Matrix Theory (EE5609) Assignment-9

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***Abstract*—This document contains the few operations on given matrix.**

Download latex-tikz codes from

https://github.com/EE20RESCH11008/Matrix−Theory/tree/master/Assignment−9

1 Problem

Let

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \tag{1.0.1}$$

be a 3×3 matrix where a,b,c,d are integers. Then, we must have:

1. If $a \neq 0$, there is a polynomial $p \in Q[x]$ such that p(A) is the inverse of A.

2. For each polynomial $q \in Z[x]$, the matrix

$$q(A) = \begin{pmatrix} q(a) & q(b) & q(c) \\ 0 & q(a) & q(d) \\ 0 & 0 & q(a) \end{pmatrix} \tag{1.0.2}$$

3. If $A^n = 0$ for some positive integer n, then $A^3 = 0$

4. A commutes with every matrix of the form $\begin{pmatrix} a' & 0 & c' \\ 0 & a' & 0 \\ 0 & 0 & a' \end{pmatrix}$

2 Solution

2.1

Given $p(\mathbf{A})$ is the inverse of $\mathbf{A}$.

$$\implies p(\mathbf{A}).\mathbf{A} = \mathbf{I} \tag{2.1.1}$$

The characteristic equation is:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{2.1.2}$$

$$\implies \det\begin{pmatrix} a-\lambda & b & c \\ 0 & a-\lambda & d \\ 0 & 0 & a-\lambda \end{pmatrix} = 0 \tag{2.1.3}$$

$$\implies \lambda^3 - 3\lambda^2 a + 3\lambda a^2 - a^3 = 0 \tag{2.1.4}$$

By Cayley-Hamilton theorem,

$$\implies \mathbf{A^3} - 3a\mathbf{A^2} + 3a^2\mathbf{A} = a^3\mathbf{I} \tag{2.1.5}$$

$$\implies (\frac{\mathbf{A^2}}{a^3} - \frac{3\mathbf{A}}{a^2} + \frac{3}{a}).\mathbf{A} = \mathbf{I} \tag{2.1.6}$$

From (2.1.1) and (2.1.6),

$$p(\mathbf{A}) = \frac{\mathbf{A^2}}{a^3} - \frac{3\mathbf{A}}{a^2} + \frac{3}{a} \tag{2.1.7}$$

$$\implies \boxed{p(\mathbf{x}) = \frac{\mathbf{x^2}}{a^3} - \frac{3\mathbf{x}}{a^2} + \frac{3}{a}} \tag{2.1.8}$$

Hence, if $a \neq 0$, then the polynomial p ∈ Q[x] exist such that p(A) is the inverse of A.

2.2

Let, $q(\mathbf{x}) = \mathbf{x^2} \in Z[x]$

$$\implies q(\mathbf{A}) = \mathbf{A^2} \tag{2.2.1}$$

From (1.0.2) and (2.2.1),

$$\implies q(\mathbf{A}) = \begin{pmatrix} a^2 & b^2 & c^2 \\ 0 & a^2 & d^2 \\ 0 & 0 & a^2 \end{pmatrix} \tag{2.2.2}$$

From (1.0.1),

$$\mathbf{A^2} = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & ac+bd+ca \\ 0 & a^2 & 2ad \\ 0 & 0 & a^2 \end{pmatrix} \tag{2.2.3}$$

From (2.2.2) and (2.2.3),

$$\boxed{q(\mathbf{A}) \neq \mathbf{A^2}} \tag{2.2.4}$$

Hence the given $q(\mathbf{A})$ matrix is not valid.

2.3

Given,

$$\mathbf{A^n} = 0; \text{ some positive integer of n} \tag{2.3.1}$$
$$\implies \mathbf{A} \text{ is Nilpotent Matrix.} \tag{2.3.2}$$

From (1.0.1) and (2.3.2),
$\mathbf{A}$ is Nilpotent Matrix with order 3.

$$\boxed{\implies \mathbf{A^3} = 0} \tag{2.3.3}$$

Hence, it is a valid option.

2.4

Let,

$$\mathbf{B} = \begin{pmatrix} a' & 0 & c' \\ 0 & a' & 0 \\ 0 & 0 & a' \end{pmatrix} \tag{2.4.1}$$

From (1.0.1) and (2.4.1),

$$\mathbf{AB} = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} a' & 0 & c' \\ 0 & a' & 0 \\ 0 & 0 & a' \end{pmatrix} = \begin{pmatrix} aa' & a'b & ac' + ca' \\ 0 & aa' & a'd \\ 0 & 0 & aa' \end{pmatrix} \tag{2.4.2}$$

Now,

$$\mathbf{BA} = \begin{pmatrix} a' & 0 & c' \\ 0 & a' & 0 \\ 0 & 0 & a' \end{pmatrix}\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} aa' & a'b & ac' + ca' \\ 0 & aa' & a'd \\ 0 & 0 & aa' \end{pmatrix} \tag{2.4.3}$$

From (2.4.2) and (2.4.3),

$$\boxed{\mathbf{AB} = \mathbf{BA}} \tag{2.4.4}$$

Hence, it is valid option.

Finally, from (2.1.8) (2.2.4) (2.3.3) and (2.4.4),

$$\boxed{\mathbf{Ans} : \mathbf{1, 3, 4}} \tag{2.4.5}$$