac{s}{(1+\frac{s}{10})(1+\frac{s}{100})}$$

$\rightarrow$ Zero at 0 !

$$20\log|j\omega| = 20\log\omega \rightarrow \text{add slope of 20 !}$$

```
numg=poly([0]);
deng=poly([-10 -100]);
G=tf(numg,deng)*1000
bode(G), grid on
axis([0.1 13000 -90 90])
```

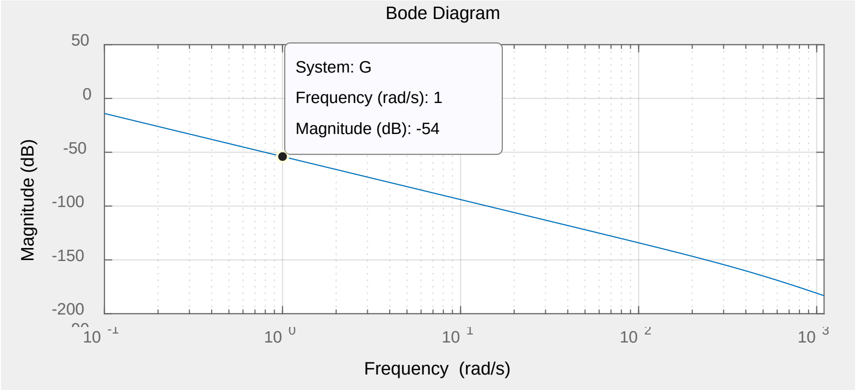


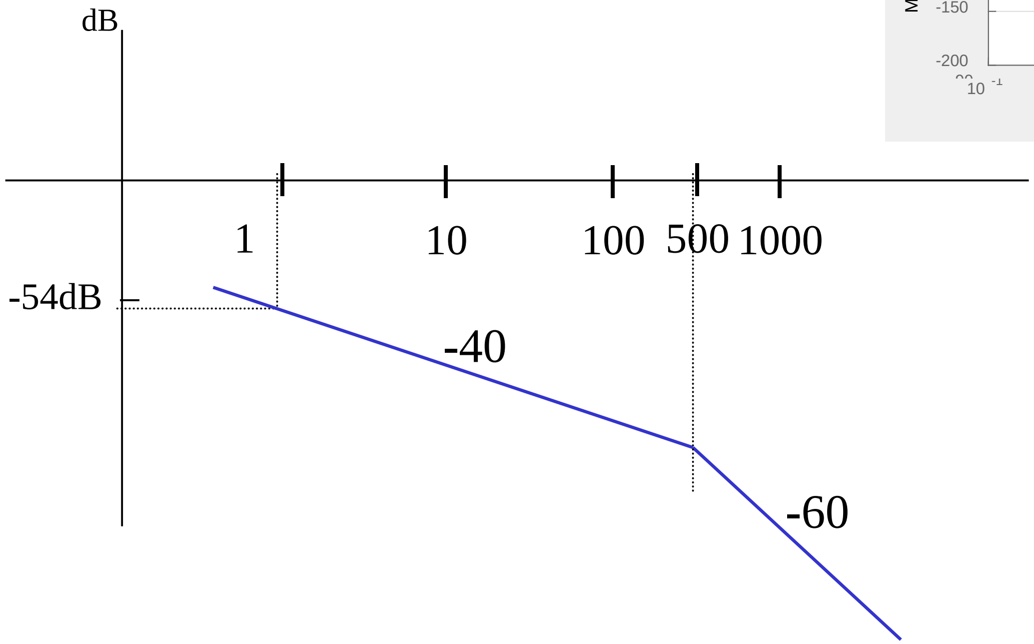


Example 5

$$G_p(s) = \frac{1}{s^2(s+500)} = \frac{1}{500}\frac{1}{s^2\left(1+\frac{s}{500}\right)}$$

$$\rightarrow 20\log\left(\frac{1}{500}\right) = -54$$






```
numg=poly([]);
deng=poly([0 0 -500]);
G=tf(numg,deng)
bode(G), grid on
axis([0.1 1100 -90 90])

G =
          1
   -------------
   s^3 + 500 s^2
```

(B) Phase Response

$$G_p(s) = A\frac{\Pi(1+s/z_j)}{\Pi(1+s/\omega_i)} \quad \rightarrow \quad \angle G_p(s) = \angle A + \sum_{j=1}^{m}\angle(1+\frac{s}{z_j}) - \sum_{i=1}^{n}\angle(1+\frac{s}{\omega_i})$$

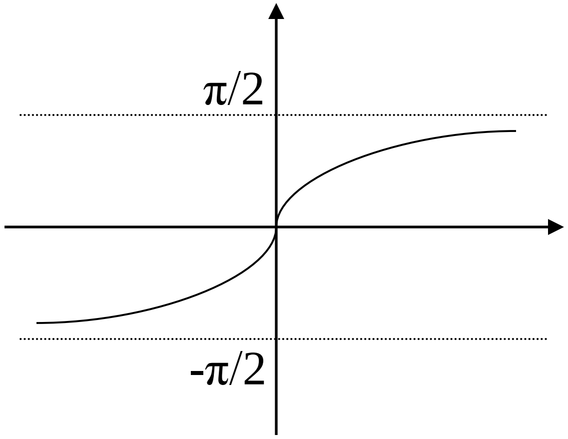
- Consider: $\angle(1+s/z_j)$

$$s = j\omega : \angle\left(1 + j\frac{\omega}{z_j}\right) = \arctan\left(\frac{\omega}{z_j}\right)$$

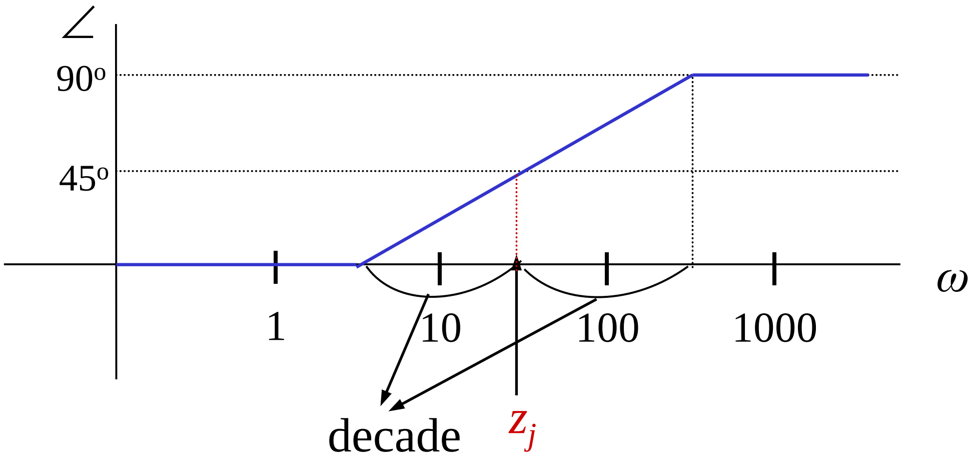
i) $z_j > 0$

$$\omega >> z_j : \arctan\left(\frac{\omega}{z_j}\right) \approx 90^\circ$$

$$\omega << z_j : \arctan\left(\frac{\omega}{z_j}\right) \approx 0^\circ$$



$\arctan(1)=45^\circ$

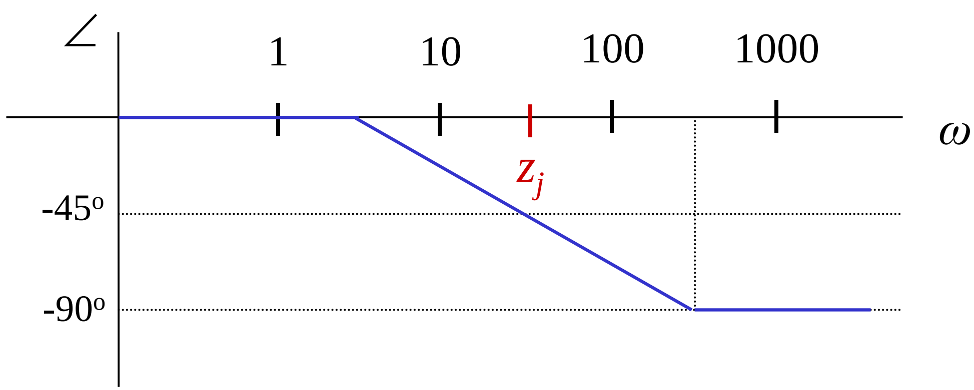


atan(1)=45°

ii) z_j<0

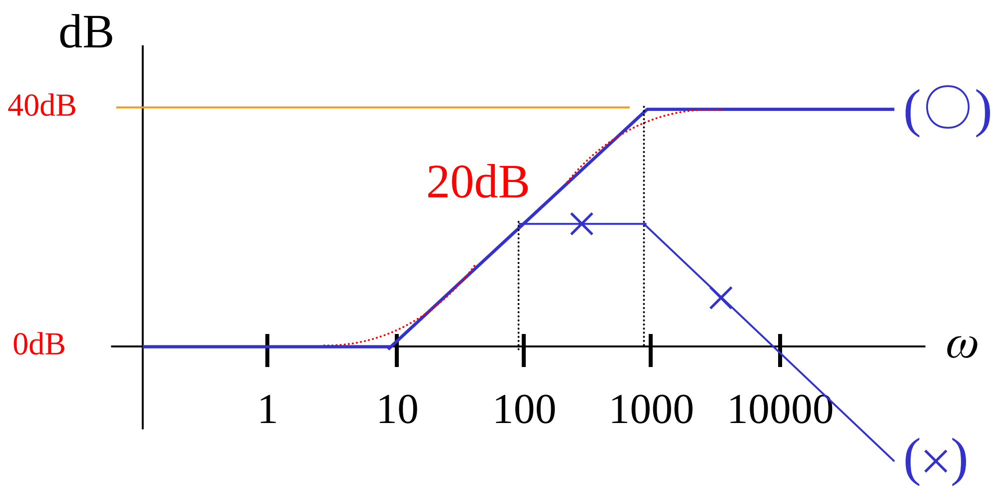
$$\omega >> |z_j|: \ \arctan\left(\frac{\omega}{z_j}\right) \approx -90^\circ$$

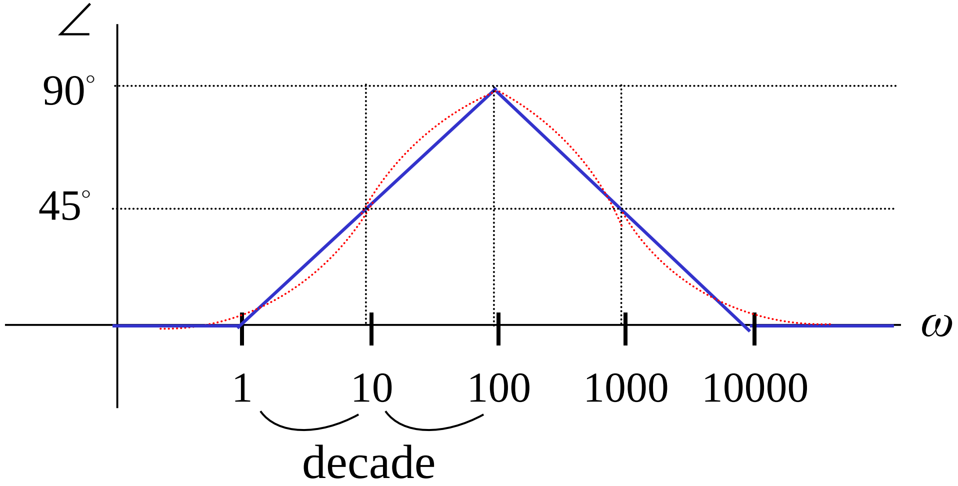
$$\omega << |z_j|: \ \arctan\left(\frac{\omega}{z_j}\right) \approx 0^\circ$$

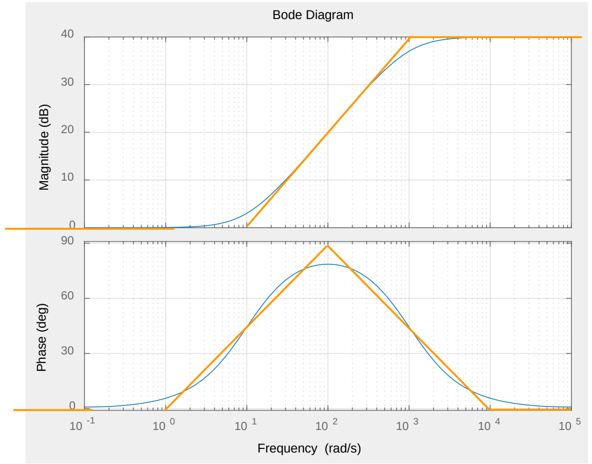


Example 6

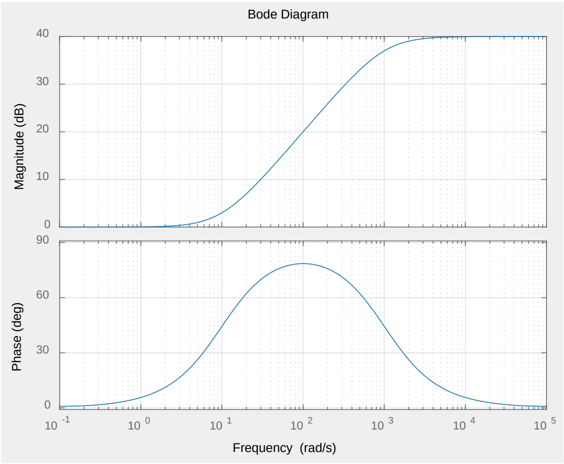
$$G_p(s)=\frac{1+s/10}{1+s/1000}$$





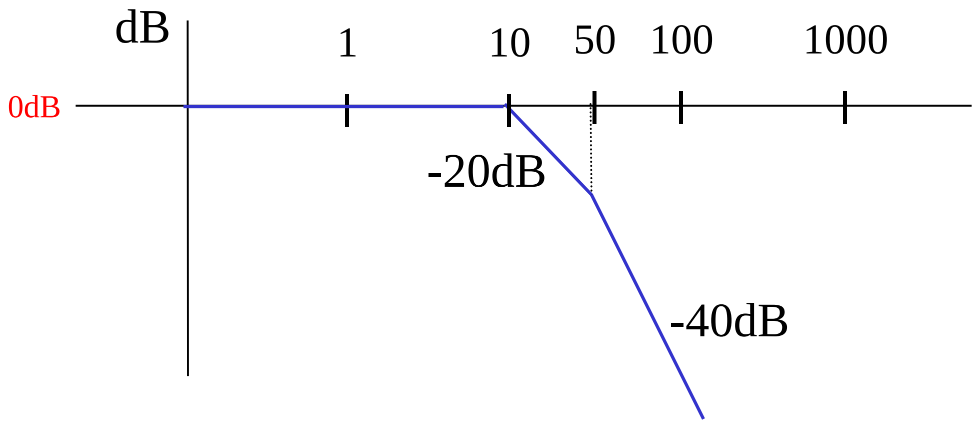


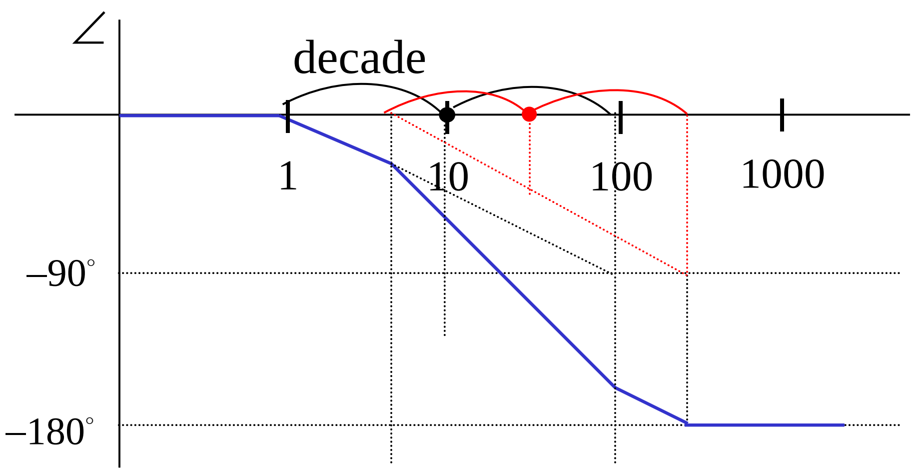

```
numg=poly([-10]);
deng=poly([-1000]);
G=tf(numg,deng)*100
bode(G),  grid on
```

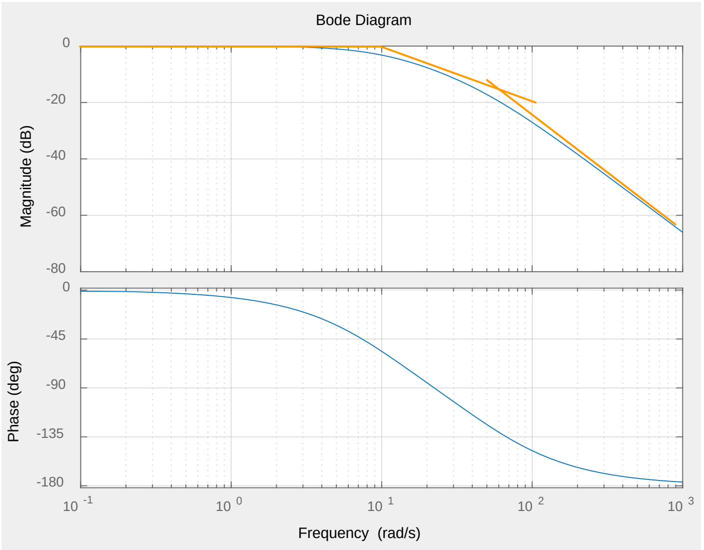


Example 7

$$G_p(s) = \frac{1}{(1+s/10)(1+s/50)}$$



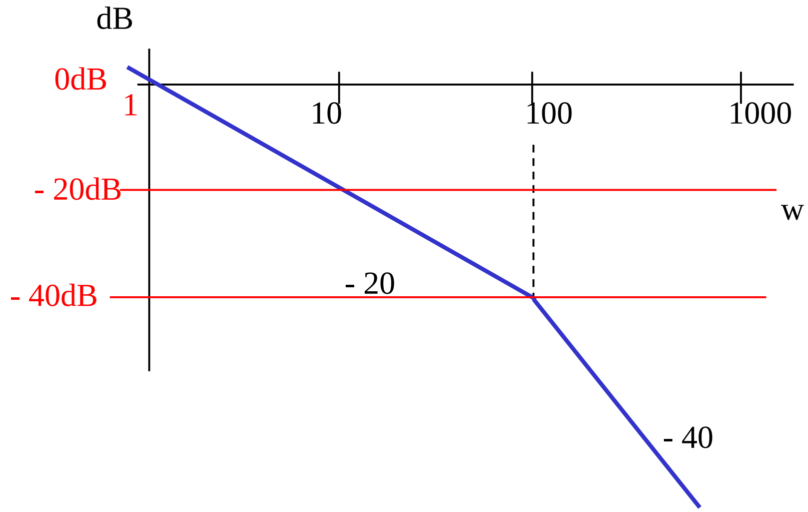


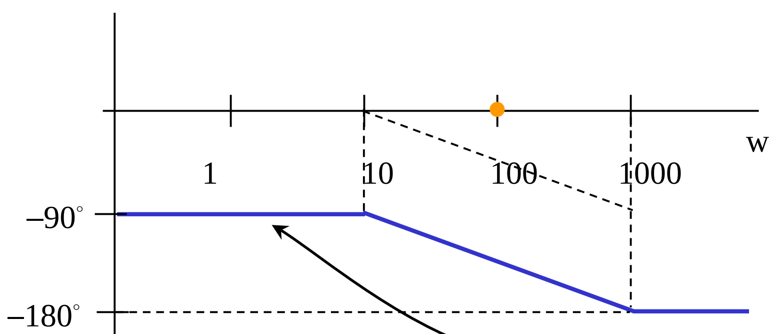


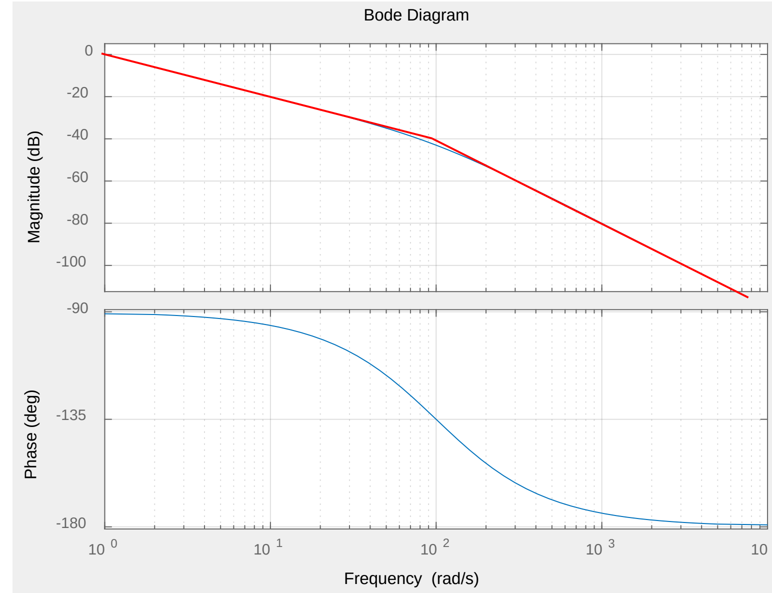

```
numg=poly([]);
deng=poly([-10 -50]);
G=tf(numg,deng)*500
bode(G), grid on

G =
          500
   ---------------
   s^2 + 60 s + 500
```

Example 8

$$G_p(s) = \frac{1}{s\left(1+\frac{s}{100}\right)}$$








```
numg=poly([]);
deng=poly([0 -100]);
G=tf(numg,deng)*100
bode(G), grid on

G =
        100
    -----------
    s^2 + 100 s
```

$$\angle(jw) = 90^\circ$$

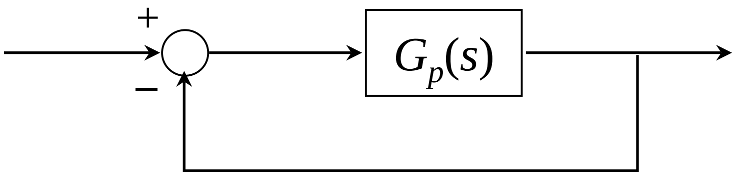
$\uparrow$

Add ±90° to the phase

$*$ Pole at 0

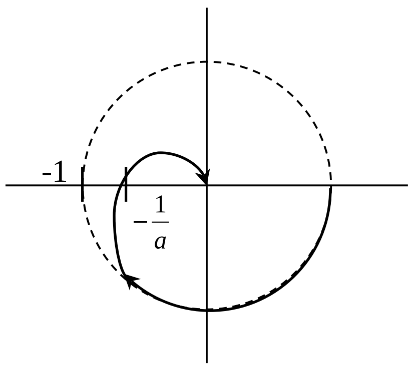
$\rightarrow$ Add -90° to the phase

Stability test, phase and gain margins (1)



Suppose that $G_p(s)$ has no RHP ploes.

The Nyquist plot of $G_p(s)$ is

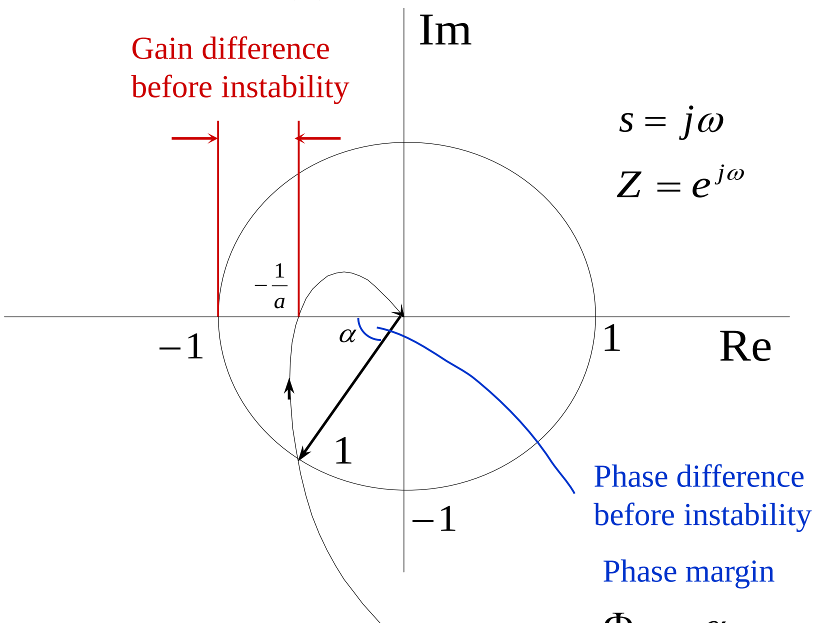


with k=1, the system is stable.

$$-\frac{1}{k} < -\frac{1}{a} \quad \rightarrow \quad k < a$$

Gain margin

$$G_M = 20\log(a)$$
$$= -20\log\left(\frac{1}{a}\right)$$



Phase margin

$$\Phi_M = \alpha$$

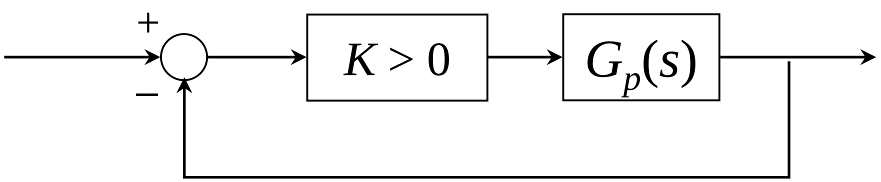
Stability test, phase and gain margins (2)

∗ Two questions:

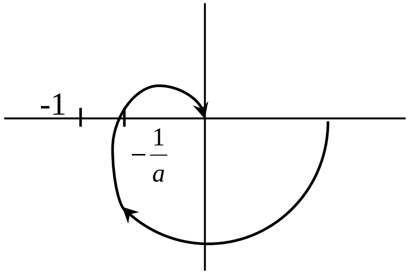
① By how much can you increase the gain of G_p before instability occurs?

② How much delay (negative phase) can you add to G_p before instability occurs?

① →



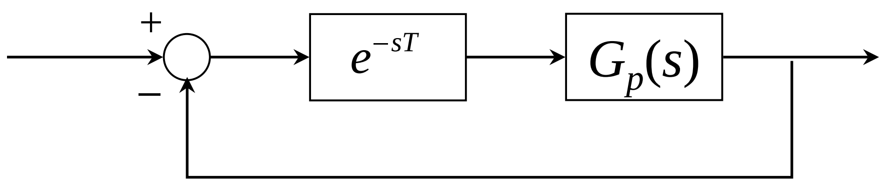
What is the maximum K with which you still have stability ? → Gain Margin



$$\text{Gain Margin} = G_M = 20\log(a) = -20\log\left(\frac{1}{a}\right) \text{ dB}$$

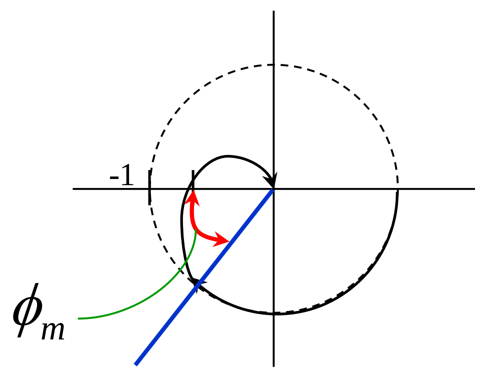
Stability test, phase and gain margins (3)

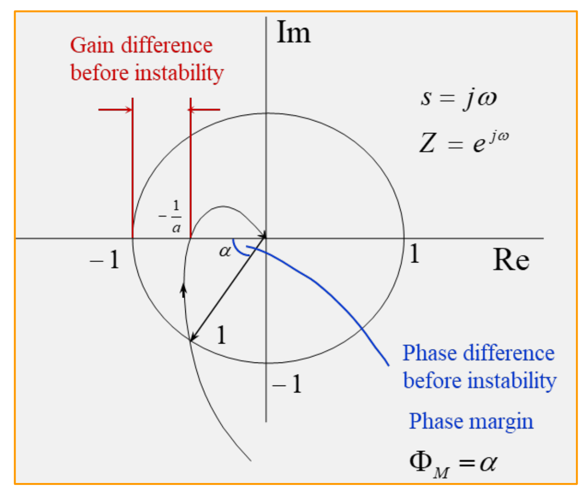
② →



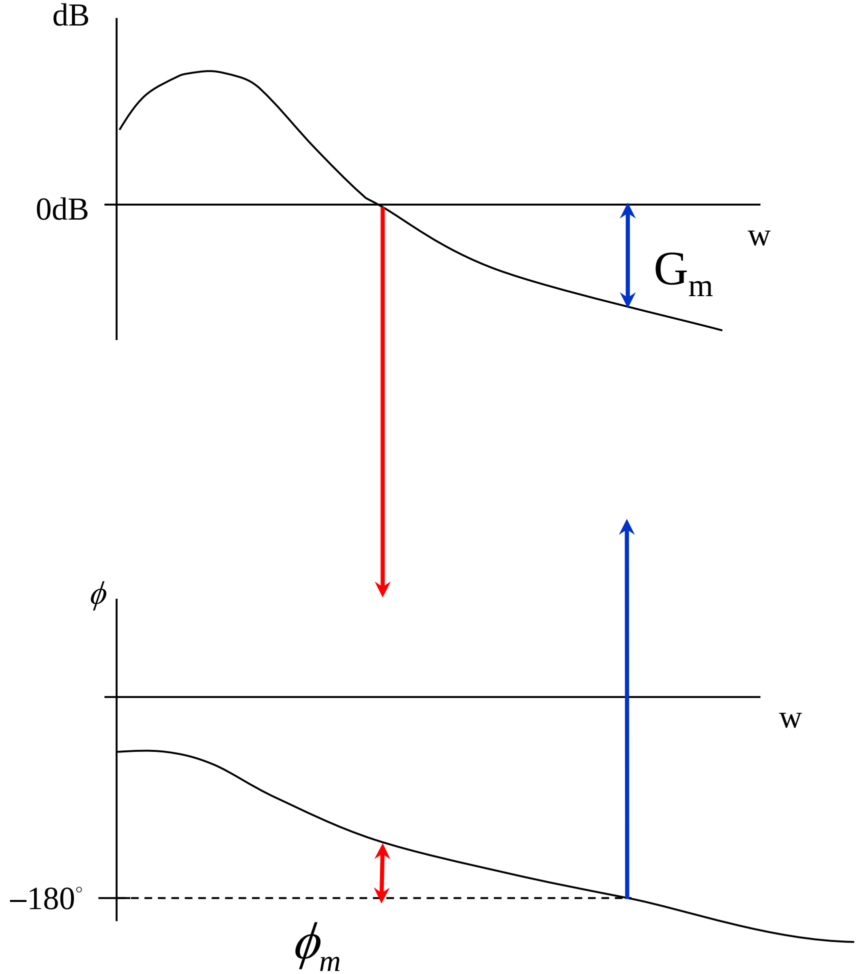
What is the maximum T with which you still have stability?

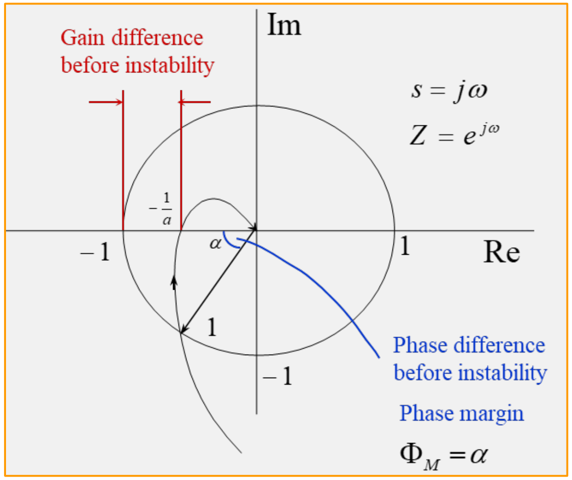
→ Phase Margin





⇒ From Bode Plots.





```
% Bode Diagrams
k=1.5; ng=1; dg=poly([0 -1 -2]); w=logspace(-1,1,100)';
[m,p]=bode(k*ng, dg,w);
figure(1)
%subplot(211); semilogx(w, 20*log10(m)); grid
%subplot(212); semilogx(w, p); grid
bode(k*ng, dg,w)
[gm,pm,wgc,wpc]=margin(m,p,w);
[gm,pm,wgc,wpc]   % 4.0002 41.5332  1.4142  0.6118
margin(m,p,w)


%   For Nyquist
w2=linspace(0, 2*pi, 100)';
ejw=exp(j*w2); r2=real(ejw); i2=imag(ejw);
[r,i]=nyquist(k*ng, dg, w);
figure(2)
plot(r2,i2, r,i, 'r-');
axis('square'); grid
axis([-1 1 -1 1])
```