

EE5609: Matrix Theory

Assignment-7

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Abstract—This document explains the QR decomposition of a 2x2 square matrix.

Download all latex-tikz codes from

<https://github.com/pavanmanesh/EE5609/tree/master/Assignment7>

and all python codes from

<https://github.com/pavanmanesh/EE5609/tree/master/Assignment7/codes>

1 PROBLEM

Find the foot of the perpendicular using svd drawn from $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ to the plane

$$(2 \quad -1 \quad -2)\mathbf{x} + 4 = 0 \quad (1.0.1)$$

2 EXPLANATION

Let us consider orthogonal vectors \mathbf{m}_1 and \mathbf{m}_2 to the given normal vector \mathbf{n} . Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (2.0.1)$$

$$\Rightarrow (a \quad b \quad c) \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0 \quad (2.0.2)$$

$$\Rightarrow 2a - b - 2c = 0 \quad (2.0.3)$$

Let a=1 and b=0 we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (2.0.4)$$

Let a=0 and b=1 we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \quad (2.0.5)$$

Let us solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (2.0.6)$$

Substituting (2.0.4) and (2.0.5) in (2.0.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad (2.0.7)$$

To solve (2.0.7), we will perform Singular Value Decomposition on \mathbf{M} as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (2.0.8)$$

Where the columns of \mathbf{V} are the eigen vectors of $\mathbf{M}^T \mathbf{M}$, the columns of \mathbf{U} are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T \mathbf{M}$.

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix} \quad (2.0.9)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{5}{4} \end{pmatrix} \quad (2.0.10)$$

Substituting (2.0.8) in (2.0.6),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (2.0.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (2.0.12)$$

Where \mathbf{S}_+ is Moore-Penrose Pseudo-Inverse of \mathbf{S} .

Let us calculate eigen values of $\mathbf{M}\mathbf{M}^T$,

$$|\mathbf{M}\mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (2.0.13)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{5}{4}-\lambda \end{vmatrix} = 0 \quad (2.0.14)$$

$$\Rightarrow \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \quad (2.0.15)$$

From equation (2.0.15) eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad (2.0.16)$$

The eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ -1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 1 \end{pmatrix} \quad (2.0.17)$$

Normalizing the eigen vectors in equation (2.0.17)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ -\frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \quad (2.0.18)$$

Hence we obtain \mathbf{U} as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \quad (2.0.19)$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get \mathbf{S} as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.20)$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (2.0.21)$$

$$\Rightarrow \begin{pmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \quad (2.0.22)$$

$$\Rightarrow \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \quad (2.0.23)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad (2.0.24)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \quad (2.0.25)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \quad (2.0.26)$$

Hence we obtain \mathbf{V} as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \quad (2.0.27)$$

From (2.0.6), the Singular Value Decomposition of \mathbf{M} is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}^T \quad (2.0.28)$$

Now, Moore-Penrose Pseudo inverse of \mathbf{S} is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.29)$$

From (2.0.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{11}{3} \end{pmatrix} \quad (2.0.30)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{8}{9\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (2.0.31)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \quad (2.0.32)$$

Verifying the solution of (2.0.32) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (2.0.33)$$

Evaluating the R.H.S in (2.0.33) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (2.0.34)$$

$$\Rightarrow \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (2.0.35)$$

Solving the augmented matrix of (2.0.35) we get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \xrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \quad (2.0.36)$$

$$\xrightarrow{R_2=R_2+\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{9}{8} & -\frac{1}{4} \end{pmatrix} \quad (2.0.37)$$

$$\xrightarrow{R_2=\frac{8}{9}R_2} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix} \quad (2.0.38)$$

$$\xrightarrow{R_1=R_1+\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & -\frac{5}{9} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix} \quad (2.0.39)$$

From equation (2.0.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \quad (2.0.40)$$

Comparing results of \mathbf{x} from (2.0.32) and (2.0.40), we can say that the solution is verified.