Assignment 11

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***Abstract*—This document explains a proof on linear transformations.**

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_11

1 Problem

Let $\mathbb{W}$ be the set of all 2×2 complex Hermitian matrices, that is the sset of 2×2 complex matrices $\mathbf{A}$ ssuch that $\mathbf{A}_{ij} = \overline{\mathbf{A_{ji}}}$ (the bar denoting complex conjugation). $\mathbb{W}$ is a vector space over the field of real numbers, under the usual operations. Verify that

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \tag{1.0.1}$$

is an isomorphism of $\mathbb{R}^4$ onto $\mathbb{W}$.

2 Solution

1) **Check for linearity:** The transformation T is given by

$$T : \mathbb{R}^4 \to \mathbb{W} \tag{2.0.1}$$

$$T \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \tag{2.0.2}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \\ t \\ z \end{pmatrix}$. Expressing R.H.S of equation (2.0.2) using Kronecker Product,

$$T(\mathbf{x}) = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \mathbf{x} & \begin{pmatrix} 0 & 1 & i & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & 1 & -i & 0 \end{pmatrix} \mathbf{x} & \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{2.0.3}$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \end{pmatrix} \mathbf{x} & \begin{pmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{2.0.4}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \\ z & 0 \\ t & 0 \\ 0 & x \\ 0 & y \\ 0 & z \\ 0 & t \end{pmatrix} \tag{2.0.5}$$

$$\implies T(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{x} & \mathbf{0}_{4\times1} \\ \mathbf{0}_{4\times1} & \mathbf{x} \end{pmatrix} \tag{2.0.6}$$

Where $\mathbf{A}$ and $\mathbf{B}$ are block matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.8}$$

The Kronecker Product of $\mathbf{I}_2$ and $\mathbf{x}$ gives the block matrix in equation (2.0.6).

$$\mathbf{I}_{2\times2} \otimes \mathbf{x}_{4\times1} = \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix}_{8\times2} \tag{2.0.9}$$

Hence we can write equation (2.0.6) as,

$$T(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}) \tag{2.0.10}$$

Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^4$ and $\alpha, \beta \in \mathbb{R}$.

$$T(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes (\alpha\mathbf{x}_1 + \beta\mathbf{x}_2)) \tag{2.0.11}$$

$$= \alpha\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{x}_1) + \beta\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{x}_2) \tag{2.0.12}$$

$$= \alpha T\mathbf{x}_1 + \beta T\mathbf{x}_2 \tag{2.0.13}$$

Therefore from equation (2.0.13), we can say T is linear transformation.

2) **Check for one-one property:** For transformation T to be one-one, we can prove if $T(\mathbf{x}) = \mathbf{0}$, that implies $\mathbf{x} = \mathbf{0}$. From the equation (2.0.10),

$$T(\mathbf{x}) = \mathbf{0} \tag{2.0.14}$$

$$\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{x}) = \mathbf{0} \tag{2.0.15}$$

$$\implies \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \\ z & 0 \\ t & 0 \\ 0 & x \\ 0 & y \\ 0 & z \\ 0 & t \end{pmatrix} = \mathbf{0}_{2\times 2} \tag{2.0.16}$$

From equation (2.0.2),

$$\begin{pmatrix} t + x & y + iz \\ y - iz & t - x \end{pmatrix} = \mathbf{0}_{2\times 2} \tag{2.0.17}$$

$$\implies x = 0, y = 0, z = 0, t = 0 \tag{2.0.18}$$

$$\implies \mathbf{x} = \mathbf{0} \tag{2.0.19}$$

Hence from (2.0.14) and (2.0.19), T is one-one and that implies $T : \mathbb{R}^4 \to \mathbb{W}$ is isomorphism.