Assignment 11

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EE20MTECH14019

Abstract—This document explains a proof on linear transformations.

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1 Problem

Let $\mathbb{W}$ be the set of all 2×2 complex Hermitian matrices, that is the sset of 2×2 complex matrices $\mathbf{A}$ ssuch that $\mathbf{A}_{ij} = \overline{\mathbf{A}_{ji}}$ (the bar denoting complex conjugation). $\mathbb{W}$ is a vector space over the field of real numbers, under the usual operations. Verify that

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} t + x & y + iz \\ y - iz & t - x \end{pmatrix} \quad (1.0.1)$$

is an isomorphism of $\mathbb{R}^4$ onto $\mathbb{W}$.

2 Solution

1) **Check for linearity:** The transformation T is given by

$$T : \mathbb{R}^4 \to \mathbb{W} \quad (2.0.1)$$

$$T\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} t + x & y + iz \\ y - iz & t - x \end{pmatrix} \quad (2.0.2)$$

Let vectors $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix} \in \mathbb{R}^4$ and scalar $\alpha \in \mathbb{R}$.

$$T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix} \right) = T\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ t_1 + t_2 \end{pmatrix} \right) \quad (2.0.3)$$

$$= \begin{pmatrix} (t_1 + t_2) + (x_1 + x_2) & (y_1 + y_2) + i(z_1 + z_2) \\ (y_1 + y_2) - i(z_1 + z_2) & (t_1 + t_2) - (x_1 + x_2) \end{pmatrix} \quad (2.0.4)$$

$$= \begin{pmatrix} t_1 + x_1 & y_1 + iz_1 \\ y_1 - iz_1 & t_1 - x_1 \end{pmatrix} + \begin{pmatrix} t_2 + x_2 & y_2 + iz_2 \\ y_2 - iz_2 & t_2 - x_2 \end{pmatrix} \quad (2.0.5)$$

$$= T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix} \right) + T\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix} \right) \quad (2.0.6)$$

$$T\left(\alpha \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix} \right) = T\left(\begin{pmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 \\ \alpha t_1 \end{pmatrix} \right) \quad (2.0.7)$$

$$= \begin{pmatrix} \alpha(t_1 + x_1) & \alpha(y_1 + iz_1) \\ \alpha(y_1 - iz_1) & \alpha(t_1 - x_1) \end{pmatrix} \quad (2.0.8)$$

$$= \alpha \begin{pmatrix} t_1 + x_1 & y_1 + iz_1 \\ y_1 - iz_1 & t_1 - x_1 \end{pmatrix} \quad (2.0.9)$$

$$= \alpha T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix} \quad (2.0.10)$$

From the equations (2.0.6) and (2.0.10), we can say that transformation T is linear.

2) **Check for invertibility of T:** Consider the following matrix representation of the transfor-

mation T,

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ -0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x + t \\ y + iz \\ y - iz \\ t - x \end{pmatrix} \qquad (2.0.11)$$

We now form a 2×2 complex Hermitian matrix from the transformed complex vector in (2.0.11) as follows.

$$\begin{pmatrix} t + x & y + iz \\ y - iz & t - x \end{pmatrix} \qquad (2.0.12)$$

Hence the transformation matrix $\mathbf{A}$ is,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ -0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.0.13)$$

Consider the agumented matrix,

$$(\mathbf{A}|\mathbf{I}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -i & 0 & 0 & 0 & 1 & 0 \\ -0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.0.14)$$

Row-reducing the equation (2.0.14), we get

$$[\mathbf{I}|\mathbf{A}^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{-i}{2} & \frac{i}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$
$$(2.0.15)$$

From the equation (2.0.15), we can say that the transformation matrix $\mathbf{A}$ is invertible. Hence we can write the inverse transformation from $\mathbb{W}$ onto $\mathbb{R}^4$ as follows,

$$\begin{pmatrix} t + x & y + iz \\ y - iz & t - x \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \qquad (2.0.16)$$

Where the matrix representation is,

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{-i}{2} & \frac{i}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x + t \\ y + iz \\ y - iz \\ t - x \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \qquad (2.0.17)$$

We know $\mathbb{R}^4$ is isomorphic to $\mathbb{W}$ if there exists a linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{W}$ that is invertible. Such a T is an isomorphism from $\mathbb{R}^4$ onto $\mathbb{W}$. Hence from equations (2.0.11) and (2.0.17), we can say that (1.0.1) is isomorphism from $\mathbb{R}^4$ onto $\mathbb{W}$.