Assignment 5

Yenigalla Samyuktha
EE20MTECH14019

***Abstract*—This document explains the concept of finding the representation of conics from the given second degree equation**

Download all latex-tikz codes from

```
https://github.com/EE20MTECH14019/EE5609/
    tree/master/Assignment_5
```

and all python codes from

```
https://github.com/EE20MTECH14019/EE5609/
    tree/master/Assignment_5/codes
```

1 Problem

What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre.

$$2x^2 - 72xy + 23y^2 - 4x - 2y - 48 = 0 \tag{1.0.1}$$

2 Solution

The second degree equation in general can be represented as

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

The equation (2.0.1) in matrix form can be represented as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 2 & -36 \\ -36 & 23 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{2.0.4}$$

$$f = -48 \tag{2.0.5}$$

$$\det(\mathbf{V}) = \begin{vmatrix} 2 & -36 \\ -36 & 23 \end{vmatrix} \tag{2.0.6}$$

$$\implies \det(\mathbf{V}) = -1250 \tag{2.0.7}$$

$$\implies \det(\mathbf{V}) < 0 \tag{2.0.8}$$

Since $\det(\mathbf{V}) < 0$ the given equation (1.0.1) represents a hyperbola. The characteristic equation of $\mathbf{V}$ is acquired by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \tag{2.0.9}$$

$$\begin{vmatrix} 2-\lambda & -36 \\ -36 & 23-\lambda \end{vmatrix} = 0 \tag{2.0.10}$$

$$\implies \lambda^2 - 25\lambda - 1250 = 0 \tag{2.0.11}$$

On solving the equation (2.0.11), the eigen values are given by

$$\lambda_1 = 50 \tag{2.0.12}$$

$$\lambda_2 = -25 \tag{2.0.13}$$

We can observe that for a hyperbola, $\lambda_1 > 0$ and $\lambda_2 < 0$. Consider the eigenvector $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.14}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.15}$$

For $\lambda_1 = 50$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} -48 & -36 \\ -36 & -27 \end{pmatrix} \tag{2.0.16}$$

By row reduction ,

$$\begin{pmatrix} -48 & -36 \\ -36 & -27 \end{pmatrix} \tag{2.0.17}$$

$$\xleftrightarrow[R_1 \leftarrow \frac{R_1}{-12}]{R_2 \leftarrow \frac{R_2}{-9}} \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} \tag{2.0.18}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix} \tag{2.0.19}$$

Subsituting equation (2.0.19) in equation (2.0.15) we get

$$\begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.20}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{-3}{4} \\ 1 \end{pmatrix} \tag{2.0.21}$$

For $\lambda_2 = -25$,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} 27 & -36 \\ -36 & 48 \end{pmatrix} \tag{2.0.22}$$

By row reduction ,

$$\begin{pmatrix} 27 & -36 \\ -36 & 48 \end{pmatrix} \tag{2.0.23}$$

$$\xleftrightarrow[R_1 \leftarrow \frac{R_1}{9}]{R_2 \leftarrow \frac{R_2}{-12}} \begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix} \tag{2.0.24}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \tag{2.0.25}$$

Subsituting equation (2.0.25) in equation (2.0.15) we get

$$\begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.26}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \tag{2.0.27}$$

By eigen decompostion, $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \tag{2.0.28}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.0.29}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.30}$$

Substituting equations (2.0.21), (2.0.27) in equation (2.0.29) we get

$$\mathbf{P} = \begin{pmatrix} \frac{-3}{4} & \frac{4}{3} \\ 1 & 1 \end{pmatrix} \tag{2.0.31}$$

Substituting equations (2.0.12), (2.0.13) in (2.0.30) we get

$$\mathbf{D} = \begin{pmatrix} 50 & 0 \\ 0 & -25 \end{pmatrix} \tag{2.0.32}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \tag{2.0.33}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{-23}{1250} & \frac{-18}{625} \\ \frac{-18}{625} & \frac{-1}{625} \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{2.0.34}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-41}{625} \\ \frac{-37}{625} \end{pmatrix} \tag{2.0.35}$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = \frac{-30119}{625} < 0 \tag{2.0.36}$$

We swap the semi-major and semi-minor axes and the respective are given by

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_1}} \end{cases} \tag{2.0.37}$$

Calculating the axes, we get

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = 1.388 \tag{2.0.38}$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_1}} = 0.981 \tag{2.0.39}$$

The figure 1 verifies the given equation (2.0.2) as hyperbola with centre $\begin{pmatrix} \frac{-41}{625} \\ \frac{-37}{625} \end{pmatrix}$

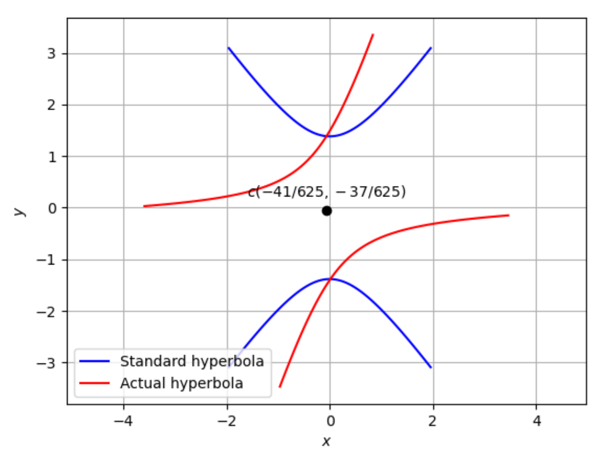

Fig. 1: Hyperbola when origin is shifted

Now (2.0.2) can be written as,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \tag{2.0.40}$$

where ,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \tag{2.0.41}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.0.42}$$

$$\mathbf{y} = \begin{pmatrix} \frac{-3}{4} & 1 \\ \frac{4}{3} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{-3}{4} & 1 \\ \frac{4}{3} & 1 \end{pmatrix}\begin{pmatrix} \frac{-41}{625} \\ \frac{-37}{625} \end{pmatrix} \tag{2.0.43}$$

$$\mathbf{y} = \begin{pmatrix} \frac{-3}{4} & 1 \\ \frac{4}{3} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{-1}{100} \\ \frac{-11}{75} \end{pmatrix} \tag{2.0.44}$$

Subsituting the eqauations (2.0.36), (2.0.32) in equation (2.0.40)

$$\mathbf{y}^T\begin{pmatrix} 50 & 0 \\ 0 & -25 \end{pmatrix}\mathbf{y} = \frac{-30119}{625} \tag{2.0.45}$$