Chapter 3. Modeling in the Time Domain

Chapter Learning Outcomes

After completing this chapter, the student will be able to:

- Find a mathematical model, called a *state-space* representation, for a linear, time-invariant system (Sections 3.1–3.3)
- Model electrical and mechanical systems in state space (Section 3.4)
- Convert a transfer function to state space (Section 3.5)
- Convert a state-space representation to a transfer function (Section 3.6)
- Linearize a state-space representation (Section 3.7)

Chapter 2. Modeling in the Frequency Domain

After completing this chapter, the student will be able to:

- Find the Laplace transform of time functions and the inverse Laplace transform (Sections 2.1–2.2)
- Find the transfer function from a differential equation and solve the differential equation using the transfer function (Section 2.3)
- Find the transfer function for linear, time-invariant electrical networks (Section 2.4)
- Find the transfer function for linear, time-invariant translational mechanical systems (Section 2.5)
- Find the transfer function for linear, time-invariant rotational mechanical systems (Section 2.6)
- Find the transfer functions for gear systems with no loss and for gear systems with loss (Section 2.7)
- Find the transfer function for linear, time-invariant electromechanical systems (Section 2.8)
- Produce analogous electrical and mechanical circuits (Section 2.9)
- Linearize a nonlinear system in order to find the transfer function (Sections 2.10–2.11)

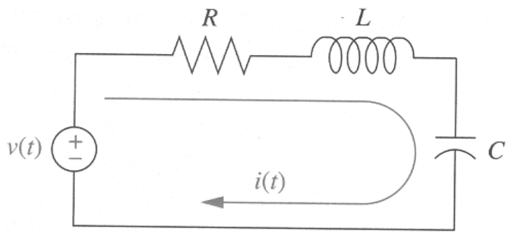
3.1 Introduction

State space (modern or time-domain) approach:

- Nonlinear system (backlash, saturation, dead zone, …)
- System with nonzero initial conditions
- Time-varying system: missiles with varying fuel levels, lift in aircraft flying
- MIMO (multiple-input, multiple-output) system
- System simulation with digital computer

Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$

3.2 Some Observations



$$\begin{cases} L\dfrac{di}{dt} + Ri + \dfrac{1}{C}\displaystyle\int i\,dt = v(t) \\ \quad \text{with} \quad i(t) = \dfrac{dq}{dt} \quad \text{and} \quad \dfrac{di}{dt} = \dfrac{d^2q}{dt^2}, \quad \displaystyle\int idt = q \\ \to L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = v(t) \end{cases}$$

$$\underbrace{\boxed{L\frac{di}{dt}}}_{v_L(t)} + Ri + \frac{1}{C}\underbrace{\boxed{\int i\,dt}}_{q} = v(t)$$

$$v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t) \quad \to \quad \begin{array}{l} L\dfrac{di}{dt} = -\dfrac{1}{C}q(t) - Ri(t) + v(t) \\ \dfrac{di}{dt} = -\dfrac{1}{LC}q - \dfrac{R}{L}i + \dfrac{1}{L}v(t) \end{array} \quad \to \begin{cases} \dfrac{dq}{dt} = i \\ \dfrac{di}{dt} = -\dfrac{1}{LC}q - \dfrac{R}{L}i + \dfrac{1}{L}v(t) \end{cases}$$

$\Rightarrow$ Output equation: linear combination of $q(t)$, $i(t)$, and input, $v(t)$

$\Rightarrow$ State equation: $\dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{B}u$

Output equation: $y = \mathrm{Cx} + \mathrm{D}u$

State x:

$$\dot{\mathrm{x}} = \begin{bmatrix} \dfrac{dq}{dt} \\ \dfrac{di}{dt} \end{bmatrix} = \begin{bmatrix} q' \\ i' \end{bmatrix}, \quad \mathrm{x} = \begin{bmatrix} q \\ i \end{bmatrix}$$

$\Rightarrow$ State-space representation

$$\begin{bmatrix} \dfrac{dq}{dt} \\ \dfrac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{-1}{LC} & \dfrac{-R}{L} \end{bmatrix} \begin{bmatrix} q \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} v(t)$$

$$v_L(t) = \begin{bmatrix} \dfrac{-1}{C} & -R \end{bmatrix} \begin{bmatrix} q \\ i \end{bmatrix} + 1 \cdot v(t)$$

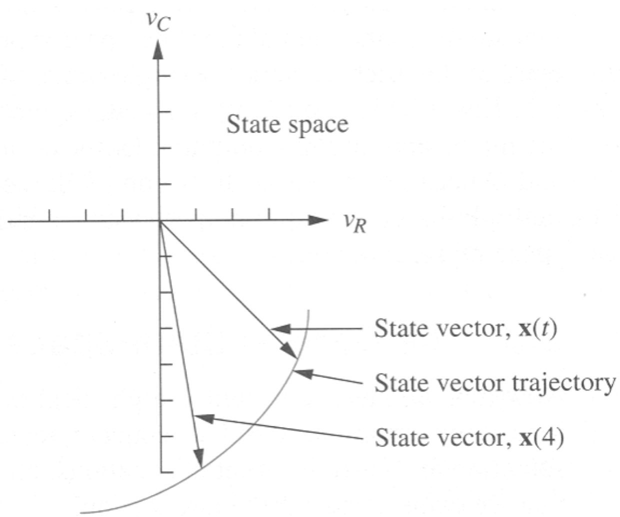
$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ \dfrac{-1}{LC} & \dfrac{-R}{L} \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix}$$

$$\mathrm{C} = \begin{bmatrix} \dfrac{-1}{C} & -R \end{bmatrix}$$

$$\mathrm{D} = 1$$

3.3 The General State-Space Representation



Definitions:

1. Linear combination:

$$S = K_n x_n + K_{n-1} x_{n-1} + \cdots + K_1 x_1$$

2. Linear independence:

S=0 <u>only if</u> every K_i=0 and no x_i=0.

Ex: Determine linearly independent or linearly dependent.

- Two vectors: $x_1 = (1,1),\ x_2 = (2,2)$

$\Downarrow$

Two vectors: $x_1 = (1,1),\ x_2 = (2,2)$

$S = K_1 x_1 + K_2 x_2 = K_1(1,1) + K_2(2,2)$

$= (K_1 + 2K_2,\ K_1 + 2K_2)$

if $K_1 = -2, K_2 = 1$ then $S = 0$

Since a and b are not zero,

the two vectors are linearly dependent.

3. System variables: Any variables that responds to an input or initial conditions.

$q,\ i$

4. State variables: Linearly independent system variables. Determine the value of all system variables for all $t \ge t_0$.

$\mathrm{x} = \begin{bmatrix} q \\ i \end{bmatrix}$

$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

5. State vector: Elements of a vector are the state variables.

6. State space: The n-dim. space whose axes are the state variables.

7. State equations: A set of n simultaneous, first-order differential equations.

8. Output equation: the output variables represented with a linear combination of the state variables and the inputs.

$\dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{B}u$

$y = \mathrm{Cx} + \mathrm{D}u$

• Linear independence: $S=0$ only if every $K_i=0$ and no $x_i=0$.

Example 1:

Consider the set of vectors $\{(1,2,0,0),(0,4,4,0),(2,0,3,0)\}$ on the $\mathbb{R}^4$ vector space.
Determine if this set is linearly independent or linearly dependent.

$$K_1(1,2,0,0)+K_2(0,4,4,0)+K_3(2,0,3,0)=0$$
$$(K_1,2K_1,0,0)+(0,4K_2,4K_2,0)+(2K_3,0,3K_3,0)=0 \rightarrow \begin{matrix} 2K_1+4K_2=0 \\ 4K_2+3K_3=0 \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 2 & 4 & 0 \\ 0 & 4 & 3 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ --- (A)}$$
$$(K_1+2K_3,\ 2K_1+4K_2,\ 4K_2+3K_3,\ 0)=(0,0,0,0)$$

$$\boxed{Ax=b \rightarrow x=A^{-1}b \quad \text{if } A \neq \text{singular matrix } (\det(A)\neq 0)}$$

$$\det\begin{pmatrix} 1 & 0 & 2 \\ 2 & 4 & 0 \\ 0 & 4 & 3 \end{pmatrix}=28$$

```
>> A=[1 0 2; 2 4 0; 0 4 3];
>> det(A)

ans =   28
```

A is invertible ($\det(A)\neq 0$). It implies that the equation (A) has only the trivial solution $K_1=K_2=K_3=0$.

$\rightarrow$ Therefore, this set of vectors is linearly independent.

Example 2:

Consider the set of vectors $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ on the vector space M_{22}.

Determine if this set is linearly independent or linearly dependent.

$$K_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+K_2\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}+K_3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=0 \rightarrow \begin{pmatrix} K_1+K_3 & -K_2 \\ 0 & K_1 \end{pmatrix}=0 \Rightarrow \begin{matrix} K_1+K_3=0 \\ -K_2=0 \\ K_1=0 \end{matrix}$$

$\rightarrow K_1=K_2=K_3=0 \rightarrow$ Therefore, this set of vectors is linearly independent.

OR

$$\begin{matrix} K_1+K_3=0 \\ -K_2=0 \\ K_1=0 \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \det\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix}=1 \ (\neq 0)$$

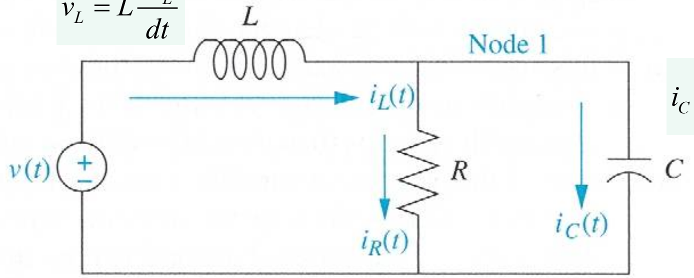
3.4 Applying the State-Space Representation

Example 3.1: Representing an electrical network (page 126)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Find a <u>state-space representation</u> if the output is i_R.

$v_L = L\dfrac{di_L}{dt}$



$i_C = C\dfrac{dv_C}{dt}$

$$\begin{cases} C\dfrac{dv_C}{dt} = i_C \\ L\dfrac{di_L}{dt} = v_L \end{cases}$$

$$i_L = i_R + i_C = \frac{v_C}{R} + C\frac{dv_C}{dt} \quad (KCL)$$
$$v(t) = v_L + v_C = L\frac{di_L}{dt} + v_C \quad (KVL)$$

$$\Rightarrow \quad C\frac{dv_C}{dt} = -\frac{v_C}{R} + i_L, \qquad L\frac{di_L}{dt} = -v_C + v(t)$$

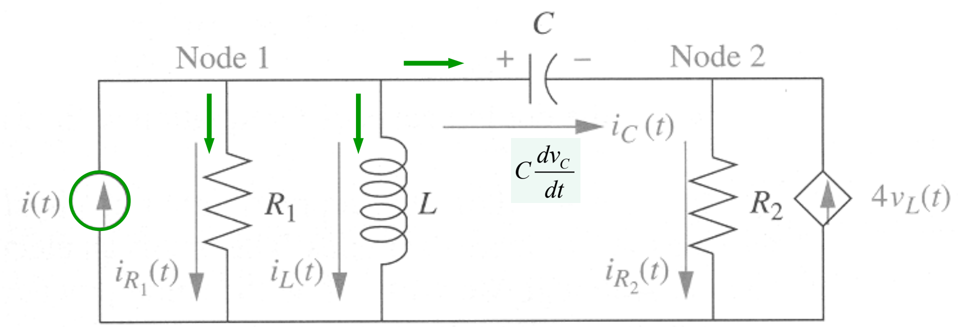
$$\Rightarrow \quad \frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L, \qquad \frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}v$$

$$\Rightarrow \quad \begin{pmatrix} \dot{v}_C \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} \dfrac{-1}{RC} & \dfrac{1}{C} \\ \dfrac{-1}{L} & 0 \end{pmatrix} \begin{pmatrix} v_C \\ i_L \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{L} \end{pmatrix} v(t)$$

$$i_R = \frac{V_C}{R} = \left(\frac{1}{R} \quad 0\right)\begin{pmatrix} v_C \\ i_L \end{pmatrix}$$

Example 3.2: Representing an electrical network (page 128)

Find the state and output equations if the output is $y=[v_{R_2} \quad i_{R_2}]^T$.



$$L\frac{di_L}{dt}=v_L, \quad C\frac{dv_C}{dt}=i_C \quad \Rightarrow \begin{pmatrix} \frac{di_L}{dt} \\ \frac{dv_C}{dt} \end{pmatrix}$$

State variables: $x_1=i_L, \quad x_2=v_C$

$$\boxed{\begin{aligned} v_{R_2} &= -v_C+v_L \\ i_{R_2} &= i_C+4v_L \end{aligned}}$$

$$\begin{cases} i=\frac{v_L}{R_1}+i_L+C\frac{dv_C}{dt} \\ C\frac{dv_C}{dt}=\frac{v_{R_2}}{R_2}-4v_L = \frac{v_L-v_C}{R_2}-4v_L=\frac{(1-4R_2)}{R_2}v_L-\frac{1}{R_2}v_C \end{cases}$$

$(v_L=v_C+v_{R_2})$

$$v_L=L\frac{di_L}{dt}$$

$$\begin{cases} i=\frac{1}{R_1}(L\frac{di_L}{dt})+i_L+C\frac{dv_C}{dt} \\ C\frac{dv_C}{dt}=\frac{(1-4R_2)}{R_2}(L\frac{di_L}{dt})-\frac{1}{R_2}v_C \end{cases} \Rightarrow \begin{cases} C(\frac{dV_C}{dt})+\frac{L}{R_1}(\frac{di_L}{dt})=i-i_L \\ C(\frac{dV_C}{dt})-\frac{L(1-4R_2)}{R_2}(\frac{di_L}{dt})=-\frac{1}{R_2}V_C \end{cases}$$

Example 3.2: (Continued)

$$\begin{cases} C(\frac{dV_C}{dt}) + \frac{L}{R_1}(\frac{di_L}{dt}) = i - i_L \\ C(\frac{dV_C}{dt}) - \frac{L(1-4R_2)}{R_2}(\frac{di_L}{dt}) = -\frac{1}{R_2}V_C \end{cases}$$

$$\downarrow$$

$$\begin{pmatrix} C & \frac{L}{R_1} \\ C & -\frac{L(1-4R_2)}{R_2} \end{pmatrix} \begin{pmatrix} \frac{dV_C}{dt} \\ \frac{di_L}{dt} \end{pmatrix} = \begin{pmatrix} i - i_L \\ -\frac{1}{R_2}V_C \end{pmatrix}$$

$$\frac{di_L}{dt} = \frac{\begin{vmatrix} C & i - i_L \\ C & -\frac{1}{R_2}V_C \end{vmatrix}}{\begin{vmatrix} C & \frac{L}{R_1} \\ C & -\frac{L(1-4R_2)}{R_2} \end{vmatrix}} = \frac{-\frac{C}{R_2}V_C - C(i - i_L)}{C[-\frac{L(1-4R_2)}{R_2} - \frac{L}{R_1}]} = \frac{+R_2 i_L - V_C - R_2 i}{L \quad \Delta}$$

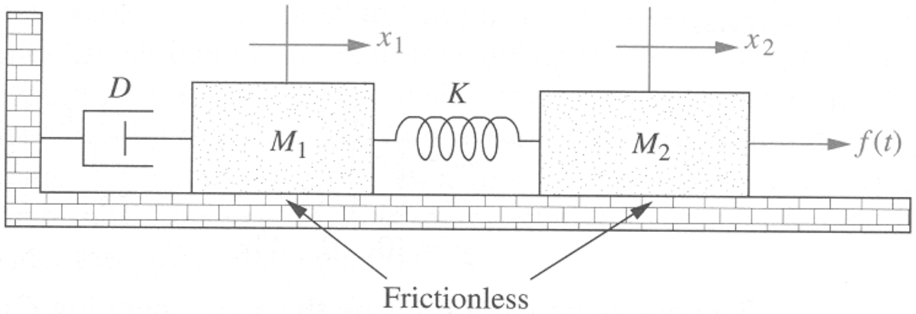
$$\frac{dV_C}{dt} = \frac{(1-4R_2)i_L + \frac{1}{R_1}V_C - (1-4R_2)i}{C \quad \Delta}, \qquad where \quad \Delta = -\left[\left(1 - 4R_2\right) + \frac{R_2}{R_1} \right]$$

The output equation is: $\Rightarrow \quad y = \begin{pmatrix} v_{R_2} \\ i_{R_2} \end{pmatrix} = \begin{pmatrix} -v_C + v_L \\ i_C + 4v_L \end{pmatrix}$

$$\Rightarrow \quad \begin{pmatrix} v_{R_2} \\ i_{R_2} \end{pmatrix} = \begin{pmatrix} R_2/\Delta & -(1 + 1/\Delta) \\ 1/\Delta & (1-4R_1)/(\Delta R_1) \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} -R_2/\Delta \\ -1/\Delta \end{pmatrix} i(t)$$

Example 3.3: Representing a translational mechanical system

Find the state equations.



$$\boxed{\dot{x} = Ax + Bu}$$

- State variables: $\left[\begin{array}{l} x_1,\ v_1 \\ x_2,\ v_2 \\ \dfrac{dx_1}{dt} = v_1 \\ \dfrac{dx_2}{dt} = v_2 \end{array}\right.$

$$\begin{cases} \dot{x}_1 = v_1 \\ M_1\dot{v}_1 = k(x_2 - x_1) - Dv_1 \quad \Rightarrow \quad \dot{v}_1 = -\dfrac{k}{M_1}x_1 - \dfrac{D}{M_1}v_1 + \dfrac{k}{M_1}x_2 \\ \dot{x}_1 = v_2 \\ M_2\dot{v}_2 = f - k(x_2 - x_1) \quad \Rightarrow \quad \dot{v}_2 = \dfrac{k}{M_2}x_1 - \dfrac{k}{M_2}x_2 + \dfrac{1}{M_2}f \end{cases}$$

$$\Rightarrow \quad \begin{pmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{pmatrix} f(t),$$

$$x_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{pmatrix}$$

3.5 Converting a Transfer Function to State Space (page 132)

- Consider differential equation:

$$\dot{x}_n \longrightarrow \boxed{\frac{d^n y}{dt^n}} + a_{n-1}\overset{x_n}{\boxed{\frac{d^{n-1} y}{dt^{n-1}}}} + \cdots + a_1 \overset{x_2}{\boxed{\frac{dy}{dt}}} + a_0 \overset{x_1}{\boxed{y}} = b_0 u$$

Choose state variables $\Rightarrow$ $\quad x_1 = y, \quad x_2 = \dfrac{dy}{dt}, \quad x_3 = \dfrac{d^2 y}{dt^2}, \quad \cdots \quad , \quad x_n = \dfrac{d^{n-1} y}{dt^{n-1}}$

Differentiating eq.: $\Rightarrow$ $\quad \dot{x}_1 = \dfrac{dy}{dt}, \quad \dot{x}_2 = \dfrac{d^2 y}{dt^2}, \quad \dot{x}_3 = \dfrac{d^3 y}{dt^3}, \quad \cdots \quad , \quad \dot{x}_n = \dfrac{d^n y}{dt^n}$

State eq. $\Rightarrow$ $\quad \dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \cdots \quad , \quad \dot{x}_{n-1} = x_n, \quad \boxed{\dot{x}_n} = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + b_0 u$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ & & & \vdots & & & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

Example 3.4: Converting a transfer function with constant term in numerator

Find the state-space representation in phase-variable form equations.

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24} \quad \rightarrow \quad (s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

- D.E. by taking the inverse Laplace transform with zero initial conditions:

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

D.E.: differential equation

- Select the state variables: $x_1 = c, \quad x_2 = \dot{c}, \quad x_3 = \ddot{c}$

$$\Rightarrow \quad \dot{x}_1 = \dot{c}, \quad \dot{x}_2 = \ddot{c}, \quad \dot{x}_3 = \dddot{c}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r \\ y = c = x \end{cases} \quad \Rightarrow \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 24 \end{pmatrix} r$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

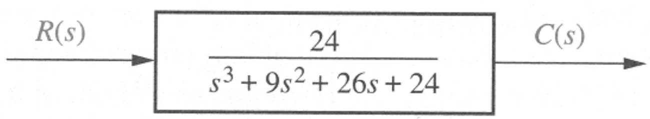
state-space representation

Example 3.4: (Continued)

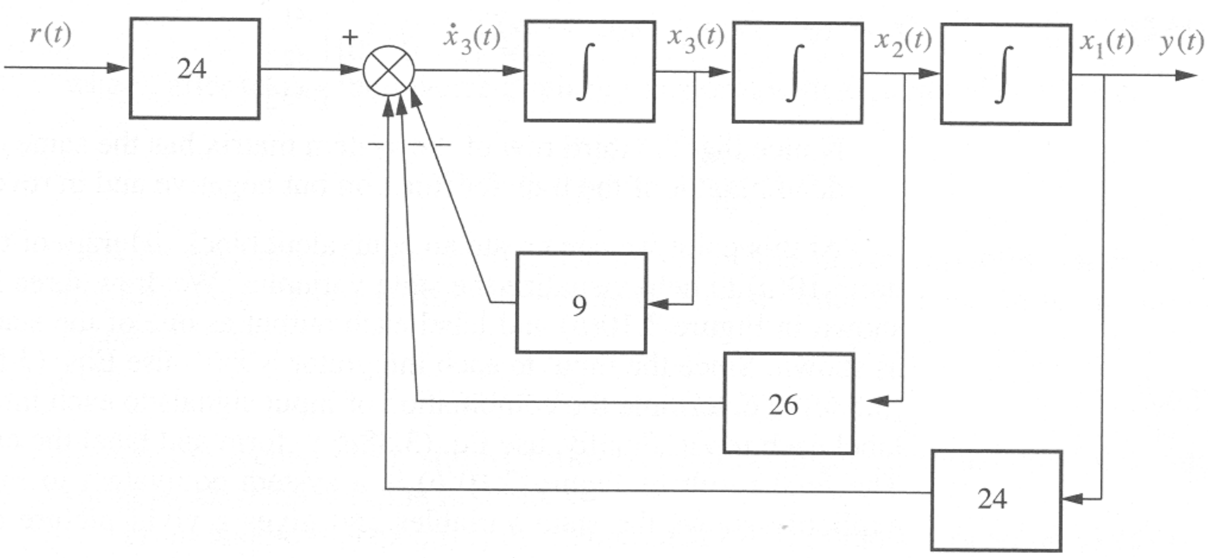
$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+\begin{pmatrix}0\\ 0\\ 24\end{pmatrix}r,\qquad y=\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$\boxed{\begin{aligned}\dot{x}&=Ax+Bu\\ y&=Cx+Du\end{aligned}}$$

state-space representation



(a)



(b)

• Appendix B: Matlab Codes & Outputs (page 793)

```
'(ch3p1)'

A=[0 1 0; 0 0 1; -9 -8 -7]

% or
A1=[0 1 0
    0 0 1
   -9 -8 -7]
```

```
A =

     0     1     0
     0     0     1
    -9    -8    -7


A1 =

     0     1     0
     0     0     1
    -9    -8    -7
```

```
% (ch3p2)

C=[2 3 4]
B=[7 8 9]'
% B=[7; 8; 9]
```

```
C =

     2     3     4


B =

     7
     8
     9
```

– state-space model

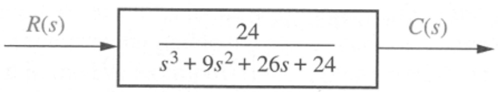
$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+\begin{pmatrix}0\\ 0\\ 24\end{pmatrix}r$$

$$y=\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

```
'(ch3p3)'
A=[0   1   0;
   0   0   1;
  -24 -26 -9];
B=[0 0 24]';
%B=[0;0;24];

C=[1 0 0];
D=0;

F=ss(A, B, C, D)
```

```
F =

  A =
          x1    x2    x3
   x1      0     1     0
   x2      0     0     1
   x3    -24   -26    -9

  B =
         u1
   x1     0
   x2     0
   x3    24

  C =
         x1   x2   x3
   y1     1    0    0

  D =
         u1
   y1     0

Continuous-time state-space model.
```

• Appendix B: Matlab Codes & Outputs (page 793)



$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

• State-Space Representation

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 24 \end{pmatrix} r,$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 0 \cdot r$$

```
'Example 3.4'

num=24;
den=[1 9 26 24];
[A, B, C, D]=tf2ss(num, den)
```

```
A =
      -9   -26   -24
       1     0     0
       0     1     0

B =
       1
       0
       0

C =
       0     0    24

D =
       0
```

⇓

Controller Canonical Form

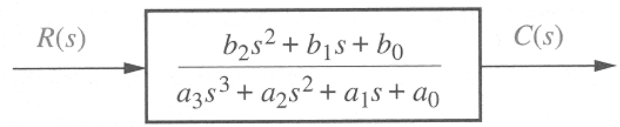
```
'Example 3.4'
num=24;
den=[1 9 26 24];
[A, B, C, D]=tf2ss(num, den);

% To 'Phase-variable form'
P=[0 0 1; 0 1 0; 1 0 0];
Ap=inv(P)*A*P
Bp=inv(P)*B
Cp=C*P
Dp=D
```

```
Ap =
       0     1     0
       0     0     1
     -24   -26    -9

Bp =
       0
       0
       1

Cp =
      24     0     0

Dp =
       0
```

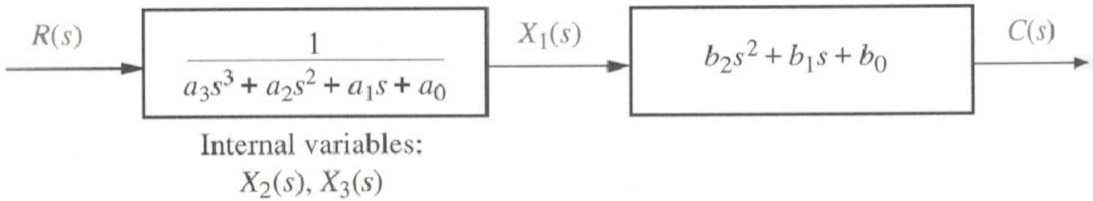
⇓

Phase-variable Form

- **Decomposing a transfer function**

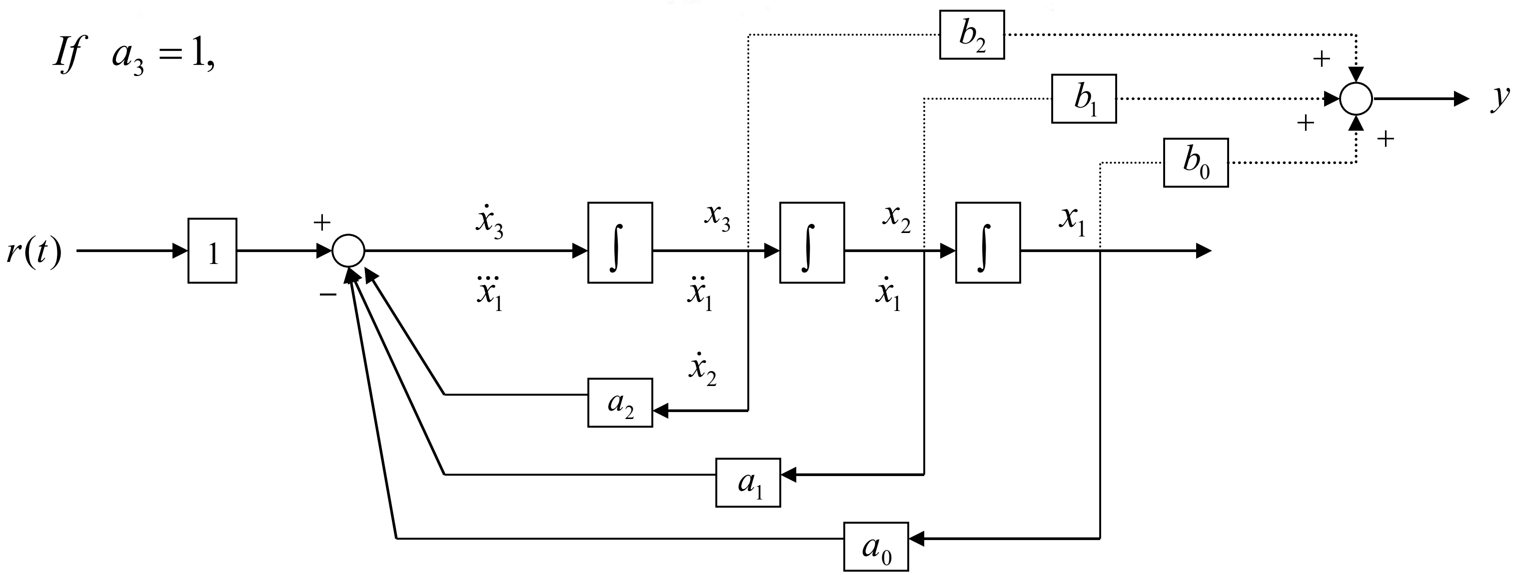


(a)



(b)

If $a_3 = 1,$



$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -a_0x_1 - a_1x_2 - a_2x_3 + r \\
y &= b_0x_1 + b_1x_2 + b_2x_3
\end{aligned}
$$

$$
\Rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r, \quad y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

Example 3.5: Converting a transfer function with polynomial in numerator

Find the state-space representation of the transfer function.

$$R(s) \longrightarrow \boxed{\frac{s^2+7s+2}{s^3+9s^2+26s+24}} \longrightarrow C(s)$$

From Example 3.4:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$C(s) = (b_2 s^2 + b_1 s + b_0) X_1(s) = (s^2 + 7s + 2) X_1(s)$$

$$c(t) = \ddot{x}_1(t) + 7\dot{x}_1(t) + 2x_1(t) \xrightarrow{\qquad\qquad} y = c(t) = b_2 x_3 + b_1 x_2 + b_0 x_1$$

$$= x_3 + 7x_2 + 2x_1$$

$$\begin{cases} x_1 = x_1 \\ \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \end{cases}$$

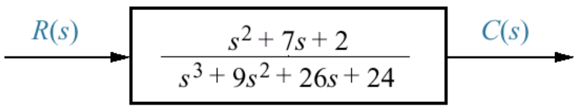
$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

state-space representation

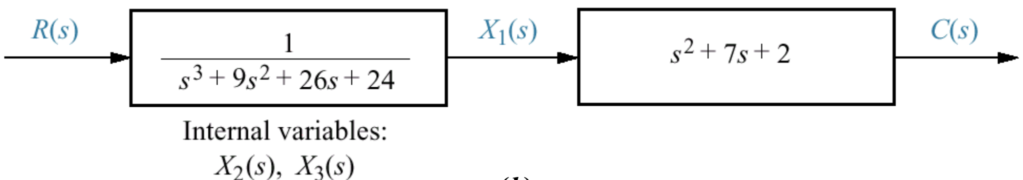
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

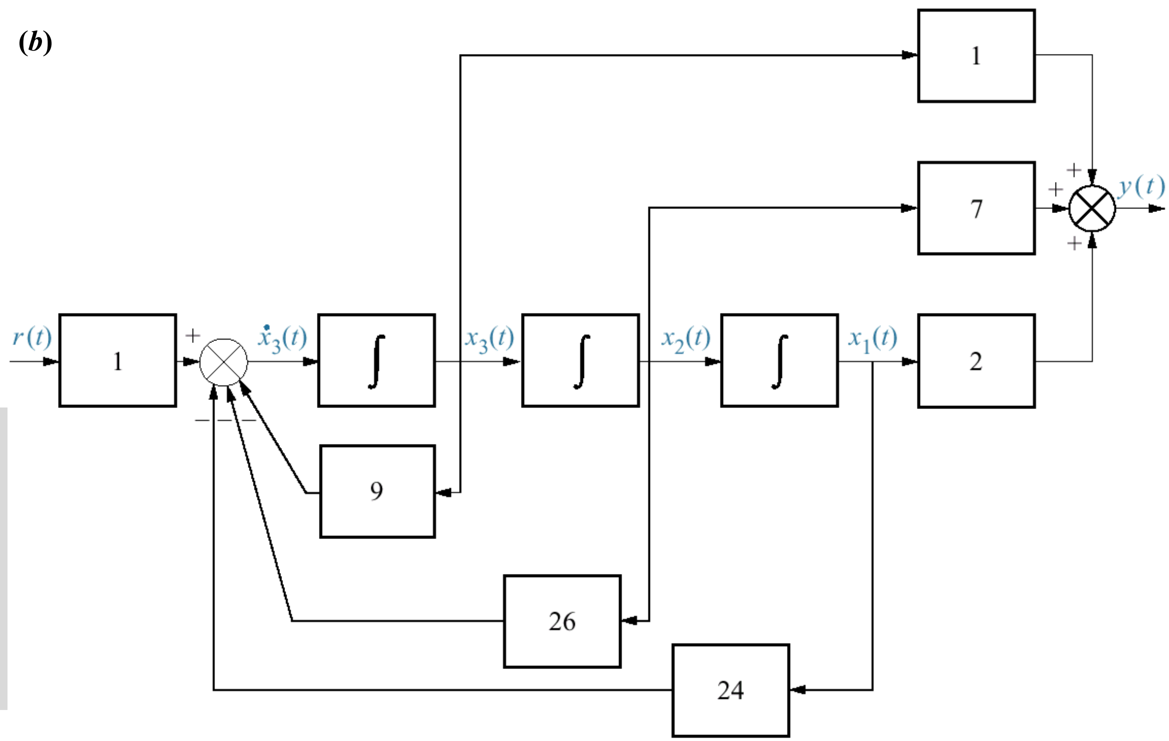
Example 3.5: (Continued)



(*a*)



(*b*)



(*c*)

Figure 3.12

a. Transfer function;

b. decomposed transfer function;

c. equivalent block diagram.

Note: $y(t) = c(t)$

* Controller Canonical Form page 260

$$G(s)=\frac{C(s)}{R(s)}=\frac{s^2+7s+2}{s^3+9s^2+26s+24}$$

- Phase-variable Form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r,\qquad y=\begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

- Renumbering the phase variables in reverse order: $x_1\rightarrow x_3,\ x_2\rightarrow x_2,\ x_3\rightarrow x_1$

$$x=\begin{bmatrix}x_3\\ x_2\\ x_1\end{bmatrix}\qquad \begin{bmatrix}\dot{x}_3\\ \dot{x}_2\\ \dot{x}_1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_3\\ x_2\\ x_1\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r,\qquad y=\begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_3\\ x_2\\ x_1\end{bmatrix}$$

- Ascending numerical order yields the ***controller canonical form***:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-9 & -26 & -24\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}r,\qquad y=\begin{bmatrix}1 & 7 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

* Matrix Transformation for Controller Canonical Form: ch3p4 (Example 3.4) and procedure in page 260

$$\begin{pmatrix} \dot{x}_p = A_P x_P + B_P r \\ y = C_P x_P \end{pmatrix} \qquad \begin{pmatrix} \dot{x}_C = A_C x_C + B_C r \\ y = C_C x_C \end{pmatrix}$$

$$\begin{cases} \dot{x}_C = A_C x_C + B_C r \\ y = C_C x_C \end{cases}$$

$$x_C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} x_P = P \cdot x_p \quad \rightarrow \quad \begin{array}{l} x_C = P \cdot x_p \\ \dot{x}_C = P \cdot \dot{x}_p \end{array}$$

$$P\dot{x}_p = A_C P x_P + B_C r, \qquad y = C_C P x_P$$

$$\dot{x}_p = \underbrace{P^{-1} A_C P}_{A_P} x_P + \underbrace{P^{-1} B_C}_{B_P} r, \quad y = \underbrace{C_C P}_{C_P} x_P$$

$$\begin{cases} \dot{x}_p = A_P x_P + B_P r \\ y = C_P x_P \end{cases}$$

$$\boxed{\begin{array}{l} A_P = P^{-1} A_C P \\ B_P = P^{-1} B_C \\ C_P = C_C P \end{array}}$$

• Appendix B: Matlab Code for ch3p4 (Example 3.4)

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

$$\left(= \frac{\text{numerator}}{\text{denominator}} \right)$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 24 \end{pmatrix} r$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

```
'Example 3.4'
num=24;
den=[1 9 26 24];

% Controller Canonical Form
[A, B, C, D]=tf2ss(num, den);

% To 'Phase-variable form'
P=[0 0 1; 0 1 0; 1 0 0];
Ap=inv(P)*A*P
Bp=inv(P)*B
Cp=C*P
Dp=D
```

$$\boxed{\begin{aligned} A_P &= P^{-1} A_C P \\ B_P &= P^{-1} B_C \\ C_P &= C_C P \\ D_P &= D \end{aligned}}$$

Controller Canonical Form

```
A =
     -9    -26    -24
      1      0      0
      0      1      0

B =
      1
      0
      0

C =
      0      0     24

D =
      0
```

Phase-variable Form

```
Ap =
      0      1      0
      0      0      1
    -24    -26     -9

Bp =
      0
      0
      1

Cp =
     24      0      0

Dp =
      0
```

```
>> T=tf(num, den)

T =
            24
   ----------------------
   s^3 + 9 s^2 + 26 s + 24

Continuous-time transfer function.


>> [N1, D1]=ss2tf(A,B,C,D)

N1 =
     0      0      0     24

D1 =

    1.0000    9.0000   26.0000   24.0000


>> [N2, D2]=ss2tf(Ap,Bp,Cp,Dp)

N2 =

     0      0      0     24


D2 =

    1.0000    9.0000   26.0000   24.0000
```

3.6 Converting from State Space to a Transfer function

- State and output equations: $\dot{x} = Ax + Bu, \quad y = Cx + Du$

- Laplace transform with zero initial conditions:
$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

- Solving for $X(s)$:
$$(sI - A)X(s) = BU(s)$$
$$X(s) = (sI - A)^{-1}BU(s)$$

- Solving for $Y(s)$:
$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$
$$= \left[C(sI - A)^{-1}B + D\right]U(s)$$

- If $U(s)$ and $Y(s)$ are scalars, the transfer functions is:
$$\boxed{T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}$$

Example 3.6: State-space representation to transfer function (page 140)

Find the transfer function, $T(s)=Y(s)/U(s)$.

$$\dot{\mathrm{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}\mathrm{x} + \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\mathrm{x} \quad \Rightarrow \quad \boxed{T(s) = \frac{Y(s)}{U(s)} = C(s\mathrm{I}-\mathrm{A})^{-1}\mathrm{B}+\mathrm{D}} \quad ?$$

$$(sI-A) = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{\mathrm{adj}(sI-\mathrm{A})}{\det(sI-\mathrm{A})} = \frac{\begin{pmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{pmatrix}}{s^3+3s^2+2s+1}$$

$$T(s) = \frac{Y(s)}{U(s)} = C(s\mathrm{I}-\mathrm{A})^{-1}\mathrm{B}+\mathrm{D}$$

$$= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}(s\mathrm{I}-\mathrm{A})^{-1}\begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + 0 = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$

$\Leftarrow$ The final result!

• Using Matlab to convert a state-space representation to a transfer function

```
A=[0 1 0;0 0 1;-1 -2 -3];
B=[10; 0; 0];
C=[1 0 0];
D=0;

[num,den]=ss2tf(A,B,C,D,1)
%G(s)=num/den,

Tss=ss(A,B,C,D)

'Polynomial form, Ttf(s)'
Ttf=tf(Tss)

'Factored form, Tzpk(s)'
Tzpk=zpk(Tss)
```

```
num =
        0    10.0000    30.0000    20.0000
den =
    1.0000     3.0000     2.0000     1.0000
```

```
a =
          x1   x2   x3
     x1    0    1    0
     x2    0    0    1
     x3   -1   -2   -3

b =
          u1
     x1   10
     x2    0
     x3    0

c =
          x1   x2   x3
     y1    1    0    0

d =
          u1
     y1    0

Continuous-time model.
```

```
ans = Polynomial form, Ttf(s)

Transfer function:
 10 s^2 + 30 s + 20
---------------------
s^3 + 3 s^2 + 2 s + 1

ans = Factored form, Tzpk(s)

Zero/pole/gain:

         10 (s+2) (s+1)
-----------------------------------
(s+2.325) (s^2  + 0.6753s + 0.4302)
```

- Using Matlab's Symbolic Math Toolbox to convert a state-space representation to a transfer function

```
'(ch3sp1) Example 3.6'          % Display label.
syms s                          % Construct symbolic object for frequency
                                % variable 's'.
A=[0 1 0;0 0 1;-1 -2 -3];       % Create matrix A.
B=[10;0;0];                     % Create vector B.
C=[1 0 0];                      % Create vector C.
D=0;                            % Create D.
I=[1 0 0;0 1 0;0 0 1];          % Create identity matrix.
'T(s)'                          % Display label.
T=C*((s*I-A)^-1)*B+D;           % Find transfer function.
pretty(T)                       % Pretty print transfer function.
```

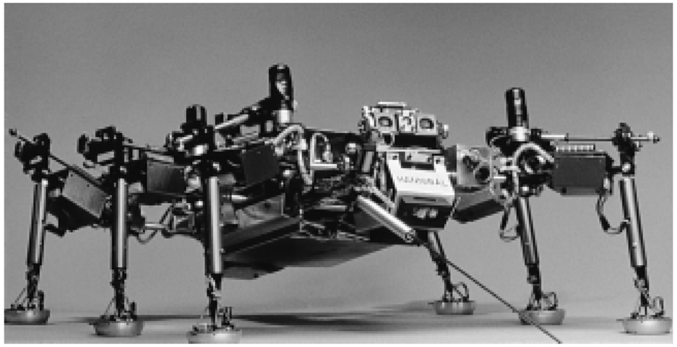
```
ans =
(ch3sp1) Example 3.6

ans =

T(s)
                        2
                       s  + 3 s + 2
                  10 -------------------
                      3      2
                     s  + 3 s  + 2 s + 1
>>
```

3.7 Linearization

Example 3.7: Representing a nonlinear system (page 142)

Linearize the state equations about the pendulum's equilibrium point

Walking robots (Hannibal): explore hostile environments and rough terrain



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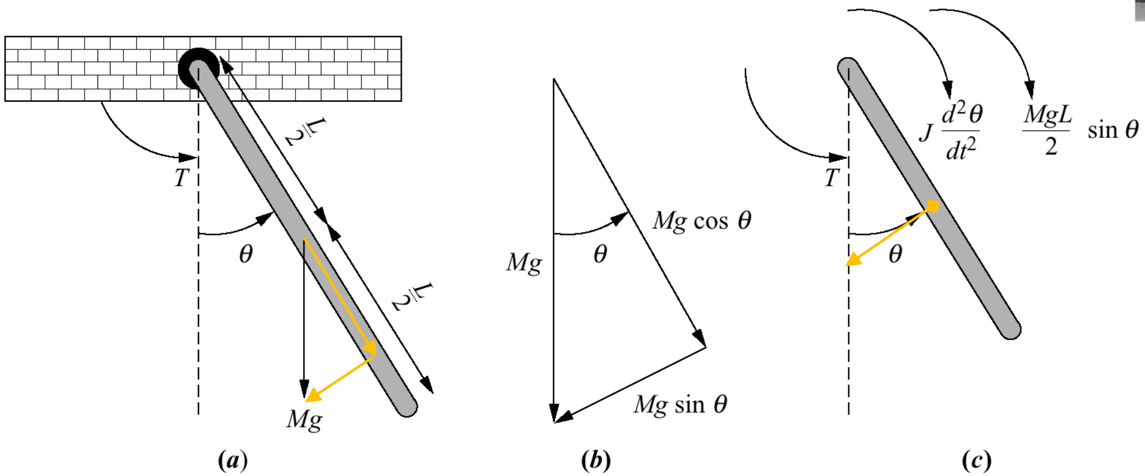

Torque angular displacement

Spring: $T(s) = K\theta(s)$

Viscous damper: $T(s) = Ds\theta(s)$

Inertia: $T(s) = Js^2\theta(s)$

$$J\frac{d^2\theta}{dt^2} + \frac{MgL}{2}\sin\theta = T$$

Let $x_1 = \theta, \quad x_2 = \dfrac{d\theta}{dt}$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{MgL}{2J}\sin x_1 + \dfrac{T}{J} \end{cases}$$

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{MgL}{2J}\sin\theta + \frac{T}{J}}$$

$$\boxed{x_1 = \theta, \quad x_2 = \frac{d\theta}{dt}}$$

- We need to linearize the equation about equilibrium point.
- The equilibrium point is: $x_1 = 0, \; x_2 = 0 \quad \rightarrow \quad \theta = 0, \; \dfrac{d\theta}{dt} = 0$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_e = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}_e = 0$$

- Let x_1 and x_2 be perturbed about the equilibrium point:

$$\left.\begin{aligned} x_1 &= 0 + \delta x_1 \\ x_2 &= 0 + \delta x_2 \end{aligned}\right\} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix}$$

- Taylor series expansion of sin(x) about equilibrium point:

$$f(x_1) = \sin(x_1) \approx f(x_0) + \left.\frac{df(x_1)}{dx}\right|_{x=x_0} \frac{(x_1 - x_0)}{1!} + \cdots \approx \sin(0) + \left.\frac{d\sin(x_1)}{dx_1}\right|_{x_1=0} (x_1 - 0) = \delta x_1$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{MgL}{2J}\sin x_1 + \dfrac{T}{J} \end{cases} \Rightarrow \begin{cases} \delta\dot{x}_1 = \delta x_2 \\ \delta\dot{x}_2 = -\dfrac{MgL}{2J}(\delta x_1) + \dfrac{T}{J} \end{cases}$$

$$\boxed{\begin{pmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{MgL}{2J} & 0 \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{T}{J} \end{pmatrix}}$$