Assignment 9

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***Abstract*—This document checks for the vectors in the subspace spanned by given vectors.**

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_9

1 Problem

Let

$$\alpha_1 = \begin{pmatrix} 1 & 1 & -2 & 1 \end{pmatrix}^T \tag{1.0.1}$$

$$\alpha_2 = \begin{pmatrix} 3 & 0 & 4 & -1 \end{pmatrix}^T \tag{1.0.2}$$

$$\alpha_3 = \begin{pmatrix} -1 & 2 & 5 & 2 \end{pmatrix}^T \tag{1.0.3}$$

Let

$$\alpha = \begin{pmatrix} 4 & -5 & 9 & -7 \end{pmatrix}^T \tag{1.0.4}$$

$$\beta = \begin{pmatrix} 3 & 1 & -4 & 4 \end{pmatrix}^T \tag{1.0.5}$$

$$\gamma = \begin{pmatrix} -1 & 1 & 0 & 1 \end{pmatrix}^T \tag{1.0.6}$$

1) Which of the vectors α, β, γ are in the subspace of $\mathbb{R}^4$ spanned by α_i?
2) Which of the vectors α, β, γ are in the subspace of $\mathbb{C}^4$ spanned by α_i?
3) Does this suggest a theorem?

2 Solution

1) The linear combination of α_i for $i = 1, 2, 3$ spans subspace S. We can write,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \text{span(S)} \tag{2.0.1}$$

where c_1,c_2,c_3 are scalars. Vectors in matrix form is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 0 & 2 \\ -2 & 4 & 5 \\ 1 & -1 & 2 \end{pmatrix} \tag{2.0.2}$$

We can observe that the columns of matrix $\mathbf{A}$ formed by vectors α_i are independent as the rank of matrix is 3. Hence α_i forms basis for subspace S.

Checking for α: To check if a solution exists for $\mathbf{AX} = \alpha$. The corresponding agumented matrix can be written as,

$$\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 1 & 0 & -2 & -5 \\ -2 & 4 & 5 & 9 \\ 1 & -1 & 2 & -7 \end{pmatrix} \tag{2.0.3}$$

On performing row-reduction on (2.0.3),

$$\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.4}$$

As Rank($\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix}$)=Rank($\mathbf{A}$)=3, the vector α can be represented as linear combination of α_i. From equation (2.0.4), we can write

$$-3 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 9 \\ -7 \end{pmatrix} \tag{2.0.5}$$

Hence α is in the subspace S.

Checking for β: To check if a solution exists for $\mathbf{AX} = \beta$. The corresponding agumented matrix can be written as,

$$\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & 3 \\ 1 & 0 & -2 & 1 \\ -2 & 4 & 5 & -4 \\ 1 & -1 & 2 & 4 \end{pmatrix} \tag{2.0.6}$$

On performing row-reduction on (2.0.6),

$$\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.7}$$

As Rank($\begin{pmatrix}\mathbf{A} & \beta\end{pmatrix}$)=4 and Rank($\mathbf{A}$)=3, Solution doesn't exist for $AX = \beta$ and hence β is not in the subspace S.

Checking for γ: To check if a solution exists for $\mathbf{AX} = \gamma$. The corresponding agumented matrix can be written as,

$$\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & -1 \\ 1 & 0 & -2 & 1 \\ -2 & 4 & 5 & 0 \\ 1 & -1 & 2 & 1 \end{pmatrix} \tag{2.0.8}$$

On performing row-reduction on (2.0.8),

$$\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.9}$$

As Rank($\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix}$)=4 and Rank($\mathbf{A}$)=3, Solution doesn't exist for $AX = \gamma$ and hence γ is not in the subspace S.

2) In part 1, we haven't considered the field to be either $\mathbb{R}$ or $\mathbb{C}$. The above equations solved holds for field $\mathbb{C}$ and that implies, they hold for field $\mathbb{R}$ also. Hence α is in the subspace and β and γ are not in the subspace.
3) **Theorem suggested:** Let $\mathbb{F}_1$ and $\mathbb{F}_2$ are two fields where $\mathbb{F}_2$ is subfield of $\mathbb{F}_1$. Let α_i, i=1,2,3...,n forms basis for subspace of $\mathbb{F}_2^n$ and a vector $\alpha \in \mathbb{F}_2^n$. Then α is in the subspace of $\mathbb{F}_2^n$ spanned by α_i, i=1,2,3...,n if only if α is in the subspace of $\mathbb{F}_1^n$ spanned by α_i, i=1,2,3...,n.

3 Complex Coordinates

As α_i for $i = 1, 2, 3$ which spans the subspace S. Then we can write,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \text{span(S)} \tag{3.0.1}$$

Considering coordinates c_1,c_2,c_3 as scalars, let us check if vectors from $\mathbb{C}^4$ are in the subspace spanned by α_i's. Let α, β, γ are vectors from $\mathbb{C}^4$.

$$\alpha = \begin{pmatrix} 1 + 3i & 1 & -2 + 4i & 1 - i \end{pmatrix}^T \tag{3.0.2}$$

$$\beta = \begin{pmatrix} i & 4 - i & -2 + 3i & 6 \end{pmatrix}^T \tag{3.0.3}$$

$$\gamma = \begin{pmatrix} 0 & 0 & i + 1 & 2i \end{pmatrix}^T \tag{3.0.4}$$

Agumented matrix for $\begin{pmatrix}\mathbf{A} & \alpha\end{pmatrix}$ is

$$\begin{pmatrix}\mathbf{A} & \alpha\end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & 1 + 3i \\ 1 & 0 & 2 & 1 \\ -2 & 4 & 5 & -2 + 4i \\ 1 & -1 & 2 & 1 - i \end{pmatrix} \tag{3.0.5}$$

Row reducting equation (3.0.5), we get,

$$\begin{pmatrix}\mathbf{A} & \alpha\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.6}$$

Hence we have the coordinates $c_1 = 1$, $c_2 = i$, $c_3 = 0$. And hence α is in the subspace S.
Similarly for β, augumented matrix $\begin{pmatrix}\mathbf{A} & \beta\end{pmatrix}$ is,

$$\begin{pmatrix}\mathbf{A} & \beta\end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & i \\ 1 & 0 & 2 & 4 - i \\ -2 & 4 & 5 & -2 + 3i \\ 1 & -1 & 2 & 6 \end{pmatrix} \tag{3.0.7}$$

Row reducing the equation (3.0.7), we get

$$\begin{pmatrix}\mathbf{A} & \beta\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.8}$$

Hence, solution doesn't exist for $\mathbf{AX} = \beta$, that implies β is not in the subspace S.
Similarly for γ, augumented matrix $\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix}$ is,

$$\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ -2 & 4 & 5 & i + 1 \\ 1 & -1 & 2 & 2i \end{pmatrix} \tag{3.0.9}$$

Row reducing the equation (3.0.9), we get

$$\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.10}$$

Hence, solution doesn't exist for $\mathbf{AX} = \gamma$, that implies γ is not in the subspace S.