1. Solution) $\mathcal{F} = \cap\{\mathcal{Q} ; \mathcal{F}_0 \subset \mathcal{Q}$. where $\mathcal{Q}$ is σ-field$\}$

① $G \in \mathcal{F}_0 \in \mathcal{Q}$ $\quad \therefore G \in \mathcal{F}$

② $C_1, C_2, \cdots \in \mathcal{F}_0$ $\quad \because \mathcal{Q}$ is σ-field

$\Rightarrow C_1^c, C_2^c, \cdots \in \mathcal{F}_0 \in \mathcal{Q}$

$\therefore C_1 \in \mathcal{F}_0 \subset \mathcal{F}$. $C_1^c \in \mathcal{F}_0 \subset \mathcal{F}$

③ $C_1, C_2, \cdots C_n \in \mathcal{F}_0 \rightarrow \bigcup_{n=1}^{n} C_n \in \mathcal{F}_0 \quad \because \mathcal{Q}$ is σ-field

$\therefore$ if $C_1, \cdots \in \mathcal{F}_0 \subset \mathcal{F}$, $\bigcup_{n=1}^{\infty} C_n \in \mathcal{F}_0 \subset \mathcal{F}$

$\therefore \mathcal{F}$ is σ-field

2. Solution) let $B_1 = C_1$ $\quad B_n = C_n - C_{n-1}$ $\quad (C_n \cap C_{n-1}^c)$ $\quad \therefore \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} C_n$

$$P(\lim_{n\to\infty} C_n) = P(\bigcap_{n=1}^{\infty} C_n) = P(\bigcap_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} P(B_n)$$

$$= \lim_{m\to\infty} \sum_{n=1}^{m} P(B_n) = \lim_{m\to\infty} \sum_{n=1}^{\infty} P(C_n) - P(C_{n-1}) = \lim_{n\to\infty} P(C_n)$$

$$\therefore P(\lim_{n\to\infty} C_n) = \lim_{n\to\infty} P(C_n)$$

3. Solution) ① $\int_{-\infty}^{\infty} e^{-|x|} dx = \int_{-\infty}^{0} e^{x} dx + \int_{0}^{\infty} e^{-x} dx$

$$= e^{x}\Big]_{-\infty}^{0} - e^{-x}\Big]_{0}^{\infty} = 2 > 0$$

② $\int_{-\infty}^{\infty} e^{-|x|} dx = P(G) = 2 \neq 1$ $\quad \therefore$ Not probability

$P(G) = 2c = 1$ $\quad \therefore c = \frac{1}{2}$

4. Solution) (a) $P(x) = \frac{4!}{x!(4-x)!} \left(\frac{1}{4}\right)^{x} \left(\frac{3}{4}\right)^{4-x}$ $\quad P(X<1) = \frac{4!}{4!} \times \left(\frac{3}{4}\right)^4 = \frac{81}{256} < \frac{1}{2}$

$P(X \leq 1) = P(0) + P(1)$

$= \frac{81}{256} + \frac{108}{256} > \frac{1}{2}$

$\therefore x = 1$

(b) $f(x) = 3x^2$

$P(X<a) = \int_0^a 3x^2 dx = \frac{1}{2}$ $\quad a^3 = \frac{1}{2}$ $\quad \therefore x = \sqrt[3]{\frac{1}{2}}$

(c) $f(x) = \frac{1}{\pi(1+x^2)}$

$P(X<x) = \int_0^x \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} (\arctan x\Big]_0^x = \frac{1}{\pi}(\arctan x + \frac{\pi}{2})$

$\therefore x = 0$

5. solution) $y = \tan x$

$x = \arctan y$ $\qquad f_Y(y) = f_X(x) \cdot J = \frac{1}{\pi} \times \frac{1}{1+y^2}$

$|J| = \left|\frac{dx}{dy}\right| = \frac{1}{1+y^2}$

$\therefore f_Y(y) = \frac{1}{\pi(1+y^2)} \quad (-\infty < y < \infty)$

6. solution) $-1 < x < 3$ $\qquad \begin{cases} -1 < x < 1 \\ 1 < x < 3 \end{cases} \xRightarrow{Y = X^2} \begin{matrix} 0 < y < 1 \\ 1 < y < 9 \end{matrix}$

① CDF : $F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

i) $y < 0$ $\quad F_Y(y) = 0$

ii) $0 < y < 1$ $\quad F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{4} dx$

$= \frac{\sqrt{y}}{2}$

iii) $1 < y < 9$ $\quad F_Y(y) = P(-1 \le X \le \sqrt{y}) = \int_{-1}^{\sqrt{y}} \frac{1}{4} dx$

$= \frac{1}{4}(1+\sqrt{y})$

iv) $y > 9$ $\quad F_Y(y) = 1$

② PDF $\quad F_Y'(y) = f_Y(y)$

$\therefore F_Y(y) = \begin{cases} 0 & y \le 0 \\ \frac{\sqrt{y}}{2} & 0 < y < 1 \\ \frac{1}{4}(1+\sqrt{y}) & 1 < y < 9 \\ 1 & y \ge 9 \end{cases}$ $\qquad f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}} & 0 < y < 1 \\ \frac{1}{8\sqrt{y}} & 1 < y < 9 \end{cases}$

7. $E(|X-b|) = \int_{-\infty}^{\infty} |x-b| f(x)\,dx = \int_{-\infty}^{b} (b-x) f(x)\,dx + \int_{b}^{\infty} (x-b) f(x)\,dx$

$= \int_{-\infty}^{m} (b-x) f(x)\,dx + \int_{m}^{b} (b-x) f(x)\,dx - \int_{m}^{b} (x-b) f(x)\,dx + \int_{m}^{\infty} (x-b) f(x)\,dx$

$= 2\int_{m}^{b} (b-x) f(x)\,dx + \underbrace{\int_{-\infty}^{m} (b-x) f(x)\,dx + \int_{m}^{\infty} (x-b) f(x)\,dx}_{(A)}$

(A) $= \int_{-\infty}^{m} (b-m+m-x) f(x)\,dx + \int_{m}^{\infty} (x-m+m-b) f(x)\,dx$

$= \int_{-\infty}^{m} (m-x) f(x)\,dx + \int_{-\infty}^{m} (b-m) f(x)\,dx + \int_{m}^{\infty} (x-m) f(x)\,dx + \int_{m}^{\infty} (m-b) f(x)\,dx$

$= \int_{-\infty}^{\infty} |x-m| f(x)\,dx + (b-m)\left(\overbrace{\cancel{\int_{-\infty}^{m} f(x)\,dx - \int_{m}^{\infty} f(x)\,dx}}^{0}\right) \quad \because$ median

$\therefore E(|X-b|) = 2\int_{m}^{b} (b-x) f(x)\,dx + E(|X-m|)$

if $b = m$, $2\int_{m}^{b} (b-x) f(x)\,dx = 0 \longrightarrow$ minimum

$\therefore$ minimum when $b = m$

8. solution) $\int_0^b (1-F(x))\,dx = (1-F(x))x\Big]_0^b + \int_0^b F'(x)x\,dx$

$\because F'(x) = f(x),\ F(b) = 1,\quad = \int_0^b x f(x)\,dx = E(x)$

$\therefore E(X) = \int_0^b (1-F(x))\,dx$

9. solution) $f(x) = \lambda e^{-\lambda x}$

$E(X) = \int_0^\infty \lambda x e^{-\lambda x}\,dx$, let $y = \lambda x$, $dx = \frac{1}{\lambda}dy$

$= \frac{1}{\lambda}\int_0^\infty y e^{-y}\,dy = \frac{1}{\lambda}\left(-ye^{-y}\Big]_0^\infty + \int_0^\infty e^{-y}\,dy\right)$

$= \frac{1}{\lambda}\left(-e^{-y}\Big]_0^\infty\right) = \frac{1}{\lambda}$

$mgf = E(e^{tx}) = \int_0^\infty \lambda e^{(t-\lambda)x}\,dx = \frac{\lambda}{t-\lambda} e^{(t-\lambda)x}\Big]_0^\infty$

$= \frac{\lambda}{\lambda - t}$

$Var(X) = E(X^2) - \mu^2 = \int_0^\infty \lambda x^2 e^{-\lambda x}\,dx$, let $y = \lambda x$, $dx = \frac{1}{\lambda}dy$

$= \frac{1}{\lambda^2}\int_0^\infty y^2 e^{-y}\,dy = \frac{1}{\lambda^2}\left(-e^{-y}y^2\Big]_0^\infty + \int_0^\infty 2ye^{-y}\,dy\right)$

$= \frac{1}{\lambda^2}\left(-e^{-y}y^2 - 2ye^{-y} + 2e^{-y}\Big]_0^\infty\right)$

$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

$\therefore mgf = \frac{\lambda}{\lambda - t}$

$E(X) = \frac{1}{\lambda}$

$Var(X) = \frac{1}{\lambda^2}$

10. solution)

$P(2X+3Y<1) \rightarrow \quad 0 < y < \frac{1-2x}{3}$

$0 < x < \frac{1}{2}$

$= 6\int_0^{\frac{1}{2}}\int_0^{\frac{1-2x}{3}} 1-x-y\;dy\,dx$

$= \quad y - xy - \frac{1}{2}y^2\Big]_0^{\frac{1-2x}{3}} = \frac{1-2x}{3} - \frac{x-2x^2}{3} - \frac{1}{2}\times\frac{4x^2-4x+1}{9}$

$= \left(\frac{2}{3} - \frac{2}{9}\right)x^2 + \left(-\frac{1}{3} - \frac{2}{3} + \frac{2}{9}\right)x + \frac{1}{3} - \frac{1}{18} = \frac{4}{9}x^2 - \frac{7}{9}x + \frac{5}{18}$

$= 6\int_0^{\frac{1}{2}} \frac{4}{9}x^2 - \frac{7}{9}x + \frac{5}{18}\,dx = \frac{4}{27}x^3 - \frac{7}{18}x^2 + \frac{5}{18}x\Big]_0^{\frac{1}{2}} = \frac{1}{54} - \frac{7}{72} + \frac{5}{36}$

$= \frac{1}{9} - \frac{7}{12} + \frac{5}{6} = \frac{4-21+30}{36} = \frac{13}{36}$

$\therefore \frac{13}{36}$

$E(XY + 2X^2) = 6\int_0^1\int_0^{1-x}(xy+2x^2)(1-x-y)\,dy\,dx = xy - x^2y - xy^2 + 2x^2 - 2x^3 - 2x^2y$

$= \int_0^{1-x} xy - 3x^2y - xy^2 + 2x^2 - 2x^3\;dy = \frac{1}{2}xy^2 - \frac{3}{2}x^2y^2 - \frac{1}{3}xy^3 + 2x^2y - 2x^3y\Big]_0^{1-x}$

$= \frac{1}{2}x(1-x)^2 - \frac{3}{2}x^2(1-x)^2 - \frac{1}{3}x(1-x)^3 + \underline{2x^2(1-x) - 2x^3(1-x)}$

$2x^2(1-x)^2$

$= \frac{1}{2}x(1-x)^2 + \frac{1}{2}x^2(1-x)^2 - \frac{1}{3}x(1-x)^3$

$$(1-x)^2\left(\frac{1}{2}x+\frac{1}{2}x^2-\frac{1}{3}x(1-x)\right) = (1-x)^2\left(\frac{1}{6}x+\frac{5}{6}x^2\right) \quad \times 6$$

$$= (x^2-2x+1)(x+5x^2)$$

$$= x^3+5x^4-2x^2-10x^3+x+5x^2$$

$$= \int_0^1 5x^4-9x^3+3x^2+x\,dx$$

$$= x^5-\frac{9}{4}x^4+x^3+\frac{1}{2}x^2\Big]_0^1$$

$$= 2+\frac{1}{2}-\frac{9}{4} = \frac{1}{4} \qquad \therefore \frac{1}{4}$$

11. Solution)

$$y_1 = x_1+x_2 \qquad |J| = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$
$$y_2 = x_2$$

$$x_1 = y_1-y_2 \qquad f_{Y_1Y_2}(y_1,y_2) = f_{X_1X_2}(x_1,x_2)|J|$$
$$x_2 = y_2$$

$$= f_{X_1X_2}(x_1,x_2) = f_{X_1X_2}(y_1-x_2,x_2)$$

$$f_{Y_1}(y_1) = \int f_{X_1X_2}(y_1-x_2,x_2)\,dx_2$$

opposit case, $f_{Y_2}(y_2) = \int f_{X_1X_2}(y_2-x_1,x_1)\,dx_1$

$$\therefore f_Y(y) = \int f_{X_1X_2}(y-x,x)\,dx$$

12. Solution) ① $E(X|Y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} x \frac{f_{XY}(x,y)}{f_Y(y)}\,dx$

$$f_Y(y) = \int_0^y 2\,dx = 2y$$

$$\int_0^y \frac{x\times 2}{2y}\,dx = \frac{1}{y}\times\frac{x^2}{2}\Big]_0^y = \frac{y}{2}$$

② $E(Y|X) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dy = \int_0^1 y f_{Y|X}(y|x)\,dy = \int_0^1 \frac{y f_{XY}(x,y)}{f_X(x)}\,dy$

$$f_X(x) = \int_x^1 2\,dy = 2(1-x)$$

$$\int_0^1 \frac{\cancel{2}y}{\cancel{2}(1-x)}\,dy = \frac{y^2}{2(1-x)}\Big]_0^1 = \frac{1}{2(1-x)}$$

③ $\rho = \frac{\text{cov}(X,Y)-\mu_1\mu_2}{\sigma_1\sigma_2} = \frac{E(XY)-E(X)*E(Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$

$$E(XY) = \int_0^1\int_0^y 2xy\,dx\,dy = x^2y\Big]_0^y = \int_0^1 y^3\,dy = \frac{1}{4}$$

$$E(X) = \int_0^1 2x-2x^2\,dx = x^2-\frac{2}{3}x^3\Big]_0^1 = \frac{1}{3}$$

$$E(Y) = \int_0^1 2y^2\,dy = \frac{2y^3}{3}\Big]_0^1 = \frac{2}{3}$$

$$Var(X) = E(X^2) - \frac{1}{9} \quad = 2\int_0^1 x^2 (1-x)\,dx = \int_0^1 x^2 - x^3\,dx = \frac{x^3}{3} - \frac{x^4}{4}\Big]_0^1 = \frac{1}{12}$$

$$= \frac{6-4}{36} = \frac{1}{18}$$

$$Var(Y) = E(Y^2) - \frac{4}{9} \quad = 2\int_0^1 y^2 y\,dy = 2\int_0^1 y^3\,dy = \frac{y^4}{2}\Big]_0^1 = \frac{1}{2}$$

$$= \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$\rho = \frac{\frac{1}{4} - \frac{2}{9}}{\sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}} = \frac{\frac{1}{36}}{\frac{1}{18}} = \frac{1}{2}$$

$\because E(X|Y) = \frac{y}{2}$

$E(Y|X) = 2(1-x)$

$\rho = \frac{1}{2}$

13. $f(x,y) = \frac{2}{\pi} \quad ((x-1)^2 + (y+2)^2 < 1) \quad 1-\sqrt{1-(y+2)^2} < x < 1+\sqrt{1-(y+2)^2}$

$-2-\sqrt{1-(x-1)^2} < y < -2+\sqrt{1-(x-1)^2}$

$$f_X(x) = \int_{-2-\sqrt{1-(x-1)^2}}^{-2+\sqrt{1-(x-1)^2}} \frac{2}{\pi}\,dy = \frac{4}{\pi}\sqrt{1-(x-1)^2}$$

$$f_Y(y) = \int_{1-\sqrt{1-(y+2)^2}}^{1+\sqrt{1-(y+2)^2}} \frac{2}{\pi}\,dx = \frac{4}{\pi}\sqrt{1-(y+2)^2}$$

$f(x,y) \neq f_X(x) f_Y(y) = \frac{16}{\pi^2}\sqrt{1-(x-1)^2}\sqrt{1-(y+2)^2}$ $\therefore$ not independent

14. $F_Y(Y) = P(Y \le y) = 1 - P(Y > y)$

$Y = \min(X_1, X_2, X_3, X_4) = P(Y > y) = P(X_1 > y, X_2 > y, X_3 > y, X_4 > y)$

$= P(X_1 > y)P(X_2 > y)P(X_3 > y)P(X_4 > y)$ $\because$ independent

$= (1-P(X_1 \le y))(1-P(X_2 \le y))(1-P(X_3 \le y))(1-P(X_4 \le y))$

$$F_X(y) = \int_0^y 3(1-x)^2\,dx = -(1-x)^3\Big]_0^y = 1-(1-y)^3$$

$\therefore F_Y(y) = (1-y)^{12} \quad 0 < y < 1$

$f_Y(y) = F_Y'(y) = 12(1-y)^{11}$

15. $y_1 = x_1/x_2 \quad y_2 = x_2/x_3 \quad y_3 = x_3/x_4 \quad y_4 = x_4$

$x_4 = y_4 \quad , x_3 = y_3 y_4 \quad , x_2 = y_2 y_3 y_4 \quad , x_1 = y_1 y_2 y_3 y_4$

$f_{Y_1 Y_2 Y_3 Y_4}(y_1, y_2 y_3 y_4) = f_{X_1 X_2 X_3 X_4}(x_1, x_2, x_3, x_4)\,|J|$

$$J = \begin{vmatrix} y_2y_3y_4 & y_1y_3y_4 & y_1y_2y_4 & y_1y_2y_3 \\ 0 & y_3y_4 & y_2y_4 & y_2y_3 \\ 0 & 0 & y_4 & y_3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = y_2y_3y_4 \begin{vmatrix} y_3y_4 & y_2y_4 & y_2y_3 \\ 0 & y_4 & y_3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= y_2 y_3^2 y_4^3$$

$\therefore$ Join PDF $= 24 y_2 y_3^2 y_4^3$

$f_{Y_1}(y_1) = \int_0^1\int_0^1\int_0^1 24 y_2 y_3^2 y_4^3\, dy_4 dy_3 dy_2 = \int_0^1\int_0^1 6y_2 y_3^2\, dy_3 dy_2 = \int_0^1 2y_2\, dy_2 = 1$

$f_{Y_2}(y_2) = \int_0^1\int_0^1\int_0^1 24 y_2 y_3^2 y_4^3\, dy_4 dy_3 dy_1 = \int_0^1\int_0^1 6y_2 y_3^2\, dy_3 dy_2 = \int_0^1 2y_2\, dy_1 = 2y_2$

$f_{Y_3}(y_3) = \int_0^1\int_0^1\int_0^1 24 y_2 y_3^2 y_4^3\, dy_4 dy_2 dy_1 = \int_0^1\int_0^1 6y_2 y_3^2\, dy_2 dy_1 = \int_0^1 3y_3^2\, dy_1 = 3y_3^2$

$f_{Y_4}(y_4) = \int_0^1\int_0^1\int_0^1 24 y_2 y_3^2 y_4^3\, dy_3 dy_2 dy_1 = \int_0^1\int_0^1 8y_2 y_4^3\, dy_2 dy_1 = \int_0^1 4y_4^3\, dy_1 = 4y_4^3$

$f_{Y_1 Y_2 Y_3 Y_4}(y_1 y_2 y_3 y_4) = f_{Y_1}(y_1) f_{Y_2}(y_2) f_{Y_3}(y_3) f_{Y_4}(y_4) = 24 y_2 y_3^2 y_4^3$

$\therefore$ independent

16. (a) $M_X(t_1, t_2, \cdots, t_{k-1}) = E(\exp(\sum_{i=1}^{k-1} t_i X_i)) = \sum_{x \in S} \binom{n}{x_2 \ldots x_{k-1}} \prod_{i=2}^{k-1} p_i^{x_i} e^{t_i x_i}$

$$= \sum_{x \in S} \binom{n}{x_2 \cdots x_{k-1}} \prod_{i=2}^{k-1} (p_i e^{t_i})^{x_i}$$

$$= \left(\sum_{i=2}^{k-1} p_i e^{t_i}\right)^n$$

(b) $P(x_2, \cdots, x_{k-1}) = \frac{n!}{x_2! \cdots x_{k-1}! (n - \sum_{i=2}^{k-1} x_i)!} p_2^{x_2} \cdots p_{k-1}^{x_{k-1}} (1 - \sum_{i=2}^{k-1} p_i)^{n - \sum_{i=1}^{k-1} x_i}$

(c) $P(X_1 = x_1 \mid X_2 = x_2, \cdots X_{k-1} = x_{k-1})$

$$= \frac{P(x_1, \cdots, x_{k-1})}{P(x_2, \cdots x_{k-1})} = \frac{\frac{n!}{x_1! \cdots x_{k-1}! (n - x_1 - \cdots - x_{k-1})!} p_1^{x_1} \cdots p_{k-1}^{x_{k-1}} (1 - p_1 - \cdots p_{k-1})^{(n - x_1 \cdots - x_{k-1})}}{\frac{n!}{x_2! \cdots x_{k-1}! (n - x_2 \cdots - x_{k-1})!} p_2^{x_2} \cdot p_{k-1}^{x_{k-1}} (1 - p_2 - \cdots - p_{k-1})^{(n - x_2 - \cdots)}}$$

$$= \frac{(n - \sum_{i=2}^{k-1} x_i)! \, p_1^{x_1} (1 - \sum_{i=1}^{k-1} p_i)^{(n - \sum_{i=1}^{k-1} x_i)}}{x_1! (n - \sum_{i=1}^{k-1} x_i)! (1 - \sum_{i=2}^{k-1} p_i)^{(n - \sum_{i=2}^{k-1} x_i)}}$$

(d) $E(X_1 \mid x_2, \cdots x_{k-1}) = \sum_{x_1} x_1 P(X_1 = x_1 \mid X_2 = x_2, \cdots X_{k-1} = x_{k-1})$

$$= \sum_{x_1} \frac{x_1 (n - x_1 - \cdots - x_{k-1})! \, p_1^{x_1} (1 -}{x_1! (n - x_2 - \cdots - x_{k-1})!}$$

17. $f(x_1, x_2) = \frac{n!}{x_1! x_2! (n - x_1 - x_2)!} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{n - x_1 - x_2}$

$E[X_1] = np_1, \; E[X_2] = np_2.$

$\mathrm{Var}(X_1 - X_2) = E[(X_1 - X_2)^2] - (E[X_1 - X_2])^2$

$$= E(X_1^2 - 2X_1X_2 + X_2^2) - (E(X_1) - E(X_2))^2$$

$$= E(X_1^2) + E(X_2^2) - 2E(X_1X_2) - E(X_1)^2 - E(X_2)^2 + 2E(X_1)E(X_2)$$

$$= E(X_1^2) - E(X_1)^2 + E(X_2^2) - E(X_2)^2 + 2E(X_1)E(X_2) - 2E(X_1X_2)$$

independent

$$= \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$$

$$= np_1(1 - p_1) + np_2(1 - p_2)$$

$$\therefore n(p_1 - p_1^2 + p_2 - p_2^2)$$