Assignment 13

Yenigalla Samyuktha
EE20MTECH14019

***Abstract*—This document explains a proof in linear transformations.**

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_13

1 Problem

Let $\mathbf{A}$ be an $m \times n$ matrix with real entries. Prove that $\mathbf{A} = \mathbf{0}$ is and only if $Trace\left(\mathbf{A}^T\mathbf{A}\right) = 0$.

2 Solution

1) Given $Trace\left(\mathbf{A}^T\mathbf{A}\right) = 0$, to prove $\mathbf{A} = \mathbf{0}$: Using Singular value decomposition of any $m \times n$ matrix with real entries, we can write

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \tag{2.0.1}$$

Now,

$$Trace\left(\mathbf{A}^T\mathbf{A}\right) \tag{2.0.2}$$
$$= Trace\left(\left(\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\right)^T\left(\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\right)\right) \tag{2.0.3}$$
$$= Trace\left(\mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\right) \tag{2.0.4}$$

As $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, $\mathbf{V}^T\mathbf{V} = \mathbf{I}$, $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{\Sigma}$ is a diagonal matrix, we can re-write equation (2.0.4) as,

$$Trace\left(\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T\right) \tag{2.0.5}$$

From the Cyclic property of trace of product of matrices, (2.0.5) becomes

$$Trace\left(\mathbf{V}^T\mathbf{V}\mathbf{\Sigma}^2\right) \tag{2.0.6}$$
$$= \sum_{i=1}^{r}\sigma_i^2 \tag{2.0.7}$$

where σ_i's are singular values of matrix $\mathbf{A}$ and r is the rank of matrix $\mathbf{A}$. Considering $trace\left(\mathbf{A}^T\mathbf{A}\right) = 0$, from (2.0.7)

$$\sum_{i=1}^{r}\sigma_i^2 = 0 \tag{2.0.8}$$
$$\implies \sigma_i^2 = 0 \;\forall i = 1, 2, \ldots r \tag{2.0.9}$$

As $\mathbf{A}$ has real entries, from (2.0.9)

$$\implies \sigma_i = 0 \;\forall i = 1, 2, \ldots r \tag{2.0.10}$$

$$\because Rank\left(\mathbf{A}\right) = \#\text{non-zero singular values} \tag{2.0.11}$$
$$= \#\text{non-zero diagonal elements of } \mathbf{\Sigma} \tag{2.0.12}$$

From (2.0.10) and (2.0.11),

$$Rank\left(\mathbf{A}\right) = 0 \tag{2.0.13}$$
$$\implies \mathbf{A} = \mathbf{0} \tag{2.0.14}$$

2) Given $\mathbf{A} = \mathbf{0}$, to prove $Trace\left(\mathbf{A}^T\mathbf{A}\right) = 0$:

$$\mathbf{A} = 0 \tag{2.0.15}$$
$$\implies \mathbf{A}^T\mathbf{A} = \mathbf{0} \tag{2.0.16}$$
$$\implies Trace\left(\mathbf{A}^T\mathbf{A}\right) = 0 \tag{2.0.17}$$