Discrete Maths: Pre Regional Maths Olympiad

G V V Sharma*

Abstract—**This book provides a collection of the Indian maths olympiad problems in discrete mathematics.**

1. Rama was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3. She got 43 as the answer. What would have been her answer if she had solved the problem correctly?
2. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle ?
3. The letters R, M and O represent whole numbers. If $R \times M \times O = 240$, $R \times O + M = 46$, $R + O \times M = 64$, What is the value of R + M + O?
4. A postman has to deliver five letters to five different houses. Mischievously, he posts one letter through each door without looking to see if it is the correct address. In how many different ways could he do this so that exactly two of the five houses receive the correct letters ?
5. If
$$\frac{1}{\sqrt{2011 + \sqrt{2011^2 - 1}}} = \sqrt{m} - \sqrt{n}$$
where m and n are positive integers, what is the value of m + n?
6. How many non-negative integral values of x satisfy the equation $[\frac{x}{5}] = [\frac{x}{7}]$? (Here [x] denotes the greatest integer less than or equal to x. For example [3.4] = 3 and [-2.3] = -3.)
7. Let N be the set of natural numbers. Suppose $f : N \rightarrow N$ is a function satisfying the following conditions.
   a) f(mn) = f(m)f(n);
   b) $f(m) < f(n)$ if $m < n$;
   c) f(2) = 2. What is the value of $\sum_{k=1}^{20}$ f(k)?
8. Three points X, Y, Z are on a straight line such that $XY = 10$ and $XZ = 3$. What is the product of all possible values of YZ?
9. There are n - 1 red balls, n green balls and n + 1 blue balls in a bag. The number of ways of chosing two balls from the bag that have different colours is 299. What is the value of n?
10. Let $S(M)$ denote the sum of the digits of a positive integer M written in base 10. Let N be the smallest positive integer such that $S(N) = 2013$. What is the value of $S(5N + 2013)$?
11. Let Akbar and Birbal togeather have n marbles, where $n > 0$. Akbar says to Birbal, " If I give you some marbles then you will have twice as many marbles as I will have." Birbal says to Akbar, " If I give you some marbles then you will have thrice as many marbles as I will have."
12. Carol was given three numbers and was asked to add the largest of the three to the product of the other two. Instead, she multiplied the largest with the sum of other two, but still got the right answer. What is the sum of the three numbers?
13. To each element of the set
$$S = \{1, 2, ...1000\}$$
a colour is assigned. Suppose that for any two elements a, b of S, if 15 divides a + b then they are both assigned the same colour. What is the maximum possible number of distinct colours used?
14. Let m be the smallest odd positive integer for

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

which

$$1 + 2 + ... + m$$

is a square of an integer and let n be the smallest even positive integer for which

$$1 + 2 + ... + n$$

is a square of an integer. What is the value of $m + n$?

15. What is the maximum possible value of k for which 2013 can be written as a sum of k consecutive positive integers?
16. What is the sum(in base 10) of all natural numbers less than 64 which have exactly three ones in their base 2 representation?
17. The first term of a sequence is 2014. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2014^{th} of the sequence?
18. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle?
19. Let S be a set of real numbers with mean M. If the means of the sets $S \cup \{15\}$ and $S \cup \{15, 1\}$ are M+2 and M + 1, respectively, then how many elements does S have?
20. For how many natural numbers n between 1 and 2014 (both inclusive) is $\frac{8n}{9999-n}$ an integer?
21. One morning, each member of Manjul's family drank an 8-ounce mixture of coffee and milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Manjul drank $(\frac{1}{7})^{th}$ of the total amount of milk and $(\frac{2}{17})^{th}$ of the total amount of coffee. How many people are there in Manjul's family?
22. Let f be a one-to-one function from the set of natural numbers to itself such that f(mn) = f(m)f(n) for all natural numbers m and n. What is the least possible value of f(999)?
23. Let

$$x_1, x_2, ..., x_{2014}$$

be real numbers different from 1, such that

$$x_1 + x_2 +, ..., + x_{2014} = 1$$

$$\frac{x_1}{1 - x_1} + \frac{x_2}{1 - x_2} + ... + \frac{x_{2014}}{1 - x_{2014}} = 1$$

What is the value of

$$\frac{x_1^2}{1 - x_1} + \frac{x_2^2}{1 - x_2} + \frac{x_3^2}{1 - x_3} ...... + \frac{x_{2014}^2}{1 - x_{2014}}?$$

24. What is the number of ordered pairs (A, B) where A and B are subsets of $\{1, 2, ..., 5\}$ such that neither $A \subseteq B$ nor $B \subseteq A$?
25. A man walks a certain distance and rides back in $3\frac{3}{4}$ hours; he could ride both ways in $2\frac{1}{2}$ hours. How many hours would it take him to walk both ways?
26. How many line segments have both their endpoints located at the vertices of a given cube?
27. Let E(n) denote the sum of the even digits of n. For example, E(123) = 2 + 4 = 6. What is the value of

$$E(1) + E(2) + E(3) + ... + E(100)$$

28. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is perfect square?
29. A 2×3 rectangle 3×4 rectangle are contained within a square without overlapping at any interior point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square?
30. What is the greatest possible perimeter of a right-angled triangle with interger side lengths if one of the sides has length?
31. Let n be the largest integer that is the product of exactly 3 distint prime numbers, x, y and 10x + y, where x and y are digits. What is the sum of the digits of n?
32. At a party, each man danced with exactly four women and each woman danced with exactly three men. Nine men attended the party. How many woman attended the party?
33. A subset B of the set of first 100 integers had the property that no two elements of B sum to 125. What is the maximum possible number of elements in B?
34. The digits of a positive integer n are four consecutive integers in decreasing order when read from left to right. What is the sum of the possible remainders when n is divided by 37?
35. There are eight rooms on the first floor of a hotel, with four rooms on each side of the

corridor. Symmetrically situated. Four guests have to be accommodated in four of the eight rooms such that no two guests are in adjacent rooms or in opposite rooms. In how many ways can the guests be accommodated?

36. How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3?
37. A contractor has two teams of workers: team A and team B. Team A can complete a job in 12 days and Team B can do the same job in 36 days. Team A starts working on the job and Team B joins team A after 4 days. The team A withdraws after two more days. For how many more days should team B work to complete the job?
38. A pen costs $11/-$ and a notebook costs $13/-$. Find the number of ways in which a person can spend exactly $1000/-$ to buy pens and notebooks.
39. There are five cities A, B, C, D, E on a certain island. Each city is connected to every other city by road. In how many ways can a person starting from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once?(The order in which he visits the cities also matters: e.g, the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different.)
40. Let
$$f(x) = \sin\frac{x}{3} + \cos\frac{3x}{10}$$
for all real x. Find the least natural number n such that $f(n\pi + x) = f(x)$ for all real x.
41. In a class, the total number of boys and girls are in the ratio of 4:3. On one day it was found that 8 boys and 14 girls were absent from the class, and that the number of boys was the square of the number of girls. What is the total number of students in the class?
42. Five distinct 2-digit numbers are in a geometric progression. Find the middle term?
43. Suppose the altitudes of a triangle are 10, 12 and 15. What is the semi-perimeter?
44. For each positive integer, consider the highest common factor h_n of the two numbers $n! + 1$ and $(n+1)!$. For $n < 100$, find the largest value of h_n?
45. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume of to the second volume to the third. The number of pages in the second volume is 50 more than that in the first column, and the number pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n?
46. Consider all 6-digit numbers of the form $abccba$ where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7.
47. A point P in the interior of a regular hexagon is at distances 8, 8, 16 units from three consecutive vertices of the haxagon respectively. If r is radius of the circumscribed circle of the hexagon, what is the integer closest to r?
48. There are several tea cups in the kitchen, some with handles and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?
49. Determine the number of 8-tuples $(\epsilon_1, \epsilon_2, ...\epsilon_8)$ such that $\epsilon_1, \epsilon_2, ...\epsilon_8 \in \{1, -1\}$ and
$$\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + ...8\epsilon_8$$
is a multiple of 3.
50. Let
$$N = 6 + 66 + 666 + ...666....66$$
where there are hundred $6's$ in the last term in the sum. How many times does the digit 7 occur in the number N?
51. A positive integer k is said to be good if there exists a partition of $\{1, 2, 3, ...20\}$ in to disjoint proper subsets such that the sum of the numbers in each subset o the partition is k. How many good numbers are there?
52. If N is the number triangles of different shapes(i.e., not similar) whose angles are all integers(in degrees), what is $N/100$?
53. Let T be the smallest positive integer which, when divided by 11, 13, 15 leaves remainders

in the sets $\{7,8,9\}$, $\{1,2,3\}$, $\{4,5,6\}$ respectively. What is the sum of the squares of the digits of T?

54. What is the number of ways in which one can choose 60 unit squares from 11×11 chessboard such that no two choosen squares have a side in common?
55. What is the number of ways in which one can colour the squares of a 4×4 chessboard with colours red and blue such that each row as well as each column has exactly two red squares and two blue squares?
56. Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N?
57. From a square with sides of lengths 5, triangular pieces from the four corners are removed to form a regular octagon. Find the area removed to the nearest integer?
58. Let x_1 be a positive real number and for every integer $n \geq 1$ let
$$x_{n+1} = 1 + x_1 x_2 ......x_{n-1} x_n$$
If $x_5 = 43$, what is the sum of digits of the largest prime factor of x_6?
59. An ant leaves the anthill for its morning exercise. It walks 4 feet and then makes a 160^o turn to the right and walks 4 more feet. It makes another 160^o turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the anthill again. What is the distance in feet it would have walked?
60. Five persons wearing badges with numbers 1, 2, 3, 4, 5 are seated on 5 chairs around a circular table. In how many ways can they be seated so that no two persons whose badges have consecutive numbers are seated next to each other? (Two arrangements obtained by rotation around the table are considered different.)
61. On a clock, there are two instants between 12 noon and 1 PM. when the hour hand and the minute hand are at right angles. The difference in minutes between these two instants is written as $a + \frac{b}{c}$, where a, b, c are positive integers, with $b < c$ and b/c in the reduced form. What is the value of a + b + c?
62. Each of the numbers
$$x_1, x_2, ..., x_{101}$$
is ± 1. What is the smallest positive value of
$$\sum_{1 \leq i < j \leq 101} x_i x_j$$
63. Find the smallest positive integer $n \geq 10$ such that n + 6 is a prime and 9n + 7 is a perfect square.
64. A pen costs 13/− and a note book costs 17/−. A school spends exactly 10000/− in the year 2017-18 to buy x pens and y note books such that x and y are as close as possible (i.e., $|x - y|$ is minimum). Next year, in 2018-19, the school spends a little more than 10000/− and buys y pens and x note books. How much more did the school pay?
65. How many ordered pairs (a, b) of positive integers with $a < b$ and $100 \leq a, b \leq 1000$ satisfy
$$gcd(a, b) : lcm(a, b) = 1 : 495$$
66. Consider the set $E = \{5, 6, 7, 8, 9\}$. For any partition $\{A, B\}$ of E, with both A and B non-empty, consider the number obtained by adding the product of elements of A to the product of elements of B. Let N be the largest prime number among these numbers. Find the sum of the digits of N.
67. Consider the set E of all natural numbers n such that when divided by 11, 12, 13, respectively, the remainders, in that order, are distinct prime numbers in an arithmetic progression. If N is the largest number in E, find the sum of digits of N.
68. What is the greatest integer not exceeding the sum
$$\sum_{n=1}^{1599} \frac{1}{\sqrt{n}}$$
69. A $1 \times n$ rectangle ($n \geq 1$) is divided into n unit 1×1 squares. Each square of this rectangle is coloured red, blue or green. Let f(n) be the number of colourings of the rectangle in which there are an even number of red squares. What is the largest prime factor of $\frac{f(9)}{f(3)}$?(The number of red squares can be zero.)
70. A village has a circular wall around it, and the

wall has four gates pointing north, south, east and west. A tree stands outside the village, 16m north of the north gate, and it can be just seen appearing on the horizon from a point 48m east of the south gate. What is the diameter in meters, of the wall that surroundings the village?

71. We will say that a rearrangement of the letters of a word has no fixed letters if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, $HBRATA$ is a rearrangement with no fixed of $BHARAT$. How man distinguishable rearrangements with no fixed letters does $BHARAT$ have?(The two $A's$ are considered identical.)
72. Let E denote the set of all natural numbers n such that $3 < n < 100$ and the set $\{1, 2, 3, ..., n\}$ can be partitioned in to 3 subsets with equal sums. Find the number of elements of E?