

Assignment 9

Yenigalla Samyuktha
EE20MTECH14019

Abstract—This document checks for the vectors in the subspace spanned by given vectors.

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_9

1 PROBLEM

Let

$$\alpha_1 = \begin{pmatrix} 1 & 1 & -2 & 1 \end{pmatrix} \quad (1.0.1)$$

$$\alpha_2 = \begin{pmatrix} 3 & 0 & 4 & -1 \end{pmatrix} \quad (1.0.2)$$

$$\alpha_3 = \begin{pmatrix} -1 & 2 & 5 & 2 \end{pmatrix} \quad (1.0.3)$$

Let

$$\alpha = \begin{pmatrix} 4 & -5 & 9 & -7 \end{pmatrix} \quad (1.0.4)$$

$$\beta = \begin{pmatrix} 3 & 1 & -4 & 4 \end{pmatrix} \quad (1.0.5)$$

$$\gamma = \begin{pmatrix} -1 & 1 & 0 & 1 \end{pmatrix} \quad (1.0.6)$$

- 1) Which of the vectors α, β, γ are in the subspace of \mathbb{R}^4 spanned by α_i ?
- 2) Which of the vectors α, β, γ are in the subspace of \mathbb{C}^4 spanned by α_i ?
- 3) Does this suggest a theorem?

2 SOLUTION

- 1) The linear combination of α_i for $i = 1, 2, 3$ spans subspace S. We can write,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \text{span}(S) \quad (2.0.1)$$

where c_1, c_2, c_3 are scalars. Vectors in matrix form is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 0 & 2 \\ -2 & 4 & 5 \\ 1 & -1 & 2 \end{pmatrix} \quad (2.0.2)$$

We can observe that the columns of matrix \mathbf{A} formed by vectors α_i are independent as the rank of matrix is 3. Hence α_i forms basis for subspace S.

Checking for α : To check if a solution exists for $\mathbf{A}\mathbf{X} = \alpha$. The corresponding augmented matrix can be written as,

$$(\mathbf{A} \ \alpha) = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 1 & 0 & -2 & -5 \\ -2 & 4 & 5 & 9 \\ 1 & -1 & 2 & -7 \end{pmatrix} \quad (2.0.3)$$

On performing row-reduction on (2.0.3),

$$(\mathbf{A} \ \alpha) = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.0.4)$$

As $\text{Rank}((\mathbf{A} \ \alpha)) = \text{Rank}(\mathbf{A}) = 3$, the vector α can be represented as linear combination of α_i . From equation (2.0.4), we can write

$$-3 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 9 \\ -7 \end{pmatrix} \quad (2.0.5)$$

Hence α is in the subspace S.

Checking for β : To check if a solution exists for $\mathbf{A}\mathbf{X} = \beta$. The corresponding augmented matrix can be written as,

$$(\mathbf{A} \ \beta) = \begin{pmatrix} 1 & 3 & -1 & 3 \\ 1 & 0 & -2 & 1 \\ -2 & 4 & 5 & -4 \\ 1 & -1 & 2 & 4 \end{pmatrix} \quad (2.0.6)$$

On performing row-reduction on (2.0.6),

$$(\mathbf{A} \ \beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.0.7)$$

As $\text{Rank}(\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix})=4$ and $\text{Rank}(\mathbf{A})=3$, Solution doesn't exist for $\mathbf{A}\mathbf{X} = \beta$ and hence β is not in the subspace S.

Checking for γ : To check if a solution exists for $\mathbf{A}\mathbf{X} = \gamma$. The corresponding augmented matrix can be written as,

$$\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & -1 \\ 1 & 0 & -2 & 1 \\ -2 & 4 & 5 & 0 \\ 1 & -1 & 2 & 1 \end{pmatrix} \quad (2.0.8)$$

On performing row-reduction on (2.0.8),

$$\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.0.9)$$

As $\text{Rank}(\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix})=4$ and $\text{Rank}(\mathbf{A})=3$, Solution doesn't exist for $\mathbf{A}\mathbf{X} = \gamma$ and hence γ is not in the subspace S.

- 2) In part 1, we haven't considered the field to be either \mathbb{R} or \mathbb{C} . The above equations solved holds for field \mathbb{C} and that implies, they hold for field \mathbb{R} also. Hence α is in the subspace and β and γ are not in the subspace.
- 3) **Theorem suggested:** Let \mathbb{F}_1 and \mathbb{F}_2 are two fields where \mathbb{F}_2 is subfield of \mathbb{F}_1 . Let α_i , $i=1,2,3,\dots,n$ forms basis for subspace of \mathbb{F}_2^n and a vector $\alpha \in \mathbb{F}_2^n$. Then α is in the subspace of \mathbb{F}_2^n spanned by α_i , $i=1,2,3,\dots,n$ if only if α is in the subspace of \mathbb{F}_1^n spanned by α_i , $i=1,2,3,\dots,n$.

3 COMPLEX COORDINATES

As α_i for $i = 1, 2, 3$ which spans the subspace S. Then we can write,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \text{span}(\mathbf{S}) \quad (3.0.1)$$

Considering coordinates c_1, c_2, c_3 as scalars, let us check if vectors from \mathbb{C}^4 are in the subspace spanned by α_i 's. Let α, β, γ are vectors from \mathbb{C}^4 .

$$\alpha = (1 + 3i \quad 1 \quad -2 + 4i \quad 1 - i) \quad (3.0.2)$$

$$\beta = (i \quad 4 - i \quad -2 + 3i \quad 6) \quad (3.0.3)$$

$$\gamma = (0 \quad 0 \quad i + 1 \quad 2i) \quad (3.0.4)$$

Augmented matrix for $\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix}$ is

$$\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & 1 + 3i \\ 1 & 0 & 2 & 1 \\ -2 & 4 & 5 & -2 + 4i \\ 1 & -1 & 2 & 1 - i \end{pmatrix} \quad (3.0.5)$$

Row reducing equation (3.0.5), we get,

$$\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.0.6)$$

Hence we have the coordinates $c_1 = 1$, $c_2 = i$, $c_3 = 0$. And hence α is in the subspace S.

Similarly for β , augmented matrix $\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix}$ is,

$$\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & i \\ 1 & 0 & 2 & 4 - i \\ -2 & 4 & 5 & -2 + 3i \\ 1 & -1 & 2 & 6 \end{pmatrix} \quad (3.0.7)$$

Row reducing the equation (3.0.7), we get

$$\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.0.8)$$

Hence, solution doesn't exist for $\mathbf{A}\mathbf{X} = \beta$, that implies β is not in the subspace S.

Similarly for γ , augmented matrix $\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix}$ is,

$$\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ -2 & 4 & 5 & i + 1 \\ 1 & -1 & 2 & 2i \end{pmatrix} \quad (3.0.9)$$

Row reducing the equation (3.0.9), we get

$$\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.0.10)$$

Hence, solution doesn't exist for $\mathbf{A}\mathbf{X} = \gamma$, that implies γ is not in the subspace S.