김은영

제3장 ARMA 모형

3.2 MA모형

3.2.1 MA(1)모형

$$Z_t = a_t - \theta_1 a_{t-1}$$

■ 자기공분산함수

$Z_t = a_t - \theta_1 a_{t-1}$ $Var(Z_t) = Var(a_t - \theta_1 a_{t-1}) = Var(a_t) + \theta_1^2 Var(a_{t-1}) - 2\theta_1 Cov(a_t,\ a_{t-1})$ $\gamma(0) = \sigma_a^2 + \theta_1^2 \sigma_a^2 = (1+\theta_1^2)\sigma_a^2$
$Z_t = a_t - \theta_1 a_{t-1}$ $Z_t Z_{t-1} = a_t Z_{t-1} - \theta_1 a_{t-1} Z_{t-1}$ $E(Z_t Z_{t-1}) = E(a_t Z_{t-1}) - \theta_1 E(a_{t-1} Z_{t-1}) = -\theta_1 E(a_{t-1}^2) = -\theta_1 \sigma_a^2$ $\gamma(1) = -\theta_1 \sigma_a^2$
$Z_t = a_t - \theta_1 a_{t-1}$ $Z_t Z_{t-k} = a_t Z_{t-k} - \theta_1 a_{t-1} Z_{t-k}$ $E(Z_t Z_{t-k}) = E(a_t Z_{t-k}) - \theta_1 E(a_{t-1} Z_{t-k})$ $\gamma(k) = 0,\ k \geq 2$

· **정상성 조건**

$\gamma(0) = (1+\theta_1^2)\sigma_a^2 < \infty \quad \rightarrow \quad \theta_1^2 < \infty$

· **가역성(invertibility) 조건**

$$Z_t = a_t - \theta_1 a_{t-1} \quad \rightarrow \quad Z_t = a_t - \theta_1 B a_t = (1-\theta_1 B)a_t = \Theta(B)a_t$$

$$\Rightarrow \quad (1-\theta_1 B)^{-1} Z_t = a_t \quad \text{혹은}$$

$$(1+\theta_1 B + \theta_1^2 B^2 + \cdots)Z_t = a_t \quad \text{혹은}$$

$$Z_t = -\sum_{i=1}^{\infty} \theta_1^i Z_{t-i} + a_t$$

$\therefore$ 이런 식으로 표현되려면 θ_1은 $-1 < \theta_1 < 1$ 조건을 만족해야 한다.

※ **MA(1)모형의 ACF**

$$\rho(1)=\frac{\gamma(1)}{\gamma(0)}=\frac{-\theta_1\sigma_a^2}{\left(1+\theta_1^2\right)\sigma_a^2}=\frac{-\theta_1}{1+\theta_1^2}$$

$$\rho(k)=\frac{\gamma(k)}{\gamma(0)}=0,\ k\geq 2$$

· MA(1)모형의 ACF: 시차 1에서 절단되는 형태

■ MA(1)모형의 PACF

[정리 2.3]을 사용

$$P(1)=\rho(1)=\frac{\gamma(1)}{\gamma(0)}=\frac{-\theta_1\sigma_a^2}{\left(1+\theta_1^2\right)\sigma_a^2}=\frac{-\theta_1}{1+\theta_1^2}$$

$$P(2)=\frac{\rho(2)-\rho^2(1)}{1-\rho^2(1)}=\frac{-\left(\frac{-\theta_1}{1+\theta_1^2}\right)^2}{1-\left(\frac{-\theta_1}{1+\theta_1^2}\right)^2}=\frac{-\theta_1^2}{1+\theta_1^2+\theta_1^4}$$

$$=\frac{-\theta_1^2(1-\theta_1^2)}{(1+\theta_1^2+\theta_1^4)(1-\theta_1^2)}=\frac{-\theta_1^2(1-\theta_1^2)}{1-\theta_1^{2(3)}}$$

$$P(k)=\frac{-\theta_1^k(1-\theta_1^2)}{1-\theta_1^{2(k+1)}},\ k=1,\ 2,\ \cdots$$

· MA(1)모형의 PACF: 시차 k가 증가할 때 0으로 점차 감소

3.2.2 MA(2)모형

$$Z_t=a_t-\theta_1a_{t-1}-\theta_2a_{t-2}$$

$$Z_t=a_t-\theta_1Ba_t-\theta_2B^2a_t=(1-\theta_1B-\theta_2B^2)a_t=\Theta_2(B)a_t$$

■ 자기공분산함수

$$Z_t=a_t-\theta_1a_{t-1}-\theta_2a_{t-2}$$

$$Var(Z_t)=Var(a_t-\theta_1a_{t-1}-\theta_2a_{t-2})=Var(a_t)+\theta_1^2Var(a_{t-1})+\theta_2^2Var\left(a_{t-2}\right)$$

$$=(1+\theta_1^2+\theta_2^2)\,Var\left(a_t\right)=(1+\theta_1^2+\theta_2^2)\sigma_a^2$$

$$\gamma(0)=(1+\theta_1^2+\theta_2^2)\sigma_a^2$$

$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$ $E(Z_t Z_{t-1}) = E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-1} - \theta_1 a_{t-2} - \theta_2 a_{t-3})]$ $= E(-\theta_1 a_{t-1}^2 + \theta_1\theta_2 a_{t-2}^2) = -\theta_1 E(a_{t-1}^2) + \theta_1\theta_2 E(a_{t-2}^2) = (-\theta_1 + \theta_1\theta_2)\sigma_a^2$ $\gamma(1) = (-\theta_1 + \theta_1\theta_2)\sigma_a^2$
$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$ $E(Z_t Z_{t-2}) = E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4})]$ $= E(-\theta_2 a_{t-2}^2) = -\theta_2 E(a_{t-2}^2) = -\theta_2 \sigma_a^2$ $\gamma(2) = -\theta_2 \sigma_a^2$
$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$ $E(Z_t Z_{t-k}) = E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-k} - \theta_1 a_{t-k-1} - \theta_2 a_{t-k-2})]$ $= 0,\ k \geq 3$ $\gamma(k) = 0,\ k \geq 3$

· 정상성 조건

$\gamma(0) = (1 + \theta_1^2 + \theta_2^2)\sigma_a^2 < \infty \quad \rightarrow \quad \theta_1^2 + \theta_2^2 < \infty$

· 가역성(invertibility) 조건

: $\Theta_2(z) = 1 - \theta_1 z - \theta_2 z^2 = 0$의 두 근의 크기가 모두 1보다 커야 한다.

→ AR(2)모형의 정상성 조건과 대응된다.

$$-1 < \theta_2 < 1,\ \theta_2 + \theta_1 < 1,\ \theta_2 - \theta_1 < 1 \qquad (3.40)$$

※ MA(2)모형의 ACF

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{(-\theta_1 + \theta_1\theta_2)\sigma_a^2}{(1 + \theta_1^2 + \theta_2^2)\sigma_a^2} = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{-\theta_2\sigma_a^2}{(1 + \theta_1^2 + \theta_2^2)\sigma_a^2} = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)} = 0,\ k \geq 3$$

· MA(2)모형의 ACF: 시차 2에서 절단되는 형태

■ MA(2)모형의 PACF

[정리 2.3]을 사용

$$P(1)=\rho(1)=\frac{\gamma(1)}{\gamma(0)}=\frac{(-\theta_1+\theta_1\theta_2)\sigma_a^2}{\left(1+\theta_1^2+\theta_2^2\right)\sigma_a^2}=\frac{-\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2}$$

$$P(2)=\frac{\rho(2)-\rho^2(1)}{1-\rho^2(1)}$$

$$P(3)=\frac{\rho^3(1)-2\rho(1)\rho(2)+\rho(1)\rho^2(2)}{1-2\rho^2(1)+2\rho^2(1)\rho(2)-\rho^2(2)}$$

$$\cdots$$

・ MA(2)모형의 PACF: 시차 k가 증가할 때 0으로 점차 감소

3.2.3 MA(q)모형

$Z_t=a_t-\theta_1 a_{t-1}-\theta_2 a_{t-2}-\cdots-\theta_q a_{t-q}$

$Z_t=a_t-\theta_1 Ba_t-\theta_2 B^2 a_t-\cdots-\theta_q B^q a_t$

$=(1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q)a_t$

$=\Theta_q(B)a_t$

여기서, $\Theta_q(B)=1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q$

다항식: $\Theta_q(z)=1-\theta_1 z-\theta_2 z^2-\cdots-\theta_q z^q$

■ 자기공분산함수

$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}$ $Var(Z_t) = Var(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q})$ $= Var(a_t) + \theta_1^2 Var(a_{t-1}) + \theta_2^2 Var(a_{t-2}) + \cdots + \theta_q^2 Var(a_{t-q})$ $= (1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2) Var(a_t) = (1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2)\sigma_a^2$ $\gamma(0) = (1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2)\sigma_a^2$
$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}$ $E(Z_t Z_{t-1})$ $= E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q})(a_{t-1} - \theta_1 a_{t-2} - \theta_2 a_{t-3} - \cdots - \theta_q a_{t-q-1})]$ $= (-\theta_1 + \theta_2\theta_1 + \theta_3\theta_2 + \cdots + \theta_q\theta_{q-1})\sigma_a^2$ $\gamma(1) = (-\theta_1 + \theta_2\theta_1 + \theta_3\theta_2 + \cdots + \theta_q\theta_{q-1})\sigma_a^2$
$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}$ $E(Z_t Z_{t-k})$ $= E[(a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q})(a_{t-k} - \theta_1 a_{t-k-1} - \cdots - \theta_q a_{t-k-q})]$ $= (-\theta_k + \theta_{k+1}\theta_1 + \theta_{k+2}\theta_2 + \cdots + \theta_q\theta_{q-k})\sigma_a^2, \quad (k=1,\ 2,\ \cdots,\ q)$ $\gamma(k) = 0,\ k \ge q+1$

· 정상성 조건

$$\gamma(0) = (1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2)\sigma_a^2 < \infty \quad \rightarrow \quad \sum_{i=1}^{q}\theta_i^2 < \infty$$

· 가역성(invertibility) 조건

: $\Theta_q(z) = 1 - \theta_1 z - \theta_2 z^2 - \cdots - \theta_q z^q = 0$의 q개 근 각각의 크기가 1보다 커야 한다.

※ MA(q)모형의 ACF

$$\rho(k) = \begin{cases} \dfrac{-\theta_k + \theta_{k+1}\theta_1 + \ldots + \theta_q\theta_{q-k}}{1+\theta_1^2+\ldots+\theta_q^2}, & k = 1,2,\cdots,q \\ 0, & k \ge q+1 \end{cases} \qquad (3.49)$$

· MA(q)모형의 ACF: 시차 q 직후 절단되는 패턴

■ MA(q)모형의 PACF

[정리 2.3]을 사용

$$P(1)=\rho(1)=\frac{\gamma(1)}{\gamma(0)}=\frac{(-\theta_1+\theta_1\theta_2)\sigma_a^2}{\left(1+\theta_1^2+\theta_2^2\right)\sigma_a^2}=\frac{-\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2}$$

$$P(2)=\frac{\rho(2)-\rho^2(1)}{1-\rho^2(1)}$$

$$P(3)=\frac{\rho^3(1)-2\rho(1)\rho(2)+\rho(1)\rho^2(2)}{1-2\rho^2(1)+2\rho^2(1)\rho(2)-\rho^2(2)}$$

$$\cdots$$

- MA(q)모형의 PACF: 시차 k가 증가할 때 0으로 점차 감소