LECTURE NOTES ON

Lebesgue Measure Theory

MATH 422

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Contents

Unit 1

Algebras of Sets, and Measures

1.1 Closed intervals in the Real line

December 11, 2019

The set $[a, b]$ holds a special value in real analysis. This set is compact in $\langle \mathbb{R}, \text{distance} \rangle$.

Remark 1.1.1 Let $E \subset [a, b]$ be a countably infinite set. Then there exists a $f : [a, b] \longrightarrow \mathbb{R}$ such that

1. f is discontinuous at every point of E
2. f is continuous at every point of $[a, b] \setminus E$

Proof. Let $E = \{x_1, x_2, \cdots\} \in [a, b]$.

Suppose $x_1 < x_2 < \cdots$

Let $c_1, c_2, \cdots$ be the jumps.

If $0 \leq f \leq M$, choose c_i's such that $\displaystyle\sum_{i>0} c_i \leq M$.

General case:

$$f(x) = \sum_{x_i \leq x} c_i$$ ■

1.2 Algebras of Sets

December 2019

Definition 1.2.1 An algebra $\mathcal{A}$ of a non-empty set X is a collection of subsets of X which holds the following properties:

1. $A^c \in \mathcal{A} \; \forall \; A \in \mathcal{A}$
2. $A \cup B \in \mathcal{A} \; \forall \; A, B \in \mathcal{A}$

i.e.,

$$\mathcal{A} = \{A \in \mathcal{P}(X) \mid A \text{ is closed under complement, } A \text{ is closed under finite union}\}$$

1.3 Set Functions

December 2019

1.3.1 The Extended Real Number System

Definition 1.3.1 The extended system of Real Numbers are defined by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

1.4 The Lebesgue Outer Measure

February, 2020

Now, we are ready to define the following:

Definition 1.4.1 The **Lebesgue Outer Measure Function** is the function $m^*\colon \mathcal{P}(\mathbb{R}) \longrightarrow [0, +\infty]$ defined by

$$m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; E \subseteq \bigcup_{k=1}^{\infty} I_k \right\}$$

for all sets $E \in \mathcal{P}(\mathbb{R})$.

Theorem 1.4.1 The Lebesgue outer measure is translation invariant:

$$m^*(E + a) = m^*(E) \quad \forall \, a \in \mathbb{R}$$

Theorem 1.4.2 The Lebesgue outer measure of singleton set is zero.

Proof. Let $E = \{x\}$. ■

Theorem 1.4.3 The Lebesgue outer measure of a finite set is zero.

Proof. Let $E \in \mathcal{P}(\mathbb{R})$ be a finite set, say $E = \{x_1, x_2, \cdots, x_n\}$. ■

Theorem 1.4.4 The Lebesgue outer measure of a countable set is zero.

Proof. Let $E \in \mathcal{P}(\mathbb{R})$ be a countable set. Define $I_n = \left[a_n - \frac{\epsilon}{2^{n+2}}, \quad a_n + \frac{\epsilon}{2^{n+2}}\right]$ ■

Now, we would like to show that m^* is actually an extension of ℓ to the subsets of $\mathbb{R}$.

Theorem 1.4.5 The Lebesgue outermeasure of an interval is its length. (12 marks)

Proof. Let I be an interval.

Case 1: I is closed and bounded

Let $I = [a, b]$, $a, b \in \mathbb{R}$. Let $\{I_n\}_{n=1}^{\infty}$ be an open interval cover of I

$$I = [a, b] \subseteq \bigcup_{n=1}^{\infty} I_n$$

- **claim:** $m^*([a, b]) \leq \ell([a, b])$

  Use the open interval cover $\left(a - \frac{\epsilon}{2},\ b + \frac{\epsilon}{2}\right)$.

  $$m^*([a, b]) \leq \ell([a, b]) + \epsilon \quad \forall\ \epsilon > 0$$

- **claim:** $m^*([a, b]) \geq \ell([a, b])$

  Use the open interval cover $\bigcup_{n=1}^{\infty} I_n = [a, b]$. $[a, b]$ is compact $\Rightarrow \exists$ a finite sub cover $\bigcup_{n=1}^{k} I_n = [a, b]$.

Case 2: I is open/half-open and bounded

W.L.O.G., Let $I = (a, b)$. $\overline{I} = [a, b]$.
- $I \subset \overline{I} \Longrightarrow m^*(I) \leq m^*(\overline{I}) = \ell(I)$.
- $\ell(I) < m^*(I)$

■

Case 3: I is unbounded

$\ell(I) = +\infty$.

1.5 Countable Sub-additivity of Lebesgue Outer Measure

February, 2020

1.6 The notion of Measure Functions

January 29, 2020

Unfortunately, there exists no non-negative, countably additive set function $\mu : \mathcal{P}(\mathbb{R}) \longrightarrow [0, +\infty]$ that preserves the length of intervals (Ulam).

■ **Example 1.1** Fix $x_0 \in \mathbb{R}$. Define $\mu_{x_0} : \mathcal{P}(\mathbb{R}) \longrightarrow [0, +\infty]$ as $\mu_{x_0}(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A \end{cases}$

μ_{x_0} is countably additive, but does not preserve the length of intervals, as $\mu_{x_0}(I) = 1$ if $I = (x_0 - 5, x_0 + 5)$.

■ **Example 1.2** Define $\mu : \mathcal{P}(\mathbb{R}) \longrightarrow [0, +\infty]$ as $\mu(A) = \begin{cases} \ell(A) & \text{if } A \text{ is an interval} \\ 0 & \text{otherwise} \end{cases}$

The image of an interval under μ is the length of the interval. So, $\mu((3,8)) = \ell(3,8) = 8-3 = 5$. However, $\mu((3,8)\cup\{9\}) = 0$, as $(3,8)\cup\{9\}$ is not an interval.

It is not possible to extend the length of intervals to $\mathcal{P}(\mathbb{R})$. Hence, we seek to find a class of subsets to which we can extend ℓ to.

extend	$m^* : \mathcal{P}(\mathbb{R}) \longrightarrow [0,+\infty]$	By **S. Ulam**, m^* cannot be countably additive
	$m^*\Big\vert_{\mathcal{M}} : \mathcal{M} \longrightarrow [0,+\infty]$	We identify a class of sets $\mathcal{M}$, and $m^*\Big\vert_{\mathcal{M}}$ is the required function
	$\ell : \mathcal{I} \longrightarrow [0,+\infty]$	We wish to extend the interval-length function to any set

$m = m*\big|_{\mathcal{M}}$ is the required function.

The set $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R})$ is expected to have the following properties:
1. Contains $\mathbb{R}$
2. Closed under arbitrary union
3. Closed under complement

$\mathcal{M}$ is a $\sigma-$algebra.

1.7 Construction of Measure Functions

January 30, 2020

Now we will develop our way to define the properties of measure functions

Theorem 1.7.1 Let X be a non-empty set and $\mathcal{A}$ be a $\sigma-$algebra on the subsets of X. If $\{A_n\}_{n=1}^{\infty}$ is a sequence of subsets from $\mathcal{A}$, then $\exists\ \{B_n\}_{n=1}^{\infty}$ in $\mathcal{A}$:
1. $B_n \subseteq A_n \quad \forall\ n \in \mathbb{N}$
2. $B_i \cap B_j = \phi \quad \forall\ i \neq j$
3. $\displaystyle\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$

Proof. Start with a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{A}$. Now,

$$\begin{aligned}
&\text{For } A_1, \text{ take } B_1 = A_1 \\
&\text{For } A_2, \text{ take } B_2 = A_2 \setminus A_1 \\
&\text{For } A_3, \text{ take } B_3 = A_3 \setminus (A_1 \cup A_2) \\
&\qquad\quad \vdots \\
&\text{For } A_n, \text{ take } B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)
\end{aligned}$$

The constructed sequence $\{B_n\}_{n=1}^{\infty}$ satisfies all the three properties ■

1.8 Measure Functions and Measure Spaces

Definition 1.8.1 Let X be a non-empty set, $\mathcal{A}$ be a $\sigma-$algebra on the subsets of X and $\mu : \mathcal{A} \longrightarrow [0, +\infty]$ be a set function such that $\mu(\phi) = 0$. We say

1. μ is monotone if $\mu(A) \leq \mu(B)\ \forall\ A \subseteq B$
2. μ is finitely subadditive if $\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i) \quad \forall\ \{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$
3. μ is finitely additive if $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \quad \forall\ \{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$
4. μ is countably subadditive if $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i) \quad \forall\ \{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$
5. μ is countably additive if $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \quad \forall\ \{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$

The study of the relation between all the above definitions will be useful.

Theorem 1.8.1 Let X be a non-empty set, $\mathcal{A}$ be a $\sigma-$algebra on the subsets of X and $\mu : \mathcal{A} \longrightarrow [0, +\infty]$ be a set function such that $\mu(\phi) = 0$. Then,

1. μ is finitely additive $\Rightarrow$ μ is monotone
2. μ is countably additive $\Leftrightarrow$ μ is finitely additive and countably subadditive

Proof. 1. Let $A \subseteq B \in \mathcal{A}$ ■

Now, we are ready to define the measure on a $\sigma-$algebra:

Definition 1.8.2 — Measure Function. Let X be a non-empty set and $\mathcal{A}$ be a $\sigma-$algebra on the subsets of X. Then, the set function

$$\mu : \mathcal{A} \longrightarrow [0, +\infty]$$

is called a measure on $\mathcal{A}$ when μ is countably additive and $\mu(\phi) = 0$.

Equipping a set with a $\sigma-$algebra on its subsets and a measure function will create a measure space:

Definition 1.8.3 — Measure Space. Let X be a non-empty set, $\mathcal{A}$ be a $\sigma-$algebra on the subsets of X and $\mu : \mathcal{A} \longrightarrow [0, +\infty]$ be a measure function. Then the triplet $(X, \mathcal{A}, \mu)$ is called a measure space.

■ **Example 1.3** The triplet $(\mathbb{R}, \mathcal{M}, m)$ is a measure space, where

$\mathcal{M}$ is the $\sigma-$algebra on real subsets, called as *Lebesgue measurable sets* and

$m = m^* \mid_{\mathcal{M}}$ is called the *Lebesgue measure*, where $m^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) \,\middle|\, A \subseteq \bigcup_{k=1}^{\infty} I_k\right\}$.

■ **Example 1.4** In probability theory, the triplet $(\Omega, \mathcal{F}, P)$ is a measure space, where

Ω is the set of all possible outcomes (sample space),

$\mathcal{F}$ is the set of all events considered, and

$P : \mathcal{F} \longrightarrow [0, 1]$ is the probability measure. This space is called *probability space.*

Definition 1.8.4 A measure space is said to be complete whenever every subset of every null set is measurable.

1.9 Lebesgue Measurable Sets and the Lebesgue Measure

February, 2020

We have seen that the Lebesgue outer measure m^* is countably additive on a σ-algbera of $\mathbb{R}$. We proceed to call the sets of the σ-algebra as *Lebesgue measurable sets* of $\mathbb{R}$. However, any σ-algebra of R will make m^* countably additive. Hence, we begin the construction of the "set of all Lebesgue measurable functions" by defining the following criterion for a set to be "measurable"

1.9.1 Carathéodory's Criterion

Definition 1.9.1 — Carathéodory's Criterion. Let E be a non-empty subset of $\mathbb{R}$. We say E satisfies Carathéodory's criterion when

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad \forall\ A \subseteq \mathbb{R}$$

The collection of all sets which satisfies the Carathéodory's criterion is denoted by $\mathcal{M}$.

$$\mathcal{M} = \{E \in \mathbb{R} \mid \quad m^*(A \cap E) + m^*(A \cap E^c) = m^*(A) \quad \forall\ A \subseteq \mathbb{R}\}$$

1.9.2 Properties of $\mathcal{M}$

Theorem 1.9.1 $E \in \mathcal{M}$ if and only if $E^c \in \mathcal{M}$

Theorem 1.9.2 ϕ and $\mathbb{R} \in \mathcal{M}$

Theorem 1.9.3 If $m^*(E) = 0$, then $E \in \mathcal{M}$

Proof. easy. ■

Theorem 1.9.4 If E_1 and $E_2 \in \mathcal{M}$, then $E_1 \cup E_2 \in \mathcal{M}$

Theorem 1.9.5 $\mathcal{M}$ is a σ-algebra on $\mathbb{R}$.

Theorem 1.9.6 The triplet $(\mathbb{R}, \mathcal{M}, m)$ is a complete measure space.

Definition 1.9.2 The measure function $m = m^*|_{\mathcal{M}} \colon \mathcal{M} \longrightarrow [0, +\infty]$ is called the Lebesgue measure.

1.9.3 Continuity Properties of Lebesgue Measure

Theorem 1.9.7 — First Continuity Property. Let $\{A_n\}_{n=1}^{\infty}$ be a collection of Lebesgue measurable sets such that $A_n \subseteq A_{n+1}\ \forall\ n \in \mathbb{N}$. Then,

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$$

Proof. We begin by dividing the proof into two cases:

Case 1: The sequence contains a set of infinte measure

Case 2: All the sets are of finite measure ■

Theorem 1.9.8 — Second Continuity Property. Let $\{A_n\}_{n=1}^{\infty}$ be a collection of Lebesgue measurable sets such that $A_{n+1} \subseteq A_n \ \forall\ n \in \mathbb{N}$ and $m(A_n) < +\infty \ \forall\ n \in \mathbb{N}$. Then,

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$$

Proof. This can be proved using de Morgan's law on the previous theorem. ■

1.9.4 Regularity Properties of Lebesgue Measure

Theorem 1.9.9 Let $E \subset \mathbb{R}$. Prove that the following are equivalent:

1. E is measurable
2. $\forall\ \epsilon > 0\ \exists$ open $G \supset E : m^*(G \setminus E) < \epsilon$.
3. $\exists$ a $\mathcal{G}_\delta$ - set $G \supset E : m^*(G \setminus E) = 0$.

Theorem 1.9.10 Let $E \subset \mathbb{R}$. Prove that the following are equivalent:

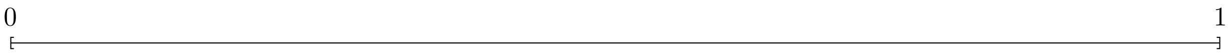
1. E is measurable
2. $\forall\ \epsilon > 0\ \exists$ closed $F \subset E : m^*(E \setminus F) < \epsilon$.
3. $\exists$ a $\mathcal{F}_\sigma$ - set $F \subset E : m^*(E \setminus F) = 0$.

1.10 The Cantor Set

December 11, 2019

> *"97.3 percent of all counter examples in Real Analysis involve the Cantor set"*
>
> Unknown

We begin with the interval $C_0 = [0, 1]$.



We split the interval into three equal halves and remove the middle open interval. The remaining set is

$$\begin{aligned} C_1 &= C_0 \setminus \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \\ &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \end{aligned}$$

Now, we split each of the disjoint intervals into three parts and remove their respective middle open intervals. The remaining set is

$$\begin{aligned} C_2 &= C_1 \setminus \ \left[\left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right)\right] \\ &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \end{aligned}$$

Proceeding further, we obtain an infinite series of closed sets

$$C_0 \supset C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$$

Definition 1.10.1 The Cantor set is given by $\mathscr{C} = C_\infty = \bigcap_{n=1}^{\infty} C_n$.

We have shown that any countable set has Lebesgue measure zero. The converse is not true. The Cantor set stands as a counter example:

Theorem 1.10.1 The Cantor set has Lebesgue measure zero.

Proof. The collection $\{C_n\}_{n=1}^{\infty}$

$$
\begin{aligned}
C_0 &= [0,1] \\
C_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
C_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\
&\vdots
\end{aligned}
$$

is measurable, and $m(C_n) = \left(\frac{2}{3}\right)^n \ \forall\ n \in \mathbb{N}$.

By second continuity property, we can show that

$$
\begin{aligned}
m(\mathscr{C}) &= m\left(\bigcap_{n=1}^{\infty} C_n\right) \\
&= \lim_{n\to\infty} m(C_n) \\
&= \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0
\end{aligned}
$$

■

Aliter:

Proof. For each n, let E_n be the union of the intervals removed in the n^{th} step. Then,

$$m(E_n) = \frac{1}{2}\left(\frac{2}{3}\right)^n$$

The total measure of the intervals removed from [0,1] are

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{2}\cdot\frac{\frac{2}{3}}{1-\frac{2}{3}} = 1$$

$$
\begin{aligned}
\therefore\ m(\mathscr{C}) &= m([0,1]) - m\left(\bigcup_{n=1}^{\infty} E_n\right) \\
&= 1 - 1 = 0
\end{aligned}
$$

■

Theorem 1.10.2 The Cantor set is uncountable.

Proof. Suppose $\mathscr{C}$ is countable, i.e., $\mathscr{C} = \{c_n\}_{n=1}^{\infty}$.
- The point c_1 is contained in one of the intervals in the union C_1. We choose the interval F_1 which does not contain c_1.
- Furthermore, at least one of the two intervals in C_2 whose union is F_1 does not contain c_2. We choose the interval F_2 which does not contain c_2.
- Proceeding further, we have a nested sequence $\{F_n\}_{n=1}^{\infty}$ which are closed and bounded.
- By nested interval theorem, $\exists\ c : \bigcap_{n=1}^{\infty} F_n = \{c\}$.
- But, by the construction, this should be empty
- Therefore, $\mathscr{C}$ is uncountable.

■

Unit 2

Lebesgue Measurable Functions

2.1 Construction of Non-Lebesgue Measurable subset of (0,1)

January 2020

All the general subsets of $\mathbb{R}$ are measurable so far. We now proceed to show the existence of a subset of $\mathbb{R}$ which is non-measurable.

Theorem 2.1.1 — Vitali. There exists a Non-Lebesgue measurable set.

Proof. Define the relation $\sim$ on [0,1] where

$$x \sim y \text{ means } x - y \in \mathbb{Q} \quad \forall\, x, y \in [0,1]$$

We observe the following:

- $x - x = 0 \in \mathbb{Q} \Rightarrow \boxed{x \sim x}$
- $x - y = 0 \in \mathbb{Q} \Rightarrow y - x \in \mathbb{Q}$. So, $\boxed{x \sim y \Rightarrow y \sim x}$
- $x-y \in \mathbb{Q}$ and $y-z \in \mathbb{Q} \Rightarrow (x-y)+(y-z) \in \mathbb{Q} \Rightarrow x-z \in \mathbb{Q}$. So, $\boxed{x \sim y \text{ and } y \sim z \Rightarrow x \sim z}$

Therefore, $\sim$ is an equivalence relation on $[0,1]$, hence, $\sim$ partitions [0,1] in to equivalence classes of $\sim$.
Define V as the set of all representatives from each $\sim$-partition of [0,1]. Note that $V \subseteq [0,1]$.

Let $\{r_1,\ r_2,\ \cdots\}$ be an enumeration of $\mathbb{Q} \cap [-1,1]$. Consider the collection

$$\{V_i\}_{n=1}^{\infty}$$

where $V_i = V + r_i = \{v + r_i \mid v \in V\}$. We now make the following claims:

- **claim:** $\boxed{\text{All } V_i\text{'s are pairwise disjoint}}$

  Suppose not, i.e., $\exists\, x \in (V + r_i) \cap (V + r_j)$ for some $i \neq j$.
  Then, x can be written as both $x = a + r_i$ and $x = b + r_j$ as well, for some $a, b \in V$.
  $x = a + r_i = b + r_j \Longrightarrow a - b = r_j - r_i$.
  But $r_j - r_i \in \mathbb{Q} \Longrightarrow a - b \in \mathbb{Q} \Longrightarrow a \sim b$
  $\Longrightarrow\Longleftarrow$ V has only one representative from each class.
  Hence, V_i's are pairwise disjoint.

- **claim:** $\boxed{[0,1] \subseteq \bigcup_{k=1}^{\infty} V_k \subseteq [-1,2]}$

  $V \subseteq [0,1] \Longrightarrow V + r_i \subseteq [-1,2]\ \forall\ r_i \in [-1,1]$. Hence, $V_i \subseteq [-1,2] \Longrightarrow \bigcup_{k=1}^{\infty} V_k \subseteq [-1,2]$.

  Let $x \in [0,1]$. Then x must contain in some equivalence class of $\sim$, i.e., $\exists\ v \in V : x \in v + r_k$

for some $r_k \in \mathbb{Q}$. Hence, $x \in V + r_k \Longrightarrow [0,1] \subseteq \bigcup_{k=1}^{\infty} V_k$.

claim: $\boxed{V \text{ is not Lebesgue measurable.}}$

Suppose V is Lebesgue measurable. Then all V_i's are Lebesgue measurable.

- Applying Lebesgue measure using σ-additivity, we have $1 \leq \sum_{k=1}^{\infty} m(V_k) \leq 3$
- Since V is translation invariant, we have $1 \leq \sum_{k=1}^{\infty} m(V) \leq 3$

$$\Longrightarrow\Longleftarrow m(V) = \begin{cases} 0 & \text{if } m(V) < 1 \\ +\infty & \text{if } m(V) \geq 1 \end{cases}$$

i.e., $m(V)$ is a constant < 1 or ≥ 1, and the infinite-sum will be 0 or $+\infty$ respectively. Hence, V cannot be Lebesgue measurable. ■

The set V is called a **Vitali set**, named after **Giuseppe Vitali**.

2.2 Measurable Functions

February, 2020

Definition 2.2.1 Let E be a Lebesgue measurable function. Then, the function $f\colon E \longrightarrow \overline{\mathbb{R}}$ is said to be ***Lebesgue measurable*** when the set

$$\{x \in E \mid f(x) < \alpha\} \text{ is measurable } \forall\, \alpha \in \mathbb{R}$$

The following theorem shows that the above criteria can be stated in other ways.

Theorem 2.2.1 Let $f : E \longrightarrow \overline{\mathbb{R}}$ be a function where E is measurable. Then, the following are equivalent:

1. the set $\{x \in E \mid f(x) < \alpha\}$ is measurable $\forall\, \alpha \in \mathbb{R}$
2. the set $\{x \in E \mid f(x) \geq \alpha\}$ is measurable $\forall\, \alpha \in \mathbb{R}$
3. the set $\{x \in E \mid f(x) > \alpha\}$ is measurable $\forall\, \alpha \in \mathbb{R}$
4. the set $\{x \in E \mid f(x) \leq \alpha\}$ is measurable $\forall\, \alpha \in \mathbb{R}$
5. the set $\{x \in E \mid f(x) = \alpha\}$ is measurable $\forall\, \alpha \in \mathbb{R}$

2.3 Properties of Measurable Functions

February, 2020

We now show that the collection of Lebesgue measurable functions has somewhat of a linear structure to it.

Theorem 2.3.1 — Linearity of Measurable Functions. Let f and g be Lebesgue measurable functions defined on the measurable set E. Then,

- $f + g$ is a Lebesgue measurable function on E.
- cf is a Lebesgue measurable function on E, where $c \in \mathbb{R}$.

Lemma 2.3.1. *Let f be a Lebesgue measurable function defined on the measurable set E. Then f^2 is a Lebesgue measurable function.*

Theorem 2.3.2 Let f and g be Lebesgue measurable functions defined on the measurable set E. Then fg is a Lebesgue measurable function.

Theorem 2.3.3 Every continuous real valued function defined on a Lebesgue measurable set is Lebesgue measurable

Proof. Let E be a Lebesgue measurable set, and $f\colon E \longrightarrow \mathbb{R}$ be a continuous function. ■

2.4 Approximation by Simple Functions

March 2, 2020

2.5 Egorov's Theorem

March 3, 2020

Consider $f\colon [0,1] \longrightarrow \mathbb{R}$ defined by $f_n(x) = x^n \ \forall\ x \in [0,1]$. The pointwise limit of this sequence is the function $f\colon [0,1] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & 0 \leq x < 1 \end{cases}$$

We have seen that this convergence is not uniform. We fix this by removing the trouble causing point 1. Then $\lim\limits_{n\to\infty} \sup\limits_{[0,1-\epsilon]} f_n(x) = \lim\limits_{n\to\infty} (1-\epsilon)^n = 0$. We attempt the same procedure of 'removing the troubling points' to any function.

Theorem 2.5.1 — Severini – Egorov. Let $m(E) < +\infty$, and the sequence $f\colon E \longrightarrow \overline{\mathbb{R}}$ converge to $f\colon E \longrightarrow \mathbb{R}$ pointwise. Then, $\forall\ \epsilon > 0\ \exists$ a measurable subset $A \subset E$ with $m(A) < \epsilon : f_n \longrightarrow f$ converges uniformly on $E \setminus A$.

Proof. For $n, k \in \mathbb{N}$, we define the set $E_{n,k}$ by the following union:

$$E_{n,k} = \bigcup_{m \geq n} \left\{ x \in A \;\middle|\; |f_m(x) - f(x)| \geq \frac{1}{k} \right\}$$

by applying σ-additivity, we conclude

$$m(A) \leq \sum_{k\in\mathbb{N}} m(E_{n_k,k}) < \sum_{k\in\mathbb{N}} \frac{\epsilon}{2^k} = \epsilon.$$

■

2.6 Previous Year Questions

1. Prove that any set of zero outer measure is Lebesgue measurable.
2. Prove that Lebesgue outer measurable sets are translation invariant.
3. If E_1 and E_2 are Lebesgue measurable, $E_1 \cup E_2$ is Lebesgue outer measurable.
4. (True or False?) Every real continuous function defined on a Lebesgue measurable set is Lebesgue measurable.
5. (True or False?) Every monotone real function on [0,1] is measurable.
6. Show that Lebesgue (outer) measure of the Cantor set is zero.
7. Show that $\mu\colon \mathcal{A} \longrightarrow [0, +\infty]$ with $\mu(\phi) = 0$ is countably additive if and only if μ is finitely additive and countably sub additive.
8. If $A_1 \subseteq A_2 \subseteq \cdots A_n \subseteq \cdots$ are measurable, then $m\left(\bigcup^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$.
9. Let $E \subset \mathbb{R}$. Prove that the following are equivalent:
   (a) E is measurable
   (b) $\forall\ \epsilon > 0\ \exists$ open $G \supset E : m^*(G \setminus E) < \epsilon$.
   (c) $\exists$ a $\mathcal{G}_\delta$ - set $G \supset E : m^*(G \setminus E) = 0$.
10. Let $E \subset \mathbb{R}$. Prove that the following are equivalent:
    (a) E is measurable
    (b) $\forall\ \epsilon > 0\ \exists$ closed $F \subset E : m^*(E \setminus F) < \epsilon$.
    (c) $\exists$ a $\mathcal{F}_\sigma$ - set $F \subset E : m^*(E \setminus F) = 0$.
11. Prove that Lebesgue outer measure of an interval is its length.
12. Construct a non-Lebesgue measurable subset of $[0, 1]$.
13. Define Simple functions and its canonical form.
14. Show that if f and g are Lebesgue measurable functions, then $f + g$ is a Lebesgue measurable function.
15. Prove that the point wise limit of a sequence of measurable functions is measurable.
16. $\forall$ bounded, non-negative $f\colon E \longrightarrow \overline{\mathbb{R}}\ \exists$ non-negative, simple measurable functions $\{f_n\}_{n=1}^{\infty}$: $f_n(x) \longrightarrow f(x)$ p.w.
17. State and prove Egorov's theorem.

Unit 3

The Lebesgue Integral

"I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral."

Henri Lebesgue,

3.1 Lusin's Theorem

February 2019

The Egoroff's theorem is a precise realization on Littlewood's last principle on real (measurable) functions. We now develop upon the theorem to state a more general version:

Theorem 3.1.1 — Lusin's theorem. Let $f\colon E \longrightarrow \mathbb{R}$ be measurable. Then,

$\forall\ \epsilon > 0\ \exists\ F \subset [a, b]$ and a continuous $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$g\big|_E = f$$
$$\text{and } m(E \setminus F) < \epsilon$$

Proof. We begin with applying Egorov's theorem. ■

3.2 The Lebesgue Integral

March 16, 2020

Upon proving several theorems, the takeaway are the three principles of Littlewood:
1. A measurable set can be almost written as an union of open intervals
2. The limit point of an uniformly converging measurable functions is almost measurable
3. A measurable function is continuous almost everywhere

Now, we are ready to define the **Lebesgue integral**. We begin with simple functions:

3.2.1 Lebesgue Integral of a Simple Function

Theorem 3.2.1 — Characterization of Simple Functions. Any simple functions can be 'characterized'.

Proof. ($\Rightarrow$) Let $\phi : E \longrightarrow \mathbb{R}$ be a simple function, i.e., ϕ is measurable and range ϕ is finite, say,

$$\phi(E) = \{a_1, a_2, \cdots, a_n\}$$

Define E_i's as follows:

$$E_i = \{x \in E \mid \phi(x) = a_i\} \quad \forall\ 1 \leq i \leq n$$

Note that $\bigcup_{i=1}^{n} E_i = E$. Hence,

$$\phi = \sum_{i=1}^{n} a_i \, \chi_{E_i}$$

■

Proof. ($\Leftarrow$) Suppose $\phi : E \longrightarrow \mathbb{R}$ is defined as

$$\phi = \sum_{i=1}^{m} b_i \, \chi_{A_i}$$

where A_i's are measurable, and $A_i \subset E \; \forall \; 1 \leq i \leq n$. Note that b_i's need not be aligned

■

Definition 3.2.1 Let $\phi : E \longrightarrow \mathbb{R}$ be a simple function defined on a finite measurable set E. If ϕ can be expressed in canonical form as

$$\phi = \sum_{i=1}^{n} a_i \, \chi_{E_i}$$

Then we define the Lebesgue integral of ϕ over E as

$$\int\limits_E \phi(x) \, \mathrm{d}x = \sum_{i=1}^{n} a_i \, m(E_i) \qquad (\, \forall \, x \in E)$$

or simply,

$$\int_E \phi = \sum_{i=1}^{n} a_i \, m(E_i)$$

The sum in the Lebesgue integral in fact need not be in canonical form.

Theorem 3.2.2 Let $\phi : E \longrightarrow \mathbb{R}$ be a simple function defined on a finite measured set. If ϕ is defined as

$$\phi = \sum_{n=1}^{\infty} b_j \, \chi_{E_j}$$

which need not be in the canonical form. Then,

$$\int_E \phi = \sum_{i=1}^{n} b_j \, m(E_j)$$

Proof. Let ■

Unit 4

Riemann vs Lebesgue Integration

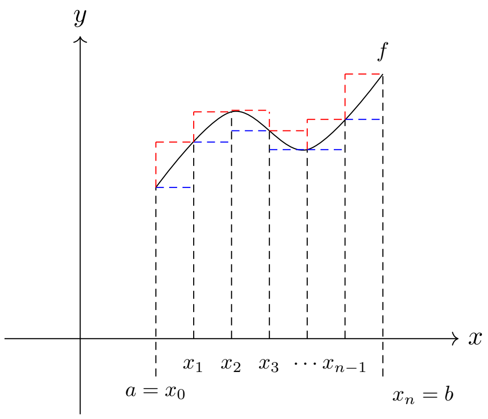
4.1 A quick review of the Riemann Integral

December 12, 2019

Let $f : [a, b] \longrightarrow \mathbb{R}$. Without loss of generality, keep $f \geq 0$.

Basic problem: To find the area of the region bounded by the graph of f and the x-axis.

Idea: Cut the region into vertical slices and approximate them into rectangular strips.



Let $\sigma \in \mathscr{P}[a, b]$ be the partition $\sigma = \{a = x_0,\ x_1,\ \cdots,\ x_n = b\}$ by which the region is split.

Let $I_i = [x_{i-1},\ x_i]$ and $\Delta x_i = \ell(I_i) = x_i - x_{i-1}$

Take $m_i = \inf\limits_{x \in I_i} f(x)$ and $M_i = \sup\limits_{x \in I_i} f(x)$.

The *infimum area* of an arbitrary rectangular strip which is cut using the partition σ, called the **lower sum** is given by

$$L(f, \sigma) = m_i\ \Delta x_i$$

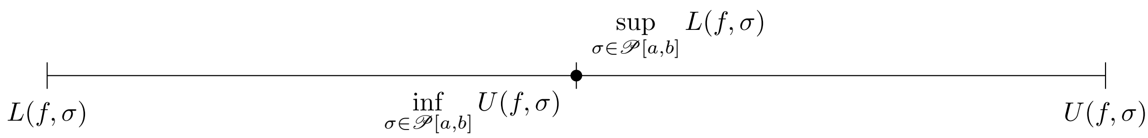
The *supremum area* of an arbitrary rectangular strip which is cut using the partition σ, called the **upper sum** is given by

$$U(f, \sigma) = M_i\ \Delta x_i$$

We observe the following:
- $m_i \leq M_i \quad \forall\ i = 1, 2, \cdots, n$
- $L(f, \sigma) \leq U(f, \sigma) \quad \forall\ \sigma \in \mathscr{P}[a, b]$

- $L(f,\sigma) \leq L(f,\tau)$ and $U(f,\sigma) \geq U(f,\tau) \quad \forall\ \sigma \subseteq \tau$
- $L(f,\sigma) \leq L(f,\ \sigma \cup \tau) \leq U(f,\ \sigma \cup \tau) \leq U(f,\sigma)$

Our intuition says that the thinner the partition is, the better the approximation will be. If the 'area' of the region makes sense, the supremum of the lower sum and the infimum of the upper sum will coincide.



Now we are ready to define the following:

Definition 4.1.1 The function f is said to be Riemann integrable on an interval $[a,b]$ when the supremum of the lower sums over all partitions of $[a,b]$ is equal to the infimum of the upper sums over all the partitions of $[a,b]$, i.e.,

$$\sup_{\sigma \in \mathscr{P}[a,b]} L(f,\sigma) = \inf_{\sigma \in \mathscr{P}[a,b]} U(f,\sigma)$$

We then define the *lower integral* of f over $[a,b]$ as

$$\underline{\int_a^b} f = \sup_{\sigma \in \mathscr{P}[a,b]} L(f,\sigma)$$

and the *upper integral* of f over $[a,b]$ as

$$\overline{\int_a^b} f = \inf_{\sigma \in \mathscr{P}[a,b]} U(f,\sigma)$$

When f is Riemann integrable over $[a,b]$, we write $f \in \mathscr{R}[a,b]$ and the integral is given as

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b}$$

This is equivalent to saying that the lower and upper sums are arbitrarily close to each other.

Definition 4.1.2 $\forall\ \epsilon > 0\ \exists\ \sigma \in \mathscr{P}[a,b]\ : U(f,\sigma) - L(f,\sigma) < \epsilon$

The above statement is called the **Cauchy's criterion** for Riemann integrability.

Examples

Example 4.1 A classic example of a function which is **not** Riemann integrable is

$$f\colon [0,1] \longrightarrow \mathbb{R}$$

$$\text{defined by } f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \qquad \forall\ x \in [0,1]$$

$\forall\ \sigma \in \mathscr{P}[a,b]$, we have $m_i = 0$ and $M_i = 1$. Which results in $L(f,\sigma) = 0$ and $U(f,\sigma) = 1$.

Choosing $\epsilon = \frac{1}{2}$ will fail the Cauchy criterion, hence, f is not Riemann integrable. ■

Example 4.2 Define $f\colon [0,1] \longrightarrow \mathbb{R}$ as

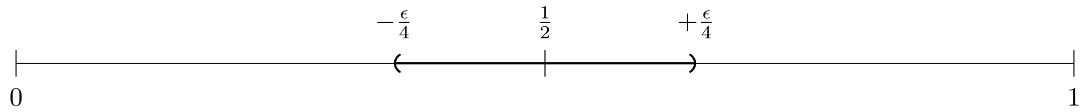
$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ x \in [0,1]$$

Take $\sigma \in \mathscr{P}[a,b]$. We have $L(f,\sigma) = 0$, hence $\underline{\int_0^1} f = 0$. Also, $U(f,\sigma) \geq 0$.

Given $\epsilon > 0$, take $\sigma_\epsilon = \left\{0, \quad \frac{1}{2} - \frac{\epsilon}{4}, \quad \frac{1}{2} + \frac{\epsilon}{4}, \quad 1\right\}$.

Then, $U(f,\sigma_\epsilon) = 0 \cdot \left(\frac{1}{2} - \frac{\epsilon}{4}\right) + 1 \cdot \epsilon + 0 \cdot \left(\frac{1}{2} - \frac{\epsilon}{4}\right) = \frac{\epsilon}{2} < \epsilon$.

By Cauchy criterion, $f \in \mathscr{R}[a,b]$, and by definition, $\displaystyle\int_0^1 f = 0$. ■



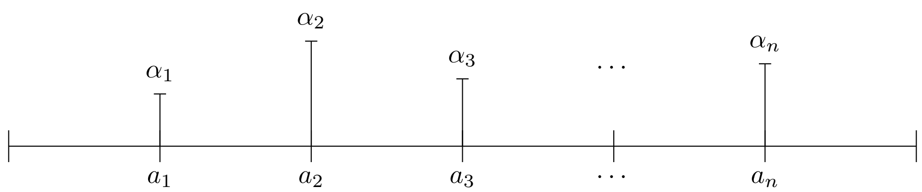
■ **Example 4.3** Define $f\colon [0,1] \longrightarrow \mathbb{R}$ as

$$f(x) = \begin{cases} 2 & \text{if } x = a_1 \\ 1 & \text{if } x = a_2 \\ 0 & \text{otherwise} \end{cases} \quad \forall\, x \in [0,1] \quad \text{where } a_1, a_2 \in [a,b]$$

$f \in \mathscr{R}[a,b]$, and $\displaystyle\int_0^1 f = 0$ (why?)

■ **Example 4.4** We can proceed further like the above examples and define $f\colon [0,1] \longrightarrow \mathbb{R}$ as

$$f(x) = \begin{cases} \alpha_i & \text{if } x = a_i \\ 0 & \text{otherwise} \end{cases} \quad \forall\, x \in [0,1] \quad \text{where } \begin{array}{l} a_1, a_2, \cdots a_n \in [a,b] \\ \alpha_1, \alpha_2, \cdots \alpha_n \in \mathbb{R} \setminus \{0\} \end{array}$$



We recall the following theorem:

Theorem 4.1.1 Let $f, g \in \mathscr{R}[a,b]$. Then, $f + g \in \mathscr{R}[a,b]$, and

$$\int_a^b f + g = \int_a^b f + \int_a^b g$$

We write our f as

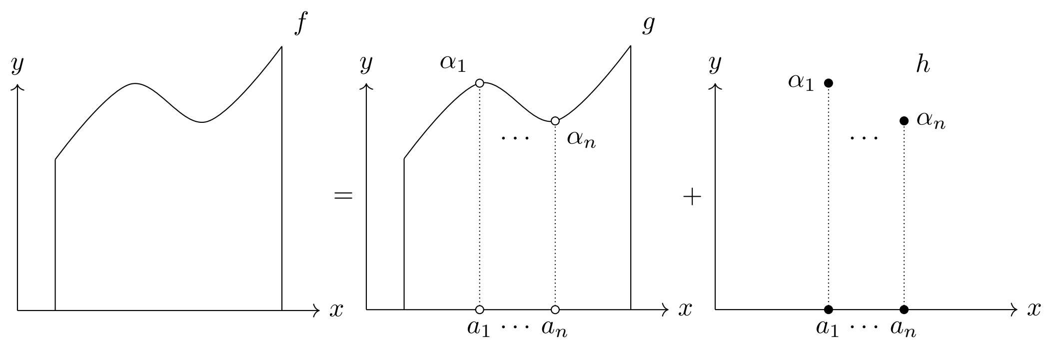
$$f = f_1 + f_2 + \cdots + f_n$$

$$\text{where each } f_i = \begin{cases} \alpha_i & \text{if } x = a_i \\ 0 & \text{otherwise} \end{cases}$$

By previous examples, each $f_i \in \mathscr{R}[a,b]$. Hence, $f \in \mathscr{R}[a,b]$.

Also, each $\displaystyle\int_0^1 f_i = 0$. Hence, $\displaystyle\int_0^1 f = 0$ ■

Observation: If $f \in \mathscr{R}[a,b]$, then redefining a finite number of points does not affect the Riemann integrability and the integral value remains the same.
We write $f = g + h$, and $\displaystyle\int_a^b h = 0$, i.e., the integral value is unaffected.



f
y
x
$=$
g
y
α_1
$\cdots$
α_n
x
$a_1 \cdots a_n$
$+$
h
y
α_1
$\cdots$
α_n
x
$a_1 \cdots a_n$

Unit 5

Absolutely Continuous Functions

5.1