Assignment 10

Yenigalla Samyuktha

EE20MTECH14019

Abstract—This document explains a proof on linear transformations.

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1 PROBLEM

Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, and let U be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Prove that the transformation UT is not invertible. Generalize the theorem.

2 SOLUTION

We have two transformations

$$T : \mathbb{R}^3 \to \mathbb{R}^2 \qquad (2.0.1)$$

$$U : \mathbb{R}^2 \to \mathbb{R}^3 \qquad (2.0.2)$$

Let $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^2$. Hence we can write the transformations in matrix form as,

$$T(\mathbf{v}) = \mathbf{Av} \qquad (2.0.3)$$

$$U(\mathbf{w}) = \mathbf{Bw} \qquad (2.0.4)$$

Where transformation matrix $\mathbf{A}$ has dimension of 2x3. Hence we can say,

$$Max(Rank(\mathbf{A})) = 2 \qquad (2.0.5)$$

And transformation matrix $\mathbf{B}$ has dimension of 3x2. Hence we can say,

$$Max(Rank(\mathbf{B})) = 2 \qquad (2.0.6)$$

Now we define the transformation UT as,

$$UT : \mathbb{R}^3 \to \mathbb{R}^3 \qquad (2.0.7)$$

And the transformation in matrix form where the dimension of transformation matrix $\mathbf{C}$ is 3x3 can be written as,

$$UT(\mathbf{x}) = \mathbf{Cx} \qquad (2.0.8)$$

$$U(T(\mathbf{x})) = \mathbf{B}(\mathbf{Ax}) \qquad (2.0.9)$$

$$\implies \mathbf{C} = \mathbf{BA} \qquad (2.0.10)$$

$$dim(\mathbf{C}) = 3 \times 3 \qquad (2.0.11)$$

Calculating the maximum rank of transformation matrix $\mathbf{C}$ can have, we know

$$Rank(\mathbf{C}) \leq min(Rank(\mathbf{B}), Rank(\mathbf{A})) \qquad (2.0.12)$$

From the equations, (2.0.5) and (2.0.6), we get

$$Rank(C) \leq min(2, 2) \qquad (2.0.13)$$

$$\implies Max(Rank(C)) = 2 \qquad (2.0.14)$$

From equations (2.0.11) and (2.0.14),

$$Rank(\mathbf{C}) < dim(\mathbf{C}) \qquad (2.0.15)$$

Therefore transformation UT is not invertible.

3 THEOREM

Generalizing the proof, for $n > m$ and considering vectors $\mathbf{v}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$. From the Table 0,

$$Rank(\mathbf{C}) = m \qquad (3.0.1)$$

$$dim(\mathbf{C}) = n \qquad (3.0.2)$$

$$\implies Rank(\mathbf{C}) < dim(\mathbf{C}) \qquad (3.0.3)$$

From equation (3.0.3)we can say that the transformation UT is not invertible.

4 EXAMPLE

Let the vectors $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ and $\mathbf{w} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \in \mathbb{R}^2$.

Transformation	Matrix Representation	Dimension	Max Rank of transformation matrix
$T : \mathbb{R}^n \to \mathbb{R}^m$	$T(\mathbf{v}) = \mathbf{Av}$	$\mathbf{A} : m \times n$	$Rank(\mathbf{A}) = m$
$U : \mathbb{R}^m \to \mathbb{R}^n$	$U(\mathbf{w}) = \mathbf{Bw}$	$\mathbf{B} : n \times m$	$Rank(\mathbf{B}) = m$
$UT : \mathbb{R}^n \to \mathbb{R}^n$	$UT(\mathbf{x}) = \mathbf{Cx}$	$\mathbf{C} : n \times n$	$Rank(\mathbf{C}) \leq min(Rank(\mathbf{B}), Rank(\mathbf{A}))$
			$Rank(\mathbf{C}) = m$

TABLE 0: Generalization of the proof

1) Calculating transformation matrix $\mathbf{A}$, from equation (2.0.3),

$$T(\mathbf{v}) = \mathbf{Av} \quad (4.0.1)$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (4.0.2)$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} = rref(\mathbf{A}) \quad (4.0.3)$$

$$\implies Rank(\mathbf{A}) = 2 \quad (4.0.4)$$

2) Calculating transformation matrix $\mathbf{B}$, from equation (2.0.4),

$$U(\mathbf{w}) = \mathbf{Bw} \quad (4.0.5)$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad (4.0.6)$$

$$\begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} = rref(\mathbf{B}) \quad (4.0.7)$$

$$\implies Rank(\mathbf{B}) = 2 \quad (4.0.8)$$

3) Now for the transformation UT, calculating the transformation matrix $\mathbf{C}$, from the equation (2.0.10),

$$\mathbf{C} = \mathbf{BA} \quad (4.0.9)$$

$$\mathbf{C} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{pmatrix}$$
$$(4.0.10)$$

$$\begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{pmatrix} \sim \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = rref(\mathbf{C})$$
$$(4.0.11)$$

$$\implies Rank(\mathbf{C}) = 2 \quad (4.0.12)$$

$$dim(\mathbf{C}) = 3 \quad (4.0.13)$$

As $Rank(\mathbf{C}) < dim(\mathbf{C})$, transformation UT (matrix $\mathbf{C}$) is not invertible.