

QR Decomposition

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Abstract—This document explains the QR decomposition of a 2x2 square matrix.

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/QR_Decomposition

and all python codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/QR_Decomposition/codes

1 PROBLEM

Perform QR decomposition of matrix $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

2 EXPLANATION

Let \mathbf{a} and \mathbf{b} are the columns of matrix \mathbf{A} . The matrix \mathbf{A} can be decomposed in the form

$$\mathbf{A} = \mathbf{Q}\mathbf{R} \quad (2.0.1)$$

where, orthogonal matrix $\mathbf{U} = (\mathbf{u}_1 \quad \mathbf{u}_2)$ (2.0.2)

Right-triangular matrix, $\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix}$ (2.0.3)

where

$$k_1 = \|\mathbf{a}\| \quad (2.0.4)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \quad (2.0.5)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (2.0.6)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1 \mathbf{u}_1}{\|\mathbf{b} - r_1 \mathbf{u}_1\|} \quad (2.0.7)$$

$$k_2 = \mathbf{u}_2^T \mathbf{b} \quad (2.0.8)$$

Then the given matrix can be represented as,

$$(\mathbf{a} \quad \mathbf{b}) = (\mathbf{u}_1 \quad \mathbf{u}_2) \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.0.9)$$

3 SOLUTION

The the columns of matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$ are \mathbf{a} and \mathbf{b} where

$$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (3.0.2)$$

Now for the given matrix, From (2.0.4) and (2.0.5)

$$k_1 = \|\mathbf{a}\| = 5 \quad (3.0.3)$$

$$\mathbf{u}_1 = \frac{1}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (3.0.4)$$

From (2.0.6)

$$r_1 = \frac{1}{5} \begin{pmatrix} 3 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{-11}{5} \quad (3.0.5)$$

From (2.0.7)

$$\mathbf{b} - r_1 \mathbf{u}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{-11}{5} \begin{pmatrix} \frac{3}{5} \\ \frac{-4}{5} \end{pmatrix} \quad (3.0.6)$$

$$\|\mathbf{b} - r_1 \mathbf{u}_1\| = \frac{2}{5} \quad (3.0.7)$$

$$\Rightarrow \mathbf{u}_2 = \frac{5}{2} \begin{pmatrix} \frac{8}{25} \\ \frac{6}{25} \end{pmatrix} \quad (3.0.8)$$

From (2.0.8)

$$k_2 = \mathbf{u}_2^T \mathbf{b} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{2}{5} \quad (3.0.9)$$

Now we can observe that $\mathbf{Q}^T \mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$

$$\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.10)$$

From (2.0.9), The matrix \mathbf{A} can now be written as,

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} -5 & \frac{-11}{5} \\ 0 & \frac{2}{5} \end{pmatrix} \quad (3.0.11)$$