Assignment 13

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***Abstract*—This document explains a proof in linear transformations.**

Download all latex-tikz codes from

https://github.com/EE20MTECH14019/EE5609/tree/master/Assignment_13

1 Problem

Let $\mathbf{A}$ be an $m \times n$ matrix with real entries. Prove that $\mathbf{A} = \mathbf{0}$ is and only if $tr\left(\mathbf{A}^T\mathbf{A}\right) = 0$.

2 Solution

The proof is given in the table 2 and the properties used for the proof are listed in the table 1

Properties Used	
SVD of matrix $\mathbf{A}$	$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$
$\mathbf{U}$ is unitary	$\mathbf{U}^T\mathbf{U} = \mathbf{I}$
$\mathbf{V}$ is unitary	$\mathbf{V}^T\mathbf{V} = \mathbf{I}$
$\mathbf{\Sigma}$ is diagonal	$\mathbf{\Sigma}^T\mathbf{\Sigma} = \mathbf{\Sigma}^2$
Cyclic property	$tr\left(\mathbf{ABC}\right) = tr\left(\mathbf{CAB}\right)$
$Rank\left(\mathbf{A}\right)$	$Rank\left(\mathbf{A}\right)$ = #non-zero singular values

TABLE 1: Properties

Statement	Proof
$tr\left(\mathbf{A}^T\mathbf{A}\right) = 0 \implies \mathbf{A} = \mathbf{0}$	$tr\left(\mathbf{A}^T\mathbf{A}\right) = tr\left(\mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\right) \quad (2.0.1)$ $= tr\left(\mathbf{V}\mathbf{\Sigma}^T\mathbf{\Sigma}\mathbf{V}^T\right) \quad (2.0.2)$ $= tr\left(\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T\right) \quad (2.0.3)$ $= tr\left(\mathbf{V}^T\mathbf{V}\mathbf{\Sigma}^2\right) \quad (2.0.4)$ $= tr\left(\mathbf{\Sigma}^2\right) \quad (2.0.5)$ $= \sum_{i=1}^{r} \sigma_i^2 \quad (2.0.6)$ $tr\left(\mathbf{A}^T\mathbf{A}\right) = 0 \quad (2.0.7)$ $\implies \sum_{i=1}^{r} \sigma_i^2 = 0 \; \forall i = 1, 2, \dots r \quad (2.0.8)$ $\implies \sigma_i = 0 \; \forall i = 1, 2, \dots r \quad (2.0.9)$ $\therefore Rank\left(\mathbf{A}\right)$ = #non-zero singular values = 0 $\quad (2.0.10)$ $\implies \mathbf{A} = \mathbf{0} \quad (2.0.11)$
$\mathbf{A} = \mathbf{0} \implies tr\left(\mathbf{A}^T\mathbf{A}\right) = 0$	$\mathbf{A} = \mathbf{0} \quad (2.0.12)$ $\implies \mathbf{A}^T\mathbf{A} = \mathbf{0} \quad (2.0.13)$ $\implies tr\left(\mathbf{A}^T\mathbf{A}\right) = 0 \quad (2.0.14)$

TABLE 2: Proofs