# Assignment 10

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***Abstract*—This document explains a proof on linear transformations.**

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_10

## 1 Problem

Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, and let U be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Prove that the transformation UT is not invertible. Generalize the theorem.

## 2 Solution

We have two transformations

$$T : \mathbb{R}^3 \to \mathbb{R}^2 \tag{2.0.1}$$

$$U : \mathbb{R}^2 \to \mathbb{R}^3 \tag{2.0.2}$$

Let $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^2$. Hence we can write the transformations in matrix form as,

$$T(\mathbf{v}) = \mathbf{A}\mathbf{v} \tag{2.0.3}$$

$$U(\mathbf{w}) = \mathbf{B}\mathbf{w} \tag{2.0.4}$$

Where transformation matrix $\mathbf{A}$ has dimension of 2x3. Hence we can say,

$$Max(Rank(\mathbf{A})) = 2 \tag{2.0.5}$$

And transformation matrix $\mathbf{B}$ has dimension of 3x2. Hence we can say,

$$Max(Rank(\mathbf{B})) = 2 \tag{2.0.6}$$

Now we define the transformation UT as,

$$UT : \mathbb{R}^3 \to \mathbb{R}^3 \tag{2.0.7}$$

And the transformation in matrix form where the dimension of transformation matrix $\mathbf{C}$ is 3x3 can be written as,

$$UT(\mathbf{x}) = \mathbf{C}\mathbf{x} \tag{2.0.8}$$

$$U(T(\mathbf{x})) = \mathbf{B}(\mathbf{A}\mathbf{x}) \tag{2.0.9}$$

$$\implies \mathbf{C} = \mathbf{B}\mathbf{A} \tag{2.0.10}$$

$$dim(\mathbf{C}) = 3 \times 3 \tag{2.0.11}$$

Calculating the maximum rank of transformation matrix $\mathbf{C}$ can have, we know

$$Rank(\mathbf{C}) \le min(Rank(\mathbf{B}), Rank(\mathbf{A})) \tag{2.0.12}$$

From the equations, (2.0.5) and (2.0.6), we get

$$Rank(\mathbf{C}) \le min(2, 2) \tag{2.0.13}$$

$$\implies Max(Rank(\mathbf{C})) = 2 \tag{2.0.14}$$

From equations (2.0.11) and (2.0.14),

$$Rank(\mathbf{C}) < dim(\mathbf{C}) \tag{2.0.15}$$

Therefore transformation UT is not invertible.

## 3 Theorem

Generalizing the proof, for $n > m$ and considering vectors $\mathbf{v}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$. From the Table 0,

$$Rank(\mathbf{C}) = m \tag{3.0.1}$$

$$dim(\mathbf{C}) = n \tag{3.0.2}$$

$$\implies Rank(\mathbf{C}) < dim(\mathbf{C}) \tag{3.0.3}$$

From equation (3.0.3)we can say that the transformation UT is not invertible.

## 4 Example

Let the vectors $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ and $\mathbf{w} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \in \mathbb{R}^2$.

| Transformation | Matrix Representation | Dimension | Max Rank of transformation matrix |
| --- | --- | --- | --- |
| $T : \mathbb{R}^n \to \mathbb{R}^m$ | $T(\mathbf{v}) = \mathbf{Av}$ | $\mathbf{A} : m \times n$ | $Rank(\mathbf{A}) = m$ |
| $U : \mathbb{R}^m \to \mathbb{R}^n$ | $U(\mathbf{w}) = \mathbf{Bw}$ | $\mathbf{B} : n \times m$ | $Rank(\mathbf{B}) = m$ |
| $UT : \mathbb{R}^n \to \mathbb{R}^n$ | $UT(\mathbf{x}) = \mathbf{Cx}$ | $\mathbf{C} : n \times n$ | $Rank(\mathbf{C}) \leq min(Rank(\mathbf{B}), Rank(\mathbf{A}))$<br>$Rank(\mathbf{C}) = m$ |

TABLE 0: Generalization of the proof

1) Calculating transformation matrix $\mathbf{A}$, from equation (2.0.3),

$$T(\mathbf{v}) = \mathbf{Av} \tag{4.0.1}$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \tag{4.0.2}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} = rref(\mathbf{A}) \tag{4.0.3}$$

$$\implies Rank(\mathbf{A}) = 2 \tag{4.0.4}$$

2) Calculating transformation matrix $\mathbf{B}$, from equation (2.0.4),

$$U(\mathbf{w}) = \mathbf{Bw} \tag{4.0.5}$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{4.0.6}$$

$$\begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} = rref(\mathbf{B}) \tag{4.0.7}$$

$$\implies Rank(\mathbf{B}) = 2 \tag{4.0.8}$$

3) Now for the transformation UT, calculating the transformation matrix $\mathbf{C}$, from the equation (2.0.10),

$$\mathbf{C} = \mathbf{BA} \tag{4.0.9}$$

$$\mathbf{C} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{pmatrix} \tag{4.0.10}$$

$$\begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{pmatrix} \sim \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = rref(\mathbf{C}) \tag{4.0.11}$$

$$\implies Rank(\mathbf{C}) = 2 \tag{4.0.12}$$

$$dim(\mathbf{C}) = 3 \tag{4.0.13}$$

As $Rank(\mathbf{C}) < dim(\mathbf{C})$, transformation UT is not invertible.