Assignment 14

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***Abstract*—This document explains a linear transmormation on polynomials.**

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_14

1 Problem

Let $\mathbb{F}$ be a subfield of complex numbers and let T be the transformation on $\mathbb{F}(x)$ defined by

$$T\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n} \frac{c_i}{i+1} x^{i+1} \tag{1.0.1}$$

Show that T is a non-singular linear operator on $\mathbb{F}[x]$. Also show that T is not invertible.

2 Solution

The transformation T does integral of a polynomial. Table 2 provides proof that the transformation T is a linear operator and non-singular. Table 3 provides proof that T is not invertible, however there exists a left inverse. The parameters used in the proof are listed in the table 1.

PARAMETER	DESCRIPTION
$\mathbb{F}$	Field of complex numbers
$\mathbb{F}^{\infty}$	Vector space defined on the field $\mathbb{F}$
$\mathbb{F}[x]$	Subspace of $\mathbb{F}^{\infty}$ spanned by $\left\{1, x, x^2, x^3, \ldots\right\}$
$T : \mathbb{F}[x] \rightarrow \mathbb{F}[x]$	Transformation T
$f = \sum_{i=0}^{n} c_i x^i$	Polynomial $f \in \mathbb{F}[x]$
$f' = \sum_{i=0}^{n} c'_i x^i$	Polynomial $f' \in \mathbb{F}[x]$
$c_i, c'_i \ \forall i = 0, 2, \ldots n$	Scalars in $\mathbb{F}$ and coefficients of polynomials f and f'
$T(f) = g = \sum_{i=0}^{n} \frac{c_i}{i+1} x^{i+1}$	Transformed polynomial $g \in \mathbb{F}[x]$
$T(f') = g' = \sum_{i=0}^{n} \frac{c'_i}{i+1} x^{i+1}$	Transformed polynomial $g' \in \mathbb{F}[x]$
$\mathbf{M}_T$	Transformation matrix for T
$N(T)$	Null Space of T

TABLE 1: Parameters

Statement	**Derivation**
$f = \sum_{i=0}^{n} c_i x^i$	$f = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \end{pmatrix}^T_{(n+1)\times 1}$
$T[f] = \mathbf{M}_T f$	$T[f] = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n+1} \end{pmatrix}_{(n+2)\times(n+1)} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_{(n+1)\times 1} = \begin{pmatrix} 0 \\ c_0 \\ \frac{c_1}{2} \\ \frac{c_2}{3} \\ \vdots \\ \frac{c_n}{n+1} \end{pmatrix}_{(n+2)\times 1} = g \in \mathbb{F}[x]$
T is a linear operator	$T[\alpha f + f'] \quad (2.0.1)$ $= \mathbf{M}_T(\alpha f + f') \quad (2.0.2)$ $= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n+1} \end{pmatrix} \begin{pmatrix} \alpha c_0 + c_0' \\ \alpha c_1 + c_1' \\ \alpha c_2 + c_2' \\ \vdots \\ \alpha c_n + c_n' \end{pmatrix} \quad (2.0.3)$ $= \begin{pmatrix} 0 \\ \alpha c_0 + c_0' \\ \frac{\alpha c_1 + c_1'}{2} \\ \frac{\alpha c_2 + c_2'}{3} \\ \vdots \\ \frac{\alpha c_n + c_n'}{n+1} \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ c_0 \\ \frac{c_1}{2} \\ \frac{c_2}{3} \\ \vdots \\ \frac{c_n}{n+1} \end{pmatrix} + \begin{pmatrix} 0 \\ c_0' \\ \frac{c_1'}{2} \\ \frac{c_2'}{3} \\ \vdots \\ \frac{c_n'}{n+1} \end{pmatrix} \quad (2.0.4)$ $= \alpha T[f] + T[f'] \quad (2.0.5)$ $= \alpha g + g' \quad (2.0.6)$ $\therefore T[\alpha f + f'] = \alpha T[f] + T[f'] \quad (2.0.7)$
T is non-singular	$T[f] = 0 \quad (2.0.8)$ $\implies \mathbf{M}_T f = \mathbf{0} \implies f = 0 \because \mathbf{M}_T \neq \mathbf{0} \quad (2.0.9)$ $\implies N(T) = \{0\} \quad (2.0.10)$

TABLE 2: Proof for Non-Singular and linear transformation T

Statement	Derivation
T is not invertible	As $\mathbf{M}_T$ is a non-square matrix with dimensions $(n+2) \times (n+1)$, the transformation T is not invertible
$\mathbf{M}_D$ is left inverse of $\mathbf{M}_T$	$$\mathbf{M}_D\mathbf{M}_T = \mathbf{I}_{n+1} \quad (2.0.11)$$ $$\implies \mathbf{M}_D = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n+1 \end{pmatrix} \quad (2.0.12)$$

TABLE 3: Non-Invertibility of transformation T