

EE5609: Matrix Theory

Assignment-5

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Abstract—This document contains solution to determine the conic representing the given equation.

Download the python codes from latex-tikz codes from

<https://github.com/pavanmanesh/EE5609/tree/master/Assignment5>

1 PROBLEM

What conic does the following equation represent.

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0$$

Find the center.

2 SOLUTION

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (2.0.3)$$

$$f = 5 \quad (2.0.4)$$

Expanding the determinant of \mathbf{V} we observe,

$$\begin{vmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{vmatrix} = 0 \quad (2.0.5)$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 3 & -\sqrt{3} & 3 \\ -\sqrt{3} & 1 & -2 \\ 3 & -2 & 5 \end{vmatrix} \quad (2.0.6)$$

$$= 12\sqrt{3} - 21 \neq 0 \quad (2.0.7)$$

Hence from (2.0.5) and (2.0.7) we conclude that given equation is an parabola. The characteristic equation of \mathbf{V} is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 3 & \sqrt{3} \\ \sqrt{3} & \lambda - 1 \end{vmatrix} = 0 \quad (2.0.8)$$

$$\Rightarrow \lambda^2 - 4\lambda = 0 \quad (2.0.9)$$

Hence the characteristic equation of \mathbf{V} is given by (2.0.9). The roots of (2.0.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 4 \quad (2.0.10)$$

The eigen vector \mathbf{p} is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.0.11)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (2.0.12)$$

for $\lambda_1 = 0$,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = \sqrt{3}R_2 + R_1} \begin{pmatrix} -1 & \frac{1}{\sqrt{3}} \\ 0 & 0 \end{pmatrix} \quad (2.0.13)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} \quad (2.0.14)$$

Again, for $\lambda_2 = 4$,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} \xrightarrow[R_2 = \frac{R_2}{\sqrt{3}}]{R_1 = R_1} \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{pmatrix} \quad (2.0.15)$$

$$\Rightarrow \mathbf{p}_2 = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \quad (2.0.16)$$

The matrix \mathbf{P} ,

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{3} \\ 1 & -1 \end{pmatrix} \quad (2.0.17)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \quad (2.0.18)$$

$$\eta = 2\mathbf{u}^T \mathbf{p}_1 = 2(\sqrt{3} - 2) \quad (2.0.19)$$

The focal length of the parabola is given by:

$$\left| \frac{\eta}{\lambda_2} \right| = \frac{2 - \sqrt{3}}{2} \quad (2.0.20)$$

When $|\mathbf{V}| = 0$, (2.0.1) can be written as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (2.0.21)$$

And the vertex \mathbf{c} is given by

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.22)$$

using equations (2.0.3), (2.0.4) and (2.0.14)

$$\begin{pmatrix} 5 - \frac{4}{\sqrt{3}} & 2\sqrt{3} - 6 \\ 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -5 \\ -1 - \frac{4}{\sqrt{3}} \\ 2\sqrt{3} - 2 \end{pmatrix} \quad (2.0.23)$$

$$\begin{pmatrix} 5 - \frac{4}{\sqrt{3}} & 2\sqrt{3} - 6 & -5 \\ 3 & -\sqrt{3} & -1 - \frac{4}{\sqrt{3}} \\ -\sqrt{3} & 1 & 2\sqrt{3} - 2 \end{pmatrix} \xleftarrow[R_1 = \frac{R_1}{5 - \frac{4}{\sqrt{3}}}]^{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & \frac{-66+6\sqrt{3}}{59} & \frac{-75-20\sqrt{3}}{59} \\ 0 & \frac{198-77\sqrt{3}}{59} & \frac{166}{59} - \frac{-56\sqrt{3}}{177} \\ -\sqrt{3} & 1 & 2\sqrt{3} - 2 \end{pmatrix} \xleftarrow[R_2 = \frac{R_2}{\frac{198-77\sqrt{3}}{59}}]^{R_3 = R_3 + \sqrt{3}R_1} \begin{pmatrix} 1 & \frac{-66+6\sqrt{3}}{59} & \frac{-75-20\sqrt{3}}{59} \\ 0 & 1 & \frac{14\sqrt{3}+44}{33} \\ 0 & \frac{77-66\sqrt{3}}{59} & \frac{-178+43\sqrt{3}}{59} \end{pmatrix} \xleftarrow[R_1 = R_1 - \frac{-66+6\sqrt{3}}{59}R_2]^{R_3 = R_3 - \frac{77-66\sqrt{3}}{59}R_2} \begin{pmatrix} 1 & 0 & \frac{1}{11} \\ 0 & 1 & \frac{14\sqrt{3}+44}{33} \\ 0 & 0 & \frac{-10+5\sqrt{3}}{3} \end{pmatrix} \quad (2.0.24)$$