Chapter 18

Integration in the Complex Plane

18.1 Contour Integrals

■ Introduction

정적분 $\int_a^b f(x)\,dx$는 x축상의 구간 $[a,b]$에서 정의된 실함수 $y=f(x)$의 적분임.

1차원에서의 구간은 2차원에서는 평면상의 곡선에 해당하므로
정적분의 정의를 평면에서 곡선 C위에서 정의된 2변수 실함수의 적분인 선적분

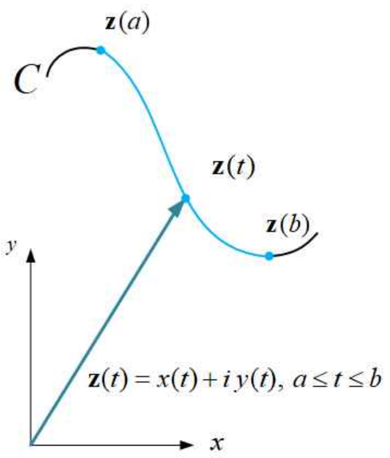
$$\int_C P(x,y)dx,\ \int_C Q(x,y)dy,\ \int_C f(x,y)ds$$

으로 일반화함.(미적분학에서)

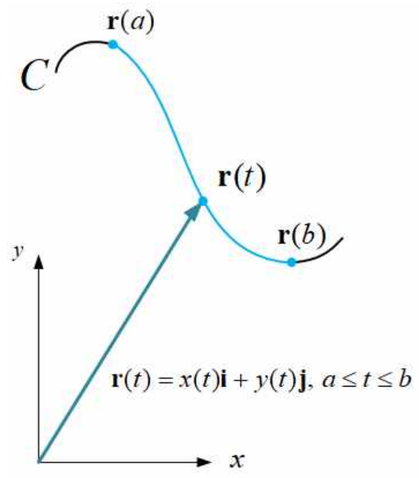
복소함수 f의 적분은 복소평면에서의 곡선, 즉 경로(contour)를 따라 정의된 $f(z)$에 대해 정의한다.

이 절에서는 복소선적분의 정의와 성질과 그 계산법이 데카르트평면에서의 실선적분과 매우 비슷함을 알 수 있을 것이다

■ Definition



2차원 평면에서의 곡선 C의 표현



복소평면에서의 곡선 C의 표현

(예) 2차원 평면에서의 원 $x = \cos t, y = \sin t, 0 \le t \le 2\pi$은 복소평면에서 $C: z(t) = \cos t + i \sin t = e^{it}, 0 \le t \le 2\pi$

매끄러운 곡선(Smooth Curve)

곡선 C의 모든 점에서 연속적으로 변해가는 접선을 가지는 경우

$\longrightarrow$ 매끄러운 곡선

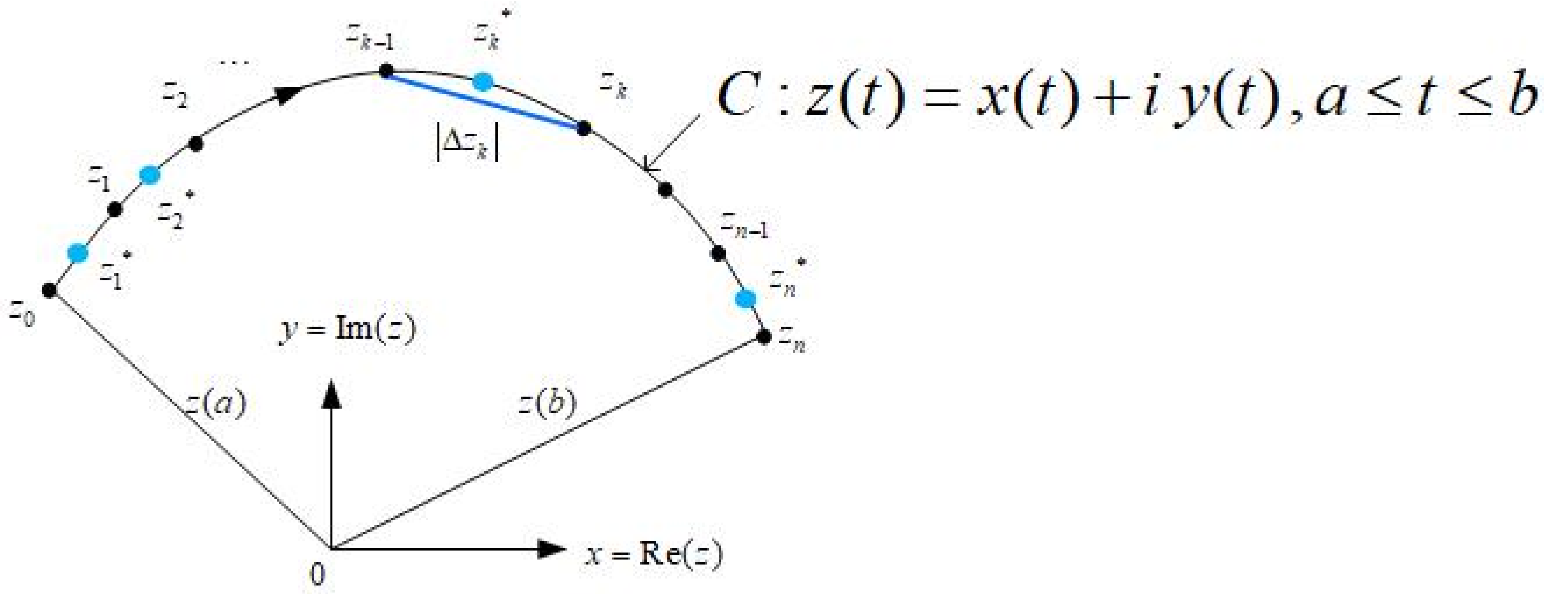
조각마다 매끄러운 곡선(piecewise smooth curve), 단순폐곡선(simple closed curve), 폐곡선(closed curve)

복소평면에서의 곡선 $C: z(t)=x(t)+iy(t), a \le t \le b$에 대해

① t가 증가하는 방향을 곡선의 양의 방향으로 정의한다.
② 복소평면에서는 조각마다 매끄러운 곡선을 경로(contour또는 path)라 부른다.
③ 복소평면에서 곡선 C 상에서의 $f(z)$의 적분 $\int_C f(z)\,dz$를

경로적분(contour integral) 또는 복소적분(complex integral)이라 부른다.

1. $f(z)=u(x,y)+iv(x,y)$가 매끄러운 곡선 $C: z(t)=x(t)+iy(t), a \le t \le b$상의 모든 점에서 정의된다고 하자.
2. 곡선 C를 구간 $[a,b]$에서 분할(partition)$a=t_0<t_1<\cdots<t_n=b$에 의해 n개의 부분호(subarc)로 나눈다.
   곡선 C상에 대응되는 점들은
   $z_1=x_0+iy_0=x(t_0)+iy(t_0), z_2=x_1+iy_1=x(t_1)+iy(t_1), \cdots, z_n=x_n+iy_n=x(t_n)+iy(t_n)$
   이고, $\triangle z_k=z_k-z_{k-1}, k=1,2,\cdots,n$
3. 분할의 놈(norm) $\|P\|$ 은 $|\triangle z_k|$의 최대값으로 정의한다.
4. 각 부분호상의 임의의 한 점 $z_k^*=x_k^*+iy_k^*$를 택한다.
5. 부분합 $\sum_{k=1}^{n} f(z_k^*)\triangle z_k$를 얻는다.



Definition 18.1.1 경로적분 (Contour Integral)

매끄러운 곡선 $C : z(t) = x(t) + iy(t), a \le t \le b$를 따라서 $f(z) = u(x,y) + i\,v(x,y)$의 경로적분(복소선적분, 복소적분)은

$$\int_C f(z)\,dz = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(z_k^*) \triangle z_k$$

적분경로 C가 폐곡선이면 $\oint_C f(z)\,dz = \int_C f(z)\,dz$으로 나타낸다.

함수 f가 경로 C의 모든 점에서 연속이고, C가 조각마다 매끄러운 곡선이면 $\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(z_k^*) \triangle z_k$는 항상 존재한다.

A Method of Evaluation

$$\int_C f(z)\,dz = \lim \Sigma(u + iv)(\Delta x + i\,\Delta y)$$

$$= \lim \{\Sigma(u\,\Delta x - v\,\Delta y) + i\,\Sigma(v\,\Delta x + u\,\Delta y)\}$$

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$C: z(t) = x(t) + iy(t),\ a \le t \le b$$

$$\int_a^b [u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)]dt + i\int_a^b [v(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)]dt$$

Theorem 18.1.1 Evaluation of a Contour Integral

$$C: z(t) = x(t) + iy(t),\ a \le t \le b$$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\frac{dz}{dt}\,dt = \int_a^b f(z(t))\,z'(t)\,dt$$

$$z'(t) \triangleq x'(t) + iy'(t)$$

Example 1 Evaluating a contour integral

$C: z(t) = 3t + t^2 i,\ -1 \le t \le 4,$ evaluate $\int_C \bar{z}\, dz$

Solution

$z'(t) = 3 + 2ti$

$$\int_C \bar{z}\, dz = \int_{-1}^{4} (3t - it^2)(3 + 2it)\, dt$$

$$= \int_{-1}^{4} (2t^3 + 9t)\, dt + i \int_{-1}^{4} 3t^2\, dt = 195 + 65i$$

Example 2 Evaluating a contour integral

$C: z(t) = \cos t + i \sin t = e^{it}, 0 \le t \le 2\pi,$ evaluate $\oint_C \frac{1}{z} dz$

Solution

$z'(t) = ie^{it}$

$$\oint_C \frac{1}{z}\, dz = \int_0^{2\pi} (e^{-it}) ie^{it} dt = i \int_0^{2\pi} dt = 2\pi i$$

■ Properties

열린영역(domain) D 에서 연속인 복소함수 $f(z),\ g(z)$와 D내에 있는 매끄러운 곡선 C에 대하여

① 선형성(linearity)

$$\int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

$$\int_C kf(z)dz = k\int_C f(z)dz \quad (k\text{는 복소상수})$$

② 경로분할(partition of a curve)

미끄러운 경로 C의 임의의 분할 $C_1,\ C_2$에 대하여

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

③ 방향성(orietation)

경로 C와 방향이 반대인 경로 $-C$에 대하여

$$\int_C f(z)dz = -\int_{-C} f(z)dz$$

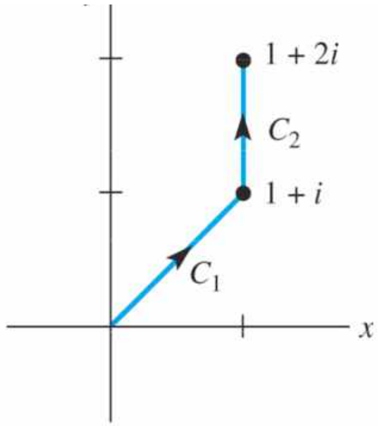
Example 2 Evaluating a contour integral

$C_1 : y = x, z = x + ix, 0 \le x \le 1$

$C_2 : x = 1, 1 \le y \le 2, z = 1 + iy,\ C = C_1 \cup C_2$

Evaluate $\int_C (x^2 + iy^2)\, dz$

Solution



$$\int_C (x^2 + iy^2)\, dz = \int_{C_1} (x^2 + iy^2)\, dz + \int_{C_2} (x^2 + iy^2)\, dz$$

$$\int_{C_1} (x^2 + iy^2)\, dz = \int_0^1 (x^2 + ix^2)(1 + i)\, dx$$

$$= (1 + i)^2 \int_0^1 x^2 dx = \frac{(1 + i)^2}{3} = \frac{2}{3} i$$

$$\int_{C_2} (x^2 + iy^2)\, dz = \int_1^2 (1 + iy^2)\, i\, dy = -\int_1^2 y^2\, dy + i \int_1^2 dy = -\frac{7}{3} + i$$

$$\int_C (x^2 + iy^2)\, dz = \tfrac{2}{3} i + (-\tfrac{7}{3} + i) = -\tfrac{7}{3} + \tfrac{5}{3} i$$

■ 곡선의 길이

데카르트 평면에서 곡선 $\mathrm{r}(t) = \langle x(t), y(t) \rangle, a \le t \le b$의 길이 $s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

복소평면에서 곡선 $z(t) = x(t) + iy(t), a \le t \le b$의 길이 $|z'(t)| = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

Theorem 18.1. A Bounding Theorem

f 가 매끄러운 곡선 C에서 연속, C상의 모든 z에 대하여 $|f(z)| \le M$

$\Rightarrow \left| \int_C f(z)\, dz \right| \le ML$ (L은 곡선 C의 길이)

Proof

$$\left| \sum_{k=1}^{n} f(z_k^*) \triangle z_k \right| \le \sum_{k=1}^{n} \left| f(z_k^*) \right| \left| \triangle z_k \right| \le M \sum_{k=1}^{n} \left| \triangle z_k \right| \le ML$$

$\leftarrow \left| \triangle z_k \right|$는 곡선 C의 점 z_k와 z_{k-1}을 연결하는 현(chord)의 길이, $\sum_{k=1}^{n} \left| \triangle z_k \right| \le L$

$$\therefore \int_C f(z)\, dz = \lim_{\| P \| \to 0} \left| \sum_{k=1}^{n} f(z_k^*) \triangle z_k \right| \le ML$$

Example 4 Bound for a contour integral

$$\oint_C \frac{e^z}{z+1} dz, C: |z| = 4$$

Solution

$$\left| \frac{e^z}{z+1} \right| \le \frac{|e^z|}{|z|-1} = \frac{|e^z|}{3} \le \frac{e^4}{3} \quad \leftarrow |e^z| = |e^{x+iy}| = e^x |e^{iy}| = e^x \le e^4$$

$$\left| \oint_C \frac{e^z}{z+1} dz \right| \le \frac{8\pi e^4}{3}$$

■ Circulation (순환) and Net Flux (순유량)

양의 방향을 가지는 단순폐곡선 $C:\mathbf{r}=\mathbf{r}(t)=\langle x(t),y(t)\rangle$

단위접선벡터 $\mathbf{T}=\dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}=\dfrac{d\mathbf{r}/dt}{ds/dt}=\dfrac{d\mathbf{r}}{ds}\rightarrow \mathbf{T}ds=d\mathbf{r}=\langle dx,dy\rangle,\ \mathbf{T}=\dfrac{d\mathbf{r}}{ds}=\left\langle \dfrac{dx}{ds},\dfrac{dy}{ds}\right\rangle$

단위법선벡터 $\mathbf{N}$은 $\mathbf{T}\cdot\mathbf{N}=0,\ \|\mathbf{N}\|=1\rightarrow\mathbf{N}=\left\langle \dfrac{dy}{ds},-\dfrac{dx}{ds}\right\rangle\rightarrow\mathbf{N}ds=\langle dy,-dx\rangle$

복소함수 $f(z)=u(x,y)+iv(x,y)$

곡선 C 주위의 순환(circulation around C) $\displaystyle\oint_C \mathbf{f}\cdot\mathbf{T}\,ds=\oint_C u\,dx+v\,dy$ $\rightarrow$ C를 회전하는 흐름의 경향

곡선 C 통과하는 순유량(net flux acroos C) $\displaystyle\oint_C \mathbf{f}\cdot\mathbf{N}\,ds=\oint_C u\,dy-v\,dx$

→ 유체가 곡선 C에 의해 둘러싸인 영역으로 단위시간당 들어가는 양과 빠져나가는 양의 차이로 정의한다.

$$\left(\oint_C \mathbf{f}\cdot\mathbf{T}\,ds\right)+i\left(\oint_C \mathbf{f}\cdot\mathbf{N}\,ds\right)=\oint_C (u-iv)(dx+i\,dy)=\oint_C \overline{f(z)}\,dz$$

순환 $=\mathrm{Re}\left(\displaystyle\oint_C \overline{f(z)}\,dz\right)$

순유량 $=\mathrm{Im}\left(\displaystyle\oint_C \overline{f(z)}\,dz\right)$

Example 5 Net flux

$f(z)=(1+i)z$. Compute net flux across and circulation around $C: |z|=1$

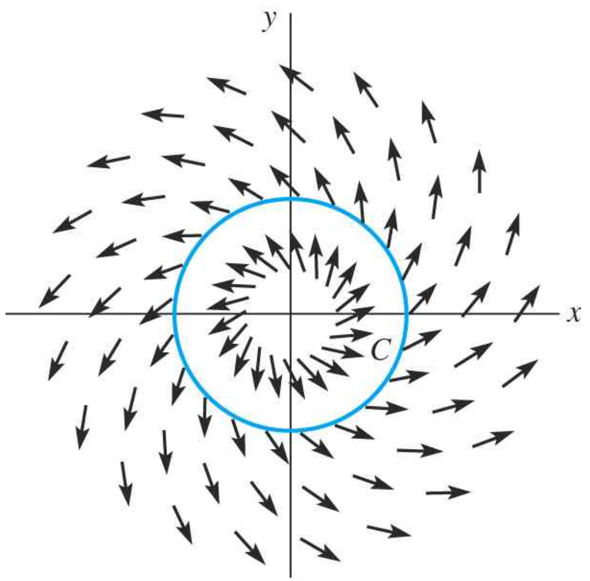
Solution

$$\overline{f(z)}=(1-i)\bar{z},\ z(t)=e^{it},\ 0\le t\le 2\pi$$

$$\oint_C \overline{f(z)}\,dz = \int_0^{2\pi}(1-i)e^{-it}ie^{it}\,dt = (1+i)\int_0^{2\pi}dt = 2\pi(1+i)$$

$2\pi i$: net flux across C

2π : circulation around $C: |z|=1$



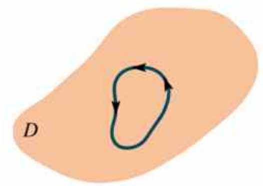
flow $f(z)=(1+i)z$

18.2 Cauchy-Goursat Theorem

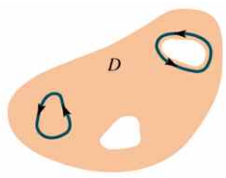
■ Introduction

열린 영역(domain)에서 양의 방향(반시계방향)의 단순 폐곡선상(simple closed curve)에서의 경로적분(contour integral)

■ Simply connected domain(단순연결 열린영역) and Multiply connected domain (다중연결 열린영역)



Simply connected domain



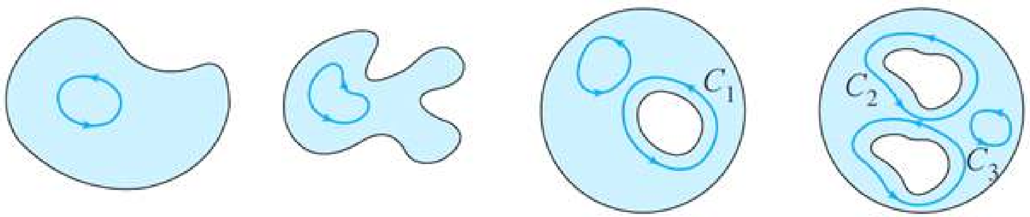
Multiply connected domain

단순연결 열린영역

:어떤 열린 영역 D내의 임의의 단순폐곡선 C가 D의 밖으로 벗어나지 않으면서 D내의 한 점으로 줄어들 수 있는 영역

다중연결 열린영역 : 단순연결 열린영역이 아닌 열린 영역

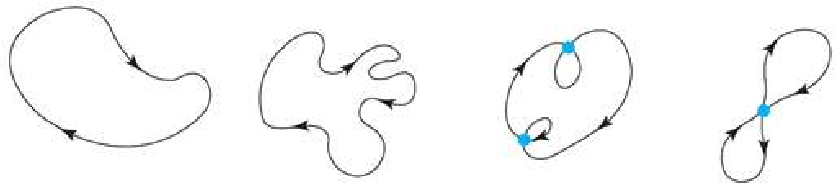
※ 다중연결 열린영역은 구멍(hole)이 있다. 구멍이 하나이면 2중 연결 열린영역(doubly connected domain),
구멍이 2개이면 3중 연결 열린영역(triply connected domain)



(a) 단순연결영역 (b) 단순연결영역 (c) 이중연결영역 (D) 삼중연결영역

단순/다중연결영역(경계 불포함)

단순폐곡선 (simple closed curve) : 자기 자신을 교차하거나 접촉하지 않는 닫힌곡선



(a) (b) (c) (d)

(a)(b) $\longrightarrow$ 단순폐곡선

■ Cauchy's Theorem

f: analytic on simply connected domain D

f' : continuous on D

$\Rightarrow$ for any simply closed contour C in D, $\oint_C f(z)dz = 0$

Proof

$$f(z) = u(x, y) + iv(x, y), \quad dz = dx + idy$$

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + idy) = \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

f: analytic $\rightarrow u, v, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$:continuous, $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$,

By Green's theorem,

$$\left.\begin{aligned} \oint_C udx - vdy &= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy = 0 \\ \oint_C vdx + udy &= \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0 \end{aligned}\right\} \therefore \quad \oint_C f(z)\,dz = 0$$

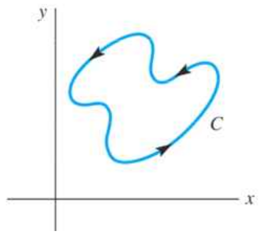
where R : region arounded by C

Theorem 18.2.1 Cauchy-Goursat Theorem

f : analytic on simply connected domain D

$\Rightarrow$ for any simply closed contour C in D, $\oint_C f(z)dz = 0$

Example 1 Applying the Cauchy-Goursat Theorem



$\oint_C e^z\,dz = 0 \leftarrow e^z$:analytic on C and C : simply closed curve (positive)

Example 2 Applying the Cauchy-Goursat Theorem

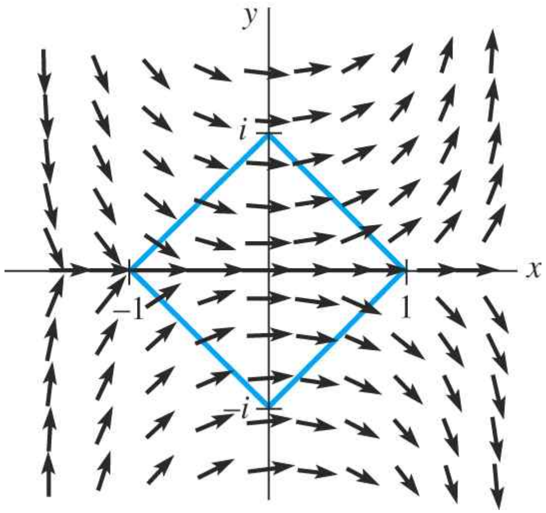
C: $(x-2)^2 + \dfrac{(y-5)^2}{4} = 1$, $\oint_C \dfrac{1}{z^2}dz = 0 \quad \leftarrow \quad \dfrac{1}{z^2}$:analytic on the region inside C

Example 3 Applying the Cauchy-Goursat Theorem

C; square with vertices $z=1, z=i, z=-1, z=-i$

$f(z)=\overline{\cos z}$: $\oint_C \overline{f(z)}\,dz = \oint_C \cos z\,dz = 0$ ← Cauchy-Gousat theorem

∴ net flux of $f(z)=0$, circulation of $f(z)=0$

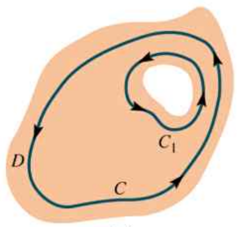


flow $f(z)=\overline{\cos z}$

■ Cauchy-Goursat Thorem for Multiply Connected Domains

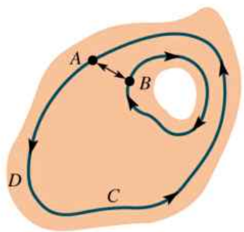
f가 다중연결 열린 영역에서 해석적일 때는 D내의 모든 단순폐곡선 C에 대해 $\oint_C f(z)dz = 0$이라고 할 수 없다.

이중연결 열린영역에서



(a)

→



(b)

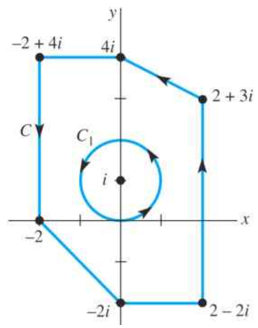
By Cauchy-Gousat theorem,

$$\oint_C f(z)dz + \int_A^B f(z)dz + \oint_{-C_1} f(z)dz + \int_B^A f(z)dz = 0 \rightarrow \oint_C f(z)dz - \oint_{C_1} f(z)dz = 0$$

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$ ← 경로변형의 원리(deformation of contour)

Example 4 Applying deformation of contour

$C_1 : |z-i| = 1,\ z-i = e^{ti},\ 0 \le t \le 2\pi,\ dz = ie^{ti}dt$



$$\oint_C \frac{dz}{z-i} = \oint_{C_1} \frac{dz}{z-i} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

■ z_0가 임의의 단순폐곡선 C의 내부의 임의의 상수 복소수일 때

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i, & n = 1 \\ 0, & n\text{은 1이 아닌 정수} \end{cases}$$

Proof

경로변형의 원리에 의해 단순폐곡선 C의 내부에 C와 겹치지 않는 임의의 단순폐경로 C_1; $|z-z_0|=r$에 대해

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$

$C_1 : z(t)-z_0 = re^{it},\ 0 \le t \le 2\pi \rightarrow z'(t) = ire^{it}$

$$\oint_C \frac{1}{(z-z_0)^n}dz = \int_0^{2\pi} \frac{1}{(re^{it})^n} ire^{it}dt = \frac{i}{r^{n-1}}\int_0^{2\pi} e^{-it(n-1)}dt = \frac{1}{r^{n-1}} i\left[-\frac{1}{i(n-1)}e^{-it(n-1)}\right]_0^{2\pi} = 0 \quad (n \ne 1\text{인 정수})$$

$$n=1 \rightarrow \oint_C \frac{1}{z-z_0}dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}dt = 2\pi i$$

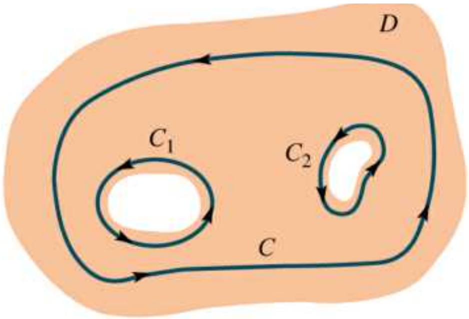
Example5

$C : |z-2| = 2$

$$\frac{5z+7}{z^2+2z-3} = \frac{3}{z-1} + \frac{2}{z+3}$$

$$\oint_C \frac{5z+7}{z^2+2z-3}dz = 3\oint_C \frac{dz}{z-1} + 2\oint_C \frac{dz}{z+3}$$

$$\oint_C \frac{5z+7}{z^2+2z-3}dz = 3(2\pi i) + 2(0) = 6\pi i$$



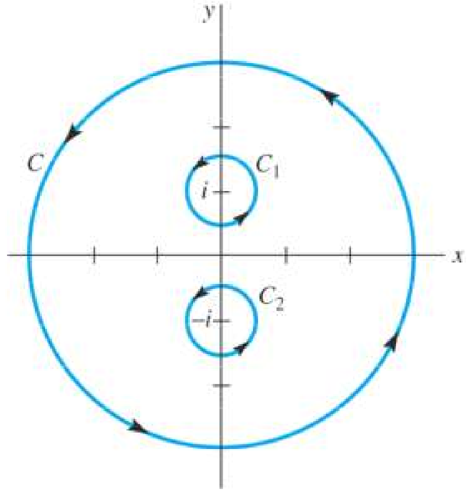
$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz$$

Theorem 18.2.2 Cauchy-Goursat Theorem for multiply connected domain
(다중연결 열린 영역에 대한 Cauchy-Goursat Theorem)

$C, C_1, \cdots, C_n$들이 양의 방향의 단순폐곡선으로서 $C_1, \cdots, C_n$은 C의 내부에 있고,
$C_1, \cdots, C_n$의 각각의 내부는 서로 겹치지 않는다고 하자.
f가 모든 경로에서 해석적이고, C의 내부에 속하면서 동시에 $C_1, \cdots, C_n$의 외부에 있는 모든 점에서 해석적

$\Rightarrow$ $$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz.$$

Example 6 Applying Theorem 18.2.2



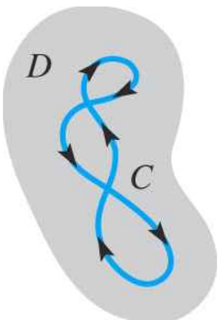
$$\frac{1}{z^2+1} = \frac{1/2i}{z-i} - \frac{1/2i}{z+i}$$

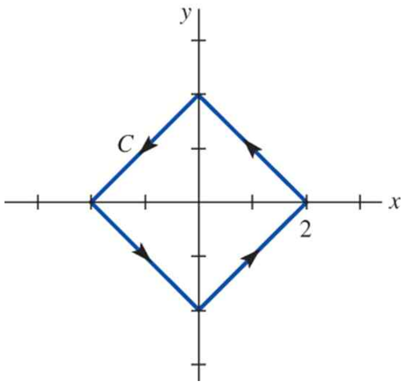
$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\oint_C \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

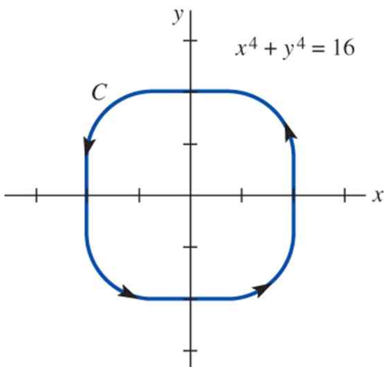
$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\oint_{C_1} \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz + \frac{1}{2i}\oint_{C_2} \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

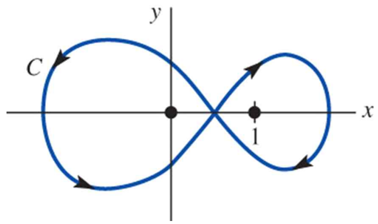
$$= \frac{1}{2i}\oint_{C_1} \frac{dz}{z-i} - \frac{1}{2i}\oint_{C_1} \frac{dz}{z+i} + \frac{1}{2i}\oint_{C_2} \frac{dz}{z-i} - \frac{1}{2i}\oint_{C_2} \frac{dz}{z+i}$$

$$\oint_{C_1} \frac{dz}{z-i} = 2\pi i\ , \qquad \oint_{C_2} \frac{dz}{z+i} = 2\pi i \qquad \oint_C \frac{dz}{z^2+1} = \pi - \pi = 0$$









18.3 Independence of Path

■ Introduction

실변수 미적분학에서 함수 f가 $F'(x)=f(x)$를 만족하는 역도함수(antiderivative) F를 가지면 미적분학의 기본정리(Fundamental theorem of Calculus)에 의해 함수 f의 정적분은

$$\int_a^b f(x)dx = F(b) - F(a)$$

이며, $\int_a^b f(x)dx$는 적분구간의 시작점과 끝점인 a와 b의 값에 의존한다.

실선적분(real line integral) $\int_C Pdx + Qdy$의 값은 일반적으로 경로 C에 따라 다르다.

만일 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$이면 $\int_C Pdx + Qdy$의 값은 경로 C와 무관하다.(경로에 독립)

(선적분의 기본정리, Fundamental theorem of line integral)

Definition 18.3.1 Independence of the Path

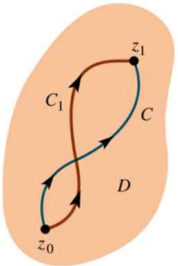
z_0와 z_1은 열린 영역 D내의 점이라 하자. 경로적분 $\int_C f(z)dz$가 시작점이 z_0이고, 끝점이 z_1인 D내에 있는 모든 경로 C에 대해 적분값이 같다면 $\int_C f(z)dz$는 경로에 독립(Independence of Path)이라 한다.

Theorem 18.3.1 Analyticity Implies Path Independence

f: analytic on simply connected domain D

$\Rightarrow$ contour integral is independent of path

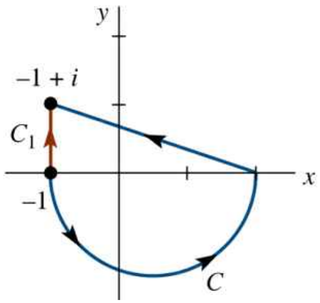
Proof



By Cauchy-Gousat theorem, $\int_C f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$ $\quad\therefore\quad$ $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$

Example 1 Choosing a different path

$$\int_C 2z\,dz = \int_{C_1} 2z\,dz = -2\int_0^1 y\,dy - 2i\int_0^1 dy = -1 - 2i$$



Definition 18.3.2 Antiderivative

f : continuous on simply connected domain D

$F'(z) = f(z)$ on D

$\Leftrightarrow$ F: 역도함수(antiderivative) of f

(예) $F(z) = -\cos z$는 $F'(z) = \sin z$이므로 $f(z) = \sin z$의 역도함수

■ $f(z)$의 역도함수 또는 부정적분(Indefinte integral)은

$$\int f(z)dz = F(z) + C \ (C\text{는 임의의 복소상수})$$

로 나타낸다.

■ 함수 f의 역도함수 F는 열린 영역(domain) D의 모든 점에서 미분가능하므로 D에서 해석적(analytic)

■ 모든 미분가능한 함수는 연속이므로 함수 f의 역도함수 F는 연속(continuous)

Theorem 18.3.2 Fundamental Theorem for Contour Integrals

f : continuous on domain D, $F'(z) = f(z)$ on D

$\Leftrightarrow$, $\int_C f(z)\, dz = \int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0)$

for any contour C in D with initial point $z_0$and terminal point z_1

[즉, 연속함수 f가 역도함수 F를 가지면 $\int_C f(z)\, dz$는 경로에 독립]

Proof

매끄러운 곡선 $C: z(t) = x(t) + iy(t), a \le t \le b$에 대하여

$$\int_C f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt = \int_a^b F'(z(t))z'(t)\, dt = \int_a^b \frac{d}{dt} F(z(t))\, dt$$

$$= F(z(t))\Big|_a^b = F(z(b)) - F(z(a)) = F(z_1) - F(z_0)$$

■ f : continuous on domain D, $F'(z) = f(z)$ on D

$\oint_C f(z)\, dz = F(z_1) - F(z_0) = 0$, for any closed contour C in D

Example 2 Using antiderivative

$$\int_{-1}^{-1+i} 2z\,dz = z^2\Big|_{-1}^{-1+i} = (-1+i)^2 - (-1)^2 = -1 - 2i$$

Example 3 Using antiderivative

$$\int_C \cos z\,dz = \int_0^{2+i} \cos z\,dz = \sin z\Big|_0^{2+i} = \sin(2+i) - \sin 0 = \sin(2+i)$$

$$= \sin(2)\cosh(1) + i\cos(2)\sinh(1) = 1.4031 - 0.4891i$$

■ Existence od Antiderivatives (역도함수의 존재성)

f : continuous on domain D, $\int_C f(z)\,dz$: independence of path $\Rightarrow f$ has antiderivative on D

Theorem 18.3.3 Existence od Antiderivatives (역도함수의 존재성)

f: analytic on a simply connected domain(단순연결 열린영역) D

$\Rightarrow$ f has antiderivative on D

[There exists a function F such that $F'(z) = f(z)$ for all z in D.]

Proof

f: analytic on a simply connected domain(단순연결 열린영역) $D \Rightarrow f(z)$: continuous on D

f: analytic on simply connected domain D ⇒ contour integral is independent of path (Th 18.3.1)

$\therefore$ f : continuous on domain D, $\int_C f(z)\,dz$: independence of path $\Rightarrow f$ has antiderivative on D

■

D : entire complex plane except $z=0$(multiply connected domain)

$\frac{1}{z}$: analytic on D

C: any simple closed contour containing $z=0$

$\Rightarrow \quad \oint_C \frac{1}{z} dz \neq 0$

[$\leftarrow C : z(t) = e^{it}, 0 \le t \le 2\pi \rightarrow \quad \oint_{|z|=1} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} (ie^{it}) dt = 2\pi i \neq 0$]

[$\leftarrow$ for any constant complex number z_0 interior to any simple closed contour C,

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i, & n=1 \\ 0, & n \text{ an integer } \neq 1 \end{cases}$$]

Recall

f:continuous on domain D, $F'(z) = f(z)$ on D

$\Rightarrow \oint_C f(z)\ dz = F(z_1) - F(z_0) = 0$, for any closed contour C in D

■

D : entire complex plane except $z=0$(multiply connected domain)

$f(z) = \frac{1}{z}$: analytic on D,

$\oint_C \frac{1}{z} dz \neq 0$ for any closed contour C in D

$\frac{d}{dz} Ln\, z = \frac{1}{z}$ except $Re z \leq 0, Im z = 0$ → $\mathrm{Ln} z$: not analytic on the nonpositive real axis

→ $\mathrm{Ln} z$: not analytic in D (entire complex plane except $z=0$)

∴ $\mathrm{Ln} z$: not antiderivative of $\frac{1}{z}$ in D

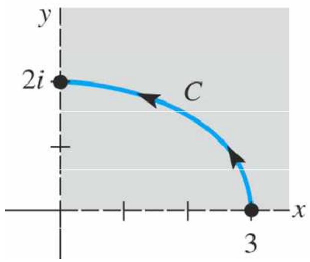
※ $\int_{|z-1| = \frac{1}{2}} \frac{1}{z} dz = 0$

Example 4 **Using the Logarithmic Function**

$$\int_3^{2i} \frac{1}{z}\,dz = \text{Ln } z\Big|_3^{2i} = \text{Ln } 2i - \text{Ln } 3$$

$$\text{Ln } 2i = \log_e 2 + \frac{\pi}{2} i \ , \qquad \text{Ln } 3 = \log_e 3$$

$$\int_3^{2i} \frac{1}{z}\,dz = \log_e \frac{2}{3} + \frac{\pi}{2} i = -0.4055 + 1.5708i$$



Remark

f, g: analytic on simply connected domain D containing contour C with initial point $z_0$and terminal point z_1:

$$(f(z)g(z))' = f'(z)g(z) + f(z)g'(z)$$

$$f(z)g(z)|_{z_0}^{z_1} = \int_{z_0}^{z_1} f'(z)g(z)dz + \int_{z_0}^{z_1} f(z)g'(z)dz$$

$$\therefore \int_{z_0}^{z_1} f'(z)g(z)dz = f(z)g(z)|_{z_0}^{z_1} - \int_{z_0}^{z_1} f(z)g'(z)dz$$

18.4 Cauchy's Integral Formulas

■ First Formula

단순연결 열린영역내에 있는 임의의 점 z_0에서의 해석함수 f의 값은 경로적분에 의해 표현할 수 있다.

(The value of an analytic function f at any point z_0 in a simply connected domain can be represented by a contour integral.)

Theorem 18.4.1 Fundamental Cauchy's Integral Formula

$f(z)$: analytic in simply connected domain D

C: simply closed contour lying entirely within D

z_0 : any points interior to C

$$\Rightarrow f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$

■ 단순연결영역이 명확히 정의되지 않은 경우, 좀 더 실용적인 Fundamental Cauchy's Integral Formula

f가 단순폐곡선 C상의 모든 점과 내부의 모든 점에서 해석적이고, z_0가 C의 내부에 있는 임의의 점이면

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$

(f : analytic at all points within and on a simple closed contour C, z_0 : any interior point to C

$$\Rightarrow f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$)

Example 1 Using Cauchy's integral formula

$C: |z| = 2$ evaluate $\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz$

Solution

$z + i = 0 \rightarrow -i$:inside C

$f(z) = \frac{z^2 - 4z + 4}{z + i}$: analytic at all points within and on a simple closed contour C

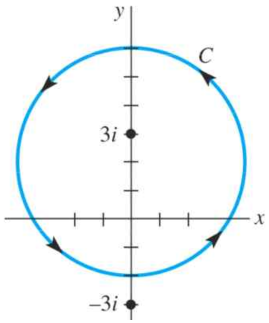
$$\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz = 2\pi i f(-i) = 2\pi i(3 + 4i) = 2\pi(-4 + 3i)$$

Example 1 Using Cauchy's integral formula

$C: |z - 2i| = 4$, evaluate $\oint_C \frac{z}{z^2 + 9}\, dz$

Solution

$z^2 + 9 = (z + 3i)(z - 3i) \rightarrow 3i$:inside C



$f(z) = \frac{z}{z + 3i}$: analytic at all points within and on a simple closed contour C

$$\oint_C \frac{z}{z^2 + 9}\, dz = \oint_C \frac{\frac{z}{z + 3i}}{z - 3i}\, dz = 2\pi i f(3i) = 2\pi i \frac{3i}{6i} = \pi i$$

Example 3 Flux and Cauchy's integral formula

$f(z) = \frac{k}{\overline{z-z_1}} = \frac{a+ib}{\bar{z}-\bar{z}_1}$: flux(흐름) in the domain $z \neq z_1$

C: simple closed contour containing $z = z_1$

By Cauchy's integral formula,

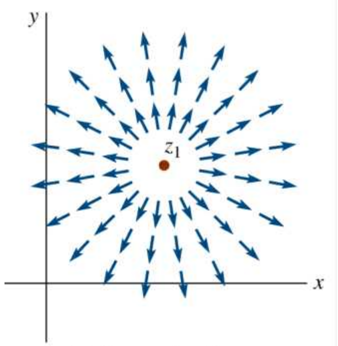
$$\oint_C \overline{f(z)}\, dz = \oint_C \frac{a-ib}{z-z_1}\, dz = 2\pi i(a-ib)$$

∴ Circulation around C (C 주위의 순환) = $Re\left(\oint_C \overline{f(z)}dz\right) = 2\pi b$

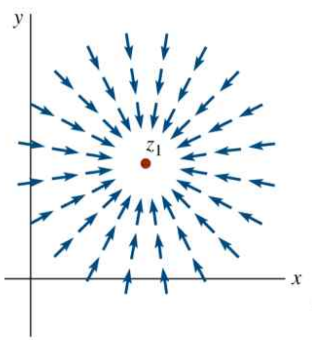
Net flux across C(C를 통과하는 순유량)= $Im\left(\oint_C \overline{f(z)}dz\right) = 2\pi a$

※ z_1: not insde C ⇒ Circulation=0, Net flux=0

※ k : real number ⇒ Circulation=0, Net flux=$2\pi k$



(a) Source: $k > 0$



(b) Sink: $k < 0$

$f(z) = \frac{k}{\overline{z-z_1}}$

■ Second Formula

단순연결 열린영역에서 해석함수는 모든 계수(order)의 도함수가 존재한다.

(An anlytic function f in a simply connected domain posseses derivatives of all orders.)

f : analytic at z_0 $\Rightarrow$ $f'(z_0), f''(z_0), \cdots, f^{(n)}(z_0)$: analytic at z_0

Theorem 18.4.2 Cauchy's Integral Formula for Derivative

$f(z)$: analytic in simply connected domain D

C: simply closed contour lying entirely within D

z_0 : any points interior to C

$\Rightarrow$ $f(z_0) = \dfrac{1}{2\pi i} \oint_C \dfrac{f(z)}{z - z_0} dz$

■ $f(z) = u(x,y) + iv(x,y)$: analytic at a point z $\Rightarrow$ its derivatives of all orders exists at z and are continuous

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y\,\partial x} - i\frac{\partial^2 u}{\partial y\,\partial x}$$

$$\vdots$$

$\therefore$ Real function $u(x,y), v(x,y)$ have continuous partial derivatives of all orders at a point z of analyticity

■ 단순연결영역이 명확히 정의되지 않은 경우, 좀 더 실용적인 Cauchy's Integral Formula for derivatives

f가 단순폐곡선 C상의 모든 점과 내부의 모든 점에서 해석적이고, z_0가 C의 내부에 있는 임의의 점이면

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

(f : analytic inside and on the boundary C of simply connected domain D

C : simply closed curve in D

z_0 is inside C

$\Rightarrow$ $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$)

Example 4 Using Cauchy's Integral Formula for derivatives

$C: |z|=1$, evaluate $\oint_C \frac{z+1}{z^4+4z^3}\,dz$

Solution

$\frac{z+1}{z^4+z^3}$: not analytic at $z=0, -4$

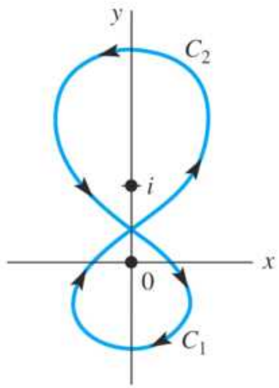
$f(z)=\frac{z+1}{z+4}$: analytic on $|z|\le 1$

$z=0$: interior point on $C: |z|=1$

$$\frac{z+1}{z^4+4z^3}=\frac{\frac{z+1}{z+4}}{z^3}$$

$$\oint_C \frac{z+1}{z^4+4z^3}\,dz=\frac{2\pi i}{2!}f''(0)=-\frac{3\pi}{32}i \quad \leftarrow f'(z)=\frac{3}{(z+4)^2}, f''(z)=\frac{-6}{(z+4)^3}$$

Example 5 Using Cauchy's Integral Formula for derivatives



$$\oint_C \frac{z^3+3}{z(z-i)^2}dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}dz$$

$$= -\oint_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z} + \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}dz = -I_1 + I_2$$

$$I_1 = \oint_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z} = 2\pi i\, f(0) = 2\pi i\left[\frac{z^3+3}{(z-i)^2}\right]_{z=0} = -6\pi i$$

$$I_2 = \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}dz = \frac{2\pi i}{1!}f'(0) = 2\pi i\left[\frac{2z^3-3}{z^2}\right]_{z=0} = 2\pi i(3+2i) = 2\pi(-2+3i)$$

$$\therefore \quad \oint_C \frac{z^3+3}{z(z-i)^2}dz = -I_1 + I_2 = 6\pi i + 2\pi(-2+3i) = 4\pi(-1+3i)$$

Theorem 18.4.3 Liouville's Theorem

The only bounded entire functions are constants

■ Fundamental Theorem of Algebra

Nonconstant polynomial $P(z)=0$ has at least one root