2.4

Convergence in Distribution

Definition 2.4.1 (Convergence in Distribution)

Let $\{X_n\}$ be a sequence of random variables with dfs $F_n(x) = P(X_n \leq X)$. Let X be a random variables with cdf $F(X)$.
X_n **converges in distribution** to X if

$$\lim n \to \infty F_n(x) = F(x),$$

for all X where F is continuous.
We denote this property as $X_n \xrightarrow{d} X$.

Convergence in Distribution

The convergence in distribution of X_n to X may also be denoted as $X_n \xrightarrow{D} X$. **Convergence in distribution** is sometimes called **convergence in law** and denoted $X_n \xrightarrow{L} X$.

Example 2.4.1

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Pareto}(1)$. Define $Y_n = nX_{(1)}$. The cdf of Y_n converges to what distribution?

Example 2.4.1 (cont)

Example 2.4.1 (cont)

Example 2.4.1 (cont)

Example 2.4.2

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Uniform}(0, 1)$. Consider $Y_n = X_{(n)}$. The cdf of Y_n converges to what?

Example 2.4.2 (cont)

Example 2.4.2 (cont)

Example 2.4.2 (cont)

Notes

CDFs are always right-continuous functions.
Convergence in distribution only must be done where $F(x)$ is continuous.

2.4.1

Convergence in Probability is Stronger

Theorem 2.4.1

$X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X.$

Proof

Proof (cont)

Proof (cont)

Example 2.4.4

$X_n \xrightarrow{P} X \not\Rightarrow$.

Example 2.4.4 (cont)

Example 2.4.4 (cont)

Example 2.4.4 (cont)

Theorem 2.4.2

Suppose that $X_n \stackrel{d}{\to} c$ where $c\in \mathbb{R}$. Then $X_n \stackrel{P}{\to} c$.

Proof

Proof (cont)

Proof (cont)

2.4.2

The Continuous Mapping Theorem

The Continuous Mapping Theorem

Suppose that $X_n \stackrel{d}{\to} X$ and g is a continuous function. Then $g(X_n) \stackrel{d}{\to} g(X)$.
The proof is omitted because it is long.

Example 2.4.4

Let $X,X_1,X_2,\ldots,X_n\stackrel{i.i.d.}{\sim}N(\mu,\sigma^2)$. Then $X_n\stackrel{d}{\to}X$.
This is why convergence in distribution is weak.

Example 2.4.4 (cont)

Example 2.4.4 (cont)

Unexpected property

Even if $X_n\stackrel{d}{\to}X$ and $Y_n\stackrel{d}{\to}Y$, it is **not necessarily true** that $X_n + Y_n \stackrel{d}{\to}X + Y$.

Example 2.4.5

Suppose that $X,X_1,X_2,\ldots,X_n\stackrel{i.i.d.}{\sim}N(\mu,\sigma^2)$ and $Y_n:=-X_n$. To what does $X_n + Y_n$ converge in distribution?

Example 2.4.5 (cont)

Example 2.4.5 (cont)

Example 2.4.5 (cont)

2.4.3

Slutsky’s Theorem: Mixing Convergence Types

Slutsky’s Theorem

Suppose that $X_n \stackrel{P}{\to} a$, where $a\in\mathbb{R}$ and $Y_n \stackrel{d}{\to} Y$. Then

1. $X_n + Y_n \stackrel{d}{\to} a + Y$.
2. $X_n Y_n \stackrel{d}{\to} a Y$.
3. $Y_n/X_n \stackrel{d}{\to} Y/a$ if $a\neq 0$.

Proof

Proof (cont)

Proof (cont)

Proof (cont)

Proof (cont)

Example 2.4.6

Let $X_1,X_2,\ldots,X_n \stackrel{i.i.d.}{\sim}\text{Pareto}(1)$. We know from Example 2.4.1 that $Y_n:= nX_{(1)}\stackrel{d}{\to}Y$ where $Y\sim\text{Exponential}(1)$. To what does $X_{(1)}$ converge in distribution?

Example 2.4.6 (cont)

Example 2.4.6 (cont)

2.4.4

Convergence in Distribution and Moment Generating Functions

Theorem 2.4.3

Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables. Let $F_n(x)$ be the cdf of X_n and $M_n(t)$ be the mgf of X_n. Let X be a random variable with cdf $F(x)$ and mgf $M(t)$.
Then $\lim_{n\to\infty} M_n(t) = M(t)\Rightarrow \lim_{n\rightarrow \infty} F_n(x) = F(x)$.
i.e., if $\lim_{n\to\infty} M_n(t) = M(t)$ then $X_n \stackrel{d}{\to} X$.

Notes

1. The first limit for the mgf in Theorem 2.4.3 only needs to hold in some open interval containing zero.
2. The result would be if and only if (go both directions) if the mgf always existed (it might diverge to infinity).

Example 2.4.7

Let $X_1,X_2,\ldots$ be a sequence of independent random variables with $X_n \sim \text{Binomial}(n, \lambda/p)$. To what does X_n converge in distribution?

Example 2.4.7 (cont)

Example 2.4.7 (cont)

Example 2.4.7 (cont)

Example 2.4.7 (cont)