

Matrix Theory (EE5609)

Assignment-6

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Abstract—This document contains the QR decomposition and SVD followed by Least Square Verification.

Download all python codes from

<https://github.com/EE20RESCH11008/Matrix-Theory/tree/master/Assignment-6/Code>

and latex-tikz codes from

<https://github.com/EE20RESCH11008/Matrix-Theory/tree/master/Assignment-6>

$$\Rightarrow \mathbf{u}_1 = \frac{1}{20} \begin{pmatrix} 16 \\ -12 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix} \quad (2.1.5)$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - r_1 \mathbf{u}_1}{\|\mathbf{v}_2 - r_1 \mathbf{u}_1\|} \quad (2.1.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{v}_2}{\|\mathbf{u}_1\|^2} = \frac{\begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \end{pmatrix} \begin{pmatrix} -12 \\ 9 \end{pmatrix}}{\left(\frac{4}{5}\right)^2 + \left(\frac{-3}{5}\right)^2} = -15 \quad (2.1.7)$$

From (2.1.2) (2.1.5) and (2.1.7),

$$\mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.8)$$

$$k_2 = \mathbf{u}_2^T \mathbf{v}_2 = 0 \quad (2.1.9)$$

The QR decomposition is given as:

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.1.10)$$

Where,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \quad (2.1.11)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.1.12)$$

From (2.1.5) (2.1.8) and (2.1.11)

$$\mathbf{Q} = \begin{pmatrix} \frac{4}{5} & 0 \\ \frac{-3}{5} & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix} \quad (2.1.13)$$

From (2.1.4) (2.1.7) (2.1.9) and (2.1.12)

$$\mathbf{R} = \begin{pmatrix} 20 & -15 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 20 & -15 \end{pmatrix} \quad (2.1.14)$$

1 PROBLEM

1). Find QR decomposition of

$$\mathbf{V} = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \quad (1.0.1)$$

2). Find the vertex \mathbf{c} of the parabola using SVD:

$$16x^2 - 24xy + 9y^2 + 32x + 86y - 39 = 0 \quad (1.0.2)$$

also verify the result using least squares.

2 SOLUTION

2.1 QR Decomposition

Let, the column vectors of \mathbf{V} be \mathbf{v}_1 and \mathbf{v}_2 :

$$\mathbf{v}_1 = \begin{pmatrix} 16 \\ -12 \end{pmatrix} \quad (2.1.1)$$

$$\mathbf{v}_2 = \begin{pmatrix} -12 \\ 9 \end{pmatrix} \quad (2.1.2)$$

To find $\mathbf{Q} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}$, we will orthonormalize the columns of \mathbf{V} using Gram-Schmidt method:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{k_1} \quad (2.1.3)$$

$$k_1 = \|\mathbf{v}_1\| = \sqrt{16^2 + (-12)^2} = 20 \quad (2.1.4)$$

Substituting the values of (2.1.5) (2.1.8) (2.1.4) (2.1.9) and (2.1.7) in (2.1.10) We get,

$$\mathbf{V} = \begin{pmatrix} \frac{4}{5} & 0 \\ \frac{-3}{5} & 0 \end{pmatrix} \begin{pmatrix} 20 & -15 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix} \begin{pmatrix} 20 & -15 \end{pmatrix} \quad (2.1.15)$$

And the vertex \mathbf{c} is:

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2} \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2} \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.2.12)$$

Substitute values of (2.2.2) (2.2.7) and (2.2.9) in (2.2.12) we get,

$$\begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \quad (2.2.13)$$

2.2 Vertex of Parabola:

The general second degree equation can be expressed as:

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.2.1)$$

From (1.0.2) and (2.2.1),

$$\mathbf{V} = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 16 \\ 43 \end{pmatrix}, \quad f = -39 \quad (2.2.2) \quad \text{where}$$

Eigen Values of \mathbf{V} is given as

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (2.2.3)$$

$$\Rightarrow \begin{vmatrix} \lambda - 16 & 12 \\ 12 & \lambda - 9 \end{vmatrix} = 0 \quad (2.2.4)$$

$$\Rightarrow \lambda^2 - 25\lambda = 0 \quad (2.2.5)$$

Hence,

$$\lambda_1 = 0, \quad \lambda_2 = 25 \quad (2.2.6)$$

Eigen-vector corresponding to $\lambda_1 = 0$,

$$\Rightarrow \mathbf{p}_1 = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (2.2.7)$$

Eigen-vector corresponding to $\lambda_2 = 25$,

$$\Rightarrow \mathbf{p}_2 = \frac{1}{5} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad (2.2.8)$$

From (2.2.2) and (2.2.7)

$$\eta = 2\mathbf{p}_1^T \mathbf{u} = 88 \quad (2.2.9)$$

Using (2.2.6) and (2.2.9), Focal length of the parabola is given by:

$$\left| \frac{\eta}{\lambda_2} \right| = \left| \frac{88}{25} \right| = 3.52 \quad (2.2.10)$$

The standard equation of parabola is given by:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (2.2.11)$$

2.3 Singular Value Decomposition:

$$\mathbf{M} \mathbf{c} = \mathbf{b} \quad (2.3.1)$$

$$\mathbf{M} = \begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \quad (2.3.2)$$

To solve (2.3.1), we perform singular value decomposition on \mathbf{M} given as

$$\mathbf{M} = \mathbf{U} \mathbf{S} \mathbf{V}^T \quad (2.3.3)$$

Substituting the value of \mathbf{M} from (2.3.3) in (2.3.1), we get

$$\mathbf{U} \mathbf{S} \mathbf{V}^T \mathbf{c} = \mathbf{b} \quad (2.3.4)$$

$$\Rightarrow \mathbf{c} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (2.3.5)$$

where, \mathbf{S}_+ is Moore-Pen-rose Pseudo-Inverse of \mathbf{S} . Columns of \mathbf{U} are eigen-vectors of $\mathbf{M} \mathbf{M}^T$, columns of \mathbf{V} are eigenvectors of $\mathbf{M}^T \mathbf{M}$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T \mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T \mathbf{M}$.

Using Python, \mathbf{U} , \mathbf{S} and \mathbf{V} for \mathbf{M} is given by:

$$\mathbf{U} = \begin{pmatrix} \frac{99}{100} & \frac{11}{250} & 0 \\ \frac{-7}{799} & \frac{250}{799} & \frac{3}{5} \\ \frac{200}{26} & \frac{1000}{-899} & \frac{4}{10} \end{pmatrix} \quad (2.3.6)$$

$$\mathbf{S} = \begin{pmatrix} \frac{8903}{100} & 0 \\ 0 & \frac{2471}{100} \\ 0 & 0 \end{pmatrix} \quad (2.3.7)$$

$$\mathbf{V} = \begin{pmatrix} \frac{93}{250} & \frac{221}{200} \\ \frac{200}{221} & \frac{250}{-93} \\ \frac{250}{200} & \frac{-93}{250} \end{pmatrix} \quad (2.3.8)$$

Now, More-Pen-Rose Pseudo inverse of \mathbf{S} is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{100}{8903} & 0 & 0 \\ 0 & \frac{100}{2471} & 0 \end{pmatrix} \quad (2.3.9)$$

Hence, we get singular value decomposition of \mathbf{M} as,

$$\mathbf{M} = \begin{pmatrix} \frac{99}{1000} & \frac{11}{250} & 0 \\ \frac{100}{7} & \frac{250}{799} & \frac{3}{5} \\ \frac{200}{26} & \frac{1000}{-599} & \frac{4}{10} \end{pmatrix} \begin{pmatrix} \frac{8903}{100} & 0 \\ 0 & \frac{2471}{100} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{93}{200} & \frac{221}{250} \\ \frac{221}{250} & \frac{-93}{200} \\ \frac{5}{250} & \frac{200}{200} \end{pmatrix} \quad (2.3.10)$$

From (2.3.2) and (2.3.6)

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{99}{100} & \frac{-7}{200} & \frac{26}{1000} \\ \frac{100}{11} & \frac{200}{799} & \frac{1000}{-599} \\ \frac{250}{0} & \frac{1000}{\frac{3}{5}} & \frac{1000}{\frac{4}{10}} \end{pmatrix} \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} = \begin{pmatrix} \frac{3838}{100} \\ \frac{1472}{100} \\ 0 \end{pmatrix} \quad (2.3.11)$$

From (2.3.8) and (2.3.9)

$$\mathbf{V} \mathbf{S}_+ = \begin{pmatrix} \frac{93}{200} & \frac{221}{250} \\ \frac{221}{250} & \frac{-93}{200} \end{pmatrix} \begin{pmatrix} \frac{100}{8903} & 0 & 0 \\ 0 & \frac{100}{2471} & 0 \end{pmatrix} = \begin{pmatrix} \frac{52}{10000} & \frac{358}{10000} & 0 \\ \frac{10000}{99} & \frac{10000}{-188} & 0 \end{pmatrix} \quad (2.3.12)$$

Substitute (2.3.12) and (2.3.11) in (2.3.5) we get,

$$\mathbf{c} = \begin{pmatrix} \frac{52}{10000} & \frac{358}{10000} & 0 \\ \frac{10000}{99} & \frac{10000}{-188} & 0 \end{pmatrix} \begin{pmatrix} \frac{3838}{100} \\ \frac{1472}{100} \\ 0 \end{pmatrix} \quad (2.3.13)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{726552}{1000000} \\ \frac{103226}{1000000} \\ \frac{4}{1000000} \end{pmatrix} = \begin{pmatrix} 0.726 \\ 0.103 \end{pmatrix} \quad (2.3.14)$$

Solving the augmented matrix, we get

$$\begin{pmatrix} \frac{54944}{25} & \frac{75392}{25} & \frac{9568}{5} \\ \frac{75392}{25} & \frac{158506}{25} & \frac{14274}{5} \end{pmatrix} \xrightarrow{R_1 = \frac{25}{54944} R_1} \begin{pmatrix} 1 & \frac{75392}{54944} & \frac{47840}{54944} \\ \frac{75392}{25} & \frac{158506}{25} & \frac{14274}{5} \end{pmatrix} \quad (2.4.4)$$

$$\xrightarrow{R_2 = R_2 - \frac{75392}{25} R_1} \begin{pmatrix} 1 & \frac{75392}{54944} & \frac{47840}{54944} \\ 0 & \frac{54944}{8809} & \frac{54944}{22903} \end{pmatrix} \quad (2.4.5)$$

$$\xrightarrow{R_2 = \frac{4}{8809} R_2} \begin{pmatrix} 1 & \frac{75392}{54944} & \frac{47840}{54944} \\ 0 & 1 & \frac{22903}{220225} \end{pmatrix} \quad (2.4.6)$$

$$\xrightarrow{R_1 = R_1 - \frac{75392}{54944} R_2} \begin{pmatrix} 1 & 0 & \frac{728}{1000} \\ 0 & 1 & \frac{22903}{220225} \end{pmatrix} \quad (2.4.7)$$

Thus,

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{728}{1000} \\ \frac{22903}{220225} \end{pmatrix} = \begin{pmatrix} 0.728 \\ 0.103 \end{pmatrix} \quad (2.4.8)$$

Hence, verified the result from SVD.

2.4 Least Square Verification

Now, verify our solution using,

$$\mathbf{M}^T \mathbf{M} \mathbf{c} = \mathbf{M}^T \mathbf{b} \quad (2.4.1)$$

From (2.3.2),

$$\begin{pmatrix} \frac{212}{5} & 16 & -12 \\ \frac{391}{5} & -12 & 9 \end{pmatrix} \begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix} \mathbf{c} = \begin{pmatrix} \frac{212}{5} & 16 & -12 \\ \frac{391}{5} & -12 & 9 \end{pmatrix} \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \quad (2.4.2)$$

$$\Rightarrow \begin{pmatrix} \frac{54944}{25} & \frac{75392}{25} \\ \frac{75392}{25} & \frac{158506}{25} \end{pmatrix} \mathbf{c} = \begin{pmatrix} \frac{9568}{5} \\ \frac{14274}{5} \end{pmatrix} \quad (2.4.3)$$