EE5609: Matrix Theory Assignment-5

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***Abstract*—This document contains solution to determine the conic representing the given equation.**

Download the python codes from latex-tikz codes from

https://github.com/pavanmanesh/EE5609/tree/master/Assignment5

1 Problem

What conic does the following equation represent.

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0$$

Find the center.

2 Solution

The general second degree equation can be expressed as follows,

$$\mathbf{x^T V x} + 2\mathbf{u^T x} + f = 0 \tag{2.0.1}$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{2.0.3}$$

$$f = 5 \tag{2.0.4}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{vmatrix} = 0 \tag{2.0.5}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 3 & -\sqrt{3} & 3 \\ -\sqrt{3} & 1 & -2 \\ 3 & -2 & 5 \end{vmatrix} \tag{2.0.6}$$

$$= 12\sqrt{3} - 21 \neq 0 \tag{2.0.7}$$

Hence from (2.0.5) and (2.0.7) we conclude that given equation is an parabola. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 3 & \sqrt{3} \\ \sqrt{3} & \lambda - 1 \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies \lambda^2 - 4\lambda = 0 \tag{2.0.9}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.0.9). The roots of (2.0.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 4 \tag{2.0.10}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{Vp} = \lambda\mathbf{p} \tag{2.0.11}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{2.0.12}$$

for $\lambda_1 = 0$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = \sqrt{3}R_2 + R_1} \begin{pmatrix} -1 & \frac{1}{\sqrt{3}} \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\implies \mathbf{p_1} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} \tag{2.0.14}$$

Again, for $\lambda_2 = 4$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} \xleftrightarrow[R_2 = \frac{1}{\sqrt{3}} + R_1]{R_1 = R_1} \begin{pmatrix} 1 & \frac{1}{-\sqrt{3}} \\ 0 & 0 \end{pmatrix} \tag{2.0.15}$$

$$\implies \mathbf{p_2} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{2.0.16}$$

The matrix P,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{3} \\ 1 & 1 \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \tag{2.0.18}$$

$$\eta = 2\mathbf{u}^T\mathbf{p_1} = 2(\sqrt{3} - 2) \tag{2.0.19}$$

The focal length of the parabola is given by:

$$\left|\frac{\eta}{\lambda_2}\right| = \frac{2 - \sqrt{3}}{2} \tag{2.0.20}$$

When $\left|\mathbf{V}\right| = 0$,(2.0.1) can be written as

$$\mathbf{y^T D y} = \eta \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \tag{2.0.21}$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix}\mathbf{u^T} + \eta\mathbf{p_1^T} \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{2.0.22}$$

using equations (2.0.3),(2.0.4) and (2.0.14)

$$\begin{pmatrix}5 - \frac{4}{\sqrt{3}} & 2\sqrt{3} - 6 \\ 3 & -\sqrt{3} \\ -\sqrt{3} & 1\end{pmatrix}\mathbf{c} = \begin{pmatrix}-5 \\ 2 - \frac{7}{\sqrt{3}} \\ 2\sqrt{3} - 2\end{pmatrix} \tag{2.0.23}$$