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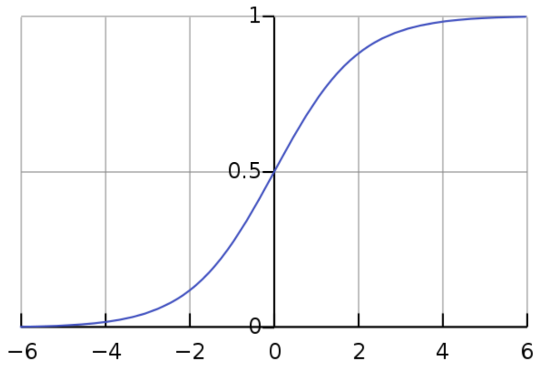
Logistic function

A **logistic function** or **logistic curve** is a common S-shaped curve (sigmoid curve) with equation

$$f(x) = \frac{L}{1 + e^{-k(x-x_0)}},$$

where

e = the natural logarithm base (also known as Euler's number),
x_0 = the x value of the sigmoid's midpoint,
L = the curve's maximum value,
k = the logistic growth rate or steepness of the curve.[1]



Standard logistic sigmoid function i.e. $L = 1, k = 1, x_0 = 0$

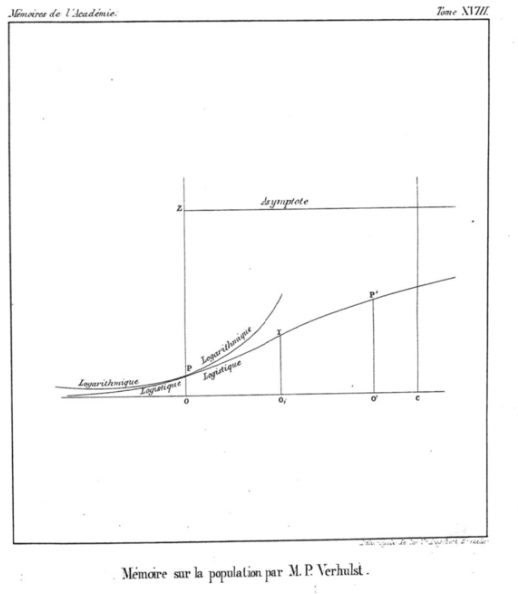
For values of x in the domain of real numbers from $-\infty$ to $+\infty$, the S-curve shown on the right is obtained, with the graph of f approaching L as x approaches $+\infty$ and approaching zero as x approaches $-\infty$.

The logistic function finds applications in a range of fields, including artificial neural networks, biology (especially ecology), biomathematics, chemistry, demography, economics, geoscience, mathematical psychology, probability, sociology, political science, linguistics, and statistics.

Contents

History

The logistic function was introduced in a series of three papers by Pierre François Verhulst between 1838 and 1847, who devised it as a model of population growth by adjusting the exponential growth model, under the guidance of Adolphe Quetelet.[2] Verhulst first devised the function in the mid 1830s, publishing a brief note in 1838,[3] then presented an expanded analysis and named the function in 1844 (published 1845);[a][4] the third paper adjusted the correction term in his model of Belgian population growth.[5]



Original image of a logistic curve, contrasted with a logarithmic curve

The initial stage of growth is approximately exponential (geometric); then, as saturation begins, the growth slows to linear (arithmetic), and at maturity, growth stops. Verhulst did not explain the choice of the term "logistic" (French: *logistique*), but it is presumably in contrast to the *logarithmic* curve,[6][b] and by analogy with arithmetic and geometric. His growth model is preceded by a discussion of arithmetic growth and geometric growth (whose curve he calls a logarithmic curve, instead of the modern term exponential curve), and thus "logistic growth" is presumably named by analogy, *logistic* being from Ancient Greek: λογῐστῐκός, romanized: *logistikós*, a traditional division of Greek mathematics.[c] The term is unrelated to the military and management term *logistics*, which is instead from French: *logis* "lodgings", though some believe the Greek term also influenced *logistics*; see Logistics § Origin for details.

Mathematical properties

The **standard logistic function** is the logistic function with parameters $k = 1$, $x_0 = 0$, $L = 1$, which yields

$$f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1} = \tfrac{1}{2} + \tfrac{1}{2}\tanh(\tfrac{x}{2}).$$

In practice, due to the nature of the exponential function e^{-x}, it is often sufficient to compute the standard logistic function for x over a small range of real numbers, such as a range contained in [−6, +6], as it quickly converges very close to its saturation values of 0 and 1.

The logistic function has the symmetry property that

$$1 - f(x) = f(-x).$$

Thus, $x \mapsto f(x) - 1/2$ is an odd function.

The logistic function is an offset and scaled hyperbolic tangent function:

$$f(x) = \tfrac{1}{2} + \tfrac{1}{2}\tanh(\tfrac{x}{2}),$$

or

$$\tanh(x) = 2f(2x) - 1.$$

This follows from

$$\begin{aligned}\tanh(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x \cdot \left(1 - e^{-2x}\right)}{e^x \cdot \left(1 + e^{-2x}\right)} \\ &= f(2x) - \frac{e^{-2x}}{1+e^{-2x}} = f(2x) - \frac{e^{-2x}+1-1}{1+e^{-2x}} = 2f(2x) - 1.\end{aligned}$$

Derivative

The standard logistic function has an easily calculated derivative. The derivative is known as the logistic distribution (not to be confused with the normal distribution).

$$f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x},$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}f(x) = \frac{e^x\cdot(1+e^x) - e^x\cdot e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} = f(x)\big(1-f(x)\big) = f(x)f(-x).$$

The derivative of the logistic function is an even function, that is,

$$f'(-x) = f'(x).$$

Integral

Conversely, its antiderivative can be computed by the substitution $u = 1 + e^x$, since $f(x) = \frac{e^x}{1+e^x} = \frac{u'}{u}$, so (dropping the constant of integration)

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du = \log u = \log(1+e^x).$$

In artificial neural networks, this is known as the *softplus* function and (with scaling) is a smooth approximation of the ramp function, just as the logistic function (with scaling) is a smooth approximation of the Heaviside step function.

Logistic differential equation

The standard logistic function is the solution of the simple first-order non-linear ordinary differential equation

$$\frac{d}{dx}f(x) = f(x)\big(1-f(x)\big)$$

with boundary condition $f(0) = 1/2$. This equation is the continuous version of the logistic map.

The qualitative behavior is easily understood in terms of the phase line: the derivative is 0 when the function is 1; and the derivative is positive for f between 0 and 1, and negative for f above 1 or less than 0 (though negative populations do not generally accord with a physical model). This yields an unstable equilibrium at 0 and a stable equilibrium at 1, and thus for any function value greater than 0 and less than 1, it grows to 1.

The logistic equation is a special case of the Bernoulli differential equation and has the following solution:

$$f(x) = \frac{e^x}{e^x + C}.$$

Choosing the constant of integration $C = 1$ gives the other well known form of the definition of the logistic curve:

$$f(x) = \frac{e^x}{e^x+1} = \frac{1}{1+e^{-x}}.$$

More quantitatively, as can be seen from the analytical solution, the logistic curve shows early exponential growth for negative argument, which slows to linear growth of slope 1/4 for an argument near 0, then approaches 1 with an exponentially decaying gap.

The logistic function is the inverse of the natural logit function and so can be used to convert the logarithm of odds into a probability. In mathematical notation the logistic function is sometimes written as *expit*[7] in the same form as *logit*. The conversion from the log-likelihood ratio of two alternatives also takes the form of a logistic curve.

The differential equation derived above is a special case of a general differential equation that only models the sigmoid function for $x > 0$. In many modeling applications, the more *general form*[8]

$$\frac{df(x)}{dx} = \frac{k}{a} f(x)\big(a - f(x)\big), \quad f(0) = a/(1 + e^{kr})$$

can be desirable. Its solution is the shifted and scaled sigmoid $aS\big(k(x-r)\big)$.

The hyperbolic-tangent relationship leads to another form for the logistic function's derivative:

$$\frac{d}{dx} f(x) = \frac{1}{4} \operatorname{sech}^2\left(\frac{x}{2}\right),$$

which ties the logistic function into the logistic distribution.

Rotational symmetry about (0, 1/2)

The sum of the logistic function and its reflection about the vertical axis, $f(-x)$, is

$$\frac{1}{1+e^{-x}} + \frac{1}{1+e^{-(-x)}} = \frac{e^x}{e^x+1} + \frac{1}{e^x+1} = 1.$$

The logistic function is thus rotationally symmetrical about the point (0, 1/2).[9]

Applications

Link[10] created an extension of Wald's theory of sequential analysis to a distribution-free accumulation of random variables until either a positive or negative bound is first equaled or exceeded. Link[11] derives the probability of first equaling or exceeding the positive boundary as $(1 + e^{-\theta A})$, the Logistic function. This is the first proof that the Logistic function may have a stochastic process as its basis. Link[12] provides a century of examples of “Logistic” experimental results and a newly derived relation between this probability and the time of absorption at the boundaries.

In ecology: modeling population growth

Pierre-François Verhulst (1804–1849)

A typical application of the logistic equation is a common model of population growth (see also population dynamics), originally due to Pierre-François Verhulst in 1838, where the rate of reproduction is proportional to both the existing population and the amount of available resources, all else being equal. The Verhulst equation was published after Verhulst had read Thomas Malthus' *An Essay on the Principle of Population*. Verhulst derived his logistic equation to describe the self-limiting growth of a biological population. The equation was rediscovered in 1911 by A. G. McKendrick for the growth of bacteria in broth and experimentally tested using a technique for nonlinear parameter estimation.[13] The equation is also sometimes called the *Verhulst-Pearl equation* following its rediscovery in 1920 by Raymond Pearl (1879–1940) and Lowell Reed (1888–1966) of the Johns Hopkins University.[14] Another scientist, Alfred J. Lotka derived the equation again in 1925, calling it the *law of population growth*.

Letting P represent population size (N is often used in ecology instead) and t represent time, this model is formalized by the differential equation:

$$\frac{dP}{dt} = rP \cdot \left(1 - \frac{P}{K}\right),$$

where the constant r defines the growth rate and K is the carrying capacity.

In the equation, the early, unimpeded growth rate is modeled by the first term $+rP$. The value of the rate r represents the proportional increase of the population P in one unit of time. Later, as the population grows, the modulus of the second term (which multiplied out is $-rP^2/K$) becomes almost as large as the first, as some members of the population P interfere with each other by competing for some critical resource, such as food or living space. This antagonistic effect is called the *bottleneck*, and is modeled by the value of the parameter K. The competition diminishes the combined growth rate, until the value of P ceases to grow (this is called *maturity* of the population). The solution to the equation (with P_0 being the initial population) is

$$P(t) = \frac{KP_0e^{rt}}{K + P_0\left(e^{rt} - 1\right)} = \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right)e^{-rt}},$$

where

$$\lim_{t\to\infty} P(t) = K.$$

Which is to say that K is the limiting value of P: the highest value that the population can reach given infinite time (or come close to reaching in finite time). It is important to stress that the carrying capacity is asymptotically reached independently of the initial value $P(0) > 0$, and also in the case that $P(0) > K$.

In ecology, species are sometimes referred to as r-strategist or K-strategist depending upon the selective processes that have shaped their life history strategies. Choosing the variable dimensions so that n measures the population in units of carrying capacity, and τ measures time in units of $1/r$, gives the dimensionless differential equation

$$\frac{dn}{d\tau} = n(1 - n).$$

Time-varying carrying capacity

Since the environmental conditions influence the carrying capacity, as a consequence it can be time-varying, with $K(t) > 0$, leading to the following mathematical model:

$$\frac{dP}{dt} = rP \cdot \left(1 - \frac{P}{K(t)}\right).$$

A particularly important case is that of carrying capacity that varies periodically with period T:

$$K(t + T) = K(t).$$

It can be shown that in such a case, independently from the initial value $P(0) > 0$, $P(t)$ will tend to a unique periodic solution $P_*(t)$, whose period is T.

A typical value of T is one year: In such case $K(t)$ may reflect periodical variations of weather conditions.

Another interesting generalization is to consider that the carrying capacity $K(t)$ is a function of the population at an earlier time, capturing a delay in the way population modifies its environment. This leads to a logistic delay equation,[15] which has a very rich behavior, with bistability in some parameter range, as well as a monotonic decay to zero, smooth exponential growth, punctuated unlimited growth (i.e., multiple S-shapes), punctuated growth or alternation to a stationary level, oscillatory approach to a stationary level, sustainable oscillations, finite-time singularities as well as finite-time death.