

# Assignment 9

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**Abstract**—This document checks for the vectors in the subspace spanned by given vectors.

Download all latex-tikz codes from

[https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment\\_9](https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_9)

## 1 PROBLEM

Let

$$\alpha_1 = (1 \ 1 \ -2 \ 1)^T \quad (1.0.1)$$

$$\alpha_2 = (3 \ 0 \ 4 \ -1)^T \quad (1.0.2)$$

$$\alpha_3 = (-1 \ 2 \ 5 \ 2)^T \quad (1.0.3)$$

Let

$$\alpha = (4 \ -5 \ 9 \ -7)^T \quad (1.0.4)$$

$$\beta = (3 \ 1 \ -4 \ 4)^T \quad (1.0.5)$$

$$\gamma = (-1 \ 1 \ 0 \ 1)^T \quad (1.0.6)$$

- 1) Which of the vectors  $\alpha, \beta, \gamma$  are in the subspace of  $\mathbb{R}^4$  spanned by  $\alpha_i$ ?
- 2) Which of the vectors  $\alpha, \beta, \gamma$  are in the subspace of  $\mathbb{C}^4$  spanned by  $\alpha_i$ ?
- 3) Does this suggest a theorem?

## 2 SOLUTION

- 1) The linear combination of  $\alpha_i$  for  $i = 1, 2, 3$  spans subspace S. We can write,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \text{span}(S) \quad (2.0.1)$$

where  $c_1, c_2, c_3$  are scalars. Vectors in matrix form is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 0 & 2 \\ -2 & 4 & 5 \\ 1 & -1 & 2 \end{pmatrix} \quad (2.0.2)$$

We can observe that the columns of matrix  $\mathbf{A}$  formed by vectors  $\alpha_i$  are independent as the rank of matrix is 3. Hence  $\alpha_i$  forms basis for subspace S.

**Checking for  $\alpha$ :** To check if a solution exists for  $\mathbf{A}\mathbf{X} = \alpha$ . The corresponding augmented matrix can be written as,

$$(\mathbf{A} \ \alpha) = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 1 & 0 & -2 & -5 \\ -2 & 4 & 5 & 9 \\ 1 & -1 & 2 & -7 \end{pmatrix} \quad (2.0.3)$$

On performing row-reduction on (2.0.3),

$$(\mathbf{A} \ \alpha) = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.0.4)$$

As  $\text{Rank}((\mathbf{A} \ \alpha)) = \text{Rank}(\mathbf{A}) = 3$ , the vector  $\alpha$  can be represented as linear combination of  $\alpha_i$ . From equation (2.0.4), we can write

$$-3 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 9 \\ -7 \end{pmatrix} \quad (2.0.5)$$

Hence  $\alpha$  is in the subspace S.

**Checking for  $\beta$ :** To check if a solution exists for  $\mathbf{A}\mathbf{X} = \beta$ . The corresponding augmented matrix can be written as,

$$(\mathbf{A} \ \beta) = \begin{pmatrix} 1 & 3 & -1 & 3 \\ 1 & 0 & 2 & 1 \\ -2 & 4 & 5 & -4 \\ 1 & -1 & 2 & 4 \end{pmatrix} \quad (2.0.6)$$

On performing row-reduction on (2.0.6),

$$(\mathbf{A} \ \beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.0.7)$$

As  $\text{Rank}(\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix})=4$  and  $\text{Rank}(\mathbf{A})=3$ , Solution doesn't exist for  $\mathbf{A}\mathbf{X} = \beta$  and hence  $\beta$  is not in the subspace S.

**Checking for  $\gamma$ :** To check if a solution exists for  $\mathbf{A}\mathbf{X} = \gamma$ . The corresponding augmented matrix can be written as,

$$\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & -1 \\ 1 & 0 & 2 & 1 \\ -2 & 4 & 5 & 0 \\ 1 & -1 & 2 & 1 \end{pmatrix} \quad (2.0.8)$$

On performing row-reduction on (2.0.8),

$$\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.0.9)$$

As  $\text{Rank}(\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix})=4$  and  $\text{Rank}(\mathbf{A})=3$ , Solution doesn't exist for  $\mathbf{A}\mathbf{X} = \gamma$  and hence  $\gamma$  is not in the subspace S.

- 2) In part 1, we haven't considered the field to be either  $\mathbb{R}$  or  $\mathbb{C}$ . The above equations solved holds for field  $\mathbb{C}$  and that implies, they hold for field  $\mathbb{R}$  also. Hence  $\alpha$  is in the subspace and  $\beta$  and  $\gamma$  are not in the subspace.
- 3) **Theorem suggested:** Let  $\mathbb{F}_1$  and  $\mathbb{F}_2$  are two fields where  $\mathbb{F}_2$  is subfield of  $\mathbb{F}_1$ . Let  $\alpha_i$ ,  $i=1,2,3,\dots,n$  forms basis for subspace of  $\mathbb{F}_2^n$  and a vector  $\alpha \in \mathbb{F}_2^n$ . Then  $\alpha$  is in the subspace of  $\mathbb{F}_2^n$  spanned by  $\alpha_i$ ,  $i=1,2,3,\dots,n$  if only if  $\alpha$  is in the subspace of  $\mathbb{F}_1^n$  spanned by  $\alpha_i$ ,  $i=1,2,3,\dots,n$ .

### 3 COMPLEX COORDINATES

As  $\alpha_i$  for  $i = 1, 2, 3$  which spans the subspace S. Then we can write,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \text{span}(\mathbf{S}) \quad (3.0.1)$$

Considering coordinates  $c_1, c_2, c_3$  as scalars, let us check if vectors from  $\mathbb{C}^4$  are in the subspace spanned by  $\alpha_i$ 's. Let  $\alpha, \beta, \gamma$  are vectors from  $\mathbb{C}^4$ .

$$\alpha = \begin{pmatrix} 1+3i \\ 1 \\ -2+4i \\ 1-i \end{pmatrix} \quad \beta = \begin{pmatrix} i \\ 4-i \\ -2+3i \\ 6 \end{pmatrix} \quad \gamma = \begin{pmatrix} 0 \\ 0 \\ i+1 \\ 2i \end{pmatrix} \quad (3.0.2)$$

Augmented matrix for  $\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix}$  is

$$\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & 1+3i \\ 1 & 0 & 2 & 1 \\ -2 & 4 & 5 & -2+4i \\ 1 & -1 & 2 & 1-i \end{pmatrix} \quad (3.0.3)$$

Row reducing equation (3.0.3), we get,

$$\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.0.4)$$

Hence we have the coordinates  $c_1 = 1$ ,  $c_2 = i$ ,  $c_3 = 0$ . And hence  $\alpha$  is in the subspace S and can be represented in terms of basis vectors as,

$$1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+3i \\ 1 \\ -2+4i \\ 1-i \end{pmatrix} \quad (3.0.5)$$

Similarly for  $\beta$ , augmented matrix  $\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix}$  is,

$$\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & i \\ 1 & 0 & 2 & 4-i \\ -2 & 4 & 5 & -2+3i \\ 1 & -1 & 2 & 6 \end{pmatrix} \quad (3.0.6)$$

Row reducing the equation (3.0.6), we get

$$\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.0.7)$$

Hence, solution doesn't exist for  $\mathbf{A}\mathbf{X} = \beta$ , that implies  $\beta$  is not in the subspace S.

Similarly for  $\gamma$ , augmented matrix  $\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix}$  is,

$$\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ -2 & 4 & 5 & i+1 \\ 1 & -1 & 2 & 2i \end{pmatrix} \quad (3.0.8)$$

Row reducing the equation (3.0.8), we get

$$\begin{pmatrix} \mathbf{A} & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.0.9)$$

Hence, solution doesn't exist for  $\mathbf{A}\mathbf{X} = \gamma$ , that implies  $\gamma$  is not in the subspace S.