EE5609: Matrix Theory Assignment-6

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***Abstract*—This document explains the QR decomposition of a 2x2 square matrix.**

Download all latex-tikz codes from

https://github.com/pavanmanesh/EE5609/tree/master/Assignment6

and all python codes from

https://github.com/pavanmanesh/EE5609/tree/master/Assignment6/codes

1 Problem

Perform QR decomposition of matrix $\begin{pmatrix} 6 & 1 \\ -8 & 2 \end{pmatrix}$

2 Explanation

Let $\mathbf{a}$ and $\mathbf{b}$ are the columns of matrix $\mathbf{A}$. The matrix $\mathbf{A}$ can be decomposed in the form

$$\mathbf{A} = \mathbf{QR} \tag{2.0.1}$$

$$\mathbf{U} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.3}$$

where

$$k_1 = \|\mathbf{a}\| \tag{2.0.4}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{2.0.5}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{b}}{\|u_1\|^2} \tag{2.0.6}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \tag{2.0.7}$$

$$k_2 = \mathbf{u_2}^T\mathbf{b} \tag{2.0.8}$$

The given matrix can be represented as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}\begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.9}$$

3 Solution

The columns of matrix $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ -8 & 2 \end{pmatrix}$ are $\mathbf{a}$ and $\mathbf{b}$ where

$$\mathbf{a} = \begin{pmatrix} 6 \\ -8 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.0.2}$$

Now for the given matrix, From (2.0.4) and (2.0.5)

$$k_1 = \|\mathbf{a}\| = 10 \tag{3.0.3}$$

$$\mathbf{u_1} = \frac{1}{10}\begin{pmatrix} 6 \\ -8 \end{pmatrix} \tag{3.0.4}$$

From (2.0.6)

$$r_1 = \frac{1}{10}\begin{pmatrix} 6 & -8 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1 \tag{3.0.5}$$

From (2.0.7)

$$\mathbf{b} - r_1\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{6}{10} \\ \frac{-8}{10} \end{pmatrix} = \begin{pmatrix} \frac{16}{10} \\ \frac{12}{10} \end{pmatrix} \tag{3.0.6}$$

$$\|\mathbf{b} - r_1\mathbf{u_1}\| = \frac{20}{10} = 2 \tag{3.0.7}$$

$$\implies \mathbf{u_2} = \frac{1}{10}\begin{pmatrix} 8 \\ 6 \end{pmatrix} \tag{3.0.8}$$

From (2.0.8)

$$k_2 = \mathbf{u_2}^T\mathbf{b} = \begin{pmatrix} \frac{8}{10} & \frac{6}{10} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{20}{10} = 2 \tag{3.0.9}$$

Now we can observe that $\mathbf{Q}^T\mathbf{Q} = \mathbf{QQ}^T = I$

$$\begin{pmatrix} \frac{6}{10} & \frac{8}{10} \\ \frac{-8}{10} & \frac{6}{10} \end{pmatrix}\begin{pmatrix} \frac{6}{10} & \frac{-8}{10} \\ \frac{8}{10} & \frac{6}{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.10}$$

From (2.0.9), The matrix $\mathbf{A}$ can now be written as,

$$\mathbf{A} = \begin{pmatrix} 6 & 1 \\ -8 & 2 \end{pmatrix} = \begin{pmatrix} \frac{6}{10} & \frac{8}{10} \\ \frac{-8}{10} & \frac{6}{10} \end{pmatrix}\begin{pmatrix} 10 & -1 \\ 0 & 2 \end{pmatrix} \tag{3.0.11}$$