EE3015 - IDP

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All codes available at

https://github.com/EE17BTECH11028/EE3015.git

1 Question 5.3 in gvv_filter.pdf

The system h(n) is said to be stable if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \tag{1.0.1}$$

Is the system defined by

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \tag{1.0.2}$$

with

$$y(n) = 0 \text{ for } n < 0 \tag{1.0.3}$$

stable for impulse response defined using the $Z-transform$,

$$H(z) = \frac{Y(z)}{X(z)} \tag{1.0.4}$$

?

2 Solution

We can verify the stability through
1. Bounded Input - Bounded Output(BIBO) stability
2. pole-zero plot.

3 BIBO stability

A system is said to be BIBO stable, if the output of the system is bounded for every input to the system that is bounded.

$$|x(n)| \le B_x \implies |y(n)| \le B_y \tag{3.0.1}$$

where

$$B_x < \infty \tag{3.0.2}$$

$$B_y < \infty \tag{3.0.3}$$

Let the input $x(n)$ be bounded,

$$|x(n)| \le B_x < \infty \tag{3.0.4}$$

For a given $x(n), y(n)$ is defined as

$$y(n) = \sum_{-\infty}^{\infty} h(k)\, x(n-k) \tag{3.0.5}$$

where h(k) is the impulse function

Considering the modulus of the equation

$$|y(n)| = |\sum_{-\infty}^{\infty} h(k)\, x(n-k)\,| \tag{3.0.6}$$

Replacing the value of x(n) with it's maximum value B_x to obtain the bounds of y(n)

$$\implies |y(n)| \le |\sum_{-\infty}^{\infty} h(k)\, B_x\,| \tag{3.0.7}$$

$$\implies |y(n)| \le B_x\,|\sum_{-\infty}^{\infty} h(k)\,| \tag{3.0.8}$$

For the system to be BIBO stable, $|y(n)| < \infty$. This holds only if

$$|y(n)| < \infty \tag{3.0.9}$$

$$\implies |\sum_{-\infty}^{\infty} h(k)\,| < \infty \tag{3.0.10}$$

since B_x is known to be a finite value

So for the system to be BIBO stable, it's impulse response in time domain must be absolutely summable to a finite value.

4 Impulse response

For the given system,

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \tag{4.0.1}$$

$$y(n) = 0 \text{ for } n < 0 \tag{4.0.2}$$

Apply Z-transform on both sides,

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.0.3}$$

$$\implies Y(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}}X(z) \tag{4.0.4}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.0.5}$$

We defined Z-transform of impulse to be

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.0.6}$$

So,

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.0.7}$$

$$\implies H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.0.8}$$

Applying $inverseZ - transform$ on both sides,

$$\implies h(n) = \left[\frac{-1}{2}\right]^{n} u(n) + \left[\frac{-1}{2}\right]^{n-2} u(n-2) \tag{4.0.9}$$

Now that we have calculated the impulse response, let's verify it's BIBO stability by verifying if

$$|\sum_{-\infty}^{\infty} h(n)| < \infty \tag{4.0.10}$$

$$|\sum_{-\infty}^{\infty} h(n)| = \sum_{n=-\infty}^{\infty} |\left[\frac{-1}{2}\right]^{n} u(n) + \left[\frac{-1}{2}\right]^{n-2} u(n-2)| \tag{4.0.11}$$

$$|\sum_{-\infty}^{\infty} h(n)| = \sum_{n=-\infty}^{\infty} |\left[\frac{1}{2}\right]^{n} u(n) + \left[\frac{1}{2}\right]^{n-2} u(n-2)| \tag{4.0.12}$$

Since n is running from $-\infty$ to ∞, the time shift can be ignored

$$|\sum_{-\infty}^{\infty} h(n)| = 2\sum_{n=-\infty}^{\infty} |\left[\frac{1}{2}\right]^{n} u(n)| \tag{4.0.13}$$

$$|\sum_{-\infty}^{\infty} h(n)| = 2\sum_{n=0}^{\infty} \left[\frac{1}{2}\right]^{n} \tag{4.0.14}$$

Using the sum of infinite length Geometric Progression,

$$|\sum_{-\infty}^{\infty} h(n)| = 2\left[\frac{1}{1-\frac{1}{2}}\right] = 4 \tag{4.0.15}$$

As impulse response sums up to a finite value,

$$|\sum_{-\infty}^{\infty} h(n)| < \infty \tag{4.0.16}$$

The system is BIBO stable.

5 Pole-Zero Plot

We obtained Z-transform of the impulse response from the system as

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.0.1}$$

$$\implies H(z) = \frac{2(z^2+1)}{z(2z+1)} \tag{5.0.2}$$

Solving for poles and zeros, we get,

$$\text{Poles} = 0, -\tfrac{1}{2}$$

$$\text{zeros} = +1j, -1j$$

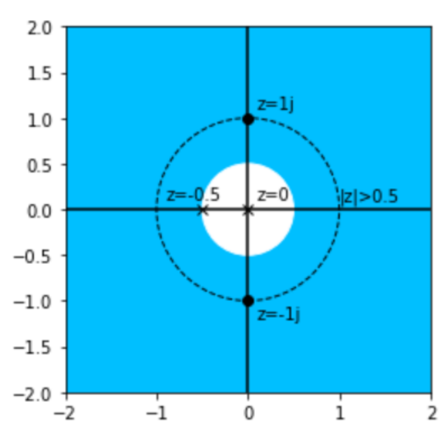

Fig. 0: Pole-Zero Plot

In the Pole-Zero plot, the poles of the impulse response lie in the left half of the s-plane which proves that the system is stable.

Let's verify the stability of the system for input from 3.1, gvv_filter.pdf Input is given as,

$$x(n) = \{1, 2, 3, 4, 2, 1\} \tag{5.0.3}$$

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \tag{5.0.4}$$
$$y(n) = 0 \text{ for } n < 0 \tag{5.0.5}$$

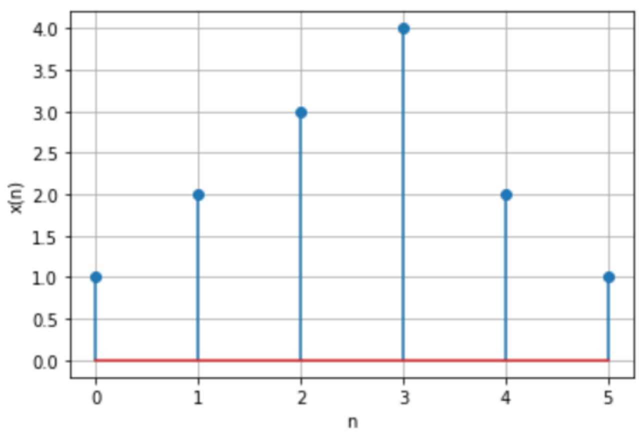

Fig. 0: Input signal, x(n)

From the above plot $B_x = 4$ for the input signal.

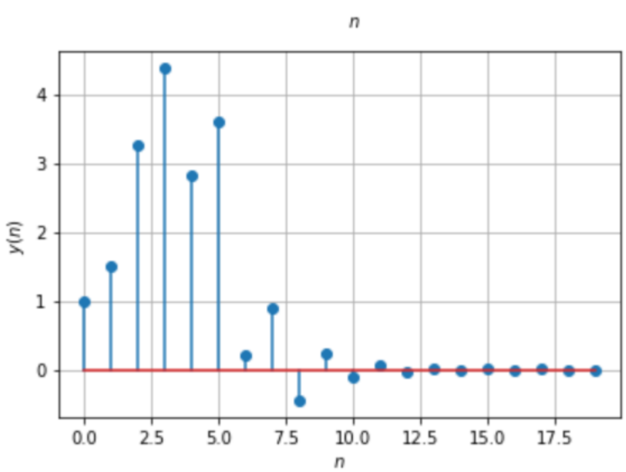

Fig. 0: Output signal, y(n)

From the above plot $B_y = 4.375$ for the output signal.

Since the input and output are bounded, the system is stable.