ASSIGNMENT 3

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***Abstract*—This document proves that triangles on the same base and having equal areas lie between the same parallel lines.**

Download latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_3

1 Problem

Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.

2 Solution

Consider $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are three points of a triangle $\triangle ABC$, $\mathbf{D}, \mathbf{B}, \mathbf{C}$ are three points of another triangle $\triangle DBC$, both triangles having same base BC and $\mathbf{B}$ at origin, then

$$\mathbf{A} - \mathbf{B} = \mathbf{A} \tag{2.0.1}$$

$$\mathbf{C} - \mathbf{B} = \mathbf{C} \tag{2.0.2}$$

$$\mathbf{D} - \mathbf{B} = \mathbf{D} \tag{2.0.3}$$

The area of the triangle $\triangle ABC$ is,

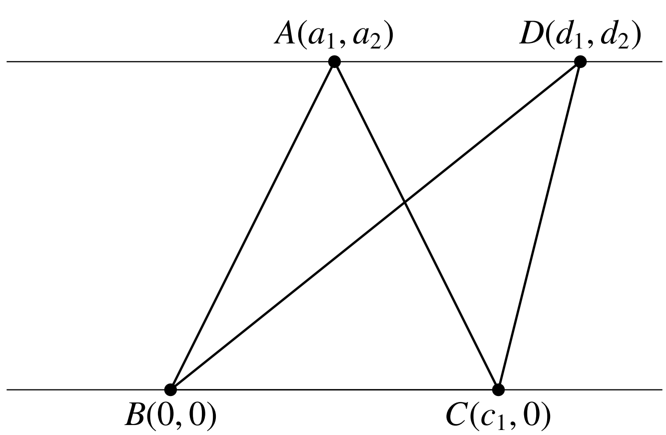

Fig. 1: $\triangle ABC$ and $\triangle DBC$ with BC as common base

$$Area\left(\triangle ABC\right) = \frac{1}{2}\left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{C} - \mathbf{B})\right\| \tag{2.0.4}$$

Substituting (2.0.1) and (2.0.2) in (2.0.4),

$$\implies Area\left(\triangle ABC\right) = \frac{1}{2}\left\|(\mathbf{A} \times \mathbf{C})\right\| \tag{2.0.5}$$

The area of the triangle $\triangle DBC$ is,

$$Area\left(\triangle ABC\right) = \frac{1}{2}\left\|(\mathbf{D} - \mathbf{B}) \times (\mathbf{C} - \mathbf{B})\right\| \tag{2.0.6}$$

Substituting (2.0.2) and (2.0.3) in (2.0.6),

$$\implies Area\left(\triangle DBC\right) = \frac{1}{2}\left\|(\mathbf{D} \times \mathbf{C})\right\| \tag{2.0.7}$$

Given the area of $\triangle ABC$ is equal to the area $\triangle DBC$, from equations (2.0.5) and (2.0.7)

$$\frac{1}{2}\left\|(\mathbf{A} \times \mathbf{C})\right\| = \frac{1}{2}\left\|(\mathbf{D} \times \mathbf{C})\right\| \tag{2.0.8}$$

Squaring on both sides of the equation (2.0.8), we get

$$\left\|(\mathbf{A} \times \mathbf{C})\right\|^2 = \left\|(\mathbf{D} \times \mathbf{C})\right\|^2 \tag{2.0.9}$$

$$\implies (\mathbf{A} \times \mathbf{C})^T (\mathbf{A} \times \mathbf{C}) = (\mathbf{D} \times \mathbf{C})^T (\mathbf{D} \times \mathbf{C}) \tag{2.0.10}$$

$$\begin{aligned} \implies \left(\mathbf{A}^T\mathbf{A}\right)\left(\mathbf{C}^T\mathbf{C}\right) - \left(\mathbf{A}^T\mathbf{C}\right)\left(\mathbf{C}^T\mathbf{A}\right) = \\ \left(\mathbf{D}^T\mathbf{D}\right)\left(\mathbf{C}^T\mathbf{C}\right) - \left(\mathbf{D}^T\mathbf{C}\right)\left(\mathbf{C}^T\mathbf{D}\right) \end{aligned} \tag{2.0.11}$$

$$\implies \|\mathbf{A}\|^2 \|\mathbf{C}\|^2 - \left(\mathbf{A}^T\mathbf{C}\right)^2 = \|\mathbf{D}\|^2 \left\|\mathbf{C}^2\right\| - \left(\mathbf{D}^T\mathbf{C}\right)^2 \tag{2.0.12}$$

Let $\mathbf{A} = \begin{pmatrix} a1 \\ a2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ .Then equation (2.0.12) can be written as,

$$\left(a_1^2 + a_2^2\right)\left(c_1^2\right) - (a_1c_1)^2 = \left(d_1^2 + d_2^2\right)\left(c_1^2\right) - (d_1c_1)^2 \tag{2.0.13}$$

$$\implies a_2 = d_2 \tag{2.0.14}$$

Now we have,

$$(\mathbf{D}-\mathbf{A}) = \begin{pmatrix} d_1 - a_1 \\ d_2 - a_2 \end{pmatrix} = \begin{pmatrix} d_1 - a_1 \\ 0 \end{pmatrix} \tag{2.0.15}$$

$$(\mathbf{C}-\mathbf{B}) = \begin{pmatrix} c_1 - 0 \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \tag{2.0.16}$$

From equations (2.0.15) and (2.0.16), we can say

$$(\mathbf{D}-\mathbf{A}) = \frac{d_1 - a_1}{c_1}(\mathbf{C}-\mathbf{B}) \tag{2.0.17}$$

$$\implies (\mathbf{D}-\mathbf{A}) = k(\mathbf{C}-\mathbf{B}) \tag{2.0.18}$$

where k is a constant. From the equation (2.0.18), we can say that $AD \parallel BC$. Hence the two triangles $\triangle ABC$ and $\triangle DBC$ lie between the same parallels AD and BC