Matrix Theory (EE5609) Assignment-6

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Abstract—**This document contains the QR decomposition.**

Download latex-tikz codes from

https://github.com/EE20RESCH11008/Matrix−Theory/tree/master/Assignment−6

1 Problem

1). Find QR decomposition of

$$\mathbf{V} = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \tag{1.0.1}$$

2). Find the vertex $\mathbf{c}$ of the parabola using SVD for

$$16x^2 - 24xy + 9y^2 + 32x + 86y - 39 = 0 \tag{1.0.2}$$

also verify the result using least squares.

2 Solution

2.1 QR Decomposition

Let, the column vectors of $\mathbf{V}$ be $\mathbf{v_1}$ and $\mathbf{v_2}$:

$$\mathbf{v_1} = \begin{pmatrix} 16 \\ -12 \end{pmatrix} \tag{2.1.1}$$

$$\mathbf{v_2} = \begin{pmatrix} -12 \\ 9 \end{pmatrix} \tag{2.1.2}$$

To find $\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}$, we will orthonormalize the columns of $\mathbf{V}$ using Gram-Schmidt method:

$$\mathbf{u_1} = \frac{\mathbf{v_1}}{k_1} \tag{2.1.3}$$

$$k_1 = \|\mathbf{v_1}\| = \sqrt{16^2 + (-12)^2} = 20 \tag{2.1.4}$$

$$\implies \mathbf{u_1} = \frac{1}{20}\begin{pmatrix} 16 \\ -12 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix} \tag{2.1.5}$$

$$\mathbf{u_2} = \frac{\mathbf{v_2} - r_1\mathbf{u_1}}{\|\mathbf{v_2} - r_1\mathbf{u_1}\|} \tag{2.1.6}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{v_2}}{\|\mathbf{u_1}\|^2} = \frac{\begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \end{pmatrix}\begin{pmatrix} -12 \\ 9 \end{pmatrix}}{(\frac{4}{5})^2 + (\frac{-3}{5})^2} = -15 \tag{2.1.7}$$

From (2.1.2) (2.1.5) and (2.1.7),

$$\mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.8}$$

$$k_2 = \mathbf{u_2}^T\mathbf{v_2} = 0 \tag{2.1.9}$$

The QR decomposition is given as:

$$\begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}\begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.1.10}$$

Where,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.1.11}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.1.12}$$

From (2.1.5) (2.1.8) and (2.1.11)

$$\mathbf{Q} = \begin{pmatrix} \frac{4}{5} & 0 \\ \frac{-3}{5} & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix} \tag{2.1.13}$$

From (2.1.4) (2.1.7) (2.1.9) and (2.1.12)

$$\mathbf{R} = \begin{pmatrix} 20 & -15 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 20 & -15 \end{pmatrix} \tag{2.1.14}$$

Substituting the values of (2.1.5) (2.1.8) (2.1.4) (2.1.9) and (2.1.7) in (2.1.10) We get,

$$\boxed{\mathbf{V} = \begin{pmatrix} \frac{4}{5} & 0 \\ \frac{-3}{5} & 0 \end{pmatrix}\begin{pmatrix} 20 & -15 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{pmatrix}\begin{pmatrix} 20 & -15 \end{pmatrix}} \tag{2.1.15}$$

The general second degree equation can be expressed as:

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.16}$$

From (1.0.2) and (2.1.16),

$$\mathbf{V} = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 16 \\ 43 \end{pmatrix}, \quad f = -39 \tag{2.1.17}$$

Eigen Values of $\mathbf{V}$ is given as

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{2.1.18}$$

$$\implies \begin{vmatrix} \lambda - 16 & 12 \\ 12 & \lambda - 9 \end{vmatrix} = 0 \tag{2.1.19}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{2.1.20}$$

Hence,

$$\lambda_1 = 0, \quad \lambda_2 = 25 \tag{2.1.21}$$

Eigen-vector corresponding to $\lambda_1 = 0$,

$$\implies \mathbf{p_1} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.1.22}$$

Eigen-vector corresponding to $\lambda_2 = 25$,

$$\implies \mathbf{p_2} = \frac{1}{5}\begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{2.1.23}$$

From (2.1.17) and (2.1.22)

$$\eta = 2\mathbf{p_1}^T\mathbf{u} = 88 \tag{2.1.24}$$

Using (2.1.21) and (2.1.24), Focal length of the parabola is given by:

$$\left|\frac{\eta}{\lambda_2}\right| = \left|\frac{88}{25}\right| = 3.52 \tag{2.1.25}$$

The standard equation of parabola is given by:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.1.26}$$

And the vertex $\mathbf{c}$ is:

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.1.27}$$

Substitute values of (2.1.17) (2.1.22) and (2.1.24) in (2.1.27) we get,

$$\begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix}\mathbf{c} = \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \tag{2.1.28}$$

2.2 Singular Value Decomposition:

$$\mathbf{Mc} = \mathbf{b} \tag{2.2.1}$$

where

$$\mathbf{M} = \begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix}, b = \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \tag{2.2.2}$$

To solve (2.2.1), we perform singular value decomposition on $\mathbf{M}$ given as

$$\mathbf{M} = \mathbf{USV^T} \tag{2.2.3}$$

Substituting the value of $\mathbf{M}$ from (2.2.3) in (2.2.1), we get

$$\mathbf{USV^Tc} = \mathbf{b} \tag{2.2.4}$$

$$\implies \mathbf{c} = \mathbf{VS_+U^Tb} \tag{2.2.5}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{MM^T}$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M^TM}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M^TM}$. First calculating the eigenvectors corresponding to $\mathbf{M^TM}$.

$$\mathbf{M^TM} = \begin{pmatrix} \frac{212}{2} & 16 & -12 \\ \frac{391}{5} & -12 & 9 \end{pmatrix}\begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix} = \begin{pmatrix} \frac{54944}{25} & \frac{75392}{25} \\ \frac{75392}{25} & \frac{158506}{25} \end{pmatrix} \tag{2.2.6}$$

Eigen values of $\mathbf{M^TM}$ can be found out as

$$\left|\mathbf{M^TM} - \lambda\mathbf{I}\right| = 0 \tag{2.2.7}$$