Chapter 6. Stability

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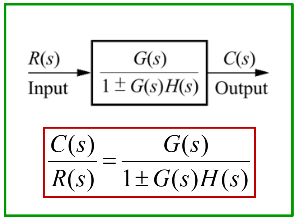
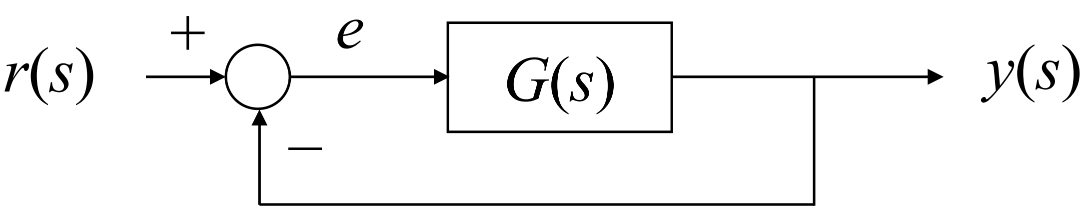
Objectives

- How to determine the stability of a system represented as a transfer function
- How to determine the stability of a system represented in state space
- How to determine system parameters to yield stability

Unity Negative Feedback

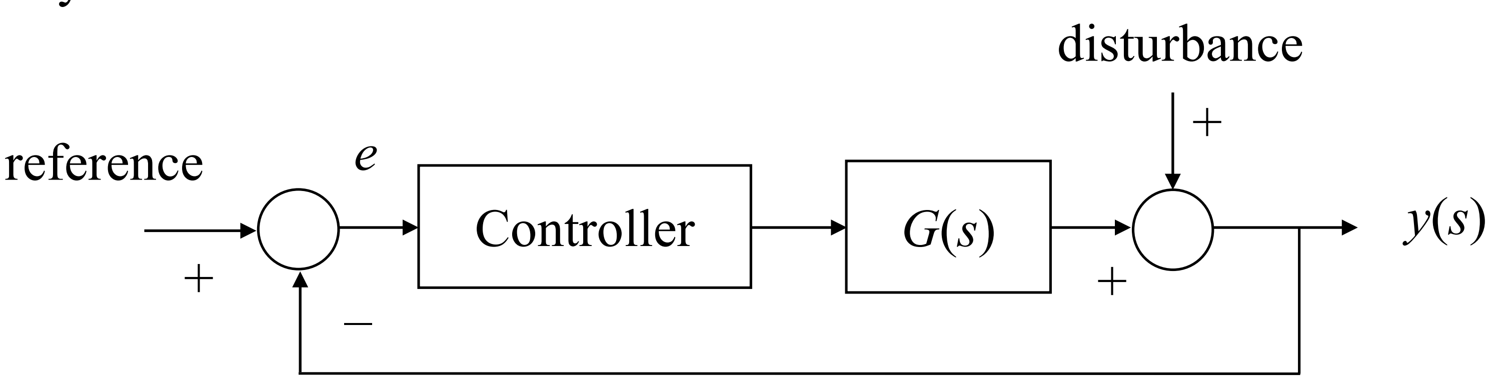




- The overall transfer function of the above system is:

$$T(s) = \frac{G(s)}{1 + G(s)}$$

- System Model:

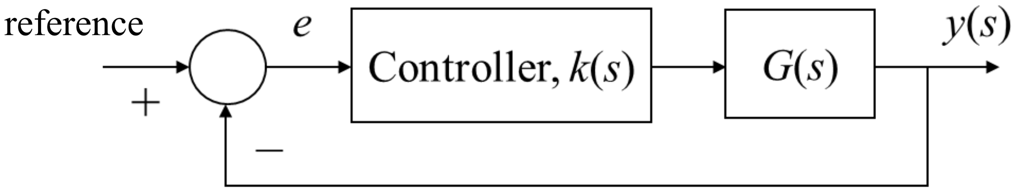


Root Locus

- The Root Locus is the graph of the closed loop poles of the system as the gain k varies from 0 to ∞.
- For certain values of k, the system goes from being stable to being unstable.

Control Strategies

- Classical Control:



- Design:
  - $k(s) = k$ (gain) $\Rightarrow$ Choose k
  - $k(s) = k\dfrac{s+z}{s+p}$ $\Rightarrow$ Choose k, z and p

- Classical control
  - Objectives: Choose $k(s)$ to satisfy:
    - Stability
    - Performance
    - Robustness
- Other kinds of control
  - Nonlinear control
  - Online control
    - Adaptive control: Identifies the plant and adapts accordingly.
  - Digital sampled data control

6.1 Introduction

- 3 requirements for control systems:
  - transient response (Chap. 4, 8)
  - stability (Chap. 6)
  - steady-state errors (Chap. 7)

- Total response of a system: $C(t) = C_{forced}(t) + C_{natural}(t)$

$$y = \frac{1}{s+1} \cdot \frac{1}{s} = \frac{1}{s} - \frac{1}{s+1} \Rightarrow y(t) = u(t) - e^{-t}u(t)$$

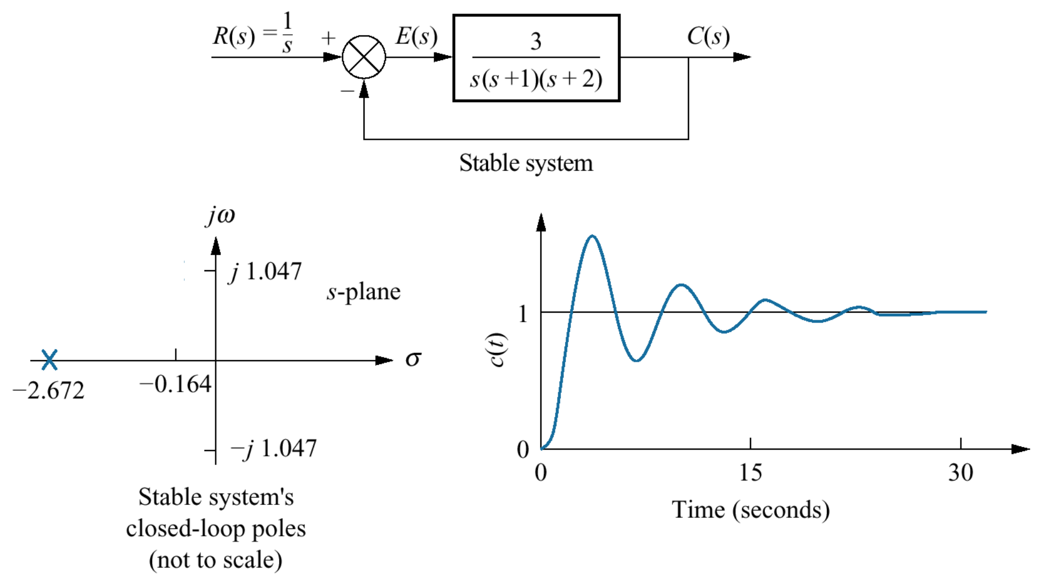
- stable system: $C_{natural}(t) \Rightarrow 0 \quad as \;\; t \to \infty$

- unstable system: $C_{natural}(t) \Rightarrow \infty \quad as \;\; t \to \infty$

- marginally stable system: $C_{natural}(t)$ remains constant *or* oscillates

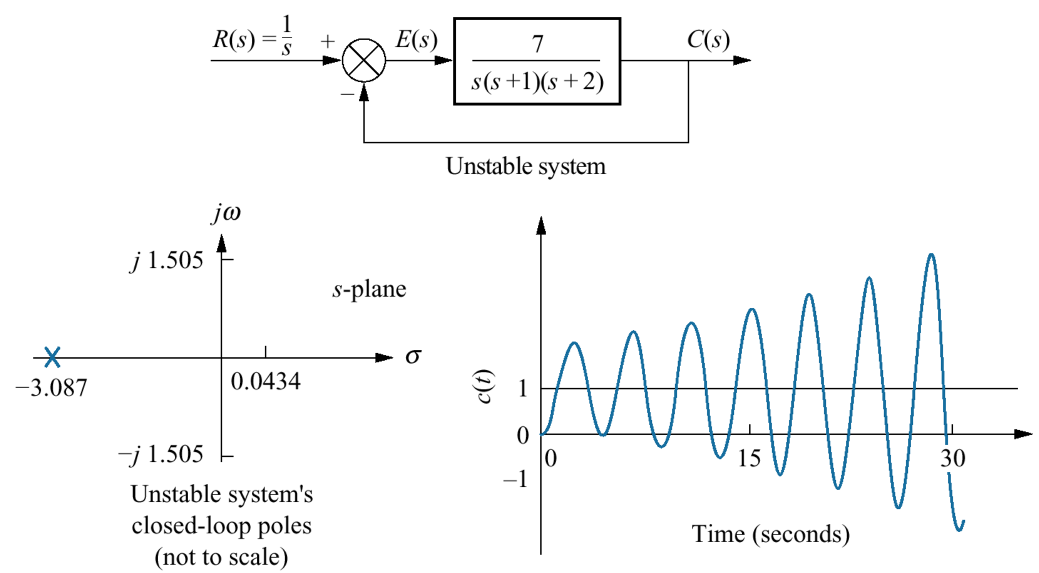
Example:

- Stable system:



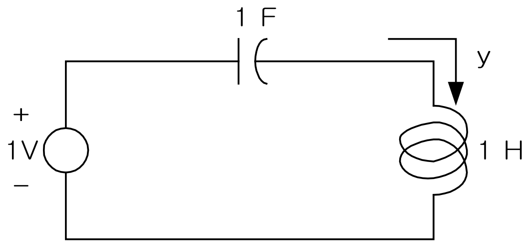
(a)

- Unstable system:



(b)

- Marginally stable system:

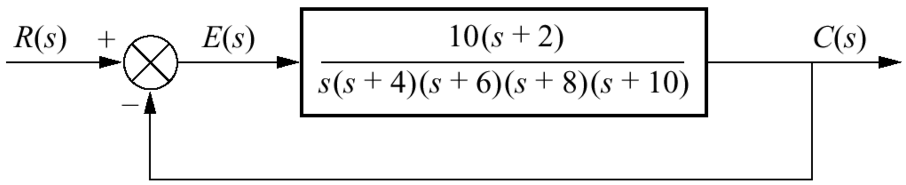


$$y=\frac{1}{s+\frac{1}{s}}\cdot\frac{1}{s}=\frac{1}{s^2+1}\Rightarrow y(t)=\sin t$$

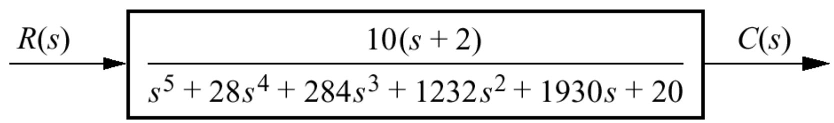
Common cause of problems in finding closed-loop poles:

(a) original system;

(b) equivalent system



(***a***)



(***b***)

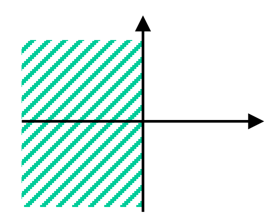
$$C(s) = \frac{Z(s)}{P(s)} R(s)$$

- Stability of a system:

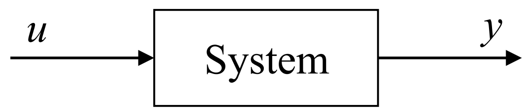
  If closed-loop transfer function has only left-half-plane poles, then

$$P(s) = \prod_{i=1}^{n} (s + a_i)$$

a_i : real and positive, or complex with a positive real part



Stability



A. Mode
B. Stability
C. Stability test: Routh criterion

$$\ddot{y} + 4\dot{y} + 3y = u$$
$$\left\{s^2Y(s) + 4sY(s) + 3Y(s)\right\} - \left\{sy(0) + 4y(0) + \dot{y}(0)\right\} = U(s)$$
$$\left(s^2 + 4s + 3\right)Y(s) - \left\{(s+4)y(0) + \dot{y}(0)\right\} = U(s)$$

$$\ddot{y} = -4\dot{y} - 3y + u$$

$$s[sY(s) - y(0)] - \dot{y}(0) + 4[sY(s) - y(0)] + 3Y(s) = U(s)$$

$$Y(s) = \underbrace{\frac{1}{s^2 + 4s + 3}U(s)}_{\boxed{\text{Zero state response}}} + \underbrace{\frac{(s+4)y(0) + \dot{y}(0)}{s^2 + 4s + 3}}_{\boxed{\text{Zero input response}}}$$

Let $Q(s) = s^2 + 4s + 3 = (s+1)(s+3) = (s - p_1)(s - p_2)$

where $p_1 = -1,\ p_2 = -3$

$p_{1,2}$ are called poles of $G(s)$, or poles of the system.

$$Y(s) = \frac{U(s)}{s^2+4s+3} - \frac{(s+4)\,y(0) + \dot{y}(0)}{s^2+4s+3}$$

$$Y(s) = \frac{U(s)}{s^2+4s+3}$$

- In the zero state response, i.e., $y(0) = \dot{y}(0) = 0$

$$Y(s) = \frac{1}{(s+1)(s+3)}U(s) = \frac{1}{(s-p_1)(s-p_2)}U(s)$$

$\Rightarrow$ $y(t)$ has terms, e^{-t} and e^{-3t} for any input!

$\Rightarrow$ e^{-t} and e^{-3t} are called modes of the system.

- In the zero input response, i.e., $u(t)=0$

$\Rightarrow$ $y(t)$ has the modes, e^{-t} and e^{-3t}.

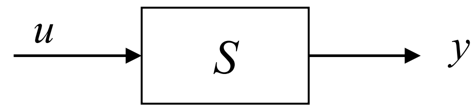
- General case: for any input u,

$$Y(s) = G(s)U(s) = \frac{P(s)}{Q(s)}U(s) = \frac{P(s)U(s)}{Q_1(s)(s-p)}$$

- In the partial fraction expansion,

$$Y(s) \to \frac{1}{s-p}, \quad y(t) \to e^{pt}$$: a mode of the system for any input.

Example: $\dot{y} = -y + u, \quad (y(0) = 0)$

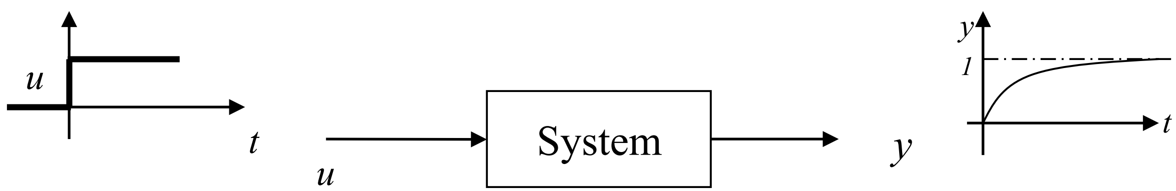


$$SY(s) = -Y(s) + U(s), \qquad Y(s) = \frac{1}{s+1} U(s)$$

$p = -1,$ a mode $\rightarrow e^{-t}$ in $y(t)$

If $u(t) = \text{step} \ \rightarrow \ U(s) = \frac{1}{s} \ \rightarrow \ Y(S) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

$y(t) = 1 - e^{-t}$



1: steady state (coming from input)

$-e^{-t}$: transient (coming from system)

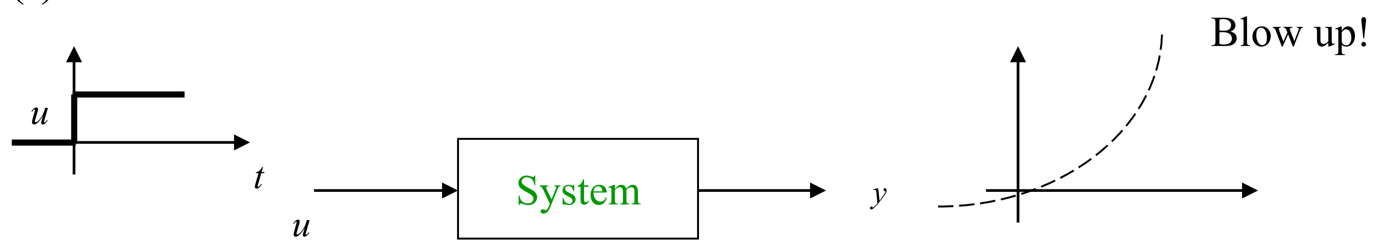
Example: $\dot{y} = y + u, \quad (y(0) = 0)$

$$SY(s) = Y(s) + U(s) \quad \rightarrow \quad Y(s) = \frac{1}{s-1} U(s)$$

$$p = 1 \rightarrow e^t \quad in \;\; y(t)$$

If $u(t) =$ step, then $\quad Y(s) = \dfrac{1}{(s-1)s} = \dfrac{1}{s-1} - \dfrac{1}{s}$

$$\rightarrow y(t) = e^t - 1$$



$\Rightarrow$ e^t is a <u>system's mode</u>: it will be the output for any input!

[Not a desirable phenomenon. Instability]

> **Definition:** A system is bounded-input bounded-output stable (BIBO stable). If whenever the input is bounded, the output is bounded.

Complex pole: $\sigma + j\omega$

$$G(s) = \frac{P(s)}{Q(s)}, \quad Q(\sigma + j\omega) = 0 \rightarrow \text{ From algebra, } Q(\sigma - j\omega) = 0$$

$$Y(s) = G(s)U(s)$$

$\rightarrow$ In PFE, $Y(s)$ will have the terms

$$\frac{1}{s-(\sigma + j\omega)} \quad \text{and} \quad \frac{1}{s+(\sigma - j\omega)}$$

$$\rightarrow y(t) : e^{(\sigma + j\omega)t} \quad \text{and} \quad e^{(\sigma - j\omega)t}$$

$$\rightarrow e^{\sigma t}(\cos \omega t + j \sin \omega t) \quad \text{and} \quad e^{\sigma t}(\cos \omega t - j \sin \omega t)$$

$$\rightarrow e^{\sigma t} \sin(\omega t + \phi) \quad \text{or} \quad e^{\sigma t}(A \cos \omega t + B \sin \omega t)$$

Example: $G(s) = \dfrac{1}{s^2 + 2s + 5}$ $\qquad u \longrightarrow \boxed{S} \longrightarrow y$

$$Y(s) = G(s)U(s)$$

Let $u(t) = \text{step} \quad \rightarrow \quad U(s) = \dfrac{1}{s}$

Poles: $s^2 + 2s + 5 = 0 \quad \rightarrow \quad s = -1 \pm 2j$

$$Y(s) = G(s)U(s) = \frac{1}{\{s-(-1+2j)\}\{s-(-1-2j)\}s}$$

After calculation (PFE),

$$\rightarrow y(t) = \frac{1}{5} - e^{-t}\left(\frac{1}{5}\cos 2t + \frac{1}{10}\sin 2t\right)$$

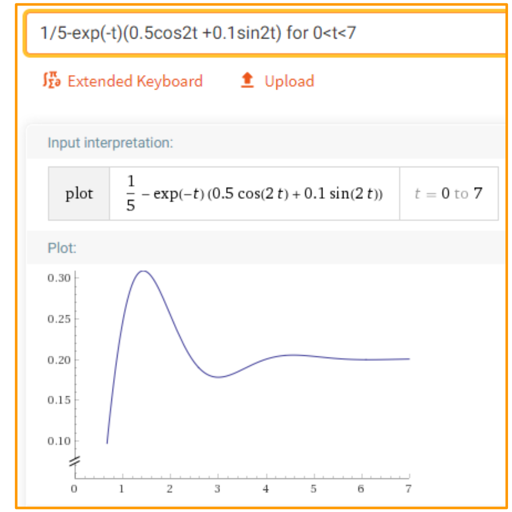
$$y(t) = \frac{1}{5} - e^{-t}C_1 \sin(2t + \phi),$$

$$\uparrow \qquad\qquad \uparrow$$

$$\phi = \arctan(2),$$

$$C_1 = \frac{1}{5\sin\phi}$$

https://www.wolframalpha.com
0.2- exp(-t)(0.5cos2t +0.1sin2t) for 0<t<7

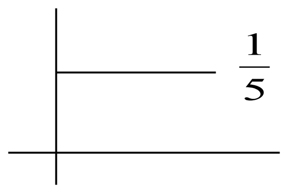


$$y(t) = \frac{1}{5} - \underline{e^{-t} C_1 \sin(2t + \phi)}$$

$\uparrow$

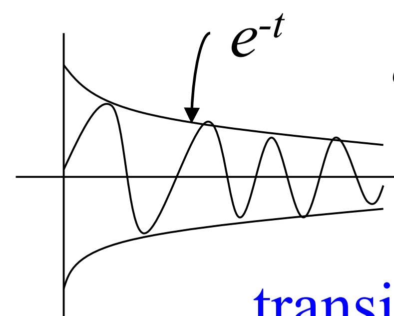
from input

from poles, $-1 \pm 2j$



steady state

$+$

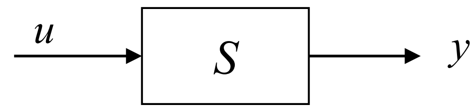


$e^{-t}\sin(2t+\phi)$

transient

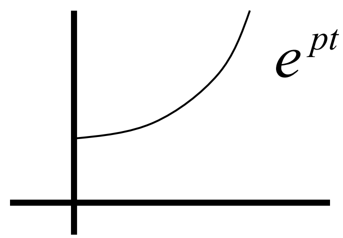
$$y(t) = \boxed{\text{steady state} + \text{transient}}$$

Summary: $Y(s) = G(s)U(s)$



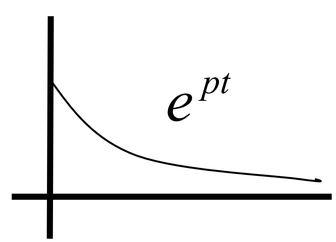
① If p is a pole of G, then the output will always have e^{pt}.

i) p is real and positive



$\Rightarrow$ the system is unstable

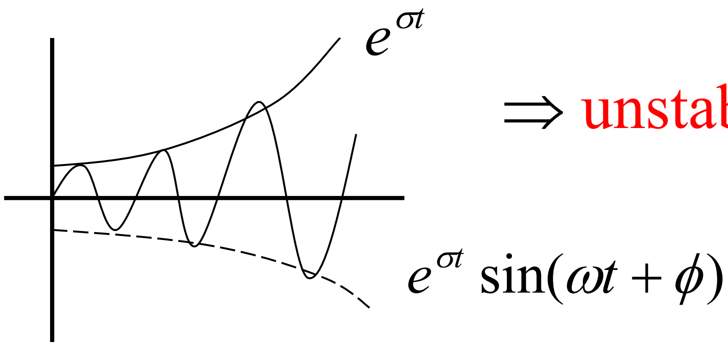
ii) p is real and negative



$\Rightarrow$ transient

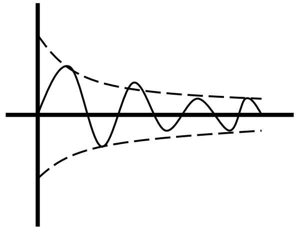
② $p = \sigma + j\omega \;\rightarrow\; \bar{p} = \sigma - j\omega \;\Rightarrow\; e^{\sigma t}\sin(\phi + \omega t)$

i) If $\sigma > 0$



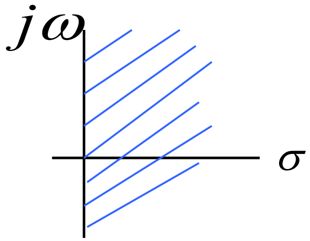
$\Rightarrow$ unstable

ii) If $\sigma < 0$



$\Rightarrow$ transient

- Thus, if $G(s)$ has pole in the open *RHP* (right half-plane) $\Rightarrow$ the system is *unstable*.

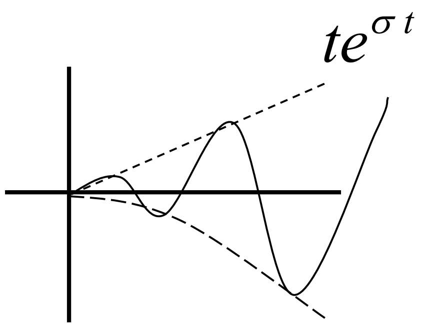


③ Double pole:

$$\sigma + j\omega, \sigma + j\omega \rightarrow \sigma - j\omega,\ \sigma - j\omega$$

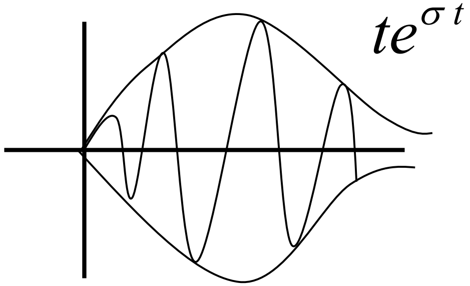
By PFE, $y(t)$ will have $te^{\sigma t}\sin(\omega t + \phi)$

i) If $\sigma > 0$



unstable

ii) If $\sigma < 0$



transient

Any pole of $G(s)$ in the open *RHP* makes the system *unstable*.
Any pole of $G(s)$ in the open *LHP* causes a *transient*.

④ Poles on the *jw*-axis ?

Example: $G(s)=\dfrac{1}{s^2+4}=\dfrac{1}{(s+2j)(s-2j)}$

i) Let $u(t)$=step

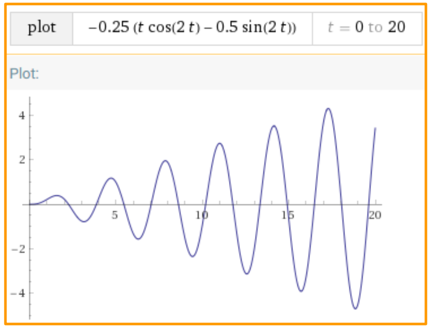
$$Y(s)=\frac{1}{(s+2j)(s-2j)s}=\frac{1}{4}\frac{1}{s}-\frac{1}{8}\frac{1}{(s+2j)}-\frac{1}{8}\frac{1}{(s-2j)}$$

$$y(t)=\frac{1}{4}-\frac{1}{8}\left(e^{-2jt}+e^{+2jt}\right)=\frac{1}{4}\left(1-\cos 2t\right)$$

$$e^{-2jt}=\cos 2t-j\sin 2t$$
$$e^{+2jt}=\cos 2t+j\sin 2t$$
$$(e^{-2jt}+e^{+2jt})=2\cos 2\mathrm{t}$$

- *y* has the term 1/4 from the input, and the term (cos2*t*)/4 from the system. → Oscillation!
- $u(t)$=step function → *y* is bounded.
- Is the system stable? → No!

https://www.wolframalpha.com
plot -0.25(tcos2t - 0.5sin2t) for 0<t<20



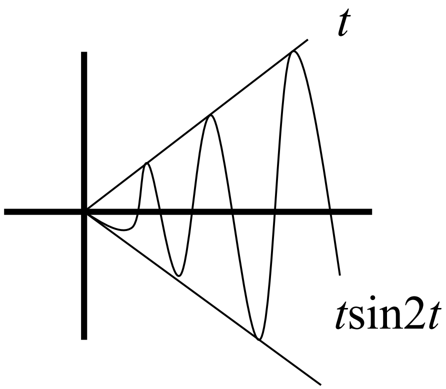
ii) Let $u(t)=\sin 2t$

$$U(s)=\frac{2}{s^2+4}\,,\qquad Y(s)=\frac{2}{(s^2+4)}\frac{1}{(s^2+4)}=\frac{2}{(s-2j)^2(s+2j)^2}$$

$$\rightarrow y(t)=-\frac{1}{4}(\underline{t\cos 2t}-\frac{1}{2}\sin 2t)$$

$\uparrow$ from double pole at $\pm 2j$

$\rightarrow$ unstable



- Pole at $\pm jw \rightarrow \sin(wt+\phi)$
  Double pole at $\pm jw \rightarrow t\sin(wt+\phi)$
  $\Rightarrow$ The system is unstable.

Fact: The system is stable if and only if all the poles of $G(s)$ are in the LHP !

Theorem: If $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s)$$

Example: $L\{e^{2t}\} = \dfrac{1}{s-2}$. Find $L\{te^{2t}\}$, $L\{t^2 e^{2t}\}$

$$\frac{d}{dt}\left(\frac{g}{f}\right) = \frac{g'\cdot f - g\cdot f'}{f^2}$$

$$L\{te^{2t}\} = (-1)\frac{d}{ds}\left(\frac{1}{s-2}\right) = (-1)\frac{-1}{(s-2)^2} = \frac{1}{(s-2)^2}$$

$$L\{t^2 e^{2t}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{d}{ds}\left(\frac{-1}{(s-2)^2}\right) = \frac{2(s-2)}{(s-2)^4} = \frac{2}{(s-2)^3}$$

$$\boxed{L\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}} \rightarrow L\{t \sin \omega t\} = (-1)\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2s\omega}{(s^2+\omega^2)^2}$$

$$L\{\cos \omega t\} = \frac{s}{s^2+\omega^2} \rightarrow L\{t \cos \omega t\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = -\frac{(s^2+\omega^2) - s(2s)}{(s^2+\omega^2)^2} = \boxed{\frac{s^2-\omega^2}{(s^2+\omega^2)^2}}$$

$$Y(s) = \frac{2}{s^2+4}\frac{1}{s^2+4} = \left(-\frac{1}{4}\right)\left(\frac{s^2-4}{(s^2+4)^2} - \frac{1}{2}\frac{2}{s^2+4}\right)$$

$$\leftarrow \quad Y(s) = \left(-\frac{1}{4}\right)\left(\frac{s^2-4-(s^2+4)}{(s^2+4)^2}\right) = \left(-\frac{1}{4}\right)\left(\frac{-8}{(s^2+4)^2}\right) = \frac{2}{(s^2+4)^2}$$

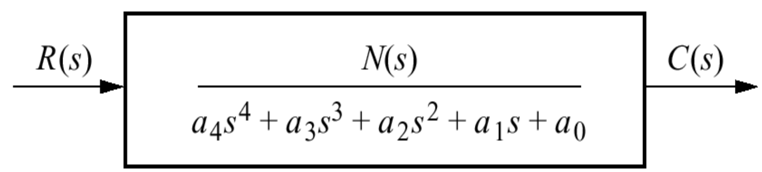
$$\rightarrow y(t) = -\frac{1}{4}\left(t\cos 2t - \frac{1}{2}\sin 2t\right)$$

6.2 Routh-Hurwitz Criterion (Routh, 1905)

- In the closed-loop system,
  - How many poles are in the left half-plane, in the right half-plane, and in the *jw*-axis.
  - *How many*, not where!
  - Routh Stability Test

- Generating a Basic Routh Table

  - A closed-loop transfer function:



  - Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

$$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$$

$$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$$

Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\boxed{\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1}$	$\boxed{\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2}$	$\boxed{\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0}$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Routh Stability Test (Routh-Hurwitz Criterion)

Given $Q(s) = s^n + a_{n\text{-}1}s^{n\text{-}1} + a_{n\text{-}2}s^{n\text{-}2} + \cdots + a_1 s + a_0$

The Routh test tells how many zeros of Q are in the RHP.

$$R(s) \longrightarrow \left(\frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \right) \longrightarrow C(s)$$

Routh Table:

$$\begin{array}{c|ccc}
s^4 & a_4 & a_2 & a_0 \\
s^3 & a_3 & a_1 & 0 \\
s^2 & -\dfrac{1}{a_3}\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix} = b_1 & b_2 & b_3 \\
s^1 & c_1 & c_2 & c_3 \\
s^0 & d_1 & d_2 & d_3
\end{array}$$

$$b_2 = -\frac{1}{a_3}\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix} \qquad b_3 = -\frac{1}{a_3}\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix} \qquad c_2 = \boxed{-\frac{1}{b_1}\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}$$

$$d_1 = \boxed{}$$

Routh Test:

The number of *RHP* zeros of $Q(s)$

= the # of sign changes in the first column of the Routh table

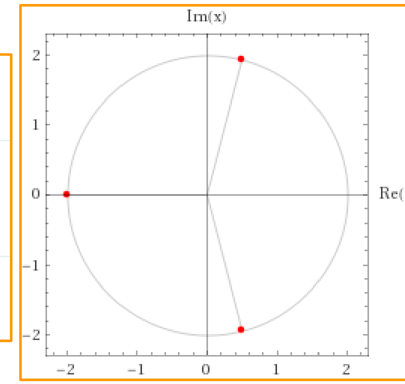
Example: $Q(s)=s^3+s^2+2s+8$

s^3	1	2
s^2	1	8
s^1	-6	
s^0	8	

$x = -2$

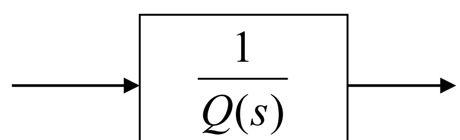
$x = \frac{1}{2}\left(1 - i\sqrt{15}\right)$

$x = \frac{1}{2}\left(1 + i\sqrt{15}\right)$



https://www.wolframalpha.com
solve s^3+s^2+2s+8=0

$\boxed{2\ RHP}$ zeros $\rightarrow$
The system is unstable since two poles exist in the RHP



https://www.wolframalpha.com
solve s^5 - 2s^4 + s^3 + 4s + 2 = 0

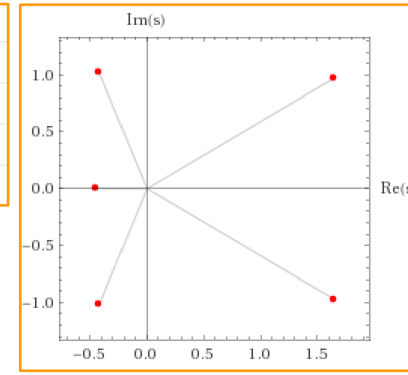
$s \approx -0.451516$

$s \approx -0.418382 - 1.020343\,i$

$s \approx -0.418382 + 1.020343\,i$

$s \approx 1.64414 - 0.96906\,i$

$s \approx 1.64414 + 0.96906\,i$



Example: $Q(s) = s^5 - 2s^4 + s^3 + 4s + 2$

$$
\begin{array}{lccc}
s^5: & 1 & 1 & 4 \\
s^4: & -2 & 0 & 2 \\
s^3: & 1 & 5 & \\
s^2: & 10 & 2 & \\
s^1: & 4.8 & & \\
s^0: & 2 & &
\end{array}
$$

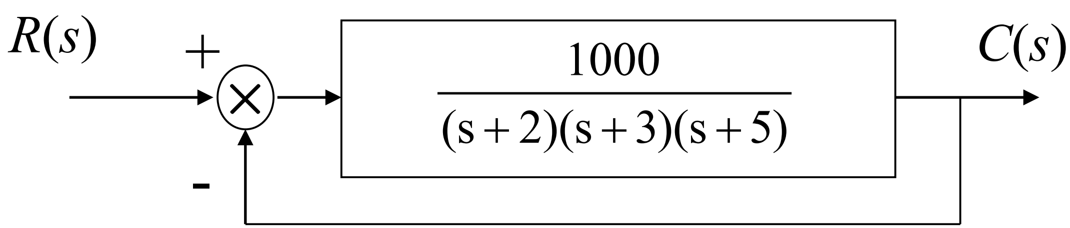
$$-\frac{1}{2}(-2)$$

$$\frac{-1}{1}\begin{vmatrix} -2 & 0 \\ 1 & 5 \end{vmatrix} = 10$$

$$\frac{-1}{1}\begin{vmatrix} -2 & 2 \\ 1 & 0 \end{vmatrix} = -1(-2) = 2$$

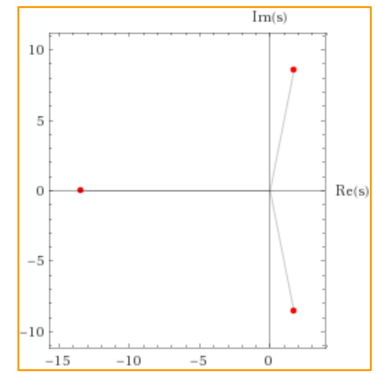
$\Rightarrow 2\ RHP$ zeros

Example:



$$\frac{K}{1+KG} \qquad R(s) \longrightarrow \left(\frac{1000}{s^3+10s^2+31s+1030}\right) \longrightarrow C(s)$$

https://www.wolframalpha.com
solve s^3 + 10s^2 + 31s + 1030 = 0



$$\mathrm{s}^3 : 1 \qquad 31$$

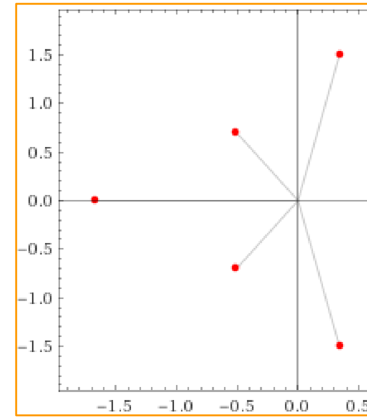
$$\mathrm{s}^3 : \cancel{10} \quad 1 \qquad \cancel{1030} \quad 103$$

$$\mathrm{s}^3 : \frac{-1}{1}(103-31) = -72, \qquad \frac{-1}{1}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$\mathrm{s}^3 : \frac{-1}{-72}\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix} = 103, \qquad \frac{-1}{-72}\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix} = 0$$

$$\boxed{2\ RHP \text{ poles} \Rightarrow \text{unstable}}$$

https://www.wolframalpha.com
solve s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3 = 0



Example: Closed-loop transfer function

$$T(s)=\frac{10}{s^5+2s^4+3s^3+6s^2+5s+3} \quad \rightarrow (1)$$

$$\begin{array}{llll}
s^5 & 1 & 3 & 5 \\
s^4 & 2 & 6 & 3 \\
s^3 & \cancel{0}\ \varepsilon & \dfrac{7}{2} & 0 \\
s^2 & \dfrac{1}{\varepsilon}(6\varepsilon-7) & \dfrac{1}{\varepsilon}(3\varepsilon-0)=3 & 0 \\
s^1 & \dfrac{42\varepsilon-49-6\varepsilon^2}{12\varepsilon-14} & 0 & 0 \\
s^0 & 3 & &
\end{array}$$

$$\begin{array}{llcc}
 & & \varepsilon=+ & \varepsilon=- \\
s^5 & 1 & + & + \\
s^4 & 2 & + & + \\
s^3 & \varepsilon & + & - \\
s^2 & \dfrac{1}{\varepsilon}(6\varepsilon-7) & - & + \\
s^1 & \dfrac{42\varepsilon-49-6\varepsilon^2}{12\varepsilon-14} & + & + \\
s^0 & 3 & + & +
\end{array}$$

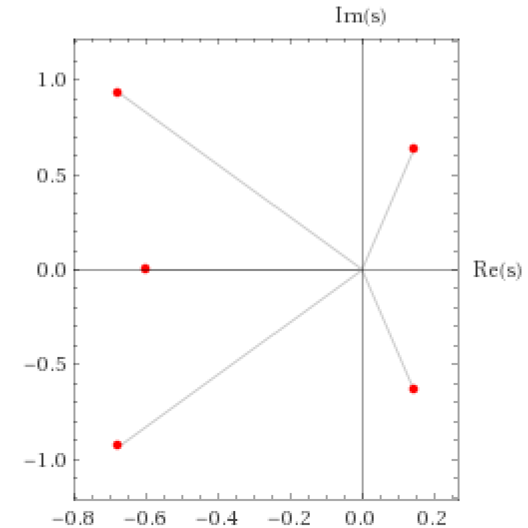
$$\boxed{2\ RHP \text{ poles} \Rightarrow \text{unstable}}$$

Example: Stability via reverse coefficient.

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

From (1) $\longrightarrow$ $D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$

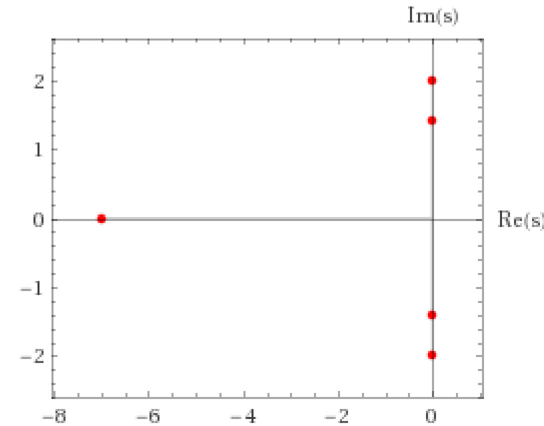
s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	-1.75		
s^0	1		



https://www.wolframalpha.com
solve 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1 = 0

$\boxed{2\ RHP \text{ poles} \Rightarrow \text{unstable}}$

https://www.wolframalpha.com
solve s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56 = 0



Example: Closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

$$\begin{array}{llll} s^5 & 1 & 6 & 8 \\ s^4 & \cancel{7}\ 1 & \cancel{42}\ 6 & \cancel{56}\ 8 \\ s^3 & 0 & 0 & 0 \end{array}$$

Zeros row

$$(s^5 + 6s^3 + 8s) + 7(s^4 + 6s^2 + 8)$$

$$= (s+7)\underbrace{(s^4 + 6s^2 + 8)}_{P(s)}$$

$$\frac{d}{ds}P(s)$$

$$\frac{d}{ds}\overbrace{\left(s^4 + 6s^2 + 8\right)}^{P(s)} = 4s^3 + 12s + 0$$

$$= 4\left(s^3 + 3s\right)$$

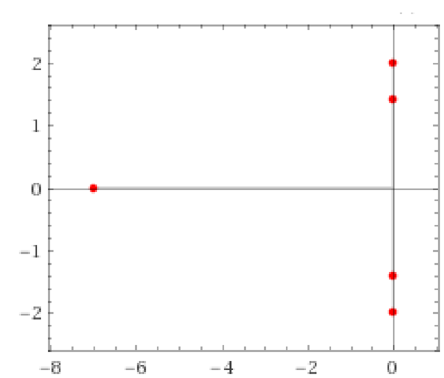
$$(s^5+6s^3+8s)+7(s^4+6s^2+8)$$
$$=(s+7)(s^4+6s^2+8)$$
$$=(s+7)(s^2+4)(s^2+2)$$

s^5	1	6	8
s^4	1	6	8
s^3	1	3	0
s^2	3	8	
s^1	$+\frac{1}{3}$	0	
s^0	8		

$$\frac{-1}{1}\begin{vmatrix}1 & 8\\ 1 & 0\end{vmatrix}=8$$

$$\frac{-1}{1}\begin{vmatrix}1 & 6\\ 1 & 3\end{vmatrix}=3$$

$$\frac{-1}{3}\begin{vmatrix}1 & 3\\ 3 & 8\end{vmatrix}$$



no sign change

$\downarrow$

No RHP

$\downarrow$

Even poly. test

No LHP $\Rightarrow \therefore$ roots are on the $j\omega$-axis

$\uparrow$ by symmetry

roots $-7,\quad \pm 2j,\quad \pm j1.14142$

Positive $\Rightarrow$ $\boxed{\text{No } RHP \text{ poles}}$

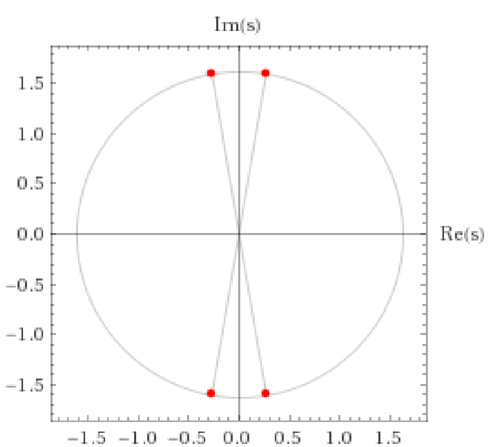
⇨ An entire row of zeros.

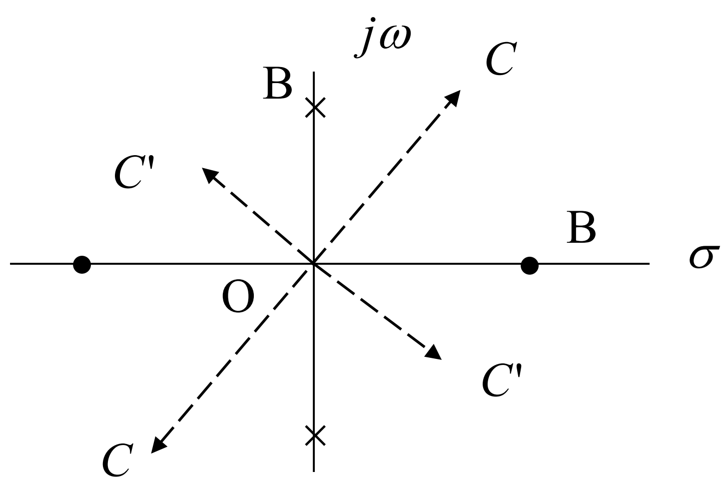
- even polynomial $s^4+5s^2+7=0 \rightarrow s = \begin{cases} -0.27 \pm j1.604 \\ +0.27 \pm j1.604 \end{cases}$

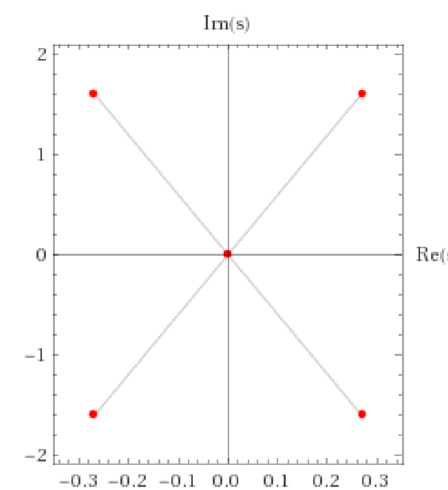
- odd polynomial $s^5+5s^3+7s=0 \rightarrow s = \begin{cases} 0 \\ -0.27 \pm j1.604 \\ +0.27 \pm j1.604 \end{cases}$

  constant term is always missing

Symmetrical about the origin







Example: $T(s) = \dfrac{20}{(s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20)}$

s^8	1		12		39		48		20
s^7	1		22		59		38		0
s^6	-10	-1	-20	-2	10	1	20	2	0
s^5	20	1	60	3	40	2	0		
s^4	1		3		2		0		
s^3	0	2	0	3	0	0			
s^2	$\frac{3}{2}$	3	2	4	0				
s^1	$\frac{1}{3}$		0						
s^0	4								

sign change

zeros row why?

even poly.test

$$s^4 + 3s^2 + 2 = P(s)$$

$$\frac{d}{ds}P(s) = 4s^3 + 6s + 0$$

$$= 2s(2s^2 + 3)$$

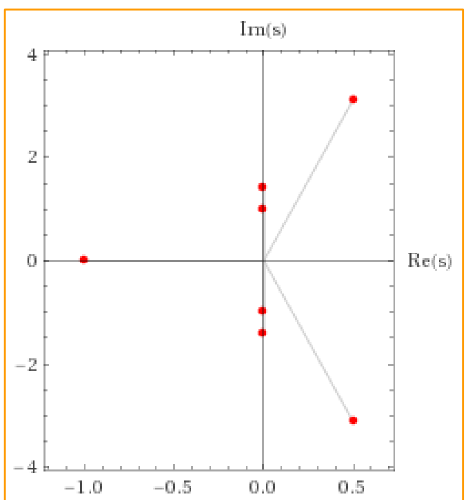
2poles. LHP
2poles. RHP

4 poles jw-axis

> 2 sign change → two poles in RHP
> No sign change after s^4 → 4 poles are on the jw-axis

- Summary of pole locations for Example

Even (4th order)	Rest (4th order)	Total (8th order)
$s^4 \sim s^0$	$s^8 \sim s^5$	
$0\ RHP$	$2RHP$	$2RHP$
$0\ LHP$	$2LHP$	$2LHP$
$4\,j\omega$	$0\,j\omega$	$4\,j\omega$



solve $s^8 + s^7 + 12 s^6 + 22 s^5 + 39 s^4 + 59 s^3 + 48 s^2 + 38 s + 20 = 0$

Results:

$s = -1$

$s = \pm i$

$s = \pm\left(i\sqrt{2}\right)$

$s = \frac{1}{2}\left(1 - i\sqrt{39}\right)$

$s = \frac{1}{2}\left(1 + i\sqrt{39}\right)$

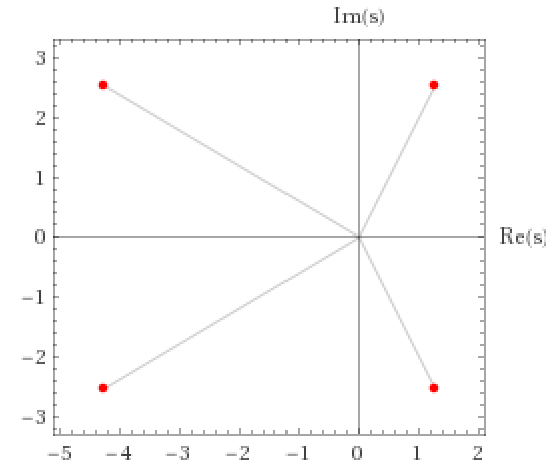
```
>> roots([1 1 12 22 39 59 48 38 20])

   0.5000 + 3.1225i
   0.5000 - 3.1225i
   0.0000 + 1.4142i
   0.0000 - 1.4142i
  -1.0000 + 0.0000i
  -1.0000 + 0.0000i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

https://www.wolframalpha.com
solve s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20 = 0

Example: Closed transfer function $T(s)$

$$T(s) = \frac{1}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

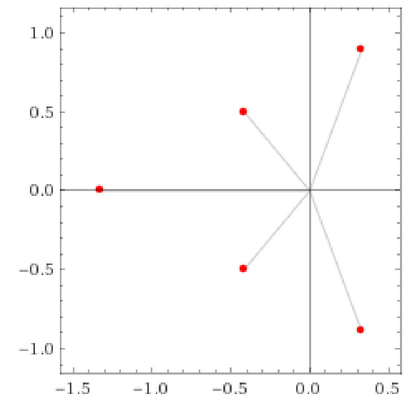
s^4	1	11	200
s^3	1	1	
s^2	1	20	
s^1	-19		
s^0	20		



https://www.wolframalpha.com
solve s^4 + 6s^3 + 11s^2 + 6s + 200 = 0

2 LHP poles
2 RHP poles $\rightarrow$ unstable

https://www.wolframalpha.com
solve 2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1 = 0



Example: $T(s) = \dfrac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$

$$
\begin{array}{cc|ccc}
 & s^5 & 2 & 2 & 2 \\
 & s^4 & 3 & 3 & 1 \\
+ & s^3 & \cancel{0}\ \varepsilon & \dfrac{4}{3} & \\
- & s^2 & \dfrac{3\varepsilon - 4}{\varepsilon} & 1 & \\
+ & s^1 & (12\varepsilon - 16 - 3\varepsilon^2)\big/(9\varepsilon - 12) & & \\
 & s^0 & 1 & &
\end{array}
$$

$\Downarrow$

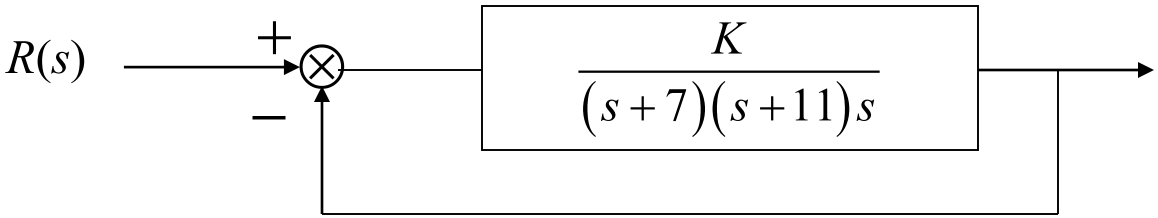
$\boxed{\begin{array}{c} 2\ RHP \text{ poles} \\ 3\ LHP \text{ poles} \end{array}}$ $\rightarrow$ unstable $\Leftarrow$

Reverse order

$$s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2$$

$$
\begin{array}{c|ccc}
s^5 & 1 & 3 & 3 \\
s^4 & 2 & 2 & 2 \\
s^3 & 2 & 2 & \\
s^2 & \cancel{0}\ \varepsilon & 2 & \\
s^1 & \dfrac{2\varepsilon - 4}{\varepsilon} & & \\
s^0 & 2 & &
\end{array}
$$

Stability Design via Routh- Hurwitz



$$\frac{G}{1+G}=\frac{\dfrac{K}{(s+7)(s+11)s}}{1+\dfrac{K}{(s+7)(s+11)s}}$$

$$=\frac{K}{(s+7)(s+11)s+K}$$

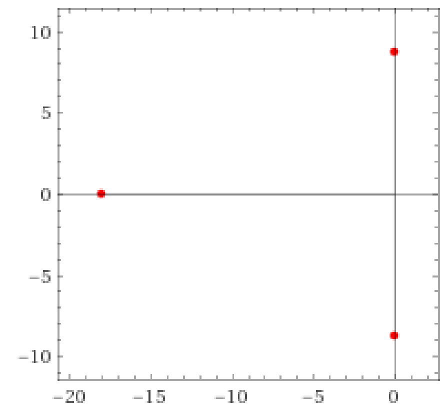
$$=\frac{K}{s^3+18s^2+77s+K}$$

$$\Rightarrow \text{Closed loop T.F.:}\quad T(s)=\frac{K}{s^3+18s^2+77s+K}$$

s^3	1	77
s^2	18	K
s^1		
s^0		

$$\frac{\begin{vmatrix}1 & 77\\ 18 & K\end{vmatrix}}{-18}=\frac{K-1386}{-18}$$

https://www.wolframalpha.com
solve s^3 + 18s^2 + 77s + 1386 = 0



- If K=1386

s^3	1	77
s^2	18	1386
s^1	$\cancel{0}$ 36	
s^0	1386	

$$p(s)=18s^2+1386$$

$$\frac{d}{ds}p(s)=36s$$

2 poles on the jw-axis $\rightarrow$ marginally stable