

ASSIGNMENT 3

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Abstract—This document proves that triangles on the same base and having equal areas lie between the same parallel lines.

Download latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_3

1 PROBLEM

Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.

2 EXPLANATION

Consider $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are three points of a triangle $\triangle ABC$,
 $\mathbf{D}, \mathbf{B}, \mathbf{C}$ are three points of another triangle $\triangle DBC$,
both triangles having same base BC and \mathbf{B} at origin,

$$\mathbf{A} - \mathbf{B} = \mathbf{A} \quad (2.0.1)$$

$$\mathbf{C} - \mathbf{B} = \mathbf{C} \quad (2.0.2)$$

$$\mathbf{D} - \mathbf{B} = \mathbf{D} \quad (2.0.3)$$

The area of the triangle $\triangle ABC$ is,

$$\text{Area}(\triangle ABC) = \frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{C} - \mathbf{B})\| \quad (2.0.4)$$

Substituting (2.0.1) and (2.0.2) in (2.0.4),

$$\Rightarrow \text{Area}(\triangle ABC) = \frac{1}{2} \|(\mathbf{A} \times \mathbf{C})\| \quad (2.0.5)$$

The area of the triangle $\triangle DBC$ is,

$$\text{Area}(\triangle DBC) = \frac{1}{2} \|(\mathbf{D} - \mathbf{B}) \times (\mathbf{C} - \mathbf{B})\| \quad (2.0.6)$$

Substituting (2.0.2) and (2.0.3) in (2.0.6),

$$\Rightarrow \text{Area}(\triangle DBC) = \frac{1}{2} \|(\mathbf{D} \times \mathbf{C})\| \quad (2.0.7)$$

Given the area of $\triangle ABC$ is equal to the area $\triangle DBC$,
from equations (2.0.5) and (2.0.7)

$$\frac{1}{2} \|(\mathbf{A} \times \mathbf{C})\| = \frac{1}{2} \|(\mathbf{D} \times \mathbf{C})\| \quad (2.0.8)$$

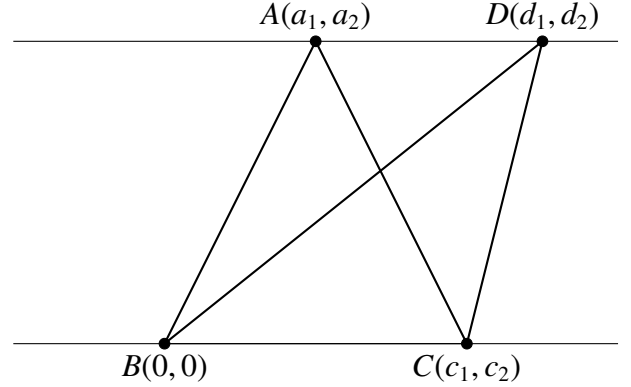


Fig. 0: $\triangle ABC$ and $\triangle DBC$ with BC as common base

Squaring on both sides of the equation (2.0.8), we get

$$\|(\mathbf{A} \times \mathbf{C})\|^2 = \|(\mathbf{D} \times \mathbf{C})\|^2 \quad (2.0.9)$$

$$\Rightarrow (\mathbf{A} \times \mathbf{C})^T (\mathbf{A} \times \mathbf{C}) = (\mathbf{D} \times \mathbf{C})^T (\mathbf{D} \times \mathbf{C}) \quad (2.0.10)$$

$$\Rightarrow (\mathbf{A}^T \mathbf{A})(\mathbf{C}^T \mathbf{C}) - (\mathbf{A}^T \mathbf{C})(\mathbf{C}^T \mathbf{A}) = (\mathbf{D}^T \mathbf{D})(\mathbf{C}^T \mathbf{C}) - (\mathbf{D}^T \mathbf{C})(\mathbf{C}^T \mathbf{D}) \quad (2.0.11)$$

$$\Rightarrow \|\mathbf{A}\|^2 \|\mathbf{C}\|^2 - (\mathbf{A}^T \mathbf{C})^2 = \|\mathbf{D}\|^2 \|\mathbf{C}\|^2 - (\mathbf{D}^T \mathbf{C})^2 \quad (2.0.12)$$

Let $\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$. Then equation (2.0.12) can be written as,

$$(a_1^2 + a_2^2)(c_1^2) - (a_1 c_1)^2 = (d_1^2 + d_2^2)(c_1^2) - (d_1 c_1)^2 \quad (2.0.13)$$

$$\Rightarrow a_2 = d_2 \quad (2.0.14)$$

Now we have,

$$(\mathbf{D} - \mathbf{A}) = \begin{pmatrix} d_1 - a_1 \\ d_2 - a_2 \end{pmatrix} = \begin{pmatrix} d_1 - a_1 \\ 0 \end{pmatrix} \quad (2.0.15)$$

$$(\mathbf{C} - \mathbf{B}) = \begin{pmatrix} c_1 - 0 \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \quad (2.0.16)$$

From equations (2.0.15) and (2.0.16), we can say

$$(\mathbf{D} - \mathbf{A}) = \frac{d_1 - a_1}{c_1} (\mathbf{C} - \mathbf{B}) \quad (2.0.17)$$

$$\implies (\mathbf{D} - \mathbf{A}) = k (\mathbf{C} - \mathbf{B}) \quad (2.0.18)$$

where k is a constant. From the equation (2.0.18), we can say that $AD \parallel BC$. Hence the two triangles $\triangle ABC$ and $\triangle DBC$ lie between the same parallels AD and BC