Matrix Theory (EE5609) Assignment-6

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Abstract—**This document contains the QR decomposition and SVD followed by Least Square Verification.**

Download all python codes from

https://github.com/EE20RESCH11008/Matrix−Theory/tree/master/Assignment−6/Code

and latex-tikz codes from

https://github.com/EE20RESCH11008/Matrix−Theory/tree/master/Assignment−6

1 Problem

1). Find QR decomposition of

$$\mathbf{V} = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \tag{1.0.1}$$

2). Find the vertex $\mathbf{c}$ of the parabola using SVD:

$$16x^2 - 24xy + 9y^2 + 32x + 86y - 39 = 0 \tag{1.0.2}$$

also verify the result using least squares.

2 Solution

2.1 QR Decomposition

Let, the column vectors of $\mathbf{V}$ be $\mathbf{v_1}$ and $\mathbf{v_2}$:

$$\mathbf{v_1} = \begin{pmatrix} 16 \\ -12 \end{pmatrix} \tag{2.1.1}$$

$$\mathbf{v_2} = \begin{pmatrix} -12 \\ 9 \end{pmatrix} \tag{2.1.2}$$

To find $\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}$, we will orthonormalize the columns of $\mathbf{V}$ using Gram-Schmidt method:

$$\mathbf{u_1} = \frac{\mathbf{v_1}}{k_1} \tag{2.1.3}$$

$$k_1 = \|\mathbf{v_1}\| = \sqrt{16^2 + (-12)^2} = 20 \tag{2.1.4}$$

$$\implies \mathbf{u_1} = \frac{1}{20}\begin{pmatrix} 16 \\ -12 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix} \tag{2.1.5}$$

$$\mathbf{u_2} = \frac{\mathbf{v_2} - r_1\mathbf{u_1}}{\|\mathbf{v_2} - r_1\mathbf{u_1}\|} \tag{2.1.6}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{v_2}}{\|\mathbf{u_1}\|^2} = \frac{\begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \end{pmatrix}\begin{pmatrix} -12 \\ 9 \end{pmatrix}}{(\frac{4}{5})^2 + (\frac{-3}{5})^2} = -15 \tag{2.1.7}$$

From (2.1.2) (2.1.5) and (2.1.7),

$$\mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.8}$$

$$k_2 = \mathbf{u_2}^T\mathbf{v_2} = 0 \tag{2.1.9}$$

The QR decomposition is given as:

$$\begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}\begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.1.10}$$

Where,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.1.11}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.1.12}$$

From (2.1.5) (2.1.8) and (2.1.11)

$$\mathbf{Q} = \begin{pmatrix} \frac{4}{5} & 0 \\ \frac{-3}{5} & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix} \tag{2.1.13}$$

From (2.1.4) (2.1.7) (2.1.9) and (2.1.12)

$$\mathbf{R} = \begin{pmatrix} 20 & -15 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 20 & -15 \end{pmatrix} \tag{2.1.14}$$

Substituting the values of (2.1.5) (2.1.8) (2.1.4) (2.1.9) and (2.1.7) in (2.1.10) We get,

$$\boxed{\mathbf{V} = \begin{pmatrix} \frac{4}{5} & 0 \\ \frac{-3}{5} & 0 \end{pmatrix}\begin{pmatrix} 20 & -15 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix}\begin{pmatrix} 20 & -15 \end{pmatrix}} \tag{2.1.15}$$

2.2 Vertex of Parabola:

The general second degree equation can be expressed as:

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.2.1}$$

From (1.0.2) and (2.2.1),

$$\mathbf{V} = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 16 \\ 43 \end{pmatrix}, \quad f = -39 \tag{2.2.2}$$

Eigen Values of $\mathbf{V}$ is given as

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{2.2.3}$$

$$\implies \begin{vmatrix} \lambda - 16 & 12 \\ 12 & \lambda - 9 \end{vmatrix} = 0 \tag{2.2.4}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{2.2.5}$$

Hence,

$$\lambda_1 = 0, \quad \lambda_2 = 25 \tag{2.2.6}$$

Eigen-vector corresponding to $\lambda_1 = 0$,

$$\implies \mathbf{p_1} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.2.7}$$

Eigen-vector corresponding to $\lambda_2 = 25$,

$$\implies \mathbf{p_2} = \frac{1}{5}\begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{2.2.8}$$

From (2.2.2) and (2.2.7)

$$\eta = 2\mathbf{p_1}^T\mathbf{u} = 88 \tag{2.2.9}$$

Using (2.2.6) and (2.2.9), Focal length of the parabola is given by:

$$\left|\frac{\eta}{\lambda_2}\right| = \left|\frac{88}{25}\right| = 3.52 \tag{2.2.10}$$

The standard equation of parabola is given by:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.2.11}$$

And the vertex $\mathbf{c}$ is:

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.2.12}$$

Substitute values of (2.2.2) (2.2.7) and (2.2.9) in (2.2.12) we get,

$$\begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix}\mathbf{c} = \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \tag{2.2.13}$$

2.3 Singular Value Decomposition:

$$\mathbf{Mc} = \mathbf{b} \tag{2.3.1}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix}, b = \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \tag{2.3.2}$$

To solve (2.3.1), we perform singular value decomposition on $\mathbf{M}$ given as

$$\mathbf{M} = \mathbf{USV^T} \tag{2.3.3}$$

Substituting the value of $\mathbf{M}$ from (2.3.3) in (2.3.1), we get

$$\mathbf{USV^Tc} = \mathbf{b} \tag{2.3.4}$$

$$\implies \mathbf{c} = \mathbf{VS_+U^Tb} \tag{2.3.5}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{MM^T}$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M^TM}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M^TM}$. First calculating the eigenvectors corresponding to $\mathbf{M^TM}$.
Using Python, $\mathbf{U}, \mathbf{S}$ and $\mathbf{V}$ for $\mathbf{M}$ is given by:

$$\mathbf{U} = \begin{pmatrix} \frac{99}{100} & \frac{11}{250} & 0 \\ \frac{-7}{200} & \frac{799}{1000} & \frac{3}{5} \\ \frac{26}{1000} & \frac{-599}{1000} & \frac{4}{10} \end{pmatrix} \tag{2.3.6}$$

$$\mathbf{S} = \begin{pmatrix} \frac{8903}{100} & 0 \\ 0 & \frac{2471}{100} \\ 0 & 0 \end{pmatrix} \tag{2.3.7}$$

$$\mathbf{V} = \begin{pmatrix} \frac{93}{200} & \frac{221}{250} \\ \frac{221}{250} & \frac{-93}{200} \end{pmatrix} \tag{2.3.8}$$

Now, More-Pen-Rose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{100}{8903} & 0 & 0 \\ 0 & \frac{100}{2471} & 0 \end{pmatrix} \tag{2.3.9}$$

Hence, we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix} \frac{99}{100} & \frac{11}{250} & 0 \\ \frac{-7}{200} & \frac{799}{1000} & \frac{3}{5} \\ \frac{26}{1000} & \frac{-599}{1000} & \frac{4}{10} \end{pmatrix} \begin{pmatrix} \frac{8903}{100} & 0 \\ 0 & \frac{2471}{100} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{93}{200} & \frac{221}{250} \\ \frac{221}{250} & \frac{-93}{200} \end{pmatrix} \tag{2.3.10}$$

From (2.3.2) and (2.3.6)

$$\mathbf{U^T b} = \begin{pmatrix} \frac{99}{100} & \frac{-7}{200} & \frac{26}{1000} \\ \frac{11}{250} & \frac{799}{1000} & \frac{-599}{1000} \\ 0 & \frac{3}{5} & \frac{4}{10} \end{pmatrix} \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} = \begin{pmatrix} \frac{3838}{100} \\ \frac{1472}{100} \\ 0 \end{pmatrix} \tag{2.3.11}$$

From (2.3.8) and (2.3.9)

$$\mathbf{VS}_+ = \begin{pmatrix} \frac{93}{200} & \frac{221}{250} \\ \frac{221}{250} & \frac{-93}{200} \end{pmatrix} \begin{pmatrix} \frac{100}{8903} & 0 & 0 \\ 0 & \frac{100}{2471} & 0 \end{pmatrix} = \begin{pmatrix} \frac{52}{10000} & \frac{358}{10000} & 0 \\ \frac{99}{10000} & \frac{-188}{10000} & 0 \end{pmatrix} \tag{2.3.12}$$

Substitute (2.3.12) and (2.3.11) in (2.3.5) we get,

$$\mathbf{c} = \begin{pmatrix} \frac{52}{10000} & \frac{358}{10000} & 0 \\ \frac{99}{10000} & \frac{-188}{10000} & 0 \end{pmatrix} \begin{pmatrix} \frac{3838}{100} \\ \frac{1472}{100} \\ 0 \end{pmatrix} \tag{2.3.13}$$

$$\implies \boxed{\mathbf{c} = \begin{pmatrix} \frac{91}{125} \\ \frac{13}{125} \end{pmatrix} = \begin{pmatrix} 0.728 \\ 0.104 \end{pmatrix}} \tag{2.3.14}$$

2.4 Least Square Verification

Now, verify our solution using,

$$\mathbf{M^T M c} = \mathbf{M^T b} \tag{2.4.1}$$

From (2.3.2),

$$\begin{pmatrix} \frac{212}{5} & 16 & -12 \\ \frac{391}{5} & -12 & 9 \end{pmatrix} \begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix} \mathbf{c} = \begin{pmatrix} \frac{212}{5} & 16 & -12 \\ \frac{391}{5} & -12 & 9 \end{pmatrix} \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \tag{2.4.2}$$

$$\implies \begin{pmatrix} \frac{54944}{25} & \frac{75392}{25} \\ \frac{75392}{25} & \frac{158506}{25} \end{pmatrix} \mathbf{c} = \begin{pmatrix} \frac{9568}{5} \\ \frac{14274}{5} \end{pmatrix} \tag{2.4.3}$$

Solving the augmented matrix, we get

$$\begin{pmatrix} \frac{54944}{25} & \frac{75392}{25} & \frac{9568}{5} \\ \frac{75392}{25} & \frac{158506}{25} & \frac{14274}{5} \end{pmatrix} \xleftrightarrow{R_1 = \frac{25}{54944} R_1} \begin{pmatrix} 1 & \frac{75392}{54944} & \frac{47840}{54944} \\ \frac{75392}{25} & \frac{158506}{25} & \frac{14274}{5} \end{pmatrix} \tag{2.4.4}$$

$$\xleftrightarrow{R_2 = R_2 - \frac{75392}{25} R_1} \begin{pmatrix} 1 & \frac{75392}{54944} & \frac{47840}{54944} \\ 0 & \frac{8809}{4} & \frac{22903}{100} \end{pmatrix} \tag{2.4.5}$$

$$\xleftrightarrow{R_2 = \frac{4}{8809} R_2} \begin{pmatrix} 1 & \frac{75392}{54944} & \frac{47840}{54944} \\ 0 & 1 & \frac{104}{1000} \end{pmatrix} \tag{2.4.6}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{75392}{54944} R_2} \begin{pmatrix} 1 & 0 & \frac{728}{1000} \\ 0 & 1 & \frac{104}{1000} \end{pmatrix} \tag{2.4.7}$$

Thus,

$$\implies \boxed{\mathbf{c} = \begin{pmatrix} \frac{728}{1000} \\ \frac{104}{1000} \end{pmatrix} = \begin{pmatrix} \frac{91}{125} \\ \frac{13}{125} \end{pmatrix} = \begin{pmatrix} 0.728 \\ 0.104 \end{pmatrix}} \tag{2.4.8}$$

Hence, verified the result from SVD.