

Assignment 10

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Abstract—This document explains a proof on linear transformations.

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_10

1 PROBLEM

Let T be a linear transformation from \mathbb{R}^3 into \mathbb{R}^2 , and let U be a linear transformation from \mathbb{R}^2 into \mathbb{R}^3 . Prove that the transformation UT is not invertible. Generalize the theorem.

2 SOLUTION

We have two transformations

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad (2.0.1)$$

$$U : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad (2.0.2)$$

Let $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^2$. Hence we can write the transformations in matrix form as,

$$T(\mathbf{v}) = \mathbf{A}\mathbf{v} \quad (2.0.3)$$

$$U(\mathbf{w}) = \mathbf{B}\mathbf{w} \quad (2.0.4)$$

Where transformation matrix \mathbf{A} has dimension of 2×3 . Hence we can say,

$$\text{Max}(\text{Rank}(\mathbf{A})) = 2 \quad (2.0.5)$$

And transformation matrix \mathbf{B} has dimension of 3×2 . Hence we can say,

$$\text{Max}(\text{Rank}(\mathbf{B})) = 2 \quad (2.0.6)$$

Now we define the transformation UT as,

$$UT : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad (2.0.7)$$

And the transformation in matrix form where the dimension of transformation matrix \mathbf{C} is 3×3 can be written as,

$$UT(\mathbf{x}) = \mathbf{C}\mathbf{x} \quad (2.0.8)$$

$$U(T(\mathbf{x})) = \mathbf{B}(\mathbf{A}\mathbf{x}) \quad (2.0.9)$$

$$\implies \mathbf{C} = \mathbf{B}\mathbf{A} \quad (2.0.10)$$

$$\dim(\mathbf{C}) = 3 \times 3 \quad (2.0.11)$$

Calculating the maximum rank of transformation matrix \mathbf{C} can have, we know

$$\text{Rank}(\mathbf{C}) \leq \min(\text{Rank}(\mathbf{B}), \text{Rank}(\mathbf{A})) \quad (2.0.12)$$

From the equations, (2.0.5) and (2.0.6), we get

$$\text{Rank}(\mathbf{C}) \leq \min(2, 2) \quad (2.0.13)$$

$$\implies \text{Max}(\text{Rank}(\mathbf{C})) = 2 \quad (2.0.14)$$

From equations (2.0.11) and (2.0.14),

$$\text{Rank}(\mathbf{C}) < \dim(\mathbf{C}) \quad (2.0.15)$$

Therefore transformation UT is not invertible.

3 THEOREM

Generalizing the proof, for $n > m$ and considering vectors $\mathbf{v}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$. From the Table 0,

$$\text{Rank}(\mathbf{C}) = m \quad (3.0.1)$$

$$\dim(\mathbf{C}) = n \quad (3.0.2)$$

$$\implies \text{Rank}(\mathbf{C}) < \dim(\mathbf{C}) \quad (3.0.3)$$

From equation (3.0.3) we can say that the transformation UT is not invertible.

4 EXAMPLE

Let the vectors $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ and $\mathbf{w} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \in \mathbb{R}^2$.

| Transformation | Matrix Representation | Dimension | Max Rank of transformation matrix |
|--|---|---------------------------|--|
| $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ | $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$ | $\mathbf{A} : m \times n$ | $Rank(\mathbf{A}) = m$ |
| $U : \mathbb{R}^m \rightarrow \mathbb{R}^n$ | $U(\mathbf{w}) = \mathbf{B}\mathbf{w}$ | $\mathbf{B} : n \times m$ | $Rank(\mathbf{B}) = m$ |
| $UT : \mathbb{R}^n \rightarrow \mathbb{R}^n$ | $UT(\mathbf{x}) = \mathbf{C}\mathbf{x}$ | $\mathbf{C} : n \times n$ | $Rank(\mathbf{C}) \leq \min(Rank(\mathbf{B}), Rank(\mathbf{A}))$ $Rank(\mathbf{C}) = m$ |

TABLE 0: Generalization of the proof

- 1) Calculating transformation matrix \mathbf{A} , from equation (2.0.3),

$$T(\mathbf{v}) = \mathbf{A}\mathbf{v} \quad (4.0.1)$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (4.0.2)$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = rref(\mathbf{A}) \quad (4.0.3)$$

$$\implies Rank(\mathbf{A}) = 2 \quad (4.0.4)$$

- 2) Calculating transformation matrix \mathbf{B} , from equation (2.0.4),

$$U(\mathbf{w}) = \mathbf{B}\mathbf{w} \quad (4.0.5)$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad (4.0.6)$$

$$\begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = rref(\mathbf{B}) \quad (4.0.7)$$

$$\implies Rank(\mathbf{B}) = 2 \quad (4.0.8)$$

- 3) Now for the transformation UT, calculating the transformation matrix \mathbf{C} , from the equation (2.0.10),

$$\mathbf{C} = \mathbf{B}\mathbf{A} \quad (4.0.9)$$

$$\mathbf{C} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \quad (4.0.10)$$

$$\begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = rref(\mathbf{C}) \quad (4.0.11)$$

$$\implies Rank(\mathbf{C}) = 2 \quad (4.0.12)$$

$$dim(\mathbf{C}) = 3 \quad (4.0.13)$$

As $Rank(\mathbf{C}) < dim(\mathbf{C})$, transformation UT is not invertible.