Matrix Theory (EE5609) Assignment-4

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***Abstract*—Proof for the tangents to the circle at given points are parallel.**

Download all python codes from

https://github.com/EE20RESCH11008/Matrix−Theory/tree/master/Assignment−4/Code

and latex-tikz codes from

https://github.com/EE20RESCH11008/Matrix−Theory/tree/master/Assignment−4

1 Problem

Find the equations of the tangents to the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 7 & 5 \end{pmatrix}\mathbf{x} + 18 = 0 \tag{1.0.1}$$

at the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, showing that they are parallel.

2 Solution

The general equation of a second degree can be expressed as:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

Comparing 1.0.1 with 2.0.1

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix} \frac{-7}{2} \\ \frac{-5}{2} \end{pmatrix}, f = 18 \tag{2.0.2}$$

The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{2.0.3}$$

2.1 Tangent-1

Given point

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{2.1.1}$$

The direction vector of the line joining the point $\mathbf{q_1}$ and the centre $\mathbf{c}$ expressed as:

$$\mathbf{n_1} = \mathbf{q_1} - \mathbf{c} \tag{2.1.2}$$

$$\implies \mathbf{n_1} = \mathbf{q_1} + \mathbf{u} \tag{2.1.3}$$

where,

$$\mathbf{c} = -\mathbf{u} \tag{2.1.4}$$

The vector $\mathbf{n_1}$ is normal to the tangent drawn at $\mathbf{q_1}$. From (2.0.2) and (2.1.1) we get,

$$\mathbf{n_1}^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{2.1.5}$$

We know,

$$\mathbf{m}^T\mathbf{n_1} = 0 \tag{2.1.6}$$

$$\implies \mathbf{m}^T \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = 0 \tag{2.1.7}$$

$$\implies \mathbf{m}^T = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{2.1.8}$$

If $\mathbf{q_1}$ be a point on the line and $\mathbf{n_1}$ is the normal vector then the equation of the line can be expressed From(2.0.3) is :

$$\mathbf{n_1}^T(\mathbf{x} - \mathbf{q_1}) = 0 \tag{2.1.9}$$

$$\implies \mathbf{n_1}^T\mathbf{x} = c \tag{2.1.10}$$

where

$$c = \mathbf{n_1}^T\mathbf{q_1} \tag{2.1.11}$$

Using the equations (2.1.1) and (2.1.5),

$$\implies c = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{7}{2} \tag{2.1.12}$$

From (2.1.10), Line equation of Tangent-1 is:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \frac{7}{2} \tag{2.1.13}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 7} \tag{2.1.14}$$

2.2 *Tangent-2*

Now,

$$\mathbf{q_2} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{2.2.1}$$

The direction vector of the line joining the point $\mathbf{q_2}$ and the centre $\mathbf{c}$ expressed as:

$$\mathbf{n_2} = \mathbf{q_2} - \mathbf{c} \tag{2.2.2}$$

$$\implies \mathbf{n_2} = \mathbf{q_2} + \mathbf{u} \tag{2.2.3}$$

where,

$$\mathbf{c} = -\mathbf{u} \tag{2.2.4}$$

The vector $\mathbf{n_2}$ is normal to the tangent drawn at $\mathbf{q_2}$. From (2.0.2) and (2.2.1),

$$\mathbf{n_2}^T = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \tag{2.2.5}$$

We know,

$$\mathbf{m}^T \mathbf{n_2} = 0 \tag{2.2.6}$$

$$\implies \mathbf{m}^T \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix} = 0 \tag{2.2.7}$$

$$\implies \mathbf{m}^T = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \tag{2.2.8}$$

If $\mathbf{q_2}$ be a point on the line and $\mathbf{n_2}$ is the normal vector, the equation of the line can be expressed From (2.0.3) is:

$$\mathbf{n_2}^T(\mathbf{x} - \mathbf{q_2}) = 0 \tag{2.2.9}$$

$$\implies \mathbf{n_2}^T \mathbf{x} = c \tag{2.2.10}$$

where

$$c = \mathbf{n_2}^T \mathbf{q_2} \tag{2.2.11}$$

Using the equations 2.2.1 and 2.2.5,

$$\implies c = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{-5}{2} \tag{2.2.12}$$

From (2.2.10), Line equation of Tangent-2 is:

$$\begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \mathbf{x} = \frac{-5}{2} \tag{2.2.13}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5} \tag{2.2.14}$$

2.3 *Result*

From the equations (2.1.14) and (2.2.14), normal vectors of Tangent-1 and Tangent-2 are equal. Hence, the two tangents are parallel.

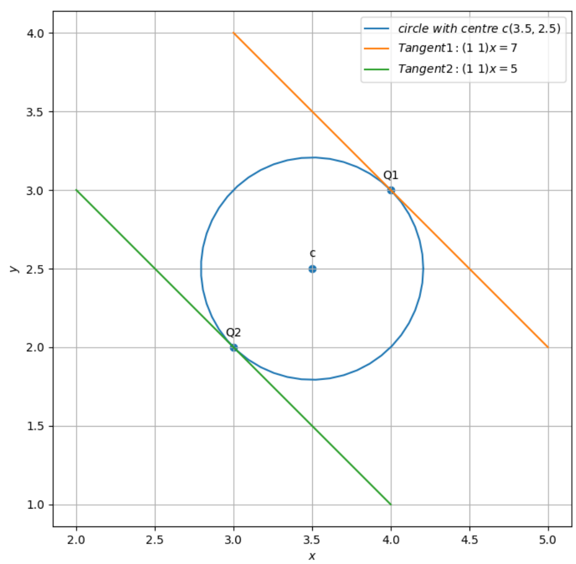

Fig. 0: Tangents to the circle at given points