2.3.2

Chebyshev’s Inequality

Chebyshev’s (Tchebychev’s) Inequality

Let X be a random variable with mean μ and variance $\sigma^2 < \infty$. For any $k > 0$,

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

Equivalently,

$$P(|X - \mu| \geq k\sigma) \leq 1 - \frac{1}{k^2}.$$

Proof

2.3.3

The Sample Mean and Convergence in Probability

The Weak Law of Large Numbers (WLLN)

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} F$ with mean μ and variance $\sigma^2 < \infty$. Then $\bar{X} \xrightarrow{P} \mu$.

Proof

Proof (cont)

Proof (cont)

Example 2.3.1

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Exponential}(\lambda)$. What does this tell us about $\bar{X}$?

Example 2.3.2

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Gamma}(\alpha, \beta)$. What does this tell us about $\bar{X}$?

2.3.4

Consistent Estimators

Definition 2.3.2 (Consistent)

Let $\hat{\theta}_n$ be a consistent estimator of θ. Then $\hat{\theta}_n$ is a **consistent estimator** of θ if $\hat{\theta}_n \xrightarrow{P} \theta$.
A question about the consistency of an estimator is related to convergence in probability.

Theorem 2.3.1

An unbiased estimator $\hat{\theta}_n$ of θ is a consitent estimator of θ if $\lim\limits_{n\to\infty} \mathrm{var}(\hat{\theta}_n) = 0.$

Proof

Proof (cont)

Proof (cont)

Definition 2.3.3 (Asymptotically unbiased)

$\hat{\theta}_n$ is an **asymptotically unbiased** estimator of θ if
$\lim_{n \to \infty} E(\hat{\theta}_n) = \theta.$

Theorem 2.3.2

An asymptotically unbiased estimator $\hat{\theta}_n$ of θ is a consistent estimator of θ if
$\lim_{n \to \infty} \operatorname{var}(\hat{\theta}_n) = 0$

Proof

Proof (cont)

Proof (cont)

Theorem

An estimator $\hat{\theta}_n$ of θ is a consistent estimator of θ if $\lim_{n \to \infty} \mathrm{MSE}(\hat{\theta}_n) = 0$

Proof

Example 2.3.3

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Uniform}(0, \theta)$. Show that $X_{(n)} \xrightarrow{P} \theta$.

Example 2.3.3 (cont)

Example 2.3.3 (cont)

Example 2.3.3 (cont)

Example 2.3.3 (cont)

Example 2.3.3 (cont)

2.3.5

Things About Convergence in Probability That Will Not Surprise You

Theorem 2.3.3

Suppose that $\{X_n\}$ and $\{Y_n\}$ are sequences of random variables such that $X_n \xrightarrow{P} (X)$ and $Y_n \xrightarrow{P} (Y)$ for some random variables X and Y. Then

1. $X_n + Y_n \xrightarrow{P} X + Y$.
2. $X_n Y_n \xrightarrow{P} XY$.

Theorem 2.3.3 (cont)

3. $X_n / Y_n \xrightarrow{P} X/Y$ as long as the denominators are non-zero with probability 1.
4. Continuous Mapping Theorem: For any continuous function g, $g(X_n) \xrightarrow{P} g(X)$.

Proof

Proof (cont)

Proof (cont)

Proof (cont)

Proof (cont)

Proof (cont)

Example 2.3.4

Suppose the $X_n \xrightarrow{P} b$ and a is a constant. Does aX_n converge?

Example 2.3.4 (cont)

Example 2.3.5

Let $\{a_n\}$ be a sequence of real number such that
$\lim_{n\to\infty} a_n = a$.

Show that $a_n \overset{P}{\to} a$.

Example 2.3.5 (cont)

Example 2.3.6

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Uniform}(0, \theta)$. Use Theorem 2.3.1 to show that $X_n \xrightarrow{P} \theta$.

Example 2.3.6 (cont)

Example 2.3.6 (cont)