QR Decomposition

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***Abstract*—This document explains the QR decomposition of a 2x2 square matrix.**

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/QR_Decomposition

and all python codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/QR_Decomposition/codes

1 Problem

Perform QR decomposition of matrix $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

2 Explanation

Let $\mathbf{a}$ and $\mathbf{b}$ are the columns of matrix $\mathbf{A}$. The matrix $\mathbf{A}$ can be decomposed in the form

$$\mathbf{A} = \mathbf{QR} \tag{2.0.1}$$

$$\mathbf{U} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.3}$$

where

$$k_1 = \|\mathbf{a}\| \tag{2.0.4}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{2.0.5}$$

$$r_1 = \frac{\mathbf{u_1}^T \mathbf{b}}{\|u_1\|^2} \tag{2.0.6}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1 \mathbf{u_1}}{\|\mathbf{b} - r_1 \mathbf{u_1}\|} \tag{2.0.7}$$

$$k_2 = \mathbf{u_2}^T \mathbf{b} \tag{2.0.8}$$

Then the given matrix can be represented as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.9}$$

3 Solution

The the columns of matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$ are $\mathbf{a}$ and $\mathbf{b}$ where

$$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{3.0.2}$$

Now for the given matrix, From (2.0.4) and (2.0.5)

$$k_1 = \|\mathbf{a}\| = 5 \tag{3.0.3}$$

$$\mathbf{u_1} = \frac{1}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{3.0.4}$$

From (2.0.6)

$$r_1 = \frac{1}{5} \begin{pmatrix} 3 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{-11}{5} \tag{3.0.5}$$

From (2.0.7)

$$\mathbf{b} - r_1 \mathbf{u_1} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{-11}{5} \begin{pmatrix} \frac{3}{5} \\ \frac{-4}{5} \end{pmatrix} \tag{3.0.6}$$

$$\|\mathbf{b} - r_1 \mathbf{u_1}\| = \frac{2}{5} \tag{3.0.7}$$

$$\implies \mathbf{u_2} = \frac{5}{2} \begin{pmatrix} \frac{8}{25} \\ \frac{6}{25} \end{pmatrix} \tag{3.0.8}$$

From (2.0.8)

$$k_2 = \mathbf{u_2}^T \mathbf{b} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{2}{5} \tag{3.0.9}$$

Now we can observe that $\mathbf{Q}^T \mathbf{Q} = \mathbf{Q} \mathbf{Q}^T = I$

$$\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.10}$$

From (2.0.9), The matrix $\mathbf{A}$ can now be written as,

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} -5 & \frac{-11}{5} \\ 0 & \frac{2}{5} \end{pmatrix} \tag{3.0.11}$$