Chapter 1

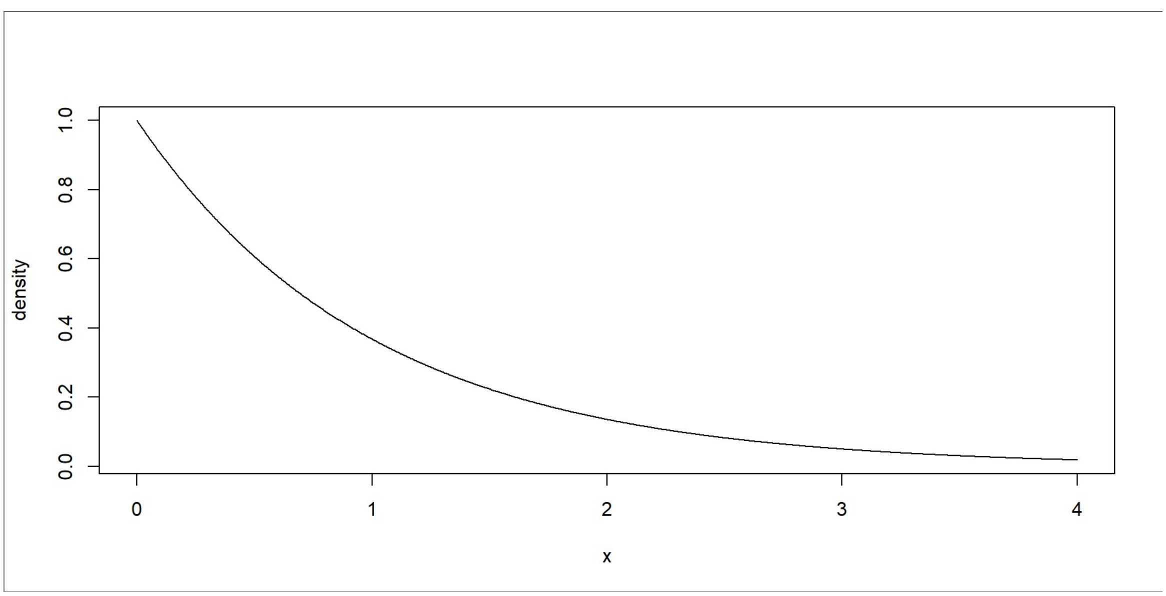
MathStat Preliminaries

1.1 Wait. Where are we going?

Exponential pdf

If $X \sim \text{Exponential}(\lambda)$, then X has the pdf $f(x) = \lambda \exp(-\lambda x) I_{[0,\infty]}(x)$.

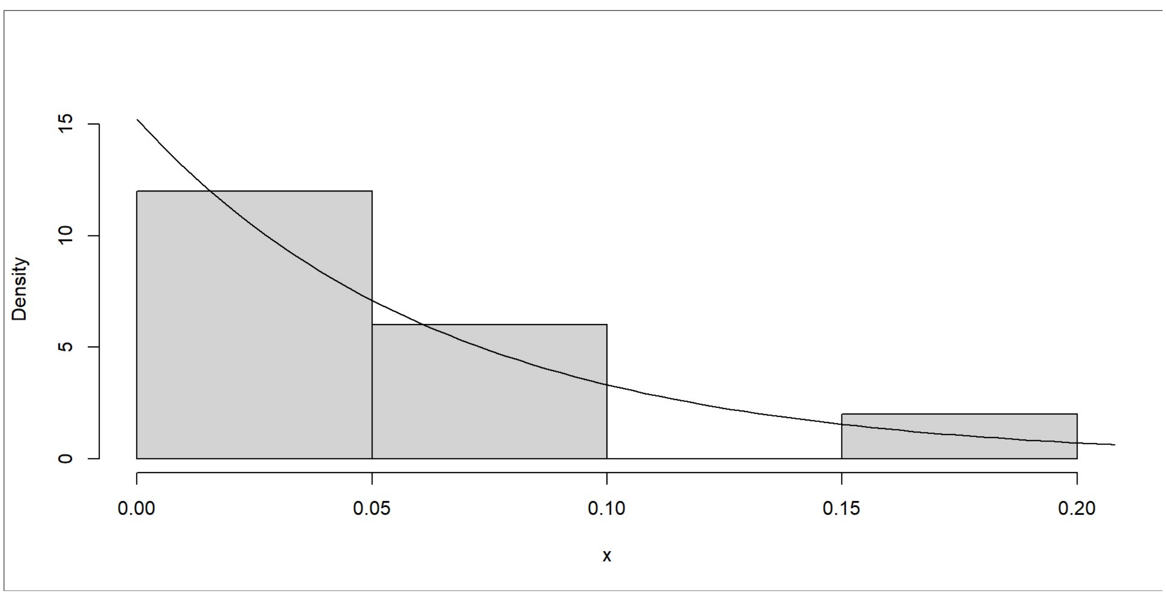
Exponential pdf visualized



Sample

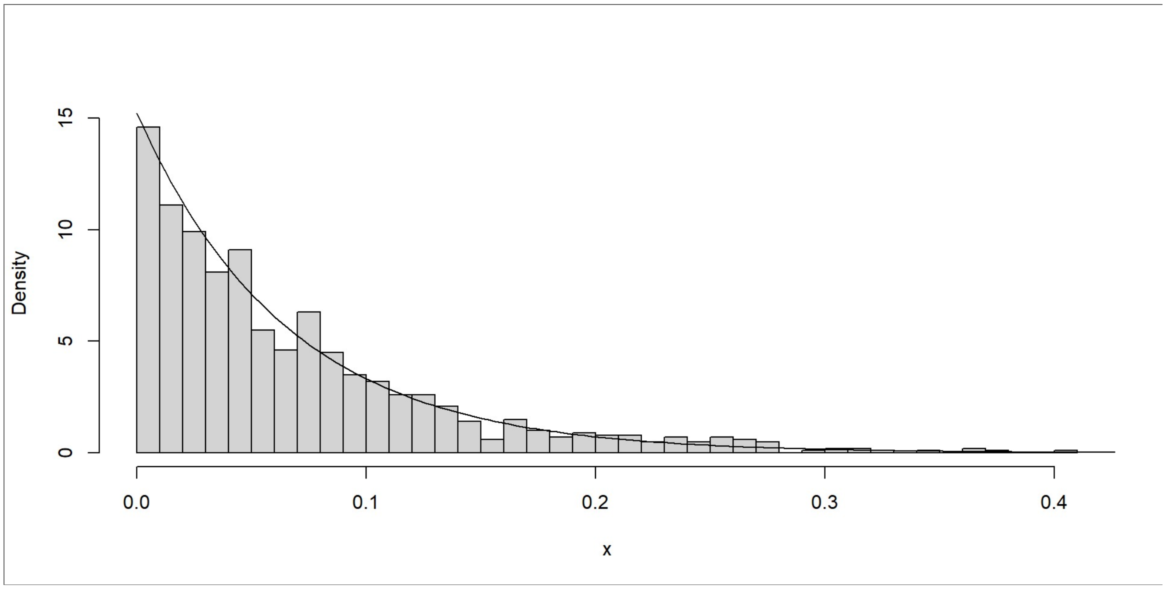
A sample of n realization of X, denoted $X_1, X_2, \ldots, X_n$, can be used to approximate the distribution of X.

Histogram for 10 realizations



A histogram of 10 realizations from an Exponential(15.2).

Histogram for 1000 realizations



A histogram of 1000 realizations from an Exponential(15.2).

1.1.1 A very special trick

Approach

We can avoid doing actual integration if we can manipulate the integrand to look more like a pdf, which must integrate to 1 of the range of the random variable.
Example: Determine

$$\int_0^\infty 3\exp(-2x)\,dx.$$

Special trick (cont)

1.2 Transformations of Random Variables

1.2.1 The discrete case and the binomial distribution

The binomial pmf

$X \sim \mathrm{Binomial}(n, p)$ has the pmf

$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} I_{\{0,1,...,n\}}(x)$.

Determine the pmf of $Y := n - X$.

1.2.2 The Continuous Case and the Gamma Distribution

Continous Transformation PDF

Let X be a continuous random variable with pdf $f_X(x)$. Let $Y = g(X)$, where g is invertible and differentiable. Then the pdf for Y is

$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|.$

Proof

Proof (cont)

Proof (cont)

Proof (cont)

Example 1.2.2

A continuous random variable $X \sim \mathrm{Gamma}(\alpha, \beta)$ has pdf
$f_X(x) = \dfrac{1}{\Gamma(\alpha)}\beta^\alpha x^{\alpha-1}\exp(-\beta x) I_{(0,\infty)}(x)$.

Example 1.2.2 (cont)

Let $Y := cX$ with $c > 0$. Determine the pdf of Y.

Example 1.2.2 (cont)

The shape parameter

The α term is the "shape" parameter of the Gamma distribution.
The kernel (non-constant) part of the pdf of the Gamma distribution is $x^{\alpha-1}\exp(-\beta x)I_{(0,\infty)}(x)$.

The shape parameter

The exponential function part is an unscaled exponential density.

- $\exp(-\beta x)$ dominates $x^{\alpha-1}$ for large x.
- For small x, the $x^{\alpha-1}$ term controls the shape.

Shape parameter visualized

The α terms controls the shape of the Gamma distribution.

The Gamma Function

The pdf for the gamma distribution is defined using the **gamma function**, denoted by $\Gamma(\alpha)$.

$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x)\, dx.$

The Gamma Function (cont)

The Gamma Function (cont)

The Gamma Function (cont)

1.3 Bivariate Transformations

Bivariate Transformation PDF

Suppose the X_1 and X_2 are jointly continuous random variables with pdf $f_{X_1,X_2}(x_1,x_2)$. Let $Y_1 = g_1(X_1,X_2)$ and $Y_2 = g_2(X_1,X_2)$. For h_1 and h_2 differentiable, let
$X_1 = h_1(Y_1,Y_2) \text{ and } X_2 = h_2(Y_1,Y_2).$
Then
$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(h_1(y_1,y_2),h_2(y_1,y_2)\right)|J|,$
where $|J|$ is the absolute value of the Jacobian.

Bivariate transformation pdf continued

The absolute value of the Jacobian is

$$J = \det \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}.$$

Example 1.3.1

Let $X_1, X_2 \overset{i.i.d.}{\sim} \mathrm{Gamma}(\alpha, \beta)$
Determine the pdf of

$$Y = \frac{X_1}{X_1 + X_2}.$$

Example 1.3.1 (cont)

Example 1.3.1 (cont)

Example 1.3.1 (cont)

Example 1.3.1 (cont)

The Beta Distribution

The continuous random variable $X \sim \text{Beta}(a, b)$ had pdf

$$f(x) = \frac{1}{\mathcal{B}(a,b)} x^{a-1}(1-x)^{b-1} I_{(0,1)}(x),$$

for $a, b > 0$, where

$$\mathcal{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx$$

denotes the **beta function**, which normalizes the kernel of the Beta distribution.

The Beta Distribution (cont)

The Beta distribution is a flexible distribution for modeling a random variable between 0 and 1.

Note: The Uniform(0, 1) distribution is a special case of the Beta distribution with

The Beta Distribution (cont)

The Beta Function

$\mathcal{B}(a, b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

Proof:

The Beta Function (cont)