

Assignment 14

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Abstract—This document explains a linear transformation on polynomials.

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_14

1 PROBLEM

Let \mathbb{F} be a subfield of complex numbers and let T be the transformation on $\mathbb{F}(x)$ defined by

$$T\left(\sum_{i=0}^n c_i x^i\right) = \sum_{i=0}^n \frac{c_i}{i+1} x^{i+1} \quad (1.0.1)$$

Show that T is a non-singular linear operator on $\mathbb{F}[x]$. Also show that T is not invertible.

2 SOLUTION

The transformation T does integral of a polynomial. Table 2 provides proof that the transformation T is a linear operator and non-singular. Table 3 provides proof that T is not invertible, however there exists a left inverse. The parameters used in the proof are listed in the table 1.

PARAMETER	DESCRIPTION
\mathbb{F}	Field of complex numbers
\mathbb{F}^∞	Vector space defined on the field \mathbb{F}
$\mathbb{F}[x]$	Subspace of \mathbb{F}^∞ spanned by $\{1, x, x^2, x^3, \dots\}$
$T: \mathbb{F}[x] \rightarrow \mathbb{F}[x]$	Transformation T
$f = \sum_{i=0}^n c_i x^i$	Polynomial $f \in \mathbb{F}[x]$
$f' = \sum_{i=0}^n c'_i x^i$	Polynomial $f' \in \mathbb{F}[x]$
$c_i, c'_i \forall i = 0, 2, \dots n$	Scalars in \mathbb{F} and coefficients of polynomials f and f'
\mathbf{M}_T	Transformation matrix for T
$N(T)$	Null Space of T

TABLE 1: Parameters

Statement	Derivation
$f = \sum_{i=0}^n c_i x^i$	$f = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \end{pmatrix}_{(n+1) \times 1}^T$
$T[f] = \mathbf{M}_T f$	$T[f] = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n+1} \end{pmatrix}_{(n+2) \times (n+1)} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_{(n+1) \times 1} = \begin{pmatrix} 0 \\ c_0 \\ \frac{c_1}{2} \\ \frac{c_2}{3} \\ \vdots \\ \frac{c_n}{n+1} \end{pmatrix}$
T is a linear operator	$T[\alpha f + f'] \quad (2.0.1)$ $= \mathbf{M}_T (\alpha f + f') \quad (2.0.2)$ $= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n+1} \end{pmatrix} \begin{pmatrix} \alpha c_0 + c'_0 \\ \alpha c_1 + c'_1 \\ \alpha c_2 + c'_2 \\ \vdots \\ \alpha c_n + c'_n \end{pmatrix} \quad (2.0.3)$ $= \begin{pmatrix} 0 \\ \alpha c_0 + c'_0 \\ \frac{\alpha c_1 + c'_1}{2} \\ \frac{\alpha c_2 + c'_2}{3} \\ \vdots \\ \frac{\alpha c_n + c'_n}{n+1} \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ c_0 \\ \frac{c_1}{2} \\ \frac{c_2}{3} \\ \vdots \\ \frac{c_n}{n+1} \end{pmatrix} + \begin{pmatrix} 0 \\ c'_0 \\ \frac{c'_1}{2} \\ \frac{c'_2}{3} \\ \vdots \\ \frac{c'_n}{n+1} \end{pmatrix} \quad (2.0.4)$ $= \alpha T[f] + T[f'] \quad (2.0.5)$ $\therefore T[\alpha f + f'] = \alpha T[f] + T[f'] \quad (2.0.6)$
T is non-singular	$T[f] = 0 \quad (2.0.7)$ $\Rightarrow \mathbf{M}_T f = \mathbf{0} \Rightarrow f = 0 \because \mathbf{M}_T \neq \mathbf{0} \quad (2.0.8)$ $\Rightarrow N(T) = \{0\} \quad (2.0.9)$

TABLE 2: Proof for Non-Singular and linear transformation T

Statement	Derivation
T is not invertible	As \mathbf{M}_T is a non-square matrix with dimensions $(n + 2) \times (n + 1)$, the transformation T is not invertible
\mathbf{M}_D is left inverse of \mathbf{M}_T	$\mathbf{M}_D \mathbf{M}_T = \mathbf{I}_{n+1} \quad (2.0.10)$ $\Rightarrow \mathbf{M}_D = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n+1 \end{pmatrix} \quad (2.0.11)$

TABLE 3: Non-Invertibility of transformation T