# Assignment 11

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*Abstract*—**This document explains a proof on linear transformations.**

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_11

## 1 Problem

Let $\mathbb{W}$ be the set of all $2 \times 2$ complex Hermitian matrices, that is the sset of $2 \times 2$ complex matrices $\mathbf{A}$ ssuch that $\mathbf{A}_{ij} = \overline{\mathbf{A_{ji}}}$ (the bar denoting complex conjugation). $\mathbb{W}$ is a vector space over the field of real numbers, under the usual operations. Verify that

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \tag{1.0.1}$$

is an isomorphism of $\mathbb{R}^4$ onto $\mathbb{W}$.

## 2 Solution

1) **Check for linearity:** The transformation T is given by

$$T : \mathbb{R}^4 \to \mathbb{W} \tag{2.0.1}$$

$$T\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \tag{2.0.2}$$

Let vectors $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix} \in \mathbb{R}^4$ and scalar $\alpha \in \mathbb{R}$.

$$T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \\ t_1+t_2 \end{pmatrix}\right) \tag{2.0.3}$$

$$= \begin{pmatrix} (t_1+t_2)+(x_1+x_2) & (y_1+y_2)+i(z_1+z_2) \\ (y_1+y_2)-i(z_1+z_2) & (t_1+t_2)-(x_1+x_2) \end{pmatrix} \tag{2.0.4}$$

$$= \begin{pmatrix} t_1+x_1 & y_1+iz_1 \\ y_1-iz_1 & t_1-x_1 \end{pmatrix} + \begin{pmatrix} t_2+x_2 & y_2+iz_2 \\ y_2-iz_2 & t_2-x_2 \end{pmatrix} \tag{2.0.5}$$

$$= T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}\right) + T\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix}\right) \tag{2.0.6}$$

$$T\left(\alpha\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}\right) = T\left(\begin{pmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 \\ \alpha t_1 \end{pmatrix}\right) \tag{2.0.7}$$

$$= \begin{pmatrix} \alpha(t_1+x_1) & \alpha(y_1+iz_1) \\ \alpha(y_1-iz_1) & \alpha(t_1-x_1) \end{pmatrix} \tag{2.0.8}$$

$$= \alpha\begin{pmatrix} t_1+x_1 & y_1+iz_1 \\ y_1-iz_1 & t_1-x_1 \end{pmatrix} \tag{2.0.9}$$

$$= \alpha T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix} \tag{2.0.10}$$

From the equations (2.0.6) and (2.0.10), we can say that transformation T is linear.

2) **Check for invertibility of T:** Consider the following matrix representation of the transfor-

mation T,

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} t+x \\ y+iz \\ y-iz \\ t-x \end{pmatrix} \tag{2.0.11}$$

We now form a $2 \times 2$ complex Hermitian matrix from the transformed complex vector in (2.0.11) as follows.

$$\begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \tag{2.0.12}$$

Hence the transformation matrix $\mathbf{A}$ is,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.13}$$

Consider the agumented matrix,

$$(\mathbf{A}|\mathbf{I}) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -i & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{2.0.14}$$

Row-reducing the equation (2.0.14), we get

$$[\mathbf{I}|\mathbf{A}^{-1}] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{-i}{2} & \frac{i}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right] \tag{2.0.15}$$

From the equation (2.0.15), we can say that the transformation matrix $\mathbf{A}$ is invertible. Hence we can write the inverse transformation from $\mathbb{W}$ onto $\mathbb{R}^4$ as follows,

$$\begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \tag{2.0.16}$$

Where the matrix representation is,

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{-i}{2} & \frac{i}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x+t \\ y+iz \\ y-iz \\ t-x \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \tag{2.0.17}$$

We know $\mathbb{R}^4$ is isomorphic to $\mathbb{W}$ if there exists a linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{W}$ that is invertible. Such a T is an isomorphism from $\mathbb{R}^4$ onto $\mathbb{W}$. Hence from equations (2.0.11) and (2.0.17), we can say that (1.0.1) is isomorphism from $\mathbb{R}^4$ onto $\mathbb{W}$.