Assignment 10

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***Abstract*—This document explains a proof on linear transformations.**

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_10

1 Problem

Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, and let U be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Prove that the transformation UT is not invertible. Generalize the theorem.

2 Solution

We have two transformations

$$T : \mathbb{R}^3 \to \mathbb{R}^2 \tag{2.0.1}$$
$$U : \mathbb{R}^2 \to \mathbb{R}^3 \tag{2.0.2}$$

Let $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^2$. Hence we can write the transformations in matrix form as,

$$T(\mathbf{v}) = \mathbf{Av} \tag{2.0.3}$$
$$U(\mathbf{w}) = \mathbf{Bw} \tag{2.0.4}$$

Where transformation matrix $\mathbf{A}$ has dimension of 2x3. Hence we can say,

$$Max(Rank(\mathbf{A})) = 2 \tag{2.0.5}$$

And transformation matrix $\mathbf{B}$ has dimension of 3x2. Hence we can say,

$$Max(Rank(\mathbf{B})) = 2 \tag{2.0.6}$$

Now we define the transformation UT as,

$$UT : \mathbb{R}^3 \to \mathbb{R}^3 \tag{2.0.7}$$

And the transformation in matrix form where the dimension of transformation matrix $\mathbf{C}$ is 3x3 can be written as,

$$UT(\mathbf{x}) = \mathbf{Cx} \tag{2.0.8}$$
$$U(T(\mathbf{x})) = \mathbf{B}(\mathbf{Ax}) \tag{2.0.9}$$
$$\implies \mathbf{C} = \mathbf{BA} \tag{2.0.10}$$
$$dim(\mathbf{C}) = 3 \times 3 \tag{2.0.11}$$

Calculating the maximum rank of transformation matrix $\mathbf{C}$ can have, we know

$$Rank(\mathbf{C}) \leq min(Rank(\mathbf{B}), Rank(\mathbf{A})) \tag{2.0.12}$$

From the equations, (2.0.5) and (2.0.6), we get

$$Rank(C) \leq min(2, 2) \tag{2.0.13}$$
$$\implies Max(Rank(C)) = 2 \tag{2.0.14}$$

From equations (2.0.11) and (2.0.14),

$$Rank(\mathbf{C}) < dim(\mathbf{C}) \tag{2.0.15}$$

Therefore transformation UT is not invertible.

3 Theorem

Generalizing the proof, for $n > m$ and considering vectors $\mathbf{v}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$. From the Table 0,

$$Rank(\mathbf{C}) = m \tag{3.0.1}$$
$$dim(\mathbf{C}) = n \tag{3.0.2}$$
$$\implies Rank(\mathbf{C}) < dim(\mathbf{C}) \tag{3.0.3}$$

From equation (3.0.3)we can say that the transformation UT is not invertible.

Transformation	**Matrix Representation**	**Dimension**	**Max Rank of transformation matrix**
$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$	$T(\mathbf{v}) = \mathbf{Av}$	$\mathbf{A} : m \times n$	$Rank(\mathbf{A}) = m$
$U : \mathbb{R}^m \rightarrow \mathbb{R}^n$	$U(\mathbf{w}) = \mathbf{Bw}$	$\mathbf{B} : n \times m$	$Rank(\mathbf{B}) = m$
$UT : \mathbb{R}^n \rightarrow \mathbb{R}^n$	$UT(\mathbf{x}) = \mathbf{Cx}$	$\mathbf{C} : n \times n$	$Rank(\mathbf{C}) \leq min(Rank(\mathbf{B}), Rank(A))$ $Rank(\mathbf{C}) = m$

TABLE 0: Generalization of the proof