Python for Math Computing

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Problem 1. For $x \in \mathbf{R}, x \neq 0, x \neq 1$, let $f_0(x) = \frac{1}{1-x}$ and $f_{n+1}(x) = f_0\left(f_n(x)\right), n = 0, 1, \ldots$. Then find the value of $f_{100}(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right)$.

Solution: From the given information,

$$f_1(x) = f_0(f_0(x)) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{-x}, \tag{1.1}$$

$$f_2(x) = f_0(f_1(x)) = \frac{1}{1-\frac{1-x}{-x}} = x, \tag{1.2}$$

$$f_3(x) = f_0(f_2(x)) = \frac{1}{1-x} = f_0(x), \tag{1.3}$$

$$f_4(x) = f_0(f_3(x)) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{-x} = f_1(x) \tag{1.4}$$

The function repeats in a similar manner for other values of n as well. From (1.1),(1.2), (1.3) and (1.4),

$$f_{100}(3) = f_1(3) = \frac{1-3}{-3} = \frac{2}{3} \tag{1.5}$$

$$f_1\left(\frac{2}{3}\right) = \frac{1-\frac{2}{3}}{-\frac{2}{3}} = \frac{-1}{2} \tag{1.6}$$

$$f_2\left(\frac{3}{2}\right) = \frac{3}{2} \tag{1.7}$$

resulting in

$$f_{100}(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right) = \frac{2}{3} + \frac{-1}{2} + \frac{3}{2} = \frac{5}{3} \tag{1.8}$$

```
#Importing numy, scipy, mpmath and
    pyplot
import numpy as np
import mpmath as mp
import scipy
import scipy.stats as sp
import matplotlib.pyplot as plt
import subprocess


def recr(n,x):
        if n%3 == 1:
                return 1.0/(1-x);
                                #f0
        elif n%3 == 2:
                return recr(1,recr
                    (1,x));  #f1


#%f100 which is equal to f1
a=recr(1,recr(100,3))
b=recr(1,recr(1,2.0/3.0))    #f1
c=recr(1,recr(2,3.0/2.0))    #f2

print a,b,c

#%Gives the required result
Sum=a+b+c

print Sum
```

Problem 2. If $P = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}, A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $Q = PAP^T$, find $P^T Q^{2015} P$.

Solution: Since $Q = PAP^T$,

$$P^T Q^{2015} P = P^T (PAP^T)^{2015} P \tag{2.1}$$

$$= \left\{(P^T P)A\right\}^{2015} \tag{2.2}$$

$$= A^{2015} \tag{2.3}$$

since $PP^T = I$. Since, A=$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, $A^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$,

$$P^T Q^{2015} P = \begin{pmatrix} 1 & 2015 \\ 0 & 1 \end{pmatrix} = A^{2015} = \begin{pmatrix} 1 & 2015 \\ 0 & 1 \end{pmatrix}$$

```
import numpy as np

P = np.matrix([[np.sqrt(3)
    /2,0.5],[-0.5,np.sqrt(3)/2]])
```

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```
A = np.matrix([[1,1],[0,1]])

B = np.matrix.transpose(P)

Q = np.dot(np.dot(P,A),B)

X = np.linalg.matrix_power(Q,2015)

print np.dot(np.dot(B,X),P)
```

Problem 3. Evaluate $\sum_{r=1}^{15} r^2 \frac{\binom{15}{r}}{\binom{15}{r-1}}$.

Solution:

$$\sum_{r=1}^{15} r^2 \frac{\binom{15}{r}}{\binom{15}{r-1}} = \sum_{r=1}^{15} r^2 \frac{15!}{r!\,(15-r)!} \times \frac{(r-1)!\,(15-r+1)!}{15!} \tag{3.1}$$

which can be expressed as

$$\sum_{r=1}^{15} r^2 \frac{(r-1)!}{r!} \frac{(16-r)!}{(15-r)!} = \sum_{r=1}^{15} r^2 \frac{(16-r)}{r} \tag{3.2}$$

$$= \sum_{r=1}^{15} (16r - r^2) \tag{3.3}$$

$$= 16 \sum_{r=1}^{15} r - \sum_{r=1}^{15} r^2$$

resulting in

$$16 \left\{ \frac{r\,(r+1)}{2} \right\} - \frac{r\,(r+1)\,(2r+1)}{6} \tag{3.4}$$

$$= \frac{(48r^2 + 48r) - (2r^3 + 3r^2 + r)}{6} \tag{3.5}$$

$$= \frac{-2r^3 + 45r^2 + 47r}{6} \tag{3.6}$$

```
from mpmath import *

#numerical
s = 0
for r in range (1,16):
        s += binomial(15,r)*r**2/
            binomial(15,r-1)
print s

#theoretical
r = 15
print (-2*r**3 +45*r**2 + 47*r)/6
```

Problem 4. If

$$\lim_{x\to\infty} \left(1 + \frac{a}{x} - \frac{4}{x^2}\right)^{2x} = e^3, \tag{4.1}$$

find a.

Solution: Since the above expression is quadratic, let

$$\left(1 + \frac{a}{x} - \frac{4}{x^2}\right)^{2x} = \left[\left(1 + \frac{\alpha}{x}\right)\left(1 - \frac{\beta}{x}\right)\right]^{2x} \tag{4.2}$$

$$= \left[\left(1 + \frac{\alpha}{x}\right)^{\frac{x}{\alpha}}\right]^{2\alpha} \left[\left(1 - \frac{\beta}{x}\right)^{\frac{\beta}{x}}\right]^{2\beta} \tag{4.3}$$

$$\Rightarrow \lim_{x\to 0} \left[\left(1 + \frac{\alpha}{x}\right)^{\frac{x}{\alpha}}\right]^{2\alpha} \left[\left(1 - \frac{\beta}{x}\right)^{\frac{\beta}{x}}\right]^{2\beta} = e^{2(\alpha-\beta)} \tag{4.4}$$

Thus, from (4.1), (5.2) and (5.3), we obtain

$$a = \alpha - \beta \tag{4.5}$$

$$2\,(\alpha - \beta) = 3 \tag{4.6}$$

$$\Rightarrow a = \frac{3}{2} \tag{4.7}$$

The following octave code yields Fig. 4 verifying the above result.

```
import numpy as np
import matplotlib.pyplot as plt

x = 1e3
a = np.linspace(0,2,100)
y = (1 + a/x -4/x**2)**(2*x)
z = ((np.exp(3)))*np.ones(100)
bx=plt.plot(a,z, label = 'e^3')
plt.plot(a,y, label = 'Limit␣value
    ')

sol = np.zeros((2,1))
sol[0] = 3.0/2.0
sol[1] = np.exp(3)

#Display solution
A = np.around(sol[0],decimals=2)
B = np.around(sol[1],decimals=2)


plt.plot(A,B,'o')
for xy in zip(A,B):
        plt.annotate('(%s,␣%s)' %
            xy, xy=xy, xytext=(30,0)
```

```
            , textcoords='offset␣
            points')

plt.grid()
plt.legend([bx[0]],['$e^3$'], loc=
    'best',prop={'size':11})
plt.xlabel('$a$')
plt.ylabel('$\\left(1␣+␣\\frac{a}{
    x}␣-␣\\frac{4}{x^2}\\right)^{2x}
    $')
plt.savefig('../figs/ee16b1004.eps
    ')
plt.show()
```

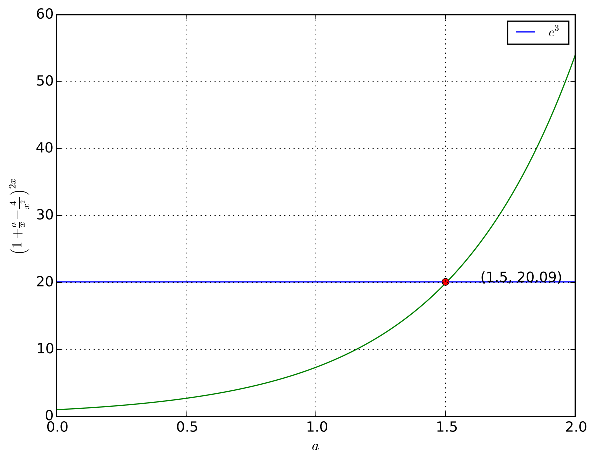

Fig. 4: LHS and RHS in (4.1)

Problem 5. The function

$$f(x) = \begin{cases} -x & x < 1 \\ a + \cos^{-1}(x+b) & 1 \le x \le 2 \end{cases} \tag{5.1}$$

is known to be differentiable at $x = 1$. What is the value of $\frac{a}{b}$?

Solution: Since the function is differentiable at $x = 1$,

$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x) \tag{5.2}$$

Also,

$$\lim_{x\to 1^-} f'(x) = -1 \tag{5.3}$$

$$\lim_{x\to 1^+} f'(x) = -\frac{1}{\sqrt{1-(x+b)^2}} \tag{5.4}$$

From (5.2) and (5.3),

$$-\frac{1}{\sqrt{1-(x+b)^2}} = -1 \Rightarrow b \quad = -x = -1 \tag{5.5}$$

Since a differentiable function is also continuous,

$$\lim_{x\to 1^+} \quad a + \cos^{-1}(x+b) = \lim_{x\to 1^-}(-x) \tag{5.6}$$

$$\Rightarrow a + \frac{\pi}{2} = -1 \tag{5.7}$$

$$\Rightarrow a = -1 - \frac{\pi}{2} \tag{5.8}$$

Then

$$c = \frac{a}{b} = 1 + \frac{\pi}{2} \tag{5.9}$$

The following octave code yields Fig. 5 verifying the above result.

```
import numpy as np
import matplotlib.pyplot as plt
x1 = 1
x2 = np.linspace(-1,1,1000)
x3 = np.linspace(1,2,1000)
b = -x1
a = -x1-np.arccos(b+x1)
y = -x2
z = a + np.arccos(b+(x3))
plt.plot(x3,z, label = '$f(x)␣=␣-
    x$')
plt.plot(x2,y, label = '$f(x)␣=␣a␣
    +␣\cos^{-1}(x+b)$')

sol = np.zeros((2,1))
sol[0] = 1
sol[1] = -1

#Display solution
A = sol[0]
B = sol[1]

plt.plot(A,B,'o')
for xy in zip(A,B):
        plt.annotate('(%s,␣%s)' %
            xy, xy=xy, xytext=(30,0)
            , textcoords='offset␣
            points')

plt.grid()
plt.legend(loc='best',prop={'size'
    :11})
plt.xlabel('$x$')
```

```
plt.ylabel('$f(x)$')
plt.savefig('../figs/ee16b1005.eps
    ')
plt.show()
```

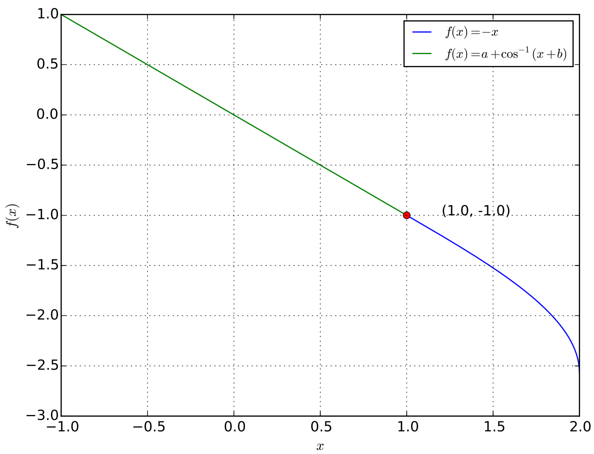

Fig. 5: Substituting the values of a and b in $f(x)$, the graph is smooth at $x = 1$. So $f(x)$ is differentiable $x = 1$.

Problem 6. The tangent at point P, for the curve $x = 4t^2 + 3, y = 8t^3 - 1$, with parameter $t \in \mathbf{R}$, meets the curve again at Q. Find the coordinates of Q.

Solution: Let P and Q be $\left(4t^2 + 3, 8t^3 - 1\right)$ and $\left(4{t_1}^2 + 3, 8{t_1}^3 - 1\right)$ respectively. At P,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{24t^2}{8t} = 3t. \tag{6.1}$$

Since the tangent at P meets the curve at Q, the equation of the tangent can be expressed as

$$\left(y - 8t_1^3 + 1\right) = 3t\left(x - 4t_1^2 - 3\right) \tag{6.2}$$

which, upon substitution of coordinates of P yields

$$\left(8t^3 - 8t_1^3\right) = 3t\left(4t^2 - 4t_1^2\right) \tag{6.3}$$

$$\Rightarrow 8\,(t_1 - t)\,({t_1}^2 + t_1 t + t^2) = 3t.4\,(t_1 - t)\,(t_1 + t)$$

$$\Rightarrow 2({t_1}^2 + t_1 t + t^2) = 3t(t_1 + t) \tag{6.4}$$

$$\Rightarrow 2{t_1}^2 - tt_1 - t^2 = 0 \tag{6.5}$$

$$\Rightarrow (t_1 - t)(2t_1 + t) = 0 \tag{6.6}$$

$$\Rightarrow t_1 = -\frac{t}{2} \tag{6.7}$$

Thus, Q can now be expressed as $\left(t^2 + 3, t^3 - 1\right)$. To demonstrate the solution of this problem, letting $t = 1$, we obtain P, Q as $(7, 7), (4, -2)$ respectively and the equation of the tangent is

$$y - 7 = 3\,(x - 7) \tag{6.8}$$

The following code generates the plot in Fig. 6 for this problem.

```
import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(-1,1.2,100)
x = 4*t**2 + 3
y = 8*t**3 - 1
z = 3*(x-7) + 7
plt.plot(x,y,label = 'curve, $x=4t
    ^2+3,y=8t^3-1$')
plt.plot(x,z,label = 'tangent, $y
    -7=3(x-7)$')
plt.grid()
plt.legend(loc='best',prop={'size'
    :11})


#Display solution
A = np.array([7, 4])
B = np.array([7, -2])

plt.plot(A,B,'o')
for xy in zip(A,B):
        plt.annotate('(%s, %s)' %
            xy, xy=xy, xytext=(30,0)
            , textcoords='offset
            points')

plt.text(7,8,'P')
plt.text(4,-4,'Q')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.savefig('../figs/ee16b1006.eps
    ')
plt.show()
```

Problem 7. Find the minimum distance of a point on the curve $y = x^2 - 4$ from the origin.

Solution: Let P be the point on the curve closest to the origin. If the coordinates of the the point be (h, k), then its distance from the origin is given by

$$d^2 = h^2 + k^2 \tag{7.1}$$

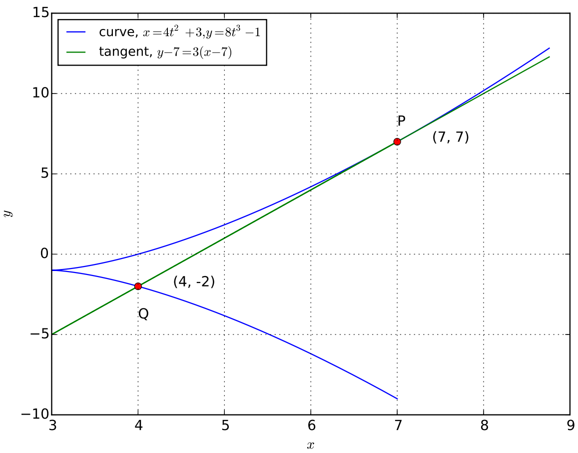

Fig. 6: The tangent at P meets the curve again at Q.

Since P lies on the curve,

$$k = h^2 - 4 \tag{7.2}$$

From (7.1) and (7.2),

$$\begin{aligned} d^2 &= k^2 + k + 4 \\ &= \left(k + \frac{1}{2}\right)^2 + \frac{15}{4} \end{aligned} \tag{7.3}$$

Thus, the smallest distance is given by the above equation as $\frac{\sqrt{15}}{2}$. The nearest point is given by (7.2) and (7.3) as $\left(\pm\sqrt{\frac{7}{2}}, -\frac{1}{2}\right)$

The following code yields Figs. 7.1 and 7.2 explaining the solution.

```
import numpy as np
import matplotlib.pyplot as plt

k = np.linspace(-5,5,100)
d = np.sqrt(k**2 + k + 4)

#dist = np.min(d)*np.ones(100)

plt.figure(1)
plt.plot(k,d,label = '$d=\\sqrt{k
    ^2+k+4}$')
plt.text(-0.5,np.sqrt(15)/2-0.2,'$
    \\left(-\\frac{1}{2},d_{min}=\\
    frac{\\sqrt{15}}{2}\\right)$')
plt.plot(-0.5,np.sqrt(15)/2,'o')
plt.grid()
plt.legend(loc='best',prop={'size'
    :11})
plt.xlabel('$k$')
plt.ylabel('$d$')
plt.savefig('../figs/ee16b1007a.
    eps')

x = np.linspace(-2.5,2.5,100)
y = x**2 - 4

plt.figure(2)
plt.plot(x,y,label = '$y=x^2-4$')
plt.ylim(-6,1)
plt.xlim(-2.5,2.5)
plt.plot(np.sqrt(3.5),-0.5,'o')
plt.plot(-np.sqrt(3.5),-0.5,'o')
plt.plot(0,0,'o')
plt.text(-np.sqrt(3.5)+0.1,-0.5,'P
    ␣$\\left(-\\sqrt{\\frac
    {7}{2}},-\\frac{1}{2}␣\\right)$'
    )
plt.text(np.sqrt(3.5)-0.8,-0.5,'$
    \\left(\\sqrt{\\frac{7}{2}},-\\
    frac{1}{2}␣\\right)$␣Q')
plt.text(0.1,0,'O␣(0,0)')
plt.xlabel('$x$')
plt.ylabel('y')
plt.grid()
plt.legend(loc='best',prop={'size'
    :11})
plt.savefig('../figs/ee16b1007b.
    eps')
plt.show()
```

Problem 8. Sketch the region

$$A = \left\{(x, y) \,|\, y \geq x^2 - 5x + 4, x + y \geq 1, y \leq 0\right\}. \tag{8.1}$$

Solution: The desired region is plotted in Fig. 8 using the following code.

```
import numpy as np
import matplotlib.pyplot as plt
from pylab import *

x = np.linspace(1,4,100)

y1 = 1-x
y2 = x**2 - 5*x + 4
X = np.concatenate([x,x[::-1]])
```

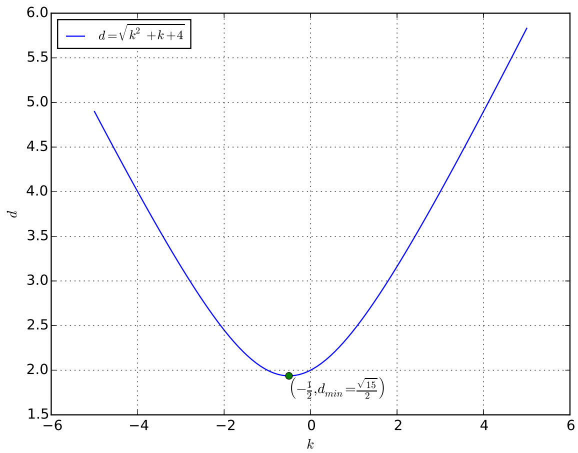

Fig. 7.1: The minimum distance is $\frac{\sqrt{17}}{2}$ for $k = -\frac{1}{2}$

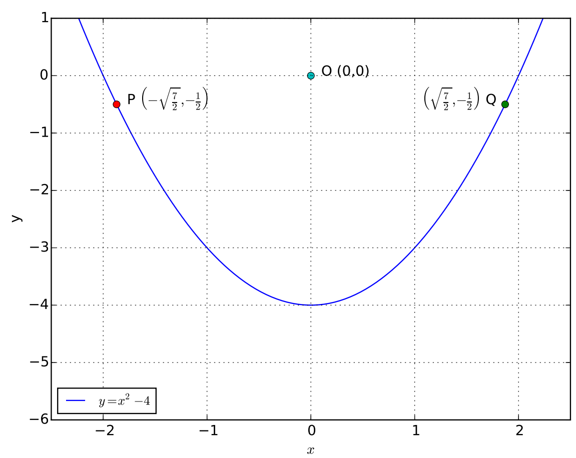

Fig. 7.2: OP and OQ represent the minimum distance of the origin from the parabola.

```
Y = np.concatenate([y1,y2[:: -1]])

f, a = plt.subplots()


p1= a.fill(x,y2,color='g')
p2= a.fill(X,Y,color = 'r')

plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')

a.legend([ p1[0]],['$A␣=␣\\left
    \\{(x,y)\\vert␣y␣\\geq␣x^2-5x+4,
```

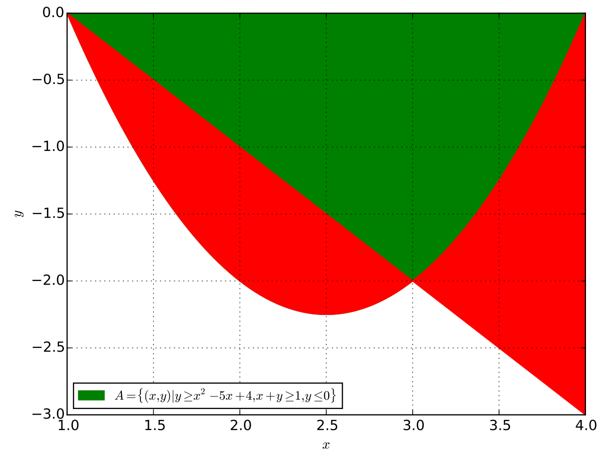

Fig. 8: The desired region is in green colour.

```
    ␣x␣+␣y␣\\geq␣1,␣y␣\\leq␣0\\right
    \\}$'], loc='best',prop={'size'
    :11})
plt.savefig('../figs/ee16b1008.eps
    ')
plt.show()
```

Problem 9. A variable line drawn through the intersection of the lines $\frac{x}{3} + \frac{y}{4} = 1$ and $\frac{x}{4} + \frac{y}{3} = 1$ meets the coordinate axes at A and $B, A \neq B$. Sketch the locus of the midpoint of AB.

Solution: The interstion of the two lines is the solution of the matrix equation

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9.1}$$

given by $\left(\frac{12}{7}, \frac{12}{7}\right)$. If A is $(a, 0)$ and B is $(0, b)$, the mid point of AB is $\left(\frac{a}{2}, \frac{b}{2}\right)$. The equation of the line AB is

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{9.2}$$

From the given information, this line passes through $\left(\frac{12}{7}, \frac{12}{7}\right)$. Substituting $h = \frac{a}{2}, y = \frac{b}{2}$ in the above and simplifying, the locus is obtained as

$$\frac{1}{h} + \frac{1}{k} = \frac{7}{6} \tag{9.3}$$

The sketch of the locus is available in Fig. 9.

```
import numpy as np
import matplotlib.pyplot as plt
```

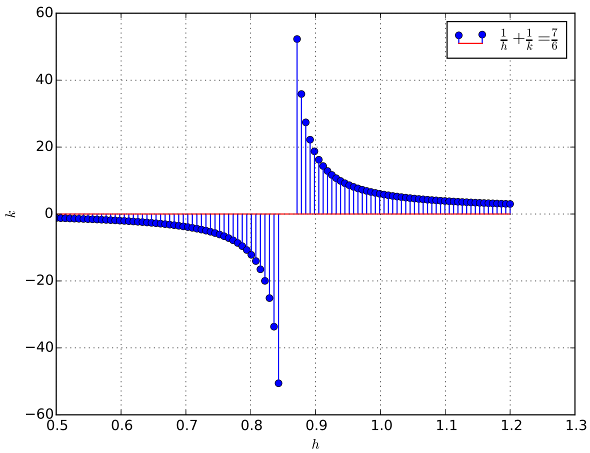

Fig. 9: Locus of the midpoint. Symmetric about the the point $\left(\frac{6}{7}, 0\right)$

```
#Point of intersection
A = np.array([[1.0/3,1.0/4] ,
    [1.0/4,1.0/3]])
B = np.array([1,1])
U = np.dot(np.linalg.inv(A),np.
    transpose(B))


#Sketching tThe curve
h = np.concatenate((np.linspace
    (0.5,5.9/7,50), np.linspace
    (6.1/7,1.2,50)), axis=0)
k = 1/(7.0/6 - 1/h)
plt.stem(h,k, label='$\\frac{1}{h}
    ␣+␣\\frac{1}{k}␣=␣\\frac{7}{6}$'
    )
plt.xlabel('$h$')
plt.ylabel('$k$')
plt.grid()
plt.legend()
plt.savefig('../figs/ee16b1009.eps
    ')
plt.show()
```

Problem 10. The point $(2, 1)$ is translated parallel to the line $L : x - y = 4$ by $2\sqrt{3}$ units to yield the point Q. If Q lies in the 3rd quadrant, sketch the line passing through Q and $\perp$ L.

Solution: The slope of the given line is 1 indicating an angle of $\frac{\pi}{4}$. Thus, the coordinates of Q are

$$\left(2 - 2\sqrt{3}\cos\frac{\pi}{4}, 1 - 2\sqrt{3}\frac{\pi}{4}\right) = \left(2 - \sqrt{6}, 1 - \sqrt{6}\right) \tag{10.1}$$

The equation of the line perpendicular to L can be expressed as

$$x + y = c \tag{10.2}$$

Since this line passes through Q, $c = 3 - 2\sqrt{6}$.

Fig. 10 illustrates this problem.

```
import numpy as np
import matplotlib.pyplot as plt

def line(a,b):
    m = (b[1]-a[1])/(b[0]-a[0])
    c = a[1]-m*a[0]
    x = np.linspace(a[0],b[0],100)
    y = m*x + c
    plt.plot(x,y, label = '$PQ$')

P = np.transpose(np.array([2,1]))
Q = P - 2*np.sqrt(3)*np.cos(0.25*
    np.pi)*np.transpose(np.array
    ([1,1]))
line(P,Q)
plt.plot(P[0],P[1],'o')
plt.plot(Q[0],Q[1],'o')
plt.text(P[0],P[1]-0.3,'P')
plt.text(Q[0]-0.3,Q[1],'Q')
plt.ylim(-5,2)
plt.xlim(-3,4)

x = np.linspace(Q[0],P[0])
y = x-4
plt.plot(x,y,label = '$x␣-␣y␣=␣4$'
    )

y = 3 - 2*np.sqrt(6) - x
plt.plot(x,y,label = '$x␣+␣y␣=␣c$'
    )
plt.grid()
plt.legend()
plt.axis('equal')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.savefig('../figs/ee16b1010.eps
    ')
plt.show()
```

Problem 11. A circle passes through $(-2, 4)$ and

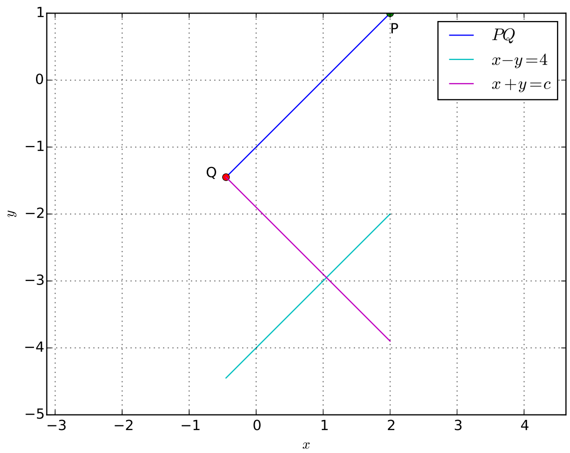

Fig. 10: $c = 3 - 2\sqrt{6}$

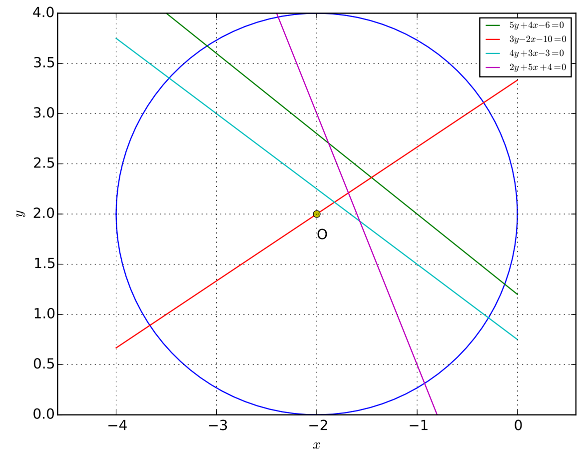

Fig. 11: $2x - 3y + 10 = 0$ is the diameter.

touches the $y-$axis at $(0, 2)$. Find out which of the following lines represents the diameter of the circle.

1) $4x + 5y - 6 = 0$
2) $2x - 3y + 10 = 0$
3) $3x + 4y - 3 = 0$
4) $5x + 2y + 4 = 0$

Solution: Let the equation of the circle be

$$(x - h)^2 + (y - k)^2 = r^2 \tag{11.1}$$

Since the circle touches the y-axis, the radius of the circle is $|h|$ and $k = 2$. Thus, the equation of the circle can be expressed as

$$(x - h)^2 + (y - 2)^2 = h^2 \tag{11.2}$$

Since the circle passes through the points $(-2, 4)$, substituting these in the above equation yields

$$(h + 2)^2 + 4 = h^2 \tag{11.3}$$

$$\Rightarrow h = -2 \tag{11.4}$$

Thus, the equation of the circle is

$$(x + 2)^2 + (y - 2)^2 = 4 \tag{11.5}$$

Fig. 11 illustrates this problem.

```
import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(-np.pi,np.pi,100)
r = 2
x = -2 + r*np.cos(t)
y = 2 + r*np.sin(t)
plt.plot(x,y)
plt.grid()
plt.axis('equal')
plt.axis([-4, 0, 0, 4])

x = np.linspace(-4,0,100)
y1 = (6 - 4*x)/5
y2 = (10 + 2*x)/3
y3 = 0.75*(1-x)
y4 = -(4 + 5*x)/2

plt.plot(x,y1,label = '$5y␣+␣4x␣-␣
    6␣=␣0$')
plt.plot(x,y2,label = '$3y␣-␣2x␣-␣
    10␣=␣0$')
plt.plot(x,y3,label = '$4y␣+␣3x␣-␣
    3␣=␣0$')
plt.plot(x,y4,label = '$2y␣+␣5x␣+␣
    4␣=␣0$')

plt.legend(loc=1,prop={'size':8})
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.plot(-2,2,'o')
plt.text(-2,1.75,'O')
plt.savefig('../figs/ee16b1011.eps
    ')
plt.show()
```

Problem 12. The eccentricity of a hyperbola satisfies the equation $9e^2 - 18e + 5 = 0$. $(5, 0)$ is a focus and the corresponding directrix is $5x = 9$. Plot the hyperbola.

Solution: The standard equation of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{12.1}$$

and the eccentricity is given by

$$e^2 = 1 + \frac{b^2}{a^2} \tag{12.2}$$

The focus of the hyperbola is at $(ae, 0), e > 1$. From the given information,

$$\begin{aligned} 9e^2 - 18e + 5 = 0, && (12.3)\\ \implies (3e-1)(3e-5) = 0 && (12.4)\\ && (12.5) \end{aligned}$$

yielding $e = \frac{1}{3}$ or $e = \frac{5}{3}$. Since $e > 1$, the desired value of the eccentricity is $e = \frac{5}{3}$. Since the focus is at $(5, 0)$, $a = 3$. From (12.2), substituting for the values of a and e,

$$1 + \left(\frac{b}{3}\right)^2 = \left(\frac{5}{3}\right)^2. \tag{12.6}$$

$$\implies b = 4 \tag{12.7}$$

Thus, the equation of the parabola is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \tag{12.8}$$

The following code plots the hyperbola in Fig. 12.

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-10,10,10000)
y1 = np.sqrt(16*((x**2)/9 - 1))
y2 = -np.sqrt(16*((x**2)/9 - 1))
plt.plot(x,y1)
plt.plot(x,y2)
plt.grid()
plt.ylabel('$y$')
plt.xlabel('$x$')
plt.savefig('../figs/ee16b1012.eps
    ')
plt.show()
```

Problem 13. Sketch the ellipse $\frac{x^2}{27} + \frac{y^2}{3} = 1$.

Solution: The following code plots the ellipse in Fig. 13

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-3*np.sqrt(3),3*np
    .sqrt(3),1000)
y = np.sqrt(3*(1-(x**2)/27))
z = -np.sqrt(3*(1-(x**2)/27))
plt.plot(x,y)
plt.plot(x,z)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.savefig('../figs/ee16b1013.eps
    ')
plt.show()
```

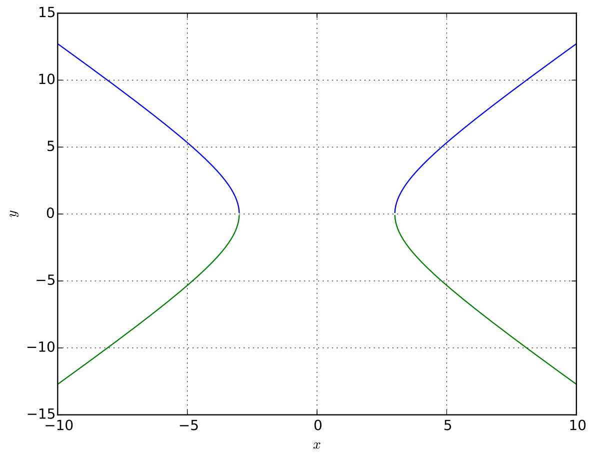

Fig. 12: Sketch of the hyperbola

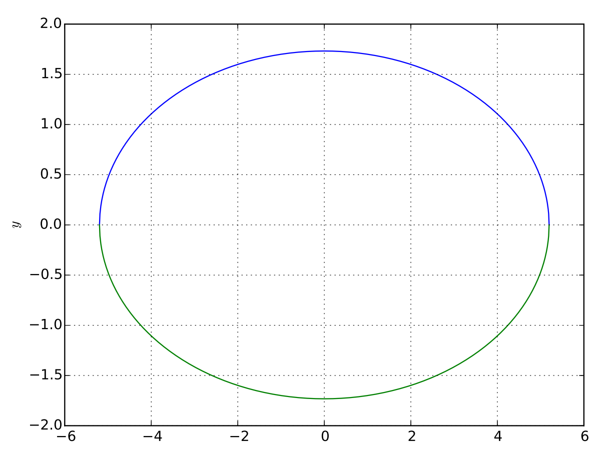

Fig. 13: Graph of ellipse $\frac{x^2}{27} + \frac{y^2}{3} = 1$

Problem 14. Find the minimum and maximum values of $4 + \frac{1}{2}\sin^2 2x - 2\cos^4 x, x \in \mathbf{R}$.

Solution: From the given information,

$$
\begin{aligned}
y &= 4 + \frac{1}{2}\sin^2 2x - 2\cos^4 x && (14.1)\\
&= 4 + 2\sin^2 x\cos^2 x - 2\cos^4 x && (14.2)\\
&= 2 + 2\sin^2 x\cos^2 x + 2\sin^2 x\left(1 + \cos^2 x\right) && (14.3)\\
&= 2 + 4\sin^2 x\cos^2 x + 2\sin^2 x && (14.4)\\
&= 4 - \cos 2x - \cos^2 2x && (14.5)\\
&= 4 + \frac{1}{4} - \left(\cos 2x + \frac{1}{2}\right) && (14.6)
\end{aligned}
$$

From the above, it is obvious that the maximum value is $4\frac{1}{4}$. From the above, we have

$$y = 2 + 2\sin^2 2x + 2\sin^2 x \quad (14.7)$$

which has the minimum value of 2 when $\sin x = 0$. The following code verifies the above result.

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-np.pi,np.pi,1000)
y = 4 + 0.5*(np.sin(2*x))**2 - 2*(
    np.cos(x))**4
plt.plot(x,y)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.savefig('../figs/ee16b1014.eps
    ')
plt.show()
```

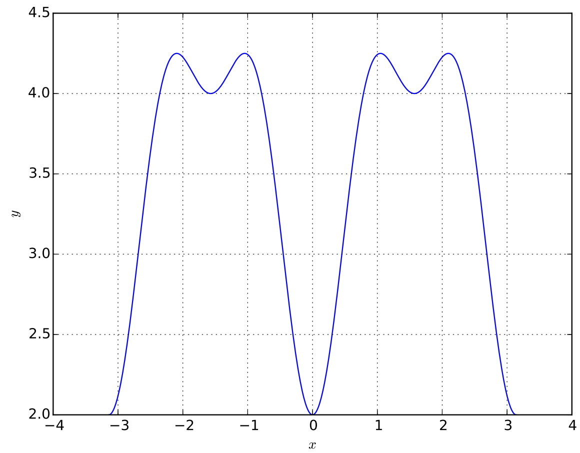

Fig. 14: Minimum value is 2 and maximum is $4\frac{1}{4}$.

Problem 15. Find the solution of the equation $\sqrt{2x+1} - \sqrt{2x-1} = 1, x \geq \frac{1}{2}$.

Solution: Since

$$
\begin{aligned}
\left(\sqrt{2x+1} - \sqrt{2x-1}\right)\left(\sqrt{2x+1} + \sqrt{2x-1}\right) &= 2,\\
\left(\sqrt{2x+1} + \sqrt{2x-1}\right) &= 2\\
\Rightarrow \sqrt{2x+1} = \frac{3}{2} \Rightarrow x &= \frac{5}{8} \quad (15.1)
\end{aligned}
$$

The graphical solution is available in Fig. 15

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0.5,5,100)
y = np.sqrt(2*x + 1) - np.sqrt(2*x
    - 1) - 1
plt.plot(x,y)
plt.plot(5.0/8,0,'o')
plt.grid()
plt.text(5.0/8 + 0.1,0,'P')
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.savefig('../figs/ee16b1015.eps
    ')
plt.show()
```

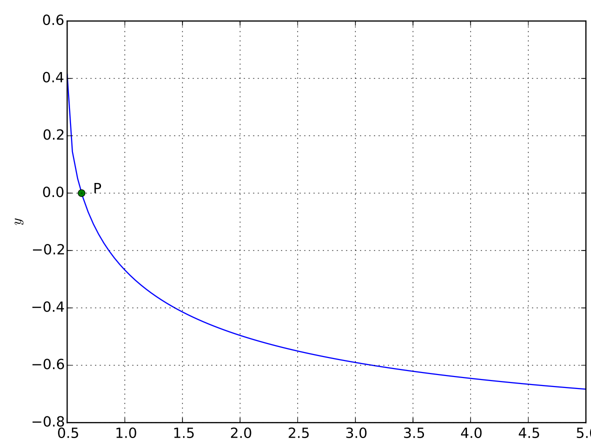

Fig. 15: $\sqrt{2x+1} - \sqrt{2x-1} - 1$ intersects the x-axis at $x = \frac{5}{8}$

Problem 16. Let $z = 1 + a\imath, a > 0$ be a complex number such that z^3 is a real number. Find $\sum_{k=0}^{11} z^k$.

Solution:

$$z^3 = (1 + ai)^3 \tag{16.1}$$
$$= (1 - 3a^2) + (3a - a^3) \tag{16.2}$$

Since z^3 is a real number,

$$\Im(z) = 0 \tag{16.3}$$
$$\Rightarrow 3a - a^3 = 0 \tag{16.4}$$

Since $a > 0$, the desired solution is $a = \sqrt{3}$. Hence $z = 1 + \sqrt{3}\imath = 2e^{\frac{\imath\pi}{3}}$ and

$$\sum_{k=0}^{11} z^k = \frac{(z^{12} - 1)}{z - 1} \tag{16.5}$$
$$= \frac{2^{12} e^{\frac{\imath 12\pi}{3}}}{1 + \sqrt{3}\imath - 1} \tag{16.6}$$
$$= \frac{2^{12}}{\sqrt{3}\imath} \tag{16.7}$$

The following code provides numerical solutions. a can be found through Fig. 16.

```
import numpy as np
import matplotlib.pyplot as plt

a = np.linspace(0,2,100)
z = 1 + 1j*a
y = np.imag(z**3)
a_t = np.roots([-1,0,3,0])
ind = np.nonzero(a_t > 0)
a_v = a_t[ind]
plt.plot(a,y)
plt.plot(a_v,0,'o')
plt.ylabel('$Im(z)$')
plt.xlabel('$a$')
plt.grid()
plt.savefig('../figs/ee16b1016.eps
   ')
plt.show()
```

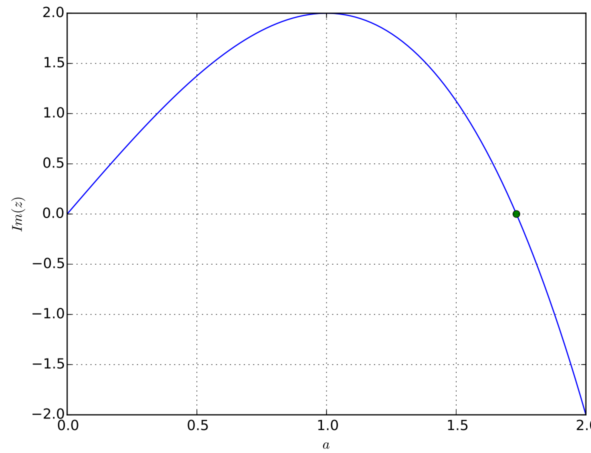

Fig. 16: For positive values, $\Im(z)$ intersects the x-axis at $a = \sqrt{3}$.

Problem 17. $A = \begin{pmatrix} -4 & -1 \\ 3 & 1 \end{pmatrix}$. Find the determinant of $A^{2016} - 2A^{2015} - A^{2014}$.

Solution: The given matrix expression can be simplified as

$$A^{2016} - 2A^{2015} - A^{2014} = A^{2014}\left(A^2 - 2A - I\right) \tag{17.1}$$

The characteristic equation for the matrix A is obtained as

$$det(A - \lambda I) = 0 \tag{17.2}$$
$$\Rightarrow (\lambda + 4)(\lambda - 1) + 3 = 0 \tag{17.3}$$
$$\Rightarrow \lambda^2 + 3\lambda - 1 = 0 \tag{17.4}$$

From the Cayley-Hamilton theorem,

$$A^2 + 3A - I = 0 \Rightarrow A^2 - 2A - I = -5A \tag{17.5}$$

Since $det(A) = -1, det(-5A) = -25$. The following code provides the numerical solution to the given problem.

```
import numpy as np

A = np.array([[-4,-1],[3,1]])
B = np.dot(A,A)
print np.linalg.det(B - 2*A - np.
   identity(2))
```

Problem 18. Find the solutions of the following equations

$$n^2 - 3n - 108 = 0$$
$$n^2 + 5n - 84 = 0$$
$$n^2 + 2n - 80 = 0$$
$$n^2 + n - 110 = 0$$

Which of these satisfy $\frac{{}^{n+2}C_6}{{}^{n-2}P_2} = 11$?

Solution: From the following code, the solution to each of the above equations are $n = 12, 7, 8$ and 10

respectively. The given condition can be expressed as

$$\frac{{}^{n+2}C_6}{{}^{n-2}P_2} = 11 \tag{18.1}$$

$$\Rightarrow \frac{(n+2)!}{(n-4)!6!}\frac{(n-4)!}{(n-2)!} = 11 \tag{18.2}$$

$$\Rightarrow \frac{n(n-1)(n+1)(n+2)}{6!} = 11 \tag{18.3}$$

From the above equation, it is obvious that the correct solution is 9. So none of the solutions of the given equations satisfy the given condition. This is verified numerically through the following code.

```
import numpy as np
import scipy as sp

A = np.array
    ([[1,-3,-108],[1,5,-84],[1,2,-80],[1,1,-110]])


q_r = []
for i in range(0,4):
    q_r.append(np.roots(A[i,:]))

q_r = np.array(q_r)
s = q_r[q_r>0]

print s*(s-1)*(s+1)*(s+2)/np.math.
    factorial(6)
```

Problem 19. Sketch

$$f(x) = \begin{cases} \frac{2x^2}{a} & 0 \le x < 1 \\ a & 1 \le x < \sqrt{2} \\ \frac{2b^2-4b}{x^3} & \sqrt{2} \le x < \infty \end{cases}$$

for (a, b) equal to
1) $\left(\sqrt{2}, 1-\sqrt{3}\right)$
2) $\left(-\sqrt{2}, 1+\sqrt{3}\right)$
3) $\left(\sqrt{2}, -1+\sqrt{3}\right)$
4) $\left(-\sqrt{2}, 1-\sqrt{3}\right)$

In which case is $f(x)$ continuous?

Solution: The following octave code generates the following figures

```

import numpy as np
import matplotlib.pyplot as plt

a = np.sqrt(2)*np.array
    ([1,-1,1,-1])
b = np.array([1,1,-1,1]) + np.sqrt
    (3)*np.array([-1,1,1,-1])
t = ord('a')
x1 = np.linspace(0,1,100)
x2 = np.linspace(1,np.sqrt(2),100)
x3 = np.linspace(np.sqrt(2),5,100)

for i in range (0,4):
    y1 = 2*(x1**2)/a[i]
    y2 = np.ones(100)*a[i]
    y3 = (2*(b[i]**2) - 4*b[i])/(
        x3**3)
    plt.figure(i)
    plt.plot(x1,y1)
    plt.plot(x2,y2)
    plt.plot(x3,y3)
    plt.grid()
    plt.savefig('../figs/ee16b1019
        '+chr(t+i)+'.eps')
```

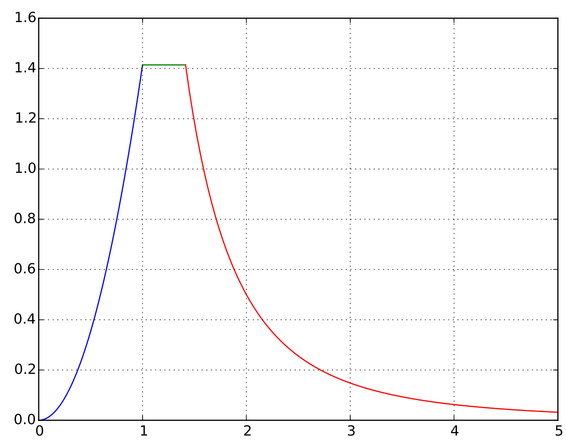

Fig. 19.1: Continuous for $\left(\sqrt{2}, 1-\sqrt{3}\right)$

Problem 20. Sketch $f(x) = \sin^4 x + \cos^4 x$. Find the intervals within $(0, \pi)$ when it is increasing.

Solution: The following code plots the graph in Fig. 20 outlining the intervals when the function is increasing.

```
import numpy as np
import matplotlib.pyplot as plt
from pylab import *

x = np.linspace(0,0.25*np.pi,100)
```

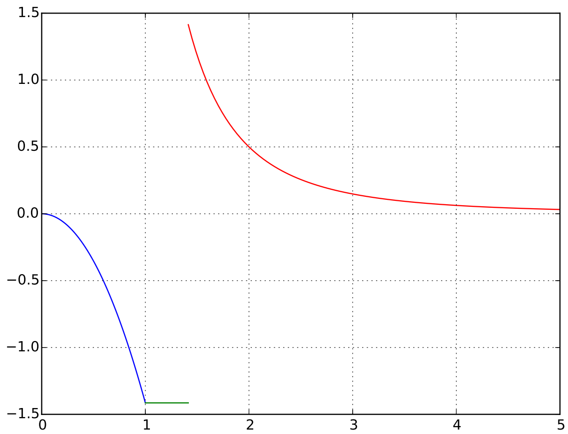

Fig. 19.2: Discontinuous for $\left(-\sqrt{2}, 1+\sqrt{3}\right)$

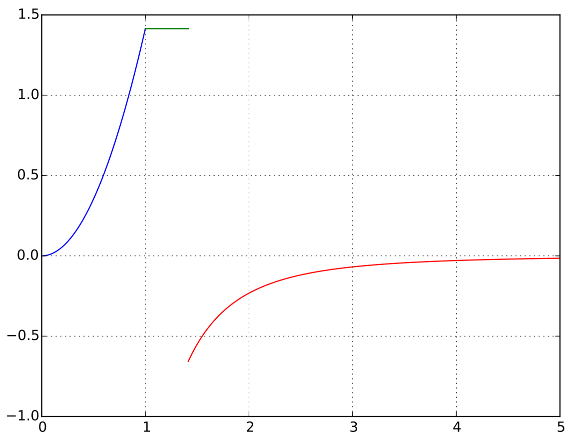

Fig. 19.3: Continuous for $\left(\sqrt{2}, 1+\sqrt{3}\right)$

```
z = np.linspace(0.25*np.pi,0.5*np.pi,100)
t = np.linspace(0.5*np.pi,0.75*np.pi,100)
s = np.linspace(0.75*np.pi,np.pi,100)

y = np.sin(x)**4 + np.cos(x)**4
u = np.sin(z)**4 + np.cos(z)**4
v = np.sin(t)**4 + np.cos(t)**4
w = np.sin(s)**4 + np.cos(s)**4

plt.plot(x,y)
fill_between(z,u,facecolor='orange')
plt.plot(t,v)
```

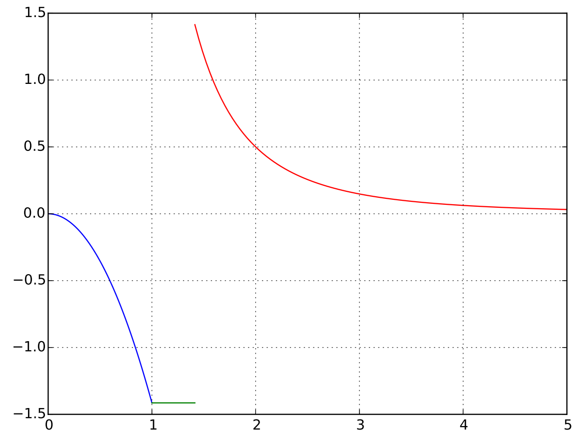

Fig. 19.4: Discontinuous for $\left(-\sqrt{2}, 1-\sqrt{3}\right)$

```
fill_between(s,w,facecolor='orange')
plt.grid()
plt.xlabel('$0 < x < \pi$')
plt.ylabel('$\sin^4(x) + \cos^4(x)$')
plt.savefig('../figs/ee16b1020.eps')
plt.show()
```

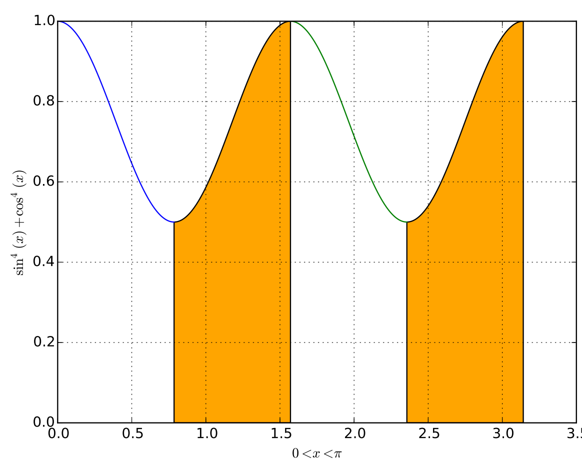

Fig. 20: The green shaded region is where the function is increasing.

Problem 21. The reflected line is given by $y+2x=1$. The surface is given by $7x-y+1=0$. Which of the following is the incident line?

1) $41x-38y+38=0$

2) $41x + 25y - 25 = 0$
3) $41x + 38y - 38 = 0$
4) $41x - 25y + 25 = 0$

Solution: The point at which the reflected line touches the surface is the solution of the equation

$$A = \begin{pmatrix} 2 & 1 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{21.1}$$

$$\tag{21.2}$$

yielding the point $(0, 1)$. Angle between the given line is given

$$\theta = \tan^{-1} \frac{(m_1 - m_2)}{(1 + m_1 * m_2)} \tag{21.3}$$

where m_1, m_2 are slopes of surface and reflected line respectively.

$$\theta = \tan^{-1} \frac{(7 - (-2))}{(1 + 7 * (-2)))} \tag{21.4}$$

$$\theta = \tan^{-1} \frac{-9}{13} \tag{21.5}$$

The slope of the incident line can be found by reversing the direction of the angle along the surface. Letting the angle that the incident line makes along the x-axis to be ϕ,

$$\phi = \tan^{-1} \frac{(m_1 - \tan\theta)}{(1 + m_1 * \tan\theta)} \tag{21.6}$$

$$m = \frac{(7 - \frac{9}{13})}{(1 + 7 * \frac{9}{13})} \tag{21.7}$$

$$m = \frac{(91 - 9)}{(63 + 13)} \tag{21.8}$$

$$m = \frac{41}{38} \tag{21.9}$$

Since m is the slope and 1 is the intercept and thus in slope form equation of line is $y = mx + 1$. Thus the equation of the incident line is

$$38y = 41x + 38 \tag{21.10}$$

The following code summarises the solution through the plot in Fig. 21

```
import numpy as np
import matplotlib.pyplot as plt

A = np.array([[2,1],[7,-1]])
B = np.transpose(np.array([1,-1]))
print np.dot(np.linalg.inv(A),B)

x = np.linspace(-3,3,10)
z = 7*x + 1

plt.plot(x,z,label = 'Surface')

x = np.linspace(-3,0,5)
y = 1 - 2*x
w = 41*x/38 + 1

plt.plot(x,y,label = 'Incident')
plt.plot(x,w,label = 'Reflected')
plt.grid()
plt.axis('equal')
plt.axis([-5,5 ,-5, 5])
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.legend()
plt.savefig('../figs/ee16b1021.eps
    ')
plt.show()
```

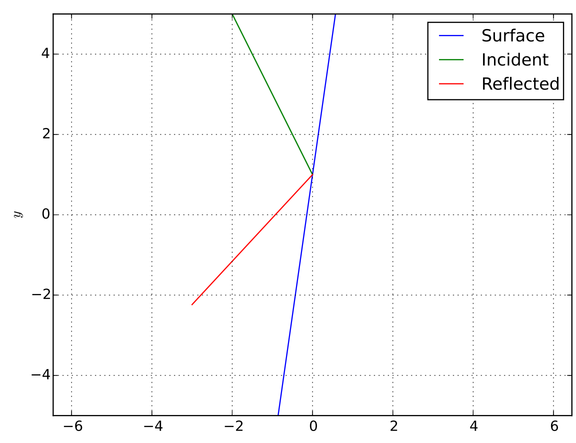

Fig. 21: $41x - 38y + 38 = 0$ is the incident line

Problem 22. The lines $x - y = 1$ and $2x + y = 3$ intersect at O. A circle with centre at point O passes through the point $(-1, 1)$. Sketch the following lines

1) $4x + y - 3 = 0$
2) $x + 4y + 3 = 0$
3) $3x - y - 4 = 0$
4) $x - 3y - 4 = 0$

Which of these is a tangent to the circle? At what point?

Solution: The lines $x-y=1$ and $2x+y=3$ intersect at the point O, whose coordinates are obtained from the following equation

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{22.1}$$

as $\left(\frac{4}{3}, \frac{1}{3}\right)$. Since the circle has centre at O and passes trough the point $(1,-1)$, its radius is

$$r = \sqrt{\left(\frac{4}{3}+1\right)^2 + \left(\frac{1}{3}-1\right)^2} = \frac{\sqrt{53}}{3} \tag{22.2}$$

The equation of the circle is then obtained as

$$\left(x-\frac{4}{3}\right)^2 + \left(y-\frac{1}{3}\right)^2 = \frac{53}{9} \tag{22.3}$$

The following octave code plots the circle as well as the various lines in Fig. 22 and shows that no line is a tangent to the circle.

```
import numpy as np
import matplotlib.pyplot as plt

#Point of intersection
A = np.array([[1,-1],[2,1]])
B = np.transpose(np.array([1,3]))
O = np.dot(np.linalg.inv(A),B)

#Finding radius
P = np.transpose(np.array([-1,1]))
r = np.linalg.norm(O-P)

#Intersection plot
x = np.linspace(-3,5,10000)
y = x - 1
z = 3 - 2*x
plt.axis('equal')
plt.plot(x,y,label = '$y - x + 1 =
    0$')
plt.plot(x,z,label = '$y + 2x = 3$
    ')
plt.grid()

y = np.sqrt(53/9 - (x-(4/3))**2) +
    1.0/3
z = -np.sqrt(53/9 - (x-(4/3))**2)
  + 1.0/3
r = 3 - 4*x
s = -(3 + x)/4
t = 3*x - 4
u = (x - 4)/3

plt.plot(x,y)
plt.plot(x,r,label = '$y + 3x = 3$
    ')
plt.plot(x,t,label = '$y - 3x + 4 
    = 0$')
plt.plot(x,u,label = '$3y - x + 4 
    = 0$')
plt.plot(x,z)
plt.plot(x,s,label = '$4y + x + 3 
    = 0$')
plt.axis([-3,5,-3,3],'equal')
plt.legend()
plt.savefig('../figs/ee16b1022.eps
    ')
plt.show()
```

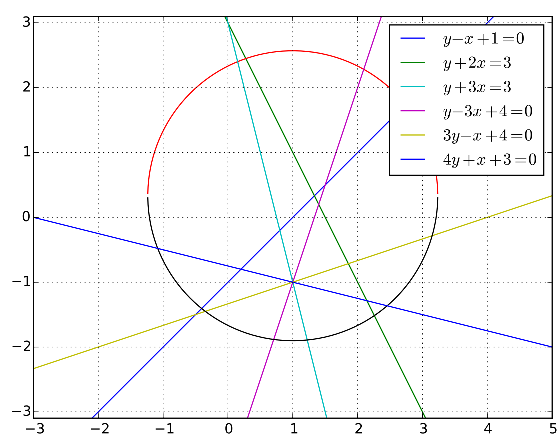

Fig. 22: Since the lines intersect the circle, they are not tangent to it and no point of tangency exists.

Problem 23. P and Q are distinct points on the parabola $y^2 = 4x$, with parameters t and t_1 respectively. The normal at P passes through Q. Find the minimum value of t_1^2.

Solution: Using the parametric form, the points P and Q can be expressed as $\left(t^2, 2t\right)$ and $\left(t_1^2, 2t_1\right)$ respectively. The slope of the normal at P is

$$-\frac{dx}{dy} = -\frac{\frac{dx}{dt}}{\frac{dy}{dt}} = -t \tag{23.1}$$

The equation of the normal is then obtained as

$$(y - 2t) = -t\left(x - t^2\right) \tag{23.2}$$
$$\Rightarrow 2(t_1 - t) = -t(t_1 - t)(t_1 + t) \tag{23.3}$$
$$\Rightarrow t_1 = -\left(t + \frac{2}{t}\right) \tag{23.4}$$

Thus,

$$t_1^2 = 4 + t^2 + \frac{4}{t^2} \tag{23.5}$$
$$= \left(t - \frac{2}{t}\right)^2 + 8 \tag{23.6}$$

and the minimum value of t_1^2 is obtained from the above as 8. For this value,

$$t - \frac{2}{t} = 0 \Rightarrow t = \pm\sqrt{2} \tag{23.7}$$

The following code provides a visualisation of the problem.

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0,20,500)
y1 = 2*np.sqrt(x)
y2 = -2*np.sqrt(x)

plt.axis('equal')
plt.plot(x,y1)
plt.plot(x,y2)
plt.grid()

temp = np.sqrt(2)*np.linspace
    (0.2,1.2,6)
for i in range(0,6):
    t = temp[i]
    P = np.array([t**2 , 2*t])
    if i==5:
        plt.plot(P[0],P[1],'o')
        plt.text(P[0]+0.2,P[1]+1,'
            P')
    else:
        plt.plot(P[0],P[1],'o')
    y = 2*t - t*(x - t**2)
    plt.plot(x,y)

t1 = -2*np.sqrt(2)
Q = np.array([t1**2,2*t1])
plt.plot(Q[0],Q[1],'o')
plt.text(Q[0]-1,Q[1]-2.5,'Q')
plt.savefig('../figs/ee16b1023.eps
    ')
plt.show()
```

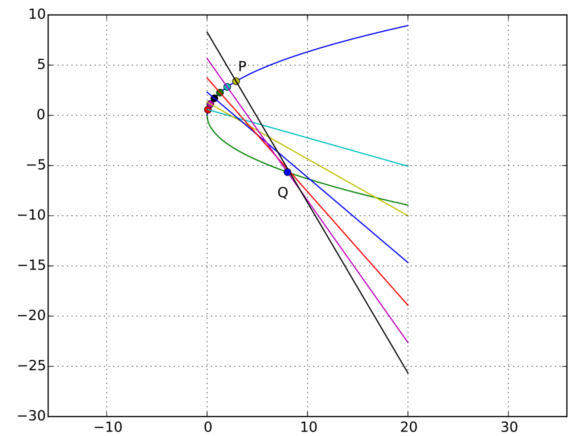

Fig. 23: Normals to the parabola for various values of t plotted. For $t = \sqrt{2}, t_1 = -2\sqrt{2}, Q\left(t_1^2 = 8, 2t_1 = -4\sqrt{2}\right)$ has the smallest x-coordinate among all the normals.

Problem 24. The transverse axis of a hyperbola is along the major axis of the conic $\frac{x^2}{3} + \frac{y^2}{4} = 4$. The vertices of the hyperbola are at the foci of this conic. The eccentricity of the hyperbola is $\frac{3}{2}$. Which of the points $(0,2)$, $\left(\sqrt{5}, 2\sqrt{2}\right)$, $\left(\sqrt{10}, 2\sqrt{3}\right)$, $\left(5, 2\sqrt{3}\right)$, do not lie on the Hyperbola?

Solution: Let the equation of the ellipse be

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1 \tag{24.1}$$

Then the semi-major and semi-minor axes of the ellipse are $q = 4, p = 2\sqrt{3}$ respectively. The eccentricity of the ellipse is

$$\varepsilon = \sqrt{1 - \left(\frac{p}{q}\right)^2} = \frac{1}{2} \tag{24.2}$$

The foci of the ellipse are at $(0, \pm q\varepsilon)$ on the y-axis. Let the equation of the hyperbola be

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \tag{24.3}$$

Then $b = q\varepsilon = 2$. Since the eccentricity $e = \frac{3}{2}$,

$$a = b\sqrt{e^2 - 1} = \sqrt{5} \tag{24.4}$$

Thus the equation of the desired hyperbola is

$$\frac{y^2}{4} - \frac{x^2}{5} = 1 \tag{24.5}$$

The following code provides a visualisation of the problem in Fig. 24.

```
import numpy as np
import matplotlib.pyplot as plt

#ellipse
t = np.linspace(-np.pi,np.pi,100)
p = 2*np.sqrt(3)
q = 4
x = p*np.cos(t)
y = q*np.sin(t)
e = np.sqrt(1 - (p/q)**2)

plt.plot(x,y)
plt.grid()
plt.axis('equal')
plt.plot(0,q*e,'o')
plt.plot(0,-q*e,'o')
plt.text(0,(q*e)-0.6,'F1')
plt.text(0,(-q*e)+0.2,'F2')

#hyperbola
b = 2
a = np.sqrt(5)

x = np.linspace(-4,4,100)
y1 = b*np.sqrt((x/a)**2 + 1)
y2 = -b*np.sqrt((x/a)**2 + 1)

plt.plot(x,y1)
plt.plot(x,y2)
plt.plot(np.sqrt(5),2*np.sqrt(2),'
    o')
plt.plot(np.sqrt(10),2*np.sqrt(3),
    'o')
plt.plot(5,2*np.sqrt(3),'o')
plt.text(np.sqrt(5),2*np.sqrt(2)
    -0.55,'A')
plt.text(np.sqrt(10)+0.2,2*np.sqrt
    (3)-0.2,'B')
plt.text(5+0.2,2*np.sqrt(3)-0.2,'C
    ')
plt.savefig('../figs/ee16b1024.eps
    ')
plt.show()
```

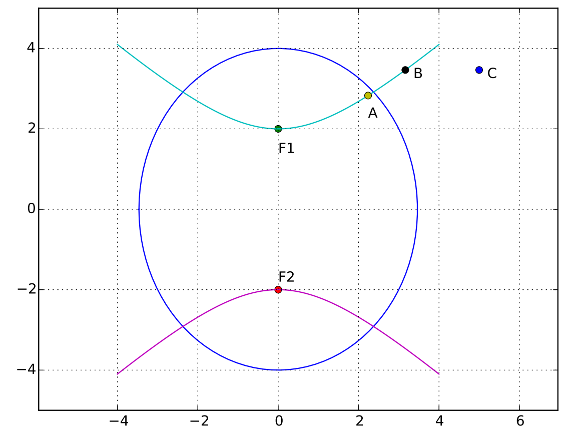

Fig. 24: The point C with coordinates $\left(5, 2\sqrt{3}\right)$ does not lie on the hyperbola.

Problem 25. Find the minimum value of $\tan A + \tan B$, given that $A + B = \frac{\pi}{6}, A > 0, B > 0$.

Solution:

$$\tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} \tag{25.1}$$

$$= \frac{\sin(A+B)}{\cos A \cos B} \tag{25.2}$$

$$= \frac{2\sin(A+B)}{\cos(A+B) + \cos(A-B)} \tag{25.3}$$

$$= \frac{2}{\sqrt{3} + 2\cos(A-B)} \tag{25.4}$$

$\because A + B = \frac{\pi}{6}$. The above expression is minimum when $\cos(A-B)$ is 1, or $A = B = \frac{\pi}{12}$.

The graph is plotted in Fig. 25

```
import numpy as np
import matplotlib.pyplot as plt

A = np.linspace(0,(np.pi)/6,100)
B = (np.pi)/6 - A

y = np.tan(A) + np.tan(B)

min_y = np.min(y)
min_x = (np.pi)/12

plt.plot(A,y)
plt.plot(min_x,min_y,'o')
plt.grid()
plt.ylabel('$tan(A) + tan(B)$')
plt.xlabel('$A (Radians)$')
```

```
plt.savefig('../figs/ee16b1025.eps
    ')
plt.show()
```

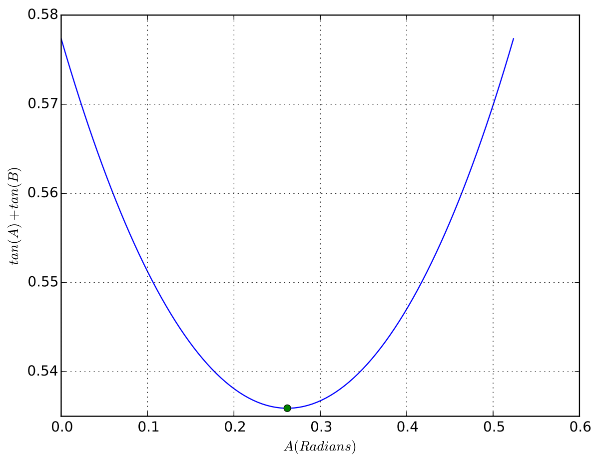

Fig. 25: Finding the minimum of $\tan A + \tan B, A + B = \frac{\pi}{6}$

Problem 26. Find θ for which $\frac{2+3\imath sin\theta}{1-2\imath \sin\theta}$ is purely imaginary.

Solution: Simplifying the complex number,

$$\frac{2+3\imath\sin\theta}{1-2\imath\sin\theta} = \frac{(2+3\iota\sin\theta)(1+2\iota\sin\theta)}{1+4(\sin\theta)^2} \tag{26.1}$$

$$= \frac{\left(2-6(\sin\theta)^2\right)+7\iota\sin\theta}{1+4(\sin\theta)^2} \tag{26.2}$$

For the number to be purely iaginary,

$$2-6(\sin\theta)^2 = 0 \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}} \tag{26.3}$$

$$\Rightarrow \theta = \arcsin\pm\left(\frac{1}{\sqrt{3}}\right) \tag{26.4}$$

The graph is plotted in Fig. 26

```
import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(-0.5*np.pi,0.5*np.
    pi,100)
im_z = (2 - 6*np.sin(t)**2)/(1 +
    4*np.sin(t)**2)
plt.plot(t,im_z)
plt.plot(-np.arcsin(1/np.sqrt(3))
    ,0,'o')
plt.plot(np.arcsin(1/np.sqrt(3))
    ,0,'o')
plt.grid()
plt.xlabel('$\\theta}$ (Radians)')
plt.ylabel('Imaginary part')
plt.savefig('../figs/ee16b1026.eps
    ')
plt.show()
```

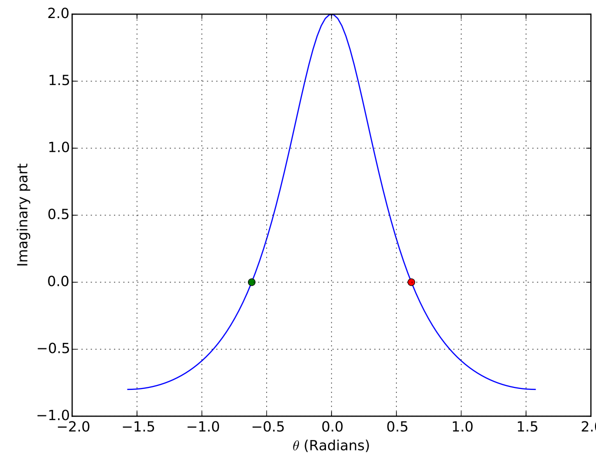

Fig. 26: Complex number is imaginary at $\theta = \arcsin\pm\left(\frac{1}{\sqrt{3}}\right)$

Problem 27. Find the sum of all the solutions of

$$\left(x^2-5x+5\right)^{x^2+4x-60} = 1$$

Solution: The solution can be obtained through the following cases

1) $\left(x^2-5x+5\right) = 1$. This yields the solution
$$(x-1)(x-4) = 0 \Rightarrow x = 1, 4. \tag{27.1}$$
2) $\left(x^2-5x+5\right) = -1, \left(x^2+4x-60\right)$ even. The first condition yields
$$\left(x^2-5x+6\right) = 0 \tag{27.2}$$
$$(x-3)(x-2) = 0 \Rightarrow x = 3, 2 \tag{27.3}$$
Testing the solutions for the second condition,
$$x^2+4x-60 = \begin{cases} -39 & x = 3 \\ -48 & x = 2 \end{cases} \tag{27.4}$$
giving $x = 2$ as the desired solution.
3) Making the power 0,
$$x^2+4x-60 = 0 \Rightarrow x = -10, 6. \tag{27.5}$$

Hence the required solutions are $x = 1, 4, 2, 6, -10$ and the sum of these roots is 3.

```
import numpy as np

#case 1 - base 1
c1 = np.roots([1,-5,4])

#case 2 - base -1, even exponent
r1 = np.roots([1,-5,6])
r2 = np.polyval([1,4,-60],r1)
c2 = r1[np.nonzero(np.round(r2)%2
    ==0)]

#case 3 - any base, 0 exponent
c3 = np.roots([1,4,-60])

print np.sum(c1) + np.sum(c2) + np
    .sum(c3)
```

Problem 28. The sum of the first 10 terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots$ is $\frac{16}{5}m$. Find m.

Solution: The given sum can be expressed as

$$\left(\frac{4}{5}\right)^2 \left(2^2 + 3^2 + 4^2 + ...11^2\right) = \frac{16}{5}m \tag{28.1}$$

$$\Rightarrow \left(\frac{4}{5}\right)^2 \left[\sum_{k=1}^{11} k^2 - 1\right] = \frac{16}{5}m \tag{28.2}$$

$$\Rightarrow \left(\frac{4}{5}\right)^2 \left(\frac{11.12.23}{6} - 1\right) = \frac{16}{5}m \tag{28.3}$$

$$\Rightarrow m = 101 \tag{28.4}$$

```
s = 0
for k in range(2,12):
    s += k**2
print s/5
```

Problem 29. $p = \lim_{x\to 0+} \left(1 + \tan^2 \sqrt{x}\right)^{\frac{1}{2x}}$. Find $\log p$.

Solution: From the given information,

$$\log p = \lim_{x\to 0+} \frac{1}{2x} \left((1 + \tan^2 \sqrt{x}\right) \tag{29.1}$$

$$= \frac{1}{2} \lim_{x\to 0+} \frac{\tan^2 \sqrt{x}}{\sqrt{x}^2} \frac{1}{\tan^2 \sqrt{x}} \left((1 + \tan^2 \sqrt{x}\right) \tag{29.2}$$

$$= \frac{1}{2} \tag{29.3}$$

The following code verifies this result in Fig. 29

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0.01,10,100)
y = (1/(2*x))*(np.log(1 + (np.tan(
    np.sqrt(x)))**2))
plt.plot(x,y)
plt.grid()
plt.xlabel('x')
plt.ylabel('log(p)')
plt.plot(0,0.5,'o')
plt.savefig('../figs/ee16b1029.eps
    ')
plt.show()
```

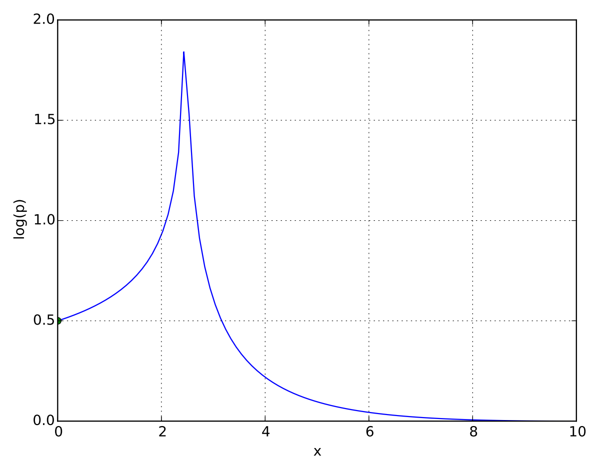

Fig. 29: $\log p = 0.5, x = 0+$

Problem 30. $f(x) = \left|\log 2 - \sin x\right|, x \in \mathbf{R}$ and $g(x) = f(f(x))$. Which of the following is true?

1) g is not differentiable at $x = 0$
2) $g'(0) = \cos(\log 2)$
3) $g'(0) = -\cos(\log 2)$

4) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$.

Solution: The function

$$g(x) = \left|\log 2 - \sin\left|\log 2 - \sin x\right|\right| \tag{30.1}$$

Sketching this function in Fig. 30 using the following octave code, it is seen that the function is continuous at $x = 0$. Computing the right and left hand limits for $g'(x)$ at $x = 0$ for $h = 10^{-10}$, the octave code shows that

$$\frac{g(h) - g(0)}{h} = \frac{g(0) - g(h)}{h} = \cos(\log 2) \tag{30.2}$$

```
import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return np.abs(np.log(2) - np.
       sin(x))
def g(x):
    return f(f(x))

x = np.linspace(-1,1,100)

plt.plot(x,g(x))
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$g(x)$')
plt.savefig('../figs/ee16b1030.eps
   ')
plt.show()

h = 10**(-10)

#right hand limit
print (g(h) - g(0))/h

#left hand limit
print (g(0) - g(-h))/h

print np.cos(np.log(2))
```

Problem 31. Consider

$$f(x) = \tan^{-1}\sqrt{\left(\frac{1+\sin x}{1-\sin x}\right)}, x \in \left(0, \frac{\pi}{2}\right)$$

Sketch the normal to $f(x)$ at $x = \frac{\pi}{6}$. Does it pass through any of the points $(0,0), \left(0, \frac{2\pi}{3}\right), \left(\frac{\pi}{6}, 0\right), \left(\frac{\pi}{4}, 0\right)$?

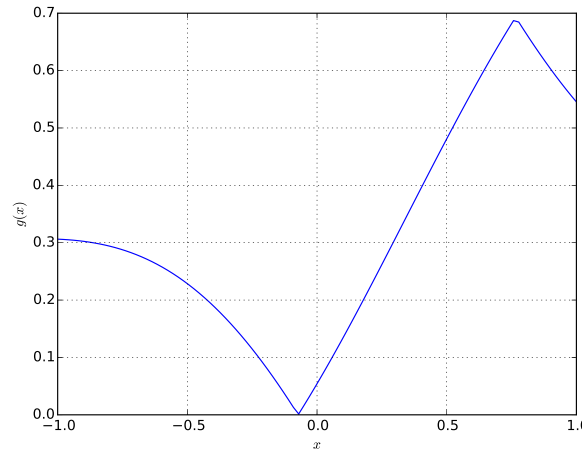

Fig. 30: $g(x)$ continuous at $x = 0$, hence differentialble. $g'(0) = \cos(\log 2)$

The given function can be simplified as

$$f(x) = \tan^{-1}\sqrt{\frac{1+\sin(x)}{1-\sin(x)}} \tag{31.1}$$

$$= \tan^{-1}\sqrt{\frac{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}} \tag{31.2}$$

$$= \tan^{-1}\cot\left(\frac{\pi}{4} - \frac{x}{2}\right) \tag{31.3}$$

$$= \frac{\pi}{4} + \frac{x}{2} \tag{31.4}$$

The normal to $f(x)$ has the equation

$$y = -2x + c, \tag{31.5}$$

where c is a constant. If $x = \frac{\pi}{6}, f(x) = \frac{\pi}{3}$. Substituting the coordinates $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ in the equation for the normal, $c = \frac{2\pi}{3}$.

The normal and the given points are plotted in Fig. 31.

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0,0.5*np.pi,100)

y = -2*x + (2*np.pi)/3

plt.plot(x,y,label='Normal␣$(y␣+␣2
   x␣=␣2\pi/3)$')
plt.plot(0,0,'o')
plt.plot(0,2*np.pi/3,'o')
```

```
plt.plot(np.pi/6,0,'o')
plt.plot(np.pi/4,0,'o')
plt.text(0+0.01,0,'(0,0)')
plt.text(0+0.01,2*np.pi/3,'(0,2$\
    pi$/3)')
plt.text(np.pi/6+0.02,0,'($\pi$
    /6,0)')
plt.text(np.pi/4+0.02,0,'($\pi$
    /4,0)')
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.legend()
plt.savefig('../figs/ee16b1031.eps
    ')
plt.show()
```

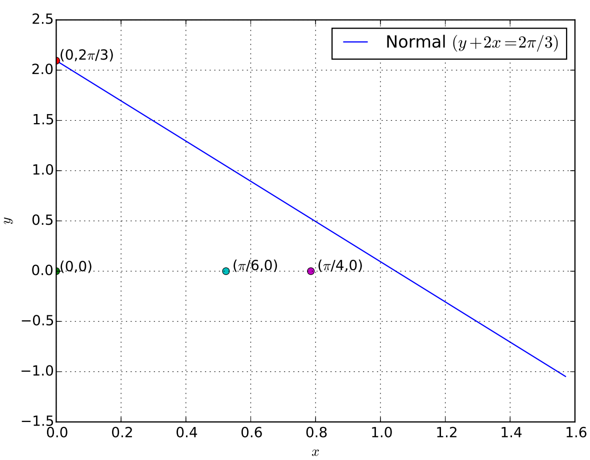

Fig. 31: The normal passes through the point $\left(0, \frac{2\pi}{3}\right)$.

Problem 32. Sketch $\left[\frac{(n+1)(n+2)\dots(3n)}{n^{2n}}\right]^{\frac{1}{n}}$ and verify if its limit at $n \to \infty$ is $\frac{18}{e^4}, \frac{27}{e^2}, \frac{9}{e^2}$ or $3\log 3 - 2$.

Solution: The given expression can be simplified as

$$p_n = \left[\frac{(n+1)(n+2)\dots(3n)}{n^{2n}}\right]^{\frac{1}{n}} = \left[\frac{(3n)!}{n!n^{2n}}\right]^{\frac{1}{n}} \tag{32.1}$$

From Stirling's formula, for large n,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{32.2}$$

Substituting the above in (32.1),

$$\left[\frac{(3n)!}{n!n^{2n}}\right]^{\frac{1}{n}} = \left[\frac{\sqrt{2\pi(3n)}\left(\frac{3n}{e}\right)^{3n}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n .n^{2n}}\right]^{\frac{1}{n}} \tag{32.3}$$

$$= \left[\frac{\sqrt{3}(3)^{3n}}{e^{2n}}\right]^{\frac{1}{n}} = \frac{27}{e^2} \tag{32.4}$$

in the limit. This solution agrees with the plot of p_n shown in Fig. 32.

```
import numpy as np
import matplotlib.pyplot as plt

maxlen = 20
p = []
x = np.linspace(1,maxlen,maxlen)
for n in range(1,maxlen+1):
    l = n
    product = []
    for k in range(1,2*n + 1):
        l = l + 1
        product.append(np.double(l
            ))
    p.append((np.prod(product)/(n
        **(2*n)))**(1.0/n))

plt.stem(x,p)

sol = np.array([18.0/np.exp(4)
    ,27.0/np.exp(2),9.0/np.exp(2),3*
    np.log(3) - 2])
y = np.ones(maxlen)
plt.plot(x,y*sol[0],label = '$18/e
    ^4$')
plt.plot(x,y*sol[1],label = '$27/e
    ^2$')
plt.plot(x,y*sol[2],label = '$9/e
    ^2$')
plt.plot(x,y*sol[3],label = '$3\
    log(3)␣-␣2$')
plt.grid()
plt.legend()
plt.xlabel('$n$')
plt.ylabel('$p_n$')
plt.savefig('../figs/ee16b1032.eps
    ')
plt.show()
```

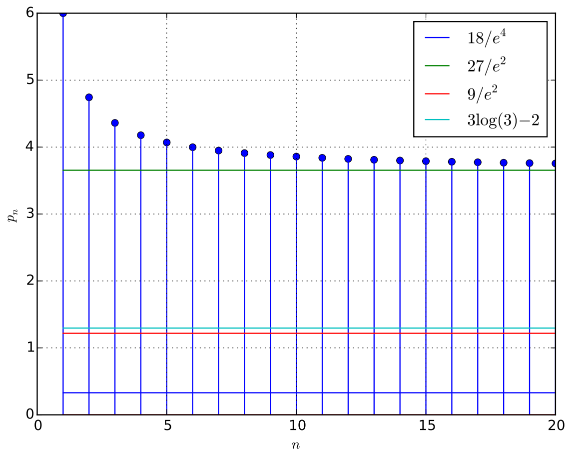

Fig. 32: In the limit, the expression converges to $\frac{27}{e^2}$

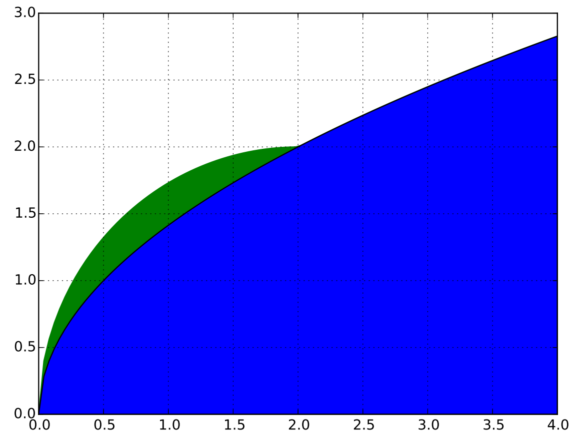

Fig. 33: Desired region is in green colour

Problem 33. Sketch the region

$$\left\{(x, y) : y^2 \geq 2x, x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\right\}$$

Solution: The following code plots the desired region in Fig. 33.

```
import numpy as np
import matplotlib.pyplot as plt
from pylab import *

x = np.linspace(0,4,100)

y1 = np.sqrt(2*x)
y2 = np.sqrt(4*x - x**2)
z = np.maximum(y1,y2)
fill_between(x,z,color='g')
fill_between(x,y1)
plt.grid()
plt.savefig('../figs/ee16b1033.eps
    ')
plt.show()
```

Problem 34. Two sides of a rhombus are along the lines $x - y + 1 = 0$ and $7x - y - 5 = 0$. Its diagonals intersect at $(-1, -2)$. Find the vertices of the rhombus.

Solution: The point of intersection of the two lines is one vertex of the rhombus. This point is obtained by soving the following matrix equation

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \tag{34.1}$$

using the octave code to obtain the point P $(1, 2)$. Since diagonals of a rhombus bisect each other and the point of intersection O is given as $(-1, -2)$ the coordinates of the opposite vertex R are given by

$$\mathbf{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ -2 \end{pmatrix} \Rightarrow \mathbf{x} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} \tag{34.2}$$

Since the sides of a rhombus are equal, if the unknown vertex Q has coordinates (x, y),

$$PQ = QR \Rightarrow (x-1)^2 + (y-2)^2 = (x+3)^2 + (y+6)^2$$
$$\Rightarrow x + 2y = -5 \tag{34.3}$$

Note that the above locus is actually the diagonal QS. Letting PQ be

$$x - y + 1 = 0, \tag{34.4}$$

Q is obtained from the following equation

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 \\ -1 \end{pmatrix} \tag{34.5}$$

as $\left(-\frac{7}{3}, -\frac{4}{3}\right)$. Similarly, letting PS to be

$$7x - y - 5 = 0, \tag{34.6}$$

S is obtained from the equation

$$\begin{pmatrix} 1 & 2 \\ 7 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \tag{34.7}$$

as $\left(\frac{1}{3}, -\frac{8}{3}\right)$.

Fig. 34 explains the problem.

```
import numpy as np
import matplotlib.pyplot as plt
```

```
def line(a,b,s):
    m = (b[1]-a[1])/(b[0]-a[0])
    c = a[1]-m*a[0]
    x = np.linspace(a[0],b[0],100)
    y = m*x + c
    plt.plot(x,y,label = s)

def point(P,T,O1,O2):
    plt.plot(P[0],P[1],'o')
    plt.text(P[0]+O1,P[1]+O2,T)

#Finding P
A = np.linalg.inv(np.array
    ([[1,-1],[7,-1]]))
B = np.array([-1,5])
P = np.dot(A,B)

#Finding Q
A = np.linalg.inv(np.array
    ([[1,2],[1,-1]]))
B = np.array([-5,-1])
Q = np.dot(A,B)

#Finding S
A = np.linalg.inv(np.array
    ([[1,2],[7,-1]]))
B = np.array([-5,5])
S = np.dot(A,B)

R = np.transpose(np.array([-3,-6])
    )
O = np.transpose(np.array([-1,-2])
    )

plt.axis('equal')
line(P,Q,'$y = x + 1$')
line(S,P,'$y = 7x - 5$')
line(Q,S,'$x + 2y + 5 = 0$')
line(Q,R,'$y = 7x + 15$')
line(R,S,'$y = x - 3$')
point(P,'P',-0.4,0)
point(Q,'Q',-0.5,0)
point(R,'R',0.2,-0.3)
point(S,'S',0.2,0)
point(O,'O',0.12,0)
plt.legend(loc=4)
plt.grid()
plt.savefig('../figs/ee16b1034.eps
    ')
plt.show()
```

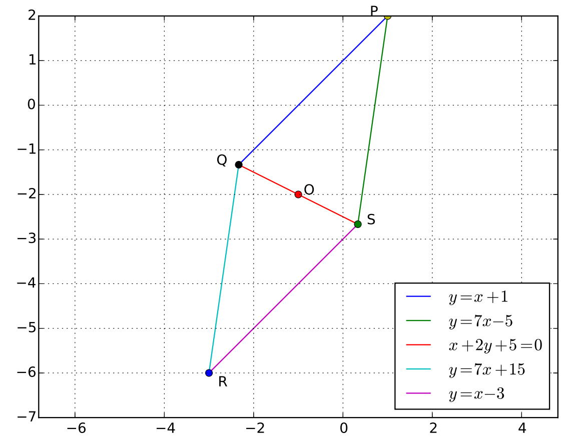

Fig. 34: Desired rhombus.

Problem 35. Sketch the locus of the centres of circles which touch the circle $x^2+y^2-8x-8y-4=0$ as well as the $x-$axis.

Solution: The given circle can be expressed in standard form as

$$(x-4)^2+(y-4)^2=6^2 \tag{35.1}$$

i.e., the cricle has centre at $(4,4)$ and radius 6. Let (h,k) be the centre of a circle that touches the given circle. Since this circle touches the x-axis, its radius is k. This circle can touch the given circle internally or externally.

1) *External:* In this case, sum of radius of two circles is equal to distance between them. Hence,

$$|k|+6=\sqrt{(h-4)^2+(k-4)^2} \tag{35.2}$$

$$\implies k^2+12\,|k|+36=(h-4)^2+k^2-8k+16 \tag{35.3}$$

$$\implies 12\,|k|+8k=(h-4)^2-20 \tag{35.4}$$

$$\implies k=\begin{cases}\frac{(h-4)^2-20}{20} & k>0\\ -\frac{(h-4)^2-20}{4} & k<0\end{cases} \tag{35.5}$$

2) *Internal:* Modulus of difference of radius of two circles is equal to distance between them.

hence

$$||k| - 6| = \sqrt{(h-4)^2 + (k-4)^2} \tag{35.6}$$

$$\implies k^2 - 12|k| + 36 = (h-4)^2 + k^2 - 8k + 16 \tag{35.7}$$

$$\implies -12|k| + 8k = (h-4)^2 - 20 \tag{35.8}$$

$$\implies k = \begin{cases} \frac{(h-4)^2-20}{20} & k < 0 \\ -\frac{(h-4)^2-20}{4} & k > 0 \end{cases} \tag{35.9}$$

Both the above cases can be combined to obtain the locus as the curves

$$y = \frac{(x-4)^2}{20} - 1 \tag{35.10}$$

$$y = 5 - \frac{(x-4)^2}{4} \tag{35.11}$$

```
import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(0,2*np.pi,100)
x = 6*np.cos(t) + 4
y = 6*np.sin(t) + 4

plt.axis('equal')
plt.plot(x,y,label = 'Circle')

x = np.linspace(-3,12,100)
y1 = ((x-4)**2)/20 - 1
y2 = 5 - ((x-4)**2)/2

plt.plot(x,y1,label = 'Locus 1')
plt.plot(x,y2,label = 'Locus 2')
plt.axis([-5,12,-5,10])
plt.grid()
plt.legend(loc=3)
plt.savefig('../figs/ee16b1035.eps
    ')
plt.show()
```

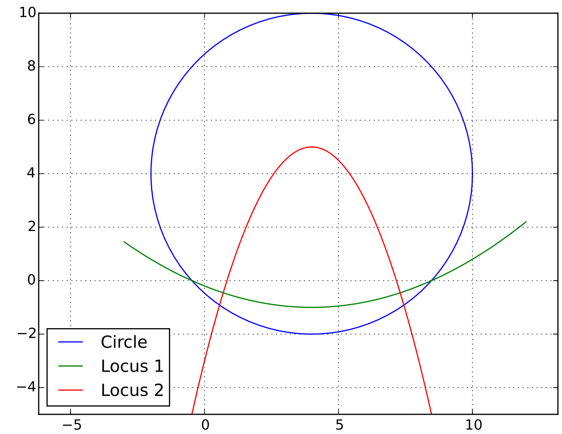

Fig. 35: Required loci.

Problem 36. One of the diameters of the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ is a chord of a circle S. The centre of S is at $(-3, 2)$. Sketch S and find its radius.

Solution: The given circle can be expressed in standard form as

$$(x-2)^2 + (y+3)^2 = 5^2 \tag{36.1}$$

i.e., the circle has centre at O with coordinates $(2, -3)$ and radius 5. Using the distance formula, $OS = 5\sqrt{2}$.Let the radius of the circle with centre at S be r. r can be obtained using Budhayana's theorem as

$$r^2 = 5^2 + OS^2 \tag{36.2}$$

$$\implies r = 5\sqrt{3} \tag{36.3}$$

The diameter of the circle with centre O and chord of circle with centre S is perpendicular to OS. The equation of the diameter is thus obtained as

$$(y+3) = (x-2) \implies y = x - 5 \tag{36.4}$$

Fig. 36 summarises the problem.

```
import numpy as np
import matplotlib.pyplot as plt

def line(a,b):
        m = (b[1]-a[1])/(b[0]-a[0])
        c = a[1]-m*a[0]
        x = np.linspace(a[0],b[0],100)
        y = m*x + c
        plt.plot(x,y)

def point(P,T,O1,O2):
        plt.plot(P[0],P[1],'o')
        plt.text(P[0]+O1,P[1]+O2,T)

t = np.linspace(0,2*np.pi,100)
x = 5*np.cos(t) + 2
y = 5*np.sin(t) - 3
```

```
plt.axis('equal')
plt.plot(x,y)

y = x - 5

plt.plot(x,y)

S = np.array([-3,2])
O = np.array([2,-3])

point(S,'S',-1,-0.1)
point(O,'O',0.45,-0.5)
line(O,S)

t = np.linspace(0,2*np.pi,100)
r = 5*np.sqrt(3)
x = r*np.cos(t) - 3
y = r*np.sin(t) + 2
plt.plot(x,y)
plt.grid()
plt.savefig('../figs/ee16b1036.eps
    ')
plt.show()
```

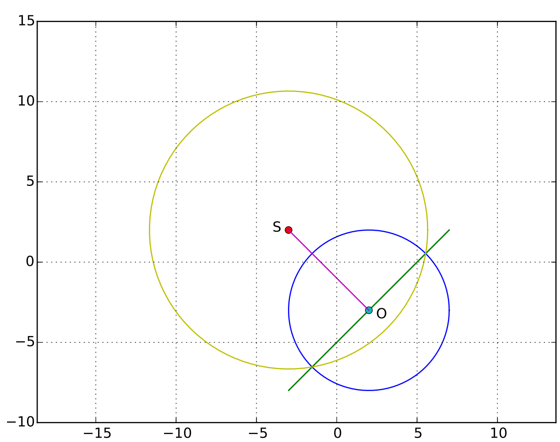

Fig. 36: The diameter of circle with centre O is a chord of the circle with centre S.

Problem 37. P is the nearest point of the parabola $y^2 = 8x$ to the centre C of the circle $x^2 + (y+6)^2 = 1$.Sketch the circle with centre P and passing through C.

Solution: Let P be denoted by $\left(2t^2, 4t\right)$. Let the centre of the circle $(0, -6)$ be O. Then

$$OP^2 = (2t^2 - 0)^2 + (4t+6)^2 = 4\left(t^4 + 4t^2 + 12t + 9\right) \tag{37.1}$$

Differentiating OP^2 with respect to t and equating to 0 results in

$$t^3 + 2t + 3 = 0 \tag{37.2}$$
$$\implies (t+1)(t^2 - t + 3) = 0 \tag{37.3}$$
$$\tag{37.4}$$

yielding $t = -1$. Thus, P is $(2, -4)$ and $OP = 2\sqrt{2}$. The equation of the desired circle is

$$(x-2)^2 + (y+4)^2 = 8 \tag{37.5}$$

```
import numpy as np
import matplotlib.pyplot as plt

def line(a,b):
    m = (b[1]-a[1])/(b[0]-a[0])
    c = a[1]-m*a[0]
    x = np.linspace(a[0],b[0],100)
    y = m*x + c
    plt.plot(x,y)

def point(P,T,O1,O2):
    plt.plot(P[0],P[1],'o')
    plt.text(P[0]+O1,P[1]+O2,T)

t = np.linspace(-np.pi,np.pi,100)
r = 1
x = r*np.cos(t)
y = r*np.sin(t) - 6

plt.figure(1)
plt.axis('equal')
plt.plot(x,y,label = 'Given␣circle
    ')

t = np.linspace(-1.5,1.5,100)
x = 2*t**2
y = 4*t

plt.plot(x,y,label = 'Given␣
    parabola')

O = np.array([0,-6])
P = np.array([2,-4])

line(O,P)
```

```
OP = np.linalg.norm(O-P)

t = np.linspace(-np.pi,np.pi,100)
r = OP
x = r*np.cos(t) + 2
y = r*np.sin(t) - 4

plt.plot(x,y,label = 'Required circle')

plt.figure(1)
point(P,'P',0.2,0.1)
point(O,'O',-0.64,-0.64)
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.grid()
plt.legend()
plt.savefig('../figs/ee16b1037a.eps')

plt.figure(2)
#Function for finding minimum
t = np.linspace(-2,2,100)
OP = 2*np.sqrt(np.polyval([1,0,4,12,9],t))
plt.figure(2)
plt.plot(t,OP)
plt.grid()
plt.xlabel('$t$')
plt.ylabel('OP')
plt.savefig('../figs/ee16b1037b.eps')

plt.show()
```

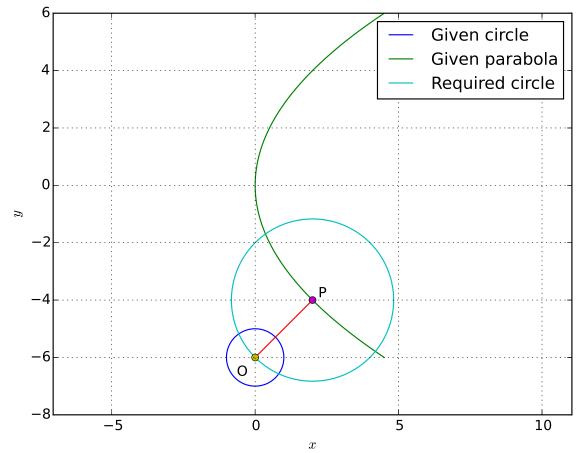

Fig. 37.1: Figures for the given problem

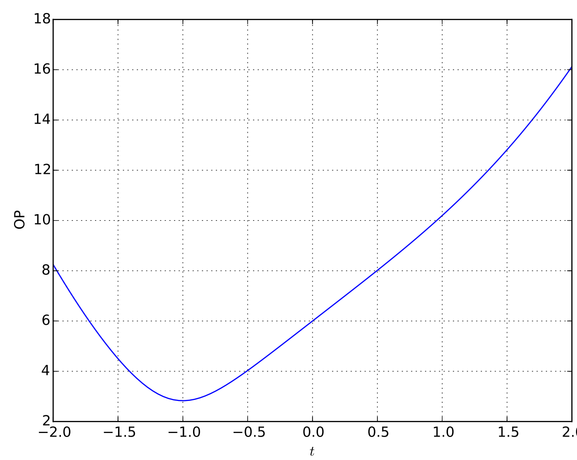

Fig. 37.2: OP has a minimum at $t = -1$.

Problem 38. The length of the latus rectum of a hyperbola is 8 and the length of its conjugate axis is half the distance between its foci. Sketch the hyperbola and find its eccentricity.

Solution: Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{38.1}$$

Since the length of the latus rectum is 8,

$$\frac{2b^2}{a} = 1 \tag{38.2}$$

The length of the conjugate axis is $2b$ and the distance between the foci is $2ae, e = \sqrt{1 + \frac{b^2}{a^2}}$, where e is the eccentricity. Given that $2b = \frac{2ae}{2}$. From the given information,

$$2b = \sqrt{a^2 + b^2} \tag{38.3}$$

Since $a = 2b^2$, from the above equation, $b = \frac{\sqrt{3}}{2}$ and $a = \frac{3}{2}$. The eccentricity

$$e = \frac{2}{\sqrt{3}} \tag{38.4}$$

The desired hyperbola is plotted in Fig. 38.

```
import numpy as np
import matplotlib.pyplot as plt

def plot_point(Q,t,O1,O2):
```

```
    plt.plot(Q[0],Q[1],'o')
    plt.text(Q[0]+O1,Q[1]+O2,t)

a = 1.5
b = np.sqrt(3)/2
e = np.sqrt(1 + (b/a)**2)
print e
x = np.linspace(-4,4,1000)
y1 = b*np.sqrt((x/a)**2 - 1)
y2 = -y1

plt.plot(x,y1)
plt.plot(x,y2)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')

F1 = [a*e,0]
F2 = [-a*e,0]
plot_point(F1,'$F_1$',0.15,-0.1)
plot_point(F2,'$F_2$',-0.45,-0.1)
plt.savefig('../figs/ee16b1038.eps
   ')
plt.show()
```

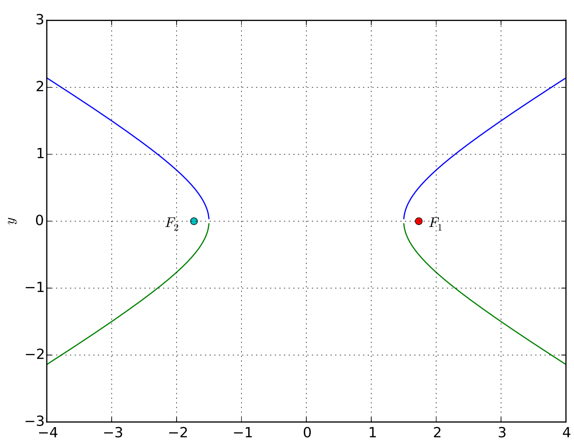

Fig. 38: Foci at F_1 and F_2. Eccentricity $e = \frac{2}{\sqrt{3}}$

Problem 39. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side x units and a circle of radius of r units. Find x if the sum of the areas of the square and the circle so formed is minimum.

Solution: From the given information, adding the perimeters of the square and the circle,

$$4x + 2\pi r = 2 \Rightarrow 2x + \pi r = 1 \tag{39.1}$$

The sum of the areas of the square and circle is

$$A = x^2 + \pi r^2 = x^2 + \frac{(1-2x)^2}{\pi} \tag{39.2}$$

$$= \frac{(\pi+4)\,x^2 - 4x + 1}{\pi} \tag{39.3}$$

$$= \left(1 + \frac{4}{\pi}\right)\left\{\left(x - \frac{2}{(\pi+4)}\right) + \frac{1}{(\pi+4)} - \left(\frac{2}{(\pi+4)}\right)^2\right\} \tag{39.4}$$

Thus, A is minimum for $x = \frac{2}{\pi+4}$. We obtain

$$r = \frac{1 - \frac{4}{4+\pi}}{\pi} = \frac{1}{\pi+4} = \frac{x}{2} \tag{39.5}$$

Let P be denoted by $\left(2t^2, 4t\right)$. Let the centre of the circle $(0, -6)$ be O. Then

$$OP^2 = (2t^2 - 0)^2 + (4t+6)^2 = 4\left(t^4 + 4t^2 + 12t + 9\right) \tag{39.6}$$

Differentiating OP^2 with respect to t and equating to 0 results in

$$t^3 + 2t + 3 = 0 \tag{39.7}$$

$$\Rightarrow (t+1)(t^2 - t + 3) = 0 \tag{39.8}$$

$$\tag{39.9}$$

yielding $t = -1$. Thus, P is $(2, -4)$ and $OP = 2\sqrt{2}$. The equation of the desired circle is

$$(x-2)^2 + (y+4)^2 = 8 \tag{39.10}$$

Fig. 39 plots A with respect to x.

```
import numpy as np
import matplotlib.pyplot as plt

def point(P,T,O1,O2):
    plt.plot(P[0],P[1],'o')
    plt.text(P[0]+O1,P[1]+O2,T)

def f(x):
    return x**2 + ((1 - 2*x)**2)/
       np.pi

x = np.linspace(0,1,20)

plt.plot(x,f(x))
```

```
x = 2/(np.pi + 4)
P = np.array([x,f(x)])
r = (1 - 2*x)/np.pi

point(P,'P',0,0.03)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$A$')
plt.savefig('../figs/ee16b1039.eps
    ')
plt.show()
```

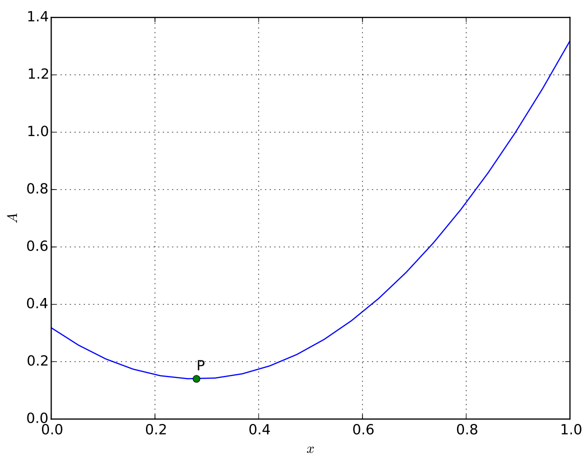

Fig. 39: Area is minimum for $x = \frac{2}{\pi+4}$

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