ECE-GY 9273 HW3

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December 8, 2019

Exercise 1 and 3 are answered directly in jupyter notebook.

Exercise 2

1. We denote the positions of four vertices by $p_i = (p_{i,x}, p_{i,y}), i = 1, \cdots, 4$, hence the constraints can be written as a function of p_i

$$g(p) = \begin{bmatrix} \|p_1 - p_2\|_2 \\ \|p_1 - p_3\|_2 \\ \|p_1 - p_4\|_2 \\ \|p_2 - p_3\|_2 \\ \|p_3 - p_4\|_2 \end{bmatrix}.$$

Therefore the Jacobian matrix is defined as

$$\frac{\partial g}{\partial p} = \begin{bmatrix} \frac{\partial g_{12}}{\partial p_{1,x}} & \frac{\partial g_{12}}{\partial p_{1,y}} & \frac{\partial g_{12}}{\partial p_{2,x}} & \frac{\partial g_{12}}{\partial p_{2,y}} & & & & \\ \frac{\partial g_{13}}{\partial p_{1,x}} & \frac{\partial g_{13}}{\partial p_{1,y}} & & & \frac{\partial g_{13}}{\partial p_{3,x}} & \frac{\partial g_{13}}{\partial p_{3,y}} & & \\ \frac{\partial g_{14}}{\partial p_{1,x}} & \frac{\partial g_{14}}{\partial p_{1,y}} & & & & & \frac{\partial g_{14}}{\partial p_{4,x}} & \frac{\partial g_{14}}{\partial p_{4,y}} \\ & & \frac{\partial g_{23}}{\partial p_{2,x}} & \frac{\partial g_{23}}{\partial p_{2,y}} & \frac{\partial g_{23}}{\partial p_{3,x}} & \frac{\partial g_{23}}{\partial p_{3,y}} & & \\ & & & & \frac{\partial g_{34}}{\partial p_{3,x}} & \frac{\partial g_{34}}{\partial p_{3,y}} & \frac{\partial g_{34}}{\partial p_{4,x}} & \frac{\partial g_{34}}{\partial p_{4,y}}. \end{bmatrix}$$

The only thing we need for computing rigidity matrix is the positions p_i. From the following figure Fig 1, we can easily see that x, y satisfy the equation:

$$x^2 + y^2 = \frac{9}{4}$$
$$y = x + 1,$$

which gives $x = \frac{-2+\sqrt{14}}{4}, y = \frac{2+\sqrt{14}}{4}$. Hence, we have $p_1 = (x, y) = (\frac{-2+\sqrt{14}}{4}, \frac{2+\sqrt{14}}{4}), p_2 = (1+x, y) = (\frac{2+\sqrt{14}}{4}, \frac{2+\sqrt{14}}{4}), p_3 = (y, x) = (\frac{2+\sqrt{14}}{4}, \frac{-2+\sqrt{14}}{4}), p_4 = (0, 0)$, and the rigidity matrix is

$$\frac{\partial g}{\partial p} = \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 2 & -2 & 0 & 0 \\ \frac{-2+\sqrt{14}}{2} & \frac{2+\sqrt{14}}{2} & 0 & 0 & 0 & 0 & \frac{2-\sqrt{14}}{2} & \frac{-2-\sqrt{14}}{2} \\ 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2+\sqrt{14}}{2} & \frac{-2+\sqrt{14}}{2} & \frac{-2-\sqrt{14}}{2} & \frac{2-\sqrt{14}}{2} \end{bmatrix}.$$

The rank is 5 which is also the dimension of its range space and thus the dimension of its kernel is 3.

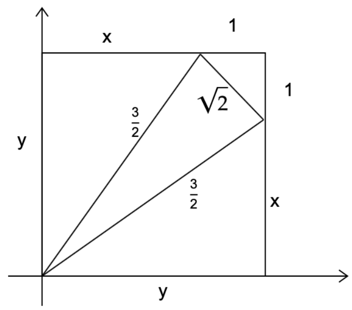

Figure 1: Framwork

2. Since the dimension of its kernel is 3, there are three directions of motions that preserve the constraints[1].

3. By matrix multiplication, we can show that

$$\frac{\partial g}{\partial p} q_x = 0$$
$$\frac{\partial g}{\partial p} q_y = 0,$$

where $q_x = [1,0,1,0,1,0,1,0]^\mathsf{T}, q_y = [0,1,0,1,0,1,0,1]^\mathsf{T}$ representing infinitesimal motion along x and y respectively. While for rotations, we fix the p_4 namely the origin and rotations for other vertices are

$$\dot{P}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} P1 = [-1.43541435, 0.43541435]^\mathsf{T}$$
$$\dot{P}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} P2 = [-1.43541435, 1.43541435]^\mathsf{T}$$
$$\dot{P}_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} P3 = [-0.43541435, 1.43541435]^\mathsf{T}$$

Then the rotation vector is

$$r = [\dot{P}_1, \dot{P}_2, \dot{P}_3, \dot{P}_4]^\mathsf{T} = [-1.43541435, 0.43541435, -1.43541435, 1.43541435, -0.43541435, 1.43541435, 0, 0]^\mathsf{T},$$

and notice that $\frac{\partial g}{\partial p} r = 0$, we conclude that this rotation also preserves the constraints.

By now, we have shown that q_x, q_y and r all lie in the kernel space of rigidity matrix and with python linear algebra lib, we can show that the three vectors are linearly independent, thus they form a basis.

[1] we use python linear algebra lib for computing ranks and matrix multiplication and codes are attached in jupyter notebook