

RANDOM VARIABLES AND DENSITIES

REVIEW:
PROBABILITY AND STATISTICS

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- Random variables X represents outcomes or states of world.
Instantiations of variables usually in lower case: x
We will write $p(x)$ to mean probability($X = x$).
- Sample Space: the space of all possible outcomes/states.
(May be discrete or continuous or mixed.)
- Probability mass (density) function $p(x) \geq 0$
Assigns a non-negative number to each point in sample space.
Sums (integrates) to unity: $\sum_x p(x) = 1$ or $\int_x p(x)dx = 1$.
Intuitively: how often does x occur, how much do we believe in x .
- Ensemble: random variable + sample space+ probability function

PROBABILITY

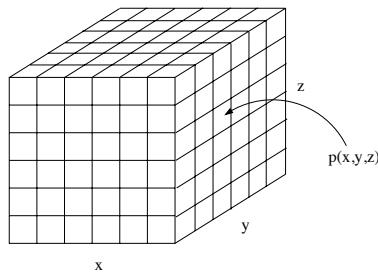
- We use probabilities $p(x)$ to represent our beliefs $B(x)$ about the states x of the world.
- There is a formal calculus for manipulating uncertainties represented by probabilities.
- Any consistent set of beliefs obeying the *Cox Axioms* can be mapped into probabilities.
 1. Rationally ordered degrees of belief:
if $B(x) > B(y)$ and $B(y) > B(z)$ then $B(x) > B(z)$
 2. Belief in x and its negation \bar{x} are related: $B(x) = f[B(\bar{x})]$
 3. Belief in conjunction depends only on conditionals:
 $B(x \text{ and } y) = g[B(x), B(y|x)] = g[B(y), B(x|y)]$

EXPECTATIONS, MOMENTS

- Expectation of a function $a(x)$ is written $E[a]$ or $\langle a \rangle$
$$E[a] = \langle a \rangle = \sum_x p(x)a(x)$$
e.g. mean = $\sum_x xp(x)$, variance = $\sum_x (x - E[x])^2 p(x)$
- Moments are expectations of higher order powers.
(Mean is first moment. Autocorrelation is second moment.)
- Centralized moments have lower moments subtracted away (e.g. variance, skew, kurtosis).
- Deep fact: Knowledge of all orders of moments completely defines the entire distribution.

JOINT PROBABILITY

- Key concept: two or more random variables may interact. Thus, the probability of one taking on a certain value depends on which value(s) the others are taking.
- We call this a joint ensemble and write $p(x, y) = \text{prob}(X = x \text{ and } Y = y)$

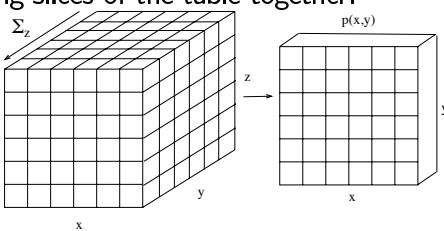


MARGINAL PROBABILITIES

- We can "sum out" part of a joint distribution to get the *marginal distribution* of a subset of variables:

$$p(x) = \sum_y p(x, y)$$

- This is like adding slices of the table together.

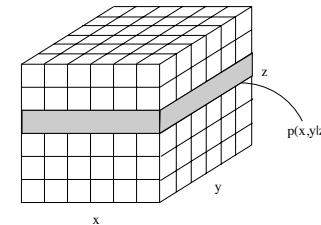


- Another equivalent definition: $p(x) = \sum_y p(x|y)p(y)$.

CONDITIONAL PROBABILITY

- If we know that some event has occurred, it changes our belief about the probability of other events.
- This is like taking a "slice" through the joint table.

$$p(x|y) = p(x, y)/p(y)$$



BAYES' RULE

- Manipulating the basic definition of conditional probability gives one of the most important formulas in probability theory:

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\sum_{x'} p(y|x')p(x')}$$

- This gives us a way of "reversing" conditional probabilities.
- Thus, all joint probabilities can be factored by selecting an ordering for the random variables and using the "chain rule":

$$p(x, y, z, \dots) = p(x)p(y|x)p(z|x, y)p(\dots | x, y, z)$$

INDEPENDENCE & CONDITIONAL INDEPENDENCE

- Two variables are independent iff their joint factors:

$$p(x, y) = p(x)p(y)$$

- Two variables are conditionally independent given a third one if for all values of the conditioning variable, the resulting slice factors:

$$p(x, y|z) = p(x|z)p(y|z) \quad \forall z$$

ENTROPY

- Measures the amount of ambiguity or uncertainty in a distribution:

$$H(p) = - \sum_x p(x) \log p(x)$$

- Expected value of $-\log p(x)$ (a function which depends on $p(x)!$).
- $H(p) > 0$ unless only one possible outcome in which case $H(p) = 0$.
- Maximal value when p is uniform.
- Tells you the expected "cost" if each event costs $-\log p(\text{event})$

BE CAREFUL!

- Watch the context:
e.g. Simpson's paradox
- Define random variables and sample spaces carefully:
e.g. Prisoner's paradox

CROSS ENTROPY (KL DIVERGENCE)

- An asymmetric measure of the distance between two distributions:

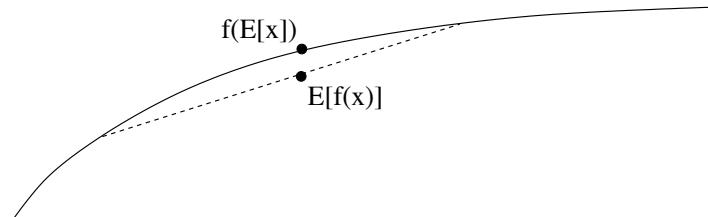
$$KL[p||q] = \sum_x p(x)[\log p(x) - \log q(x)]$$

- $KL > 0$ unless $p = q$ then $KL = 0$
- Tells you the extra cost if events were generated by $p(x)$ but instead of charging under $p(x)$ you charged under $q(x)$.

JENSEN'S INEQUALITY

- For any convex function $f()$ and any distribution on x ,

$$E[f(x)] \geq f(E[x])$$



- e.g. $\log()$ and $\sqrt{}$ are convex

STATISTICS

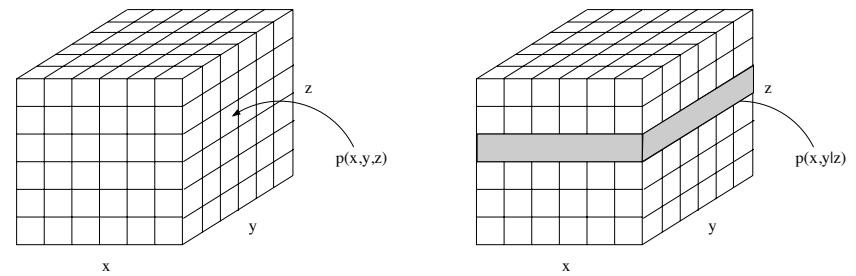
- Probability: inferring probabilistic quantities for data given fixed models (e.g. prob. of events, marginals, conditionals, etc.).
- Statistics: inferring a model given fixed data observations (e.g. clustering, classification, regression).
- Many approaches to statistics:
frequentist, Bayesian, decision theory, ...

SOME (CONDITIONAL) PROBABILITY FUNCTIONS

- Probability density functions $p(x)$ (for continuous variables) or probability mass functions $p(x = k)$ (for discrete variables) tell us how likely it is to get a particular value for a random variable (possibly conditioned on the values of some other variables.)
- We can consider various types of variables: binary/discrete (categorical), continuous, interval, and integer counts.
- For each type we'll see some basic *probability models* which are parametrized families of distributions.

(CONDITIONAL) PROBABILITY TABLES

- For discrete (categorical) quantities, the most basic parametrization is the probability table which lists $p(x_i = k^{th} \text{ value})$.
- Since PTs must be nonnegative and sum to 1, for k -ary variables there are $k - 1$ free parameters.
- If a discrete variable is conditioned on the values of some other discrete variables we make one table for each possible setting of the parents: these are called *conditional probability tables* or CPTs.



EXPONENTIAL FAMILY

- For (continuous or discrete) random variable \mathbf{x}

$$\begin{aligned} p(\mathbf{x}|\eta) &= h(\mathbf{x}) \exp\{\eta^\top T(\mathbf{x}) - A(\eta)\} \\ &= \frac{1}{Z(\eta)} h(\mathbf{x}) \exp\{\eta^\top T(\mathbf{x})\} \end{aligned}$$

is an exponential family distribution with *natural parameter* η .

- Function $T(\mathbf{x})$ is a *sufficient statistic*.
- Function $A(\eta) = \log Z(\eta)$ is the log normalizer.
- Key idea: all you need to know about the data is captured in the summarizing function $T(\mathbf{x})$.

POISSON

- For an integer count variable with rate λ :

$$\begin{aligned} p(x|\lambda) &= \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \frac{1}{x!} \exp\{x \log \lambda - \lambda\} \end{aligned}$$

- Exponential family with:

$$\begin{aligned} \eta &= \log \lambda \\ T(x) &= x \\ A(\eta) &= \lambda = e^\eta \\ h(x) &= \frac{1}{x!} \end{aligned}$$

- e.g. number of photons \mathbf{x} that arrive at a pixel during a fixed interval given mean intensity λ
- Other count densities: binomial, exponential.

BERNOULLI

- For a binary random variable with $p(\text{heads})=\pi$:

$$\begin{aligned} p(x|\pi) &= \pi^x (1-\pi)^{1-x} \\ &= \exp \left\{ \log \left(\frac{\pi}{1-\pi} \right) x + \log(1-\pi) \right\} \end{aligned}$$

- Exponential family with:

$$\eta = \log \frac{\pi}{1-\pi}$$

$$T(x) = x$$

$$A(\eta) = -\log(1-\pi) = \log(1+e^\eta)$$

$$h(x) = 1$$

- The logistic function relates the natural parameter and the chance of heads

$$\pi = \frac{1}{1+e^{-\eta}}$$

MULTINOMIAL

- For a set of integer counts on k trials

$$p(\mathbf{x}|\pi) = \frac{k!}{x_1!x_2!\cdots x_n!} \pi_1^{x_1} \pi_2^{x_2} \cdots \pi_n^{x_n} = h(\mathbf{x}) \exp \left\{ \sum_i x_i \log \pi_i \right\}$$

- But the parameters are constrained: $\sum_i \pi_i = 1$.
So we define the last one $\pi_n = 1 - \sum_{i=1}^{n-1} \pi_i$.

$$p(\mathbf{x}|\pi) = h(\mathbf{x}) \exp \left\{ \sum_{i=1}^{n-1} \log \left(\frac{\pi_i}{\pi_n} \right) x_i + k \log \pi_n \right\}$$

- Exponential family with:

$$\eta_i = \log \pi_i - \log \pi_n$$

$$T(x_i) = x_i$$

$$A(\eta) = -k \log \pi_n = k \log \sum_i e^{\eta_i}$$

$$h(\mathbf{x}) = k! / x_1! x_2! \cdots x_n!$$

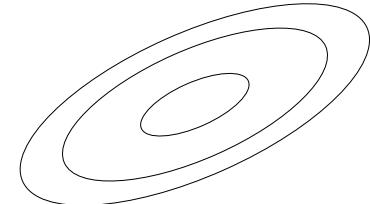
- The softmax function relates the basic and natural parameters:

$$\pi_i = \frac{e^{\eta_i}}{\sum_j e^{\eta_j}}$$

MULTIVARIATE GAUSSIAN DISTRIBUTION

- For a continuous vector random variable:

$$p(\mathbf{x}|\mu, \Sigma) = |2\pi\Sigma|^{-1/2} \exp \left\{ -\frac{1}{2}(\mathbf{x} - \mu)^\top \Sigma^{-1} (\mathbf{x} - \mu) \right\}$$



- Exponential family with:

$$\eta = [\Sigma^{-1}\mu ; -1/2\Sigma^{-1}]$$

$$T(x) = [\mathbf{x} ; \mathbf{x}\mathbf{x}^\top]$$

$$A(\eta) = \log |\Sigma|/2 + \mu^\top \Sigma^{-1} \mu / 2$$

$$h(x) = (2\pi)^{-n/2}$$

- Sufficient statistics: mean vector and correlation matrix.
- Other densities: Student-t, Laplacian.
- For non-negative values use exponential, Gamma, log-normal.

GAUSSIAN (NORMAL)

- For a continuous univariate random variable:

$$\begin{aligned} p(x|\mu, \sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2\sigma^2}(x - \mu)^2 \right\} \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ \frac{\mu x}{\sigma^2} - \frac{x^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} - \log \sigma \right\} \end{aligned}$$

- Exponential family with:

$$\eta = [\mu/\sigma^2 ; -1/2\sigma^2]$$

$$T(x) = [x ; x^2]$$

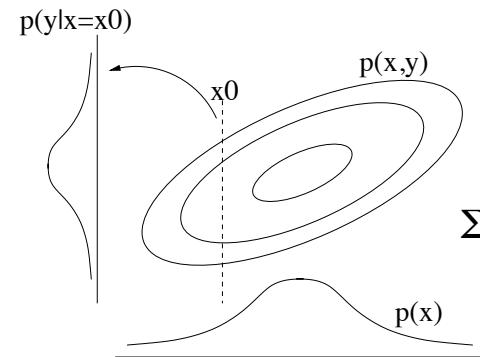
$$A(\eta) = \log \sigma + \mu/2\sigma^2$$

$$h(x) = 1/\sqrt{2\pi}$$

- Note: a univariate Gaussian is a two-parameter distribution with a two-component vector of sufficient statistics.

IMPORTANT GAUSSIAN FACTS

- All marginals of a Gaussian are again Gaussian.
Any conditional of a Gaussian is again Gaussian.



GAUSSIAN MARGINALS/CONDITIONALS

- To find these parameters is mostly linear algebra:

Let $\mathbf{z} = [\mathbf{x}^\top \mathbf{y}^\top]^\top$ be normally distributed according to:

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}; \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{bmatrix} \right)$$

where \mathbf{C} is the (non-symmetric) cross-covariance matrix between \mathbf{x} and \mathbf{y} which has as many rows as the size of \mathbf{x} and as many columns as the size of \mathbf{y} .

The marginal distributions are:

$$\mathbf{x} \sim \mathcal{N}(\mathbf{a}; \mathbf{A})$$

$$\mathbf{y} \sim \mathcal{N}(\mathbf{b}; \mathbf{B})$$

and the conditional distributions are:

$$\mathbf{x}|\mathbf{y} \sim \mathcal{N}(\mathbf{a} + \mathbf{C}\mathbf{B}^{-1}(\mathbf{y} - \mathbf{b}); \mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top)$$

$$\mathbf{y}|\mathbf{x} \sim \mathcal{N}(\mathbf{b} + \mathbf{C}^\top\mathbf{A}^{-1}(\mathbf{x} - \mathbf{a}); \mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C})$$

PARAMETERIZING CONDITIONALS

- When the variable(s) being conditioned on (parents) are discrete, we just have one density for each possible setting of the parents. e.g. a table of natural parameters in exponential models or a table of tables for discrete models.
- When the conditioned variable is continuous, its value sets some of the parameters for the other variables.
- A very common instance of this for regression is the “linear-Gaussian”: $p(\mathbf{y}|\mathbf{x}) = \text{gauss}(\boldsymbol{\theta}^\top \mathbf{x}; \Sigma)$.
- For discrete children and continuous parents, we often use a Bernoulli/multinomial whose parameters are some function $f(\boldsymbol{\theta}^\top \mathbf{x})$.

MOMENTS

- For continuous variables, moment calculations are important.
- We can easily compute moments of any exponential family distribution by taking the derivatives of the log normalizer $A(\eta)$.
- The q^{th} derivative gives the q^{th} centred moment.

$$\frac{dA(\eta)}{d\eta} = \text{mean}$$

$$\frac{d^2A(\eta)}{d\eta^2} = \text{variance}$$

...

- When the sufficient statistic is a vector, partial derivatives need to be considered.

GENERALIZED LINEAR MODELS (GLMs)

- Generalized Linear Models: $p(\mathbf{y}|\mathbf{x})$ is exponential family with conditional mean $\mu = f(\boldsymbol{\theta}^\top \mathbf{x})$.
- The function f is called the *response function*.
- If we chose f to be the inverse of the mapping b/w conditional mean and natural parameters then it is called the *canonical response function*.

$$\eta = \psi(\mu)$$

$$f(\cdot) = \psi^{-1}(\cdot)$$

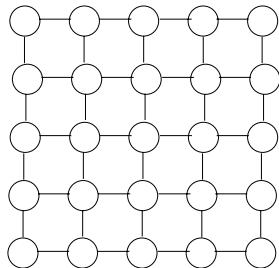
POTENTIAL FUNCTIONS

- We can be even more general and define distributions by arbitrary energy functions proportional to the log probability.

$$p(\mathbf{x}) \propto \exp\left\{-\sum_k H_k(\mathbf{x})\right\}$$

- A common choice is to use pairwise terms in the energy:

$$H(\mathbf{x}) = \sum_i a_i x_i + \sum_{\text{pairs } ij} w_{ij} x_i x_j$$



BASIC STATISTICAL PROBLEMS

- Let's remind ourselves of the basic problems we discussed on the first day: *density estimation, clustering, classification* and *regression*.

- Density estimation is hardest. If we can do joint density estimation, then we can always condition to get what we want:

Regression: $p(\mathbf{y}|\mathbf{x}) = p(\mathbf{y}, \mathbf{x})/p(\mathbf{x})$

Classification: $p(c|x) = p(c,x)/p(x)$
 Clustering: $p(c|x) \equiv p(c,x)/p(x)$ c unobserved

SPECIAL VARIABLES

- If certain variables are *always observed* we may not want to model their density. For example inputs in regression or classification. This leads to conditional density estimation.
 - If certain variables are *always unobserved*, they are called *hidden* or *latent* variables. They can always be marginalized out, but can make the density modeling of the observed variables easier.
(We'll see more on this later.)

BASIC STATISTICAL PROBLEMS

- ## FUNDAMENTAL OPERATIONS WITH DISTRIBUTIONS

- **Generate data:** draw samples from the distribution. This often involves generating a uniformly distributed variable in the range $[0,1]$ and transforming it. For more complex distributions it may involve an iterative procedure that takes a long time to produce a single sample (e.g. Gibbs sampling, MCMC).
 - **Compute log probabilities.**
When all variables are either observed or marginalized the result is a single number which is the log prob of the configuration.
 - **Inference:** Compute expectations of some variables given others which are observed or marginalized.
 - **Learning.**
Set the parameters of the density functions given some (partially) observed data to maximize likelihood or penalized likelihood.

LEARNING

- In AI the bottleneck is often knowledge acquisition.
- Human experts are rare, expensive, unreliable, slow.
- But we have lots of data.
- Want to build systems automatically based on data and a small amount of prior information (from experts).

MULTIPLE OBSERVATIONS, COMPLETE DATA, IID SAMPLING

- A single observation of the data \mathbf{X} is rarely useful on its own.
- Generally we have data including many observations, which creates a set of random variables: $\mathcal{D} = \{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^M\}$
- Two very common assumptions:
 1. Observations are independently and identically distributed according to joint distribution of graphical model: IID samples.
 2. We observe all random variables in the domain on each observation: complete data.

KNOWN MODELS

- Many systems we build will be essentially probability models.
- Assume the prior information we have specifies type & structure of the model, as well as the form of the (conditional) distributions or potentials.
- In this case learning \equiv setting parameters.
- Also possible to do “structure learning” to learn model.

LIKELIHOOD FUNCTION

- So far we have focused on the (log) probability function $p(\mathbf{x}|\theta)$ which assigns a probability (density) to any joint configuration of variables \mathbf{x} given fixed parameters θ .
- But in learning we turn this on its head: we have some fixed data and we want to find parameters.
- Think of $p(\mathbf{x}|\theta)$ as a function of θ for fixed \mathbf{x} :

$$L(\theta; \mathbf{x}) = p(\mathbf{x}|\theta)$$
$$\ell(\theta; \mathbf{x}) = \log p(\mathbf{x}|\theta)$$

This function is called the (log) “likelihood”.

- Choose θ to maximize some cost function $c(\theta)$ which includes $\ell(\theta)$:
 $c(\theta) = \ell(\theta; \mathcal{D})$ maximum likelihood (ML)
 $c(\theta) = \ell(\theta; \mathcal{D}) + r(\theta)$ maximum a posteriori (MAP)/penalizedML
(also cross-validation, Bayesian estimators, BIC, AIC, ...)

MAXIMUM LIKELIHOOD

- For IID data:

$$p(\mathcal{D}|\theta) = \prod_m p(\mathbf{x}^m|\theta)$$

$$\ell(\theta; \mathcal{D}) = \sum_m \log p(\mathbf{x}^m|\theta)$$

- Idea of maximum likelihood estimation (MLE): pick the setting of parameters most likely to have generated the data we saw:

$$\theta_{\text{ML}}^* = \operatorname{argmax}_{\theta} \ell(\theta; \mathcal{D})$$

- Very commonly used in statistics.

Often leads to “intuitive”, “appealing”, or “natural” estimators.

EXAMPLE: MULTINOMIAL

- We observe M iid die rolls (K -sided): $\mathcal{D}=3,1,K,2,\dots$

- Model: $p(k) = \theta_k \quad \sum_k \theta_k = 1$

- Likelihood (for binary indicators $[\mathbf{x}^m = k]$):

$$\begin{aligned} \ell(\theta; \mathcal{D}) &= \log p(\mathcal{D}|\theta) \\ &= \log \prod_m \theta_{\mathbf{x}^m} = \log \prod_m \theta_1^{[\mathbf{x}^m=1]} \dots \theta_K^{[\mathbf{x}^m=K]} \\ &= \sum_k \log \theta_k \sum_m [\mathbf{x}^m = k] = \sum_k N_k \log \theta_k \end{aligned}$$

- Take derivatives and set to zero (enforcing $\sum_k \theta_k = 1$):

$$\begin{aligned} \frac{\partial \ell}{\partial \theta_k} &= \frac{N_k}{\theta_k} - M \\ \Rightarrow \theta_k^* &= \frac{N_k}{M} \end{aligned}$$

EXAMPLE: BERNOULLI TRIALS

- We observe M iid coin flips: $\mathcal{D}=H,H,T,H,\dots$

- Model: $p(H) = \theta \quad p(T) = (1 - \theta)$

- Likelihood:

$$\begin{aligned} \ell(\theta; \mathcal{D}) &= \log p(\mathcal{D}|\theta) \\ &= \log \prod_m \theta^{\mathbf{x}^m} (1 - \theta)^{1 - \mathbf{x}^m} \\ &= \log \theta \sum_m \mathbf{x}^m + \log(1 - \theta) \sum_m (1 - \mathbf{x}^m) \\ &= \log \theta N_H + \log(1 - \theta) N_T \end{aligned}$$

- Take derivatives and set to zero:

$$\begin{aligned} \frac{\partial \ell}{\partial \theta} &= \frac{N_H}{\theta} - \frac{N_T}{1 - \theta} \\ \Rightarrow \theta_{\text{ML}}^* &= \frac{N_H}{N_H + N_T} \end{aligned}$$

EXAMPLE: UNIVARIATE NORMAL

- We observe M iid real samples: $\mathcal{D}=1.18,-.25,.78,\dots$

- Model: $p(x) = (2\pi\sigma^2)^{-1/2} \exp\{-(x - \mu)^2/2\sigma^2\}$

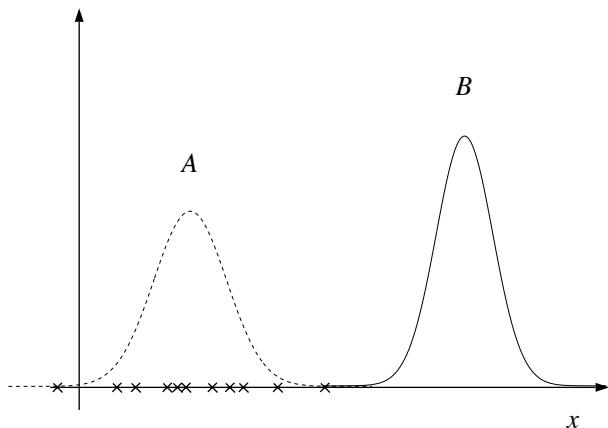
- Likelihood (using probability density):

$$\begin{aligned} \ell(\theta; \mathcal{D}) &= \log p(\mathcal{D}|\theta) \\ &= -\frac{M}{2} \log(2\pi\sigma^2) - \frac{1}{2} \sum_m \frac{(x^m - \mu)^2}{\sigma^2} \end{aligned}$$

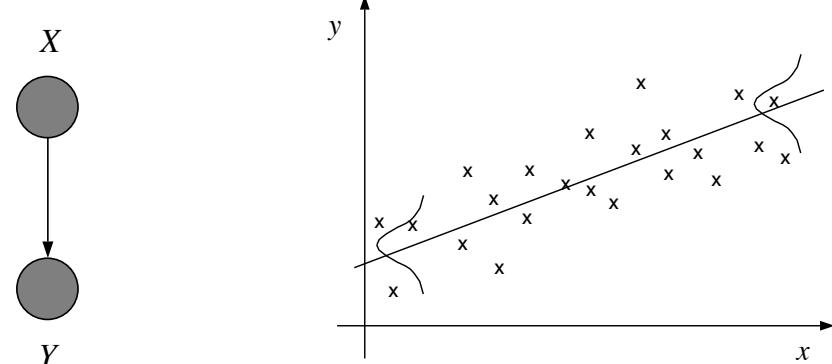
- Take derivatives and set to zero:

$$\begin{aligned} \frac{\partial \ell}{\partial \mu} &= (1/\sigma^2) \sum_m (x_m - \mu) \\ \frac{\partial \ell}{\partial \sigma^2} &= -\frac{M}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_m (x_m - \mu)^2 \\ \Rightarrow \mu_{\text{ML}} &= (1/M) \sum_m x_m \\ \sigma_{\text{ML}}^2 &= (1/M) \sum_m x_m^2 - \mu_{\text{ML}}^2 \end{aligned}$$

EXAMPLE: UNIVARIATE NORMAL



EXAMPLE: LINEAR REGRESSION



EXAMPLE: LINEAR REGRESSION

- In linear regression, some inputs (covariates, parents) and all outputs (responses, children) are continuous valued variables.
- For each child and setting of discrete parents we use the model:

$$p(y|\mathbf{x}, \theta) = \text{gauss}(y|\theta^\top \mathbf{x}, \sigma^2)$$

- The likelihood is the familiar “squared error” cost:

$$\ell(\theta; \mathcal{D}) = -\frac{1}{2\sigma^2} \sum_m (y^m - \theta^\top \mathbf{x}^m)^2$$

- The ML parameters can be solved for using linear least-squares:

$$\begin{aligned} \frac{\partial \ell}{\partial \theta} &= - \sum_m (y^m - \theta^\top \mathbf{x}^m) \mathbf{x}^m \\ \Rightarrow \theta_{\text{ML}}^* &= (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{Y} \end{aligned}$$

SUFFICIENT STATISTICS

- A statistic is a function of a random variable.
 - $T(\mathbf{X})$ is a “sufficient statistic” for \mathbf{X} if
- $$T(\mathbf{x}^1) = T(\mathbf{x}^2) \Rightarrow L(\theta; \mathbf{x}^1) = L(\theta; \mathbf{x}^2) \quad \forall \theta$$
- Equivalently (by the Neyman factorization theorem) we can write:
- $$p(\mathbf{x}|\theta) = h(\mathbf{x}, T(\mathbf{x})) g(T(\mathbf{x}), \theta)$$
- Example: exponential family models:

$$p(\mathbf{x}|\theta) = h(\mathbf{x}) \exp\{\eta^\top T(\mathbf{x}) - A(\eta)\}$$

SUFFICIENT STATISTICS ARE SUMS

- In the examples above, the sufficient statistics were merely sums (counts) of the data:
Bernoulli: # of heads, tails
Multinomial: # of each type
Gaussian: mean, mean-square
Regression: correlations
- As we will see, this is true for all exponential family models: sufficient statistics are average natural parameters.
- Only exponential family models have simple sufficient statistics.

MLE FOR EXPONENTIAL FAMILY MODELS

- Recall the probability function for exponential models:

$$p(\mathbf{x}|\theta) = h(\mathbf{x}) \exp\{\eta^\top T(\mathbf{x}) - A(\eta)\}$$

- For iid data, sufficient statistic is $\sum_m T(\mathbf{x}^m)$:

$$\ell(\eta; \mathcal{D}) = \log p(\mathcal{D}|\eta) = \left(\sum_m \log h(\mathbf{x}^m) \right) - M A(\eta) + \left(\eta^\top \sum_m T(\mathbf{x}^m) \right)$$

- Take derivatives and set to zero:

$$\begin{aligned} \frac{\partial \ell}{\partial \eta} &= \sum_m T(\mathbf{x}^m) - M \frac{\partial A(\eta)}{\partial \eta} \\ \Rightarrow \frac{\partial A(\eta)}{\partial \eta} &= \frac{1}{M} \sum_m T(\mathbf{x}^m) \\ \eta_{\text{ML}} &= \frac{1}{M} \sum_m T(\mathbf{x}^m) \end{aligned}$$

recalling that the natural moments of an exponential distribution are the derivatives of the log normalizer.