Assignment 10

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***Abstract*—This document explains a proof on linear transformations.**

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_10

1 Problem

Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, and let U be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Prove that the transformation UT is not invertible. Generalize the theorem.

2 Proof

Let $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^2$. Table 1 shows that maximum rank the transformation matrix $\mathbf{C}$ can have is 2.

$$Rank(\mathbf{C}) = 2 \tag{2.0.1}$$
$$dim(\mathbf{C}) = 3 \tag{2.0.2}$$
$$\implies Rank(\mathbf{C}) < dim(\mathbf{C}) \tag{2.0.3}$$

Therefore from the equation (2.0.3), we can say transformation UT is not invertible.

3 Theorem

Generalizing the proof, for $n > m$ and considering vectors $\mathbf{v}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$. From the Table 2,

$$Rank(\mathbf{C}) = m \tag{3.0.1}$$
$$dim(\mathbf{C}) = n \tag{3.0.2}$$
$$\implies Rank(\mathbf{C}) < dim(\mathbf{C}) \tag{3.0.3}$$

From equation (3.0.3)we can say that the transformation UT is not invertible.

4 Example

Let the vectors $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ and $\mathbf{w} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \in \mathbb{R}^2$.

1) Calculating transformation matrix $\mathbf{A}$,

$$T(\mathbf{v}) = \mathbf{A}\mathbf{v} \tag{4.0.1}$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \tag{4.0.2}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = rref(\mathbf{A}) \tag{4.0.3}$$

$$\implies Rank(\mathbf{A}) = 2 \tag{4.0.4}$$

2) Calculating transformation matrix $\mathbf{B}$,

$$U(\mathbf{w}) = \mathbf{B}\mathbf{w} \tag{4.0.5}$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{4.0.6}$$

$$\begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = rref(\mathbf{B}) \tag{4.0.7}$$

$$\implies Rank(\mathbf{B}) = 2 \tag{4.0.8}$$

3) Now for the transformation UT, calculating the transformation matrix $\mathbf{C}$,

$$UT : \mathbb{R}^3 \to \mathbb{R}^3 \tag{4.0.9}$$
$$\implies UT(\mathbf{x}) = \mathbf{C}\mathbf{x} \tag{4.0.10}$$
$$U(T(\mathbf{x})) = \mathbf{B}(\mathbf{A}\mathbf{x}) \tag{4.0.11}$$
$$\implies \mathbf{C} = \mathbf{B}\mathbf{A} \tag{4.0.12}$$

$$\mathbf{C} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{pmatrix} \tag{4.0.13}$$

$$\begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = rref(\mathbf{C}) \tag{4.0.14}$$

Transformation	Matrix Representation	Dimension	Max Rank of transformation matrix
$T:\mathbb{R}^3 \to \mathbb{R}^2$	$T(\mathbf{v}) = \mathbf{Av}$	$\mathbf{A}: 2\times 3$	$Rank(\mathbf{A}) = 2$
$U:\mathbb{R}^2 \to \mathbb{R}^3$	$U(\mathbf{w}) = \mathbf{Bw}$	$\mathbf{B}: 3\times 2$	$Rank(\mathbf{B}) = 2$
$UT:\mathbb{R}^3 \to \mathbb{R}^3$	$UT(\mathbf{x}) = \mathbf{Cx}$ $U(T(\mathbf{x})) = \mathbf{B}(\mathbf{Ax})$ $\mathbf{C} = \mathbf{AB}$	$\mathbf{C}: 3\times 3$	$Rank(\mathbf{C}) \le min(Rank(\mathbf{B}), Rank(\mathbf{A}))$ $Rank(\mathbf{C}) = 2$

TABLE 1: Proof for non-invertibility of the transformation UT where $T:\mathbb{R}^3 \to \mathbb{R}^2$ and $U:\mathbb{R}^2 \to \mathbb{R}^3$

Transformation	Matrix Representation	Dimension	Max Rank of transformation matrix
$T:\mathbb{R}^n \to \mathbb{R}^m$	$T(\mathbf{v}) = \mathbf{Av}$	$\mathbf{A}: m\times n$	$Rank(\mathbf{A}) = m$
$U:\mathbb{R}^m \to \mathbb{R}^n$	$U(\mathbf{w}) = \mathbf{Bw}$	$\mathbf{B}: n\times m$	$Rank(\mathbf{B}) = m$
$UT:\mathbb{R}^n \to \mathbb{R}^n$	$UT(\mathbf{x}) = \mathbf{Cx}$ $U(T(\mathbf{x})) = \mathbf{B}(\mathbf{Ax})$ $\mathbf{C} = \mathbf{AB}$	$\mathbf{C}: n\times n$	$Rank(\mathbf{C}) \le min(Rank(\mathbf{B}), Rank(\mathbf{A}))$ $Rank(\mathbf{C}) = m$

TABLE 2: Generalization of the proof

$$\implies Rank(\mathbf{C}) = 2 \tag{4.0.15}$$

$$dim(\mathbf{C}) = 3 \tag{4.0.16}$$

As $Rank(\mathbf{C}) < dim(\mathbf{C})$, transformation UT is not invertible.