Math Document Template

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***Abstract*—This is a document explaining for a question on the concept of linear algebra.**

Download all python codes from

```
svn co https://github.com/Ashuwin/summer_20/
    trunk/linear_algebra/codes
```

and latex-tikz codes from

```
svn co https://github.com/Ashuwin/summer_20/
    trunk/linear_algebra/figs
```

1 Triangle

1.1 Problem

1. Find the area of triangle whose vertices are
   a) $\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}2\\-4\end{pmatrix}$
   b) $\begin{pmatrix}-5\\-1\end{pmatrix}, \begin{pmatrix}3\\-5\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}$

1.2 Solution

1. The area of triangle ABC:
   Solution: The area of triangle ABC using cross product is obtained as:

$$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\| \tag{1.2.1.1}$$

$$\frac{1}{2}\left\|(\begin{pmatrix}-1\\0\end{pmatrix}-\begin{pmatrix}2\\3\end{pmatrix})\times(\begin{pmatrix}2\\-4\end{pmatrix}-\begin{pmatrix}2\\3\end{pmatrix})\right\| \tag{1.2.1.2}$$

$$\frac{1}{2}\left\|\begin{pmatrix}-3\\-3\end{pmatrix}\times\begin{pmatrix}0\\-7\end{pmatrix}\right\| = \frac{21}{2} \tag{1.2.1.3}$$

Area of $\triangle ABC = 10.5 units^2$ and it is found in the following python code:

```
codes/triangle/tri_area_ABC.py
```

$\triangle ABC$ in Fig.1.2.1 is generated using the following python code

```
codes/triangle/triangle1.py
```

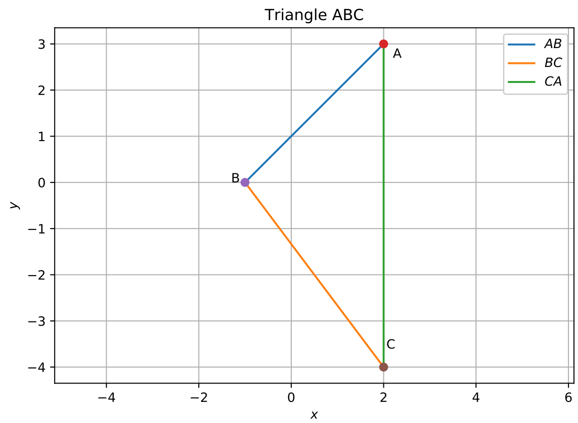

Fig. 1.2.1: Triangle ABC using python

2. The area of triangle PQR:
   Solution: The area of triangle PQR using Heron's formula is obtained as:

$$\frac{1}{2}\left\|(\mathbf{Q}-\mathbf{P})\times(\mathbf{R}-\mathbf{P})\right\| \tag{1.2.2.1}$$

$$\frac{1}{2}\left\|(\begin{pmatrix}3\\-5\end{pmatrix}-\begin{pmatrix}-5\\-1\end{pmatrix})\times(\begin{pmatrix}5\\2\end{pmatrix}-\begin{pmatrix}-5\\-1\end{pmatrix})\right\| \tag{1.2.2.2}$$

$$\frac{1}{2}\left\|\begin{pmatrix}8\\-4\end{pmatrix}\times\begin{pmatrix}10\\3\end{pmatrix}\right\| = \frac{64}{2} \tag{1.2.2.3}$$

Area of $\triangle PQR = 32 units^2$ and it is found in the following python code:

```
codes/triangle/tri_area_PQR.py
```

$\triangle PQR$ in Fig.1.2.2 is generated using the following python code

```
codes/triangle/triangle2.py
```

2 Quadrilateral

2.1 Problem

1. Find the area of the quadrilateral whose vertices are, taken in order, are $\begin{pmatrix}-4\\2\end{pmatrix}, \begin{pmatrix}-3\\-5\end{pmatrix}, \begin{pmatrix}3\\-2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}$

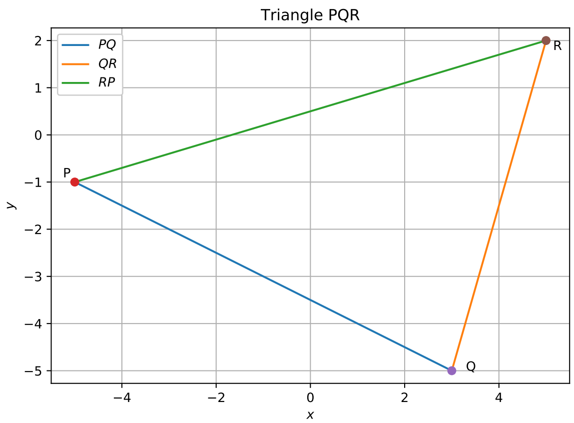

Fig. 1.2.2: Triangle PQR using python

2.2 Solution

1. The area of triangle ABC:
Solution: The area of triangle ABC using cross product is obtained as:

$$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\| \tag{2.2.1.1}$$

$$\frac{1}{2}\left\|(\begin{pmatrix}-3\\-5\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix})\times(\begin{pmatrix}3\\-2\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix})\right\| \tag{2.2.1.2}$$

$$\frac{1}{2}\left\|\begin{pmatrix}1\\-7\end{pmatrix}\times\begin{pmatrix}7\\-4\end{pmatrix}\right\| = \frac{45}{2} \tag{2.2.1.3}$$

Area of $\triangle ABC = 22.5 units^2$ and it is found in the following python code:

```
codes/tri_area_ABC.py
```

2. The area of triangle ACD:
Solution: The area of triangle ACD using Heron's formula is obtained as:

$$\frac{1}{2}\left\|(\mathbf{C}-\mathbf{A})\times(\mathbf{D}-\mathbf{A})\right\| \tag{2.2.2.1}$$

$$\frac{1}{2}\left\|(\begin{pmatrix}3\\-2\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix})\times(\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix})\right\| \tag{2.2.2.2}$$

$$\frac{1}{2}\left\|\begin{pmatrix}7\\-4\end{pmatrix}\times\begin{pmatrix}6\\1\end{pmatrix}\right\| = \frac{31}{2} \tag{2.2.2.3}$$

Area of $\triangle ACD = 15.5 units^2$ and it is found in the following python code:

```
codes/tri_area_ACD.py
```

3. The area of quadrilateral $ABCD$:
Solution: Area of Quadrilateral $ABCD$ = Area of $\triangle ABC$ + Area of $\triangle ACD$ = $38 units^2$

4. Quadrilateral $ABCD$ in Fig.2.2.4 is generated using the following python code

```
codes/quadrilateral/quad.py
```

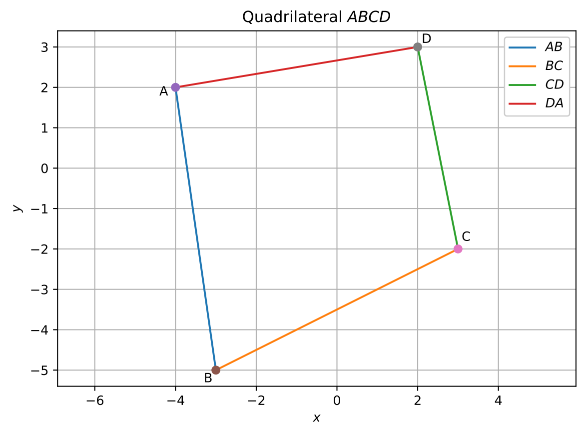

Fig. 2.2.4: Quadrilateral $ABCD$ using python

3 Line exercises

3.1 Complex numbers

3.1.1 Problem:

1. Find the conjugate of $\dfrac{\begin{pmatrix}3\\-2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}}{\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}}$

3.1.2 Solution:

1. A complex number $\begin{pmatrix}a\\b\end{pmatrix}$ can be represented as 2 x 2 matrix:

$$\begin{pmatrix}a & -b\\ b & a\end{pmatrix}$$

Multiplying the given complex numbers after converting them to a 2 x 2 matrix,

$$\frac{\begin{pmatrix}3 & 2\\ -2 & 3\end{pmatrix}\begin{pmatrix}2 & -3\\ 3 & 2\end{pmatrix}}{\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}} \tag{3.1.2.1.1}$$

$$\frac{\begin{pmatrix}12 & -5\\ 5 & 12\end{pmatrix}}{\begin{pmatrix}4 & -3\\ 3 & 4\end{pmatrix}} \tag{3.1.2.1.2}$$

Converting the matrices back to complex numbers,

$$\frac{\begin{pmatrix}12\\5\end{pmatrix}}{\begin{pmatrix}4\\3\end{pmatrix}} \tag{3.1.2.1.3}$$

Multiplying the conjugate of denominator to both numerator and denominator,

$$\frac{\begin{pmatrix}12\\5\end{pmatrix}\begin{pmatrix}4\\-3\end{pmatrix}}{\begin{pmatrix}4\\3\end{pmatrix}\begin{pmatrix}4\\-3\end{pmatrix}} \tag{3.1.2.1.4}$$

Multiplying the complex numbers after converting them to a 2 x 2 matrix,

$$\frac{\begin{pmatrix}12 & -5\\5 & 12\end{pmatrix}\begin{pmatrix}4 & 3\\-3 & 4\end{pmatrix}}{\begin{pmatrix}4 & -3\\3 & 4\end{pmatrix}\begin{pmatrix}4 & 3\\-3 & 4\end{pmatrix}} \tag{3.1.2.1.5}$$

$$\frac{\begin{pmatrix}63 & 16\\-16 & 63\end{pmatrix}}{\begin{pmatrix}25 & 0\\0 & 25\end{pmatrix}} \tag{3.1.2.1.6}$$

$$\frac{1}{25}\begin{pmatrix}63\\-16\end{pmatrix} \tag{3.1.2.1.7}$$

Conjugate of the complex number = $\frac{1}{25}\begin{pmatrix}63\\16\end{pmatrix}$

3.2 Points and Vectors

3.2.1 Problem: Find the point on the x-axis which is equidistant from

$$\begin{pmatrix}2\\-5\end{pmatrix}, \begin{pmatrix}-2\\9\end{pmatrix}$$

3.2.2 Solution:

1. From the given information,

$$\left\|\mathbf{x} - \begin{pmatrix}2\\-5\end{pmatrix}\right\|^2 = \left\|\mathbf{x} - \begin{pmatrix}-2\\9\end{pmatrix}\right\|^2 \tag{3.2.2.1.1}$$

$$\|\mathbf{x}\|^2 + \left\|\begin{pmatrix}2\\-5\end{pmatrix}\right\|^2 - 2\begin{pmatrix}2 & -5\end{pmatrix}\mathbf{x} = \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}-2\\9\end{pmatrix}\right\|^2 - 2\begin{pmatrix}-2 & 9\end{pmatrix}\mathbf{x}$$

which can be simplified to obtain

$$\begin{pmatrix}8 & -28\end{pmatrix}\mathbf{x} = -56 \tag{3.2.2.1.2}$$

Choose $\mathbf{x} = \begin{pmatrix}x\\0\end{pmatrix}$ as the point lies on the x-axis

$$\begin{pmatrix}8 & -28\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = -56 \tag{3.2.2.1.3}$$

$$\implies x = -7 \tag{3.2.2.1.4}$$

The point is $\begin{pmatrix}-7\\0\end{pmatrix}$

3.3 Points on a line

3.3.1 Problem: If $\mathbf{A} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\-4\end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB

3.3.2 Solution:

1. $\mathbf{A} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\-4\end{pmatrix}$
   Then $\mathbf{P}$ that divides $\mathbf{A}, \mathbf{B}$ in the ratio k:1 is

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{3.3.2.1.1}$$

For the given problem,$k = \frac{3}{4}$
Using the equation 3.3.2.1.1, the desired point is

$$\mathbf{P} = \frac{\frac{3}{4}\begin{pmatrix}2\\-4\end{pmatrix} + 1\begin{pmatrix}-2\\-2\end{pmatrix}}{\frac{3}{4}+1} \tag{3.3.2.1.2}$$

$$\mathbf{P} = \begin{pmatrix}-2/7\\-20/7\end{pmatrix} \tag{3.3.2.1.3}$$

The following python code plots the Fig.**??**

```
codes/point_line/int_sec.py
```

3.4 Lines and Planes

3.4.1 Problem: Write four solutions for each of the following equations
a)$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 7$
b)$\begin{pmatrix}\pi & 1\end{pmatrix}\mathbf{x} = 9$
c)$\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} = 0$

3.4.2 Solution:

1. $\mathbf{x}$ are randomly chosen and substituted in the equation and solutions are found.

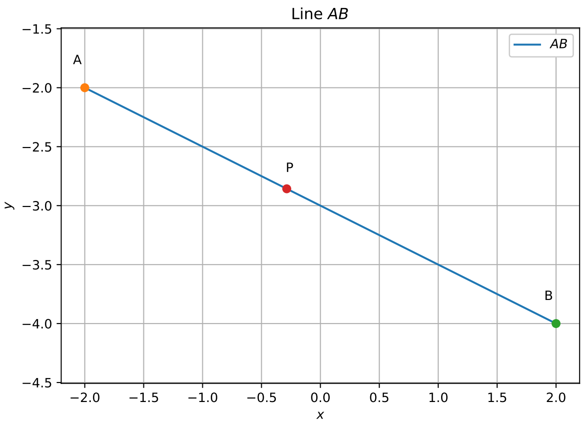

Fig. 3.3.2.1: Line AB using python

a)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 7 \quad (3.4.2.1.1)$$

Solution: Let $\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ Substituting in equation 3.4.2.1.1, $\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = 7$
$\implies a = \frac{7}{2}$
Let $\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}$ Substituting in equation 3.4.2.1.1, $\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = 7$
$\implies b = 7$
Let $\mathbf{x} = \begin{pmatrix} c \\ 1 \end{pmatrix}$ Substituting in equation 3.4.2.1.1, $\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = 7$
$\implies c = 3$
Let $\mathbf{x} = \begin{pmatrix} 1 \\ d \end{pmatrix}$ Substituting in equation 3.4.2.1.1, $\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ d \end{pmatrix} = 7$
$\implies d = 5$

b)

$$\begin{pmatrix} \pi & 1 \end{pmatrix} \mathbf{x} = 9 \quad (3.4.2.1.2)$$

Solution: Let $\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ Substituting in equation 3.4.2.1.2, $\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = 9$
$\implies a = \frac{9}{\pi}$
Let $\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}$ Substituting in equation 3.4.2.1.2, $\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = 9$
$\implies b = 9$
Let $\mathbf{x} = \begin{pmatrix} c \\ 1 \end{pmatrix}$ Substituting in equation 3.4.2.1.2, $\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = 9$
$\implies c = \frac{8}{\pi}$
Let $\mathbf{x} = \begin{pmatrix} 1 \\ d \end{pmatrix}$ Substituting in equation 3.4.2.1.2, $\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} 1 \\ d \end{pmatrix} = 9$
$\implies d = 9 - \pi$

c)

$$\begin{pmatrix} 1 & -4 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.2.1.3)$$

Solution: Let $\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ Substituting in equation 3.4.2.1.3, $\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = 0$
$\implies a = 0$
Let $\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}$ Substituting in equation 3.4.2.1.3, $\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = 0$
$\implies b = 0$
Let $\mathbf{x} = \begin{pmatrix} c \\ 1 \end{pmatrix}$ Substituting in equation 3.4.2.1.3, $\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = 0$
$\implies c = 4$
Let $\mathbf{x} = \begin{pmatrix} 1 \\ d \end{pmatrix}$ Substituting in equation 3.4.2.1.3, $\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ d \end{pmatrix} = 0$
$\implies d = \frac{1}{4}$

3.5 Motion in a plane

3.5.1 Problem:

1. A man can swim with a speed of $4.0km/h$ in still water. How long does he take to cross a river $1.0km$ wide if the river flows steadily at $3.0km/h$ and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank?

3.5.2 Solution:

1. Let the man be at point $\mathbf{M} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
   The speed of the man is $\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

The speed of the river is $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

Since the swimmer dive the river normal to the flow of river, therefore time taken by swimmer to cross the river,

$$t = \frac{d}{\|\mathbf{u}\|} = \frac{1km}{4km/h} = 15mins$$

Distance covered down the river $= t \times \|\mathbf{v}\|$

$$x = \frac{1}{4}hr \times 3km/h = 750m$$

The code for diagrammatic representation(3.5.2.1) of the solution is

```
codes/line/motion_plane/man_river.py
```

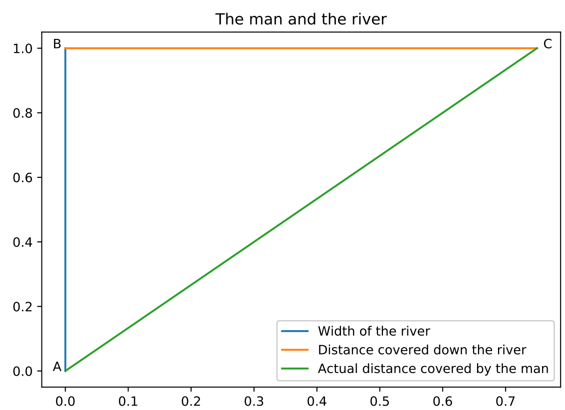

Fig. 3.5.2.1

3.6 Matrix

3.6.1 Problem:

1. Find the values of x,y and z from the following equations:
   a) $\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix}$
   b) $\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$
   c) $\begin{pmatrix} x+y+z \\ x+y \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

3.6.2 Solution:

1. This problem is solved by comparing the respective elements in both the matrices

a)

$$\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix} \tag{3.6.2.1.1}$$

$$x = 1, y = 4, z = 3 \tag{3.6.2.1.2}$$

b)

$$\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix} \tag{3.6.2.1.3}$$

$$5 + z = 5 \tag{3.6.2.1.4}$$

$$\implies z = 0 \tag{3.6.2.1.5}$$

$$x + y = 6 \tag{3.6.2.1.6}$$

$$xy = 8 \tag{3.6.2.1.7}$$

$$x = 4, y = 2 \tag{3.6.2.1.8}$$

$$x = 2, y = 4 \tag{3.6.2.1.9}$$

$x = 4, y = 2, z = 0$ or $x = 2, y = 4, z = 0$

c) $\begin{pmatrix} x+y+z \\ x+y \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

Expressing it as $Ax = b$ and $x = A^{-1}b$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix} \tag{3.6.2.1.10}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix} \tag{3.6.2.1.11}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \tag{3.6.2.1.12}$$

$$x = 2, y = 3, z = 4 \tag{3.6.2.1.13}$$

3.7 Determinants

3.7.1 Problem:

1. If $A = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$, find $|A|$.

3.7.2 Solution:

1. To find the value of the determinant,

$$\begin{pmatrix}1 & 1 & -2\\2 & 1 & -3\\5 & 4 & -9\end{pmatrix} \tag{3.7.2.1.1}$$

$$\xleftrightarrow{R_2\leftarrow R_2-2R_1}\begin{pmatrix}1 & 1 & -2\\0 & -1 & 1\\5 & 4 & -9\end{pmatrix} \tag{3.7.2.1.2}$$

$$\xleftrightarrow{R_3\leftarrow R_3-5R_1}\begin{pmatrix}1 & 1 & -2\\0 & -1 & 1\\0 & -1 & 1\end{pmatrix} \tag{3.7.2.1.3}$$

$$\xleftrightarrow{R_3\leftarrow R_3-R_2}\begin{pmatrix}1 & 1 & -2\\0 & -1 & 1\\0 & 0 & 0\end{pmatrix} \tag{3.7.2.1.4}$$

Det $|A| = 1 \times -1 \times 0 = 0$
The value of the determinant is found in the following python code

```
codes/line/determinants/det.py
```

3.8 Linear Inequalities

3.8.1 Problem:

1. Solve $3x + 2y > 6$ graphically

3.8.2 Solution:

1. Let $3x+2y = 6$ intersects the x-axis and y-axis at **A** and **B** respectively.
   a) Let $\mathbf{A} = \begin{pmatrix}x\\0\end{pmatrix}$

$$3x = 6 \tag{3.8.2.1.1}$$
$$\implies x = 2 \tag{3.8.2.1.2}$$
$$\mathbf{A} = \begin{pmatrix}2\\0\end{pmatrix} \tag{3.8.2.1.3}$$

   b) Let $\mathbf{B} = \begin{pmatrix}0\\y\end{pmatrix}$

$$2y = 6 \tag{3.8.2.1.4}$$
$$\implies y = 3 \tag{3.8.2.1.5}$$
$$\mathbf{B} = \begin{pmatrix}0\\3\end{pmatrix} \tag{3.8.2.1.6}$$

   c) Origin $= \begin{pmatrix}0\\0\end{pmatrix}$ does not satisfy the equation $3x + 2y < 6$.
   $\implies$ The solution is the right side of the line $3x + 2y = 6$

2. The following python code is the diagrammatic representation of the solution in Fig.3.8.2.2

```
codes/linear_inequalities/linear_inequalities.py
```

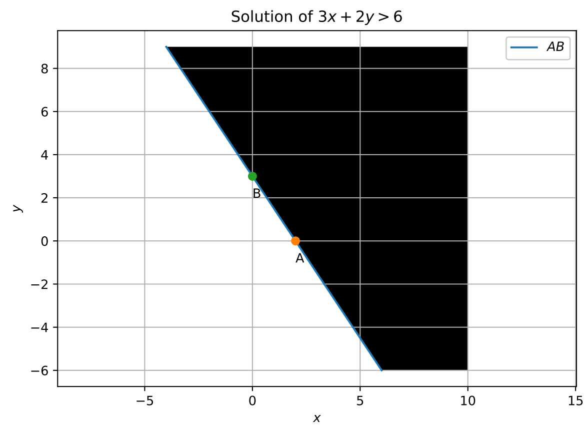

Fig. 3.8.2.2

3.9 Miscellaneous

3.9.1 Problem:

1. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid- point of the base is at the origin. Find vertices of the triangle.

3.9.2 Solution:

1. Let $\triangle ABC$ be an equilateral triangle. Let AB be the base and **M** be the midpoint.
   a) $\mathbf{A} = \begin{pmatrix}0\\m\end{pmatrix}$ (as point **A** lies on the y-axis)
   $\mathbf{B} = \begin{pmatrix}0\\n\end{pmatrix}$ (as point **B** lies on the y-axis)
   $\mathbf{M} = \begin{pmatrix}0\\0\end{pmatrix}$ (as point **M** lies on the origin)
   $\mathbf{C} = \begin{pmatrix}p\\0\end{pmatrix}$ (as point **A** lies on the x-axis)

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{C}-\mathbf{A}\| = 2a \tag{3.9.2.1.1}$$

b) $\mathbf{M}$ is the mid-point of $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{3.9.2.1.2}$$
$$\implies m = -n \tag{3.9.2.1.3}$$
$$\|\mathbf{A}-\mathbf{B}\| = 2a \tag{3.9.2.1.4}$$
$$\implies m = -n = a \tag{3.9.2.1.5}$$

c)

$$\|\mathbf{C}-\mathbf{A}\| = 2a \tag{3.9.2.1.6}$$
$$\sqrt{p^2+a^2} = 2a \tag{3.9.2.1.7}$$
$$\implies p = \pm\sqrt{3}a \tag{3.9.2.1.8}$$

d) The vertices are,
$\mathbf{A} = \begin{pmatrix} 0 \\ a \end{pmatrix}$
$\mathbf{B} = \begin{pmatrix} 0 \\ -a \end{pmatrix}$
$\mathbf{C} = \begin{pmatrix} \pm\sqrt{3}a \\ 0 \end{pmatrix}$
$\triangle ABC$ in Fig.3.9.2.1 is generated using the following python code

```
codes/line/miscellaneous/tri_equi.py
```

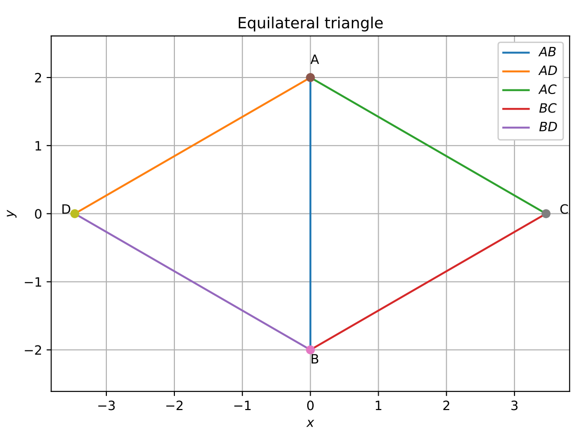

Fig. 3.9.2.1: Triangles ABC and ABD using python

4 Circle

4.1 Problem

1. Find the equation of the circle passing through the points $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{x} = 16$.

4.2 Solution

1. The vector form of general equation of circle,
$$\mathbf{x}^T\mathbf{x} - 2\mathbf{O}^T\mathbf{x} + F = 0 \tag{4.2.1.1}$$
whose centre is $\mathbf{O}$.
2. Point $\mathbf{A} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ lies on the circle. So, point A satisfies the equation 4.2.1.1
$\begin{pmatrix} 4 \\ 1 \end{pmatrix}^T \begin{pmatrix} 4 \\ 1 \end{pmatrix} - 2\mathbf{O}^T \begin{pmatrix} 4 \\ 1 \end{pmatrix} + F = 0$
$$2\mathbf{O}^T \begin{pmatrix} 4 \\ 1 \end{pmatrix} - F = 17 \tag{4.2.2.1}$$
3. Point $\mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ lies on the circle. So, point B satisfies the equation 4.2.1.1
$$\begin{pmatrix} 6 \\ 5 \end{pmatrix}^T \begin{pmatrix} 6 \\ 5 \end{pmatrix} - 2\mathbf{O}^T \begin{pmatrix} 6 \\ 5 \end{pmatrix} + F = 0 \tag{4.2.3.1}$$
$$2\mathbf{O}^T \begin{pmatrix} 6 \\ 5 \end{pmatrix} - F = 61 \tag{4.2.3.2}$$
4. Centre $\mathbf{O}$ lies on the line $\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{x} = 16$
$$\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{O} = 16 \tag{4.2.4.1}$$
5. Solving equations 4.2.2.1, 4.2.3.2 and 4.2.4.1
$$\mathbf{O} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, F = 15$$
$\implies$ Equation of the circle is $x^2 + y^2 - 6x - 8y + 15 = 0$
The vector form is
$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} + 15 = 0$$
6. The circle in Fig.4.2.6 is generated using the following python code

```
codes/circle/circle.py
```

5 Conics

5.1 Problem

1. Find the roots of the quadratic equations:
a) $x^2 - 3x - 10 = 0$
b) $2x^2 + x - 6 = 0$
c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
d) $2x^2 - x + \frac{1}{8} = 0$
e) $100x^2 - 20x + 1 = 0$

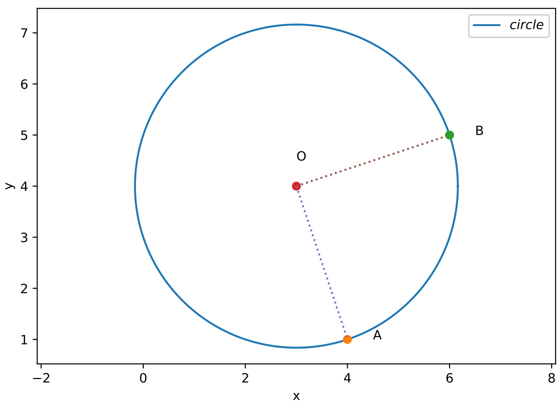

Fig. 4.2.6: Circle generated using python

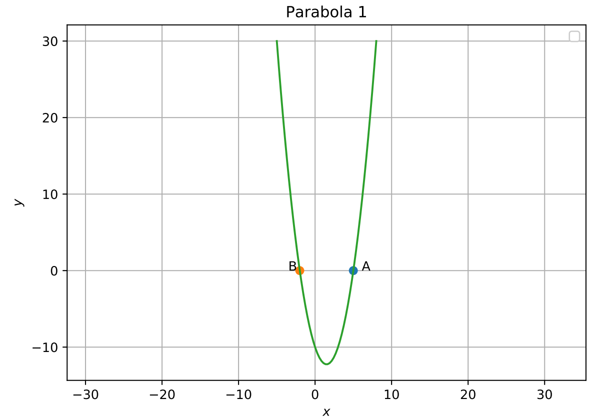

Fig. 5.2.1: Parabola 1

5.2 Solution

1. For a general polynomial equation of degree 2 ,
$p(x, y) \implies Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
The vector form is
$$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \tag{5.2.1.1}$$

   a) $x^2 - 3x - 10 = 0$
   The vector form from the equation is 5.2.1.1 is
   $$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -3 & 0 \end{pmatrix} \mathbf{x} - 10 = 0 \tag{5.2.1.2}$$
   The values of $\mathbf{x}$ are found in the following python code

   ```
   codes/conics/conics_1.py
   ```

   $\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
   which can be verified from the Fig.5.2.1. The following python code generates the fig.5.2.1

   ```
   codes/conics/conics_1.py
   ```

   b) $2x^2 + x - 6 = 0$
   The vector form from the equation is 5.2.1.1 is
   $$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{5.2.1.3}$$
   The values of $\mathbf{x}$ are found in the following python code

   ```
   codes/conics/conics_2.py
   ```

   $\mathbf{x} = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
   which can be verified from the Fig.5.2.1. The following python code generates the fig.5.2.1

   ```
   codes/conics/conics_2.py
   ```

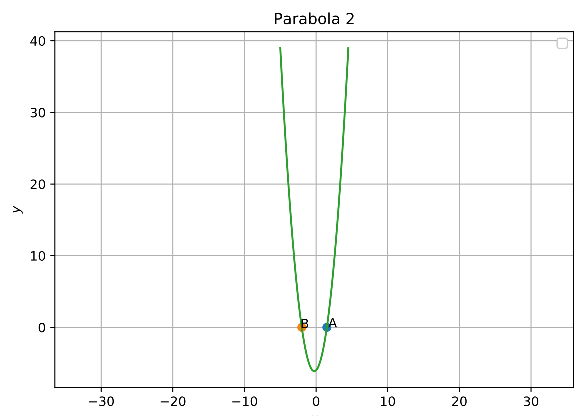

Fig. 5.2.1: Parabola 2

   c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
   The vector form from the equation is 5.2.1.1 is
   $$\mathbf{x}^T \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & 0 \end{pmatrix} \mathbf{x} + 5\sqrt{2} = 0 \tag{5.2.1.4}$$
   The values of $\mathbf{x}$ are found in the following python code

   ```
   codes/conics/conics_3.py
   ```

$\mathbf{x} = \begin{pmatrix} -1.414 \\ 0 \end{pmatrix}, \begin{pmatrix} -3.535 \\ 0 \end{pmatrix}$ which can be verified from the Fig.5.2.1. The following python code generates the fig.5.2.1

```
codes/conics/conics_3.py
```

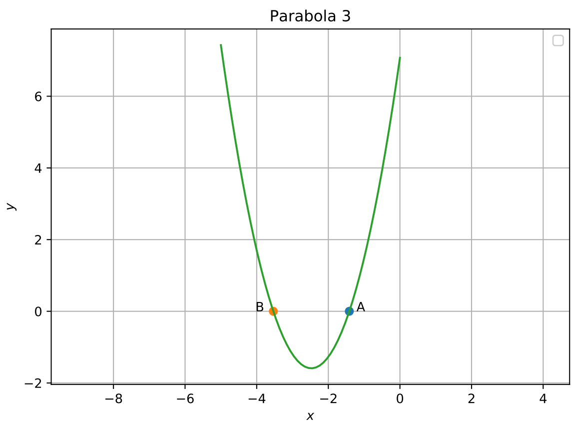

Fig. 5.2.1: Parabola 3

d) $2x^2 - x + \frac{1}{8} = 0$
The vector form from the equation is 5.2.1.1 is

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} + \frac{1}{8} = 0 \quad (5.2.1.5)$$

The values of $\mathbf{x}$ are found in the following python code

```
codes/conics/conics_4.py
```

$\mathbf{x} = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}$
which can be verified from the Fig.5.2.1. The following python code generates the fig.5.2.1

```
codes/conics/conics_4.py
```

e) $100x^2 - 20x + 1 = 0$
The vector form from the equation is 5.2.1.1 is

$$\mathbf{x}^T \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -20 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (5.2.1.6)$$

The values of $\mathbf{x}$ are found in the following python code

```
codes/conics/conics_5.py
```

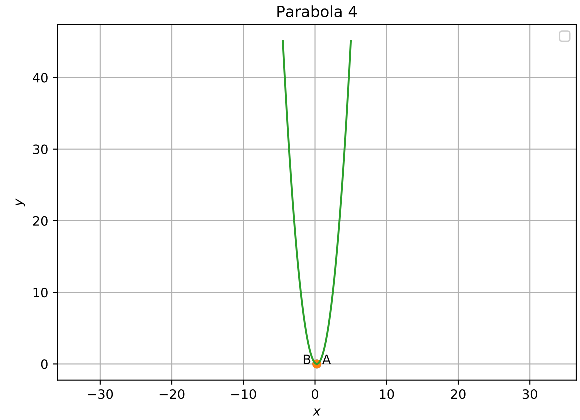

Fig. 5.2.1: Parabola 4

$\mathbf{x} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$ which can be verified from the Fig.5.2.1. The following python code generates the fig.5.2.1

```
codes/conics/conics_5.py
```

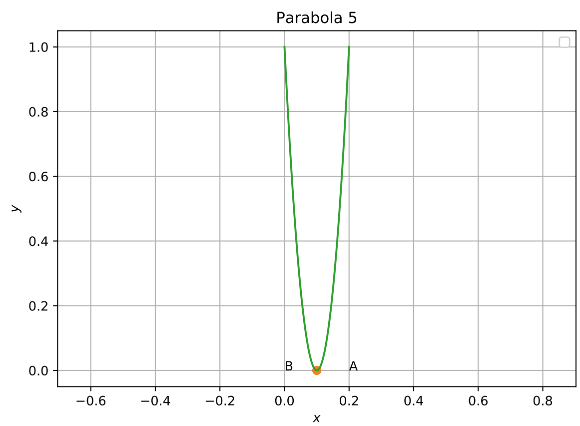

Fig. 5.2.1: Parabola 5