Matrix Theory (EE5609) Assignment-5

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Abstract—**This document contains traces the parabola when it's general second degree equation is given.**

Download all python codes from

https://github.com/EE20RESCH11008/Matrix−Theory/tree/master/Assignment−5/Code

and latex-tikz codes from

https://github.com/EE20RESCH11008/Matrix−Theory/tree/master/Assignment−5

1 Problem

Trace the parabola

$$16x^2 - 24xy + 9y^2 + 32x + 86y - 39 = 0 \tag{1.0.1}$$

2 Solution

The general equation of a second degree can be expressed as:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

Comparing (1.0.1) and (2.0.1)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 16 \\ 43 \end{pmatrix}, \quad f = -39 \tag{2.0.2}$$

2.1 Eigen Values:

The characteristic equation of $\mathbf{V}$ is given as

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{2.1.1}$$

$$\implies \begin{vmatrix} \lambda - 16 & 12 \\ 12 & \lambda - 9 \end{vmatrix} = 0 \tag{2.1.2}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{2.1.3}$$

The eigenvalues are the roots of the equation (2.1.3), which are as follows:

$$\lambda_1 = 0, \quad \lambda_2 = 25 \tag{2.1.4}$$

2.2 Eigen Vectors:

The eigen vector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.2.1}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{2.2.2}$$

For $\lambda_1 = 0$

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -16 & 12 \\ 12 & -9 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + 3R_1]{R_1 \leftarrow \frac{1}{4}R_1} \begin{pmatrix} -4 & 3 \\ 0 & 0 \end{pmatrix} \tag{2.2.3}$$

$$\implies \mathbf{p_1} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.2.4}$$

For $\lambda_2 = 25$

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 9 & 12 \\ 12 & 1 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 4R_1]{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \tag{2.2.5}$$

$$\implies \mathbf{p_2} = \frac{1}{5}\begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{2.2.6}$$

2.3 Eigen Value Decomposition:

Using EVD, we can write

$$\mathbf{D} = \mathbf{P}\mathbf{V}\mathbf{P}^T \tag{2.3.1}$$

From (2.2.4) and (2.2.6)

$$\mathbf{P} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \tag{2.3.2}$$

From (2.1.4)

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{2.3.3}$$

2.4 Parabola

$$\text{Focal Length} = \left|\frac{2\eta}{\lambda_2}\right| \tag{2.4.1}$$

From (2.2.4) and (2.0.2)

$$\eta = \mathbf{p}_1^T \mathbf{u} = 44 \tag{2.4.2}$$

Substituting values of (2.4.2) and (2.1.4) in (2.4.1), we get

$$\text{Focal Length} = \left|\frac{88}{25}\right| = 3.52 \tag{2.4.3}$$

The standard equation of parabola is given by:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \tag{2.4.4}$$

And the vertex $\mathbf{c}$ is:

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.4.5}$$

From (2.0.2) (2.4.2) and (2.2.4),

$$\begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \tag{2.4.6}$$

To find $\mathbf{c}$, perform row reduction on the augmented matrix as follows:

$$\begin{pmatrix} \frac{212}{5} & \frac{391}{5} & 39 \\ 16 & -12 & \frac{52}{5} \\ -12 & 9 & \frac{-39}{5} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{5}{212} R_1]{R_3 \leftarrow R_3 + \frac{3}{4} R_2} \begin{pmatrix} 1 & \frac{391}{212} & \frac{195}{212} \\ 16 & -12 & \frac{52}{5} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4.7}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 16 R_1} \begin{pmatrix} 1 & \frac{391}{212} & \frac{195}{212} \\ 0 & \frac{-2200}{53} & \frac{-1144}{265} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4.8}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-53}{2200} R_2} \begin{pmatrix} 1 & \frac{391}{212} & \frac{195}{212} \\ 0 & 1 & \frac{13}{125} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4.9}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{391}{212} R_2} \begin{pmatrix} 1 & 0 & \frac{4823}{6625} \\ 0 & 1 & \frac{13}{125} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4.10}$$

Hence,

$$\mathbf{c} = \begin{pmatrix} \frac{4823}{6625} \\ \frac{13}{125} \end{pmatrix} = \begin{pmatrix} 0.728 \\ 0.104 \end{pmatrix} \tag{2.4.11}$$

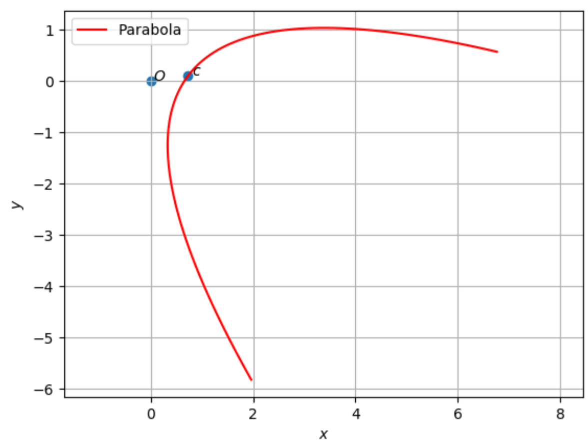

Fig. 0: Parabola with vertex c