

Linear Algebra

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Abstract—This document has solutions of Problem 3 of each section of the Linear Algebra manual.

Download python codes from

svn co <https://github.com/krishnajakodali/Summer2020/trunk/geometry/linalg/codes>

1 TRIANGLE

1.1 Problem

Draw graphs of equations

$$5x - y = 5 \quad (1.1.1)$$

$$3x - y = 3 \quad (1.1.2)$$

Determine the coordinates of vertices of triangle formed by these lines and y axis.

1.2 Solution

1.1. Line $5x - y = 5$ can be represented in vector form as,

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{X} = 5 \quad (1.1.1)$$

1.2. Line $3x - y = 3$ can be represented in vector form as,

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{X} = 3 \quad (1.2.1)$$

Also the equation of y axis is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{X} = 0 \quad (1.2.2)$$

Let line 1.1.1 and line 1.2.1 meet at point A. Then,

$$\begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad (1.2.3)$$

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad (1.2.4)$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.2.5)$$

Let line 1.1.1 and line 1.2.2 meet at point B. Then,

$$\begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (1.2.6)$$

$$\mathbf{B} = \begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (1.2.7)$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \quad (1.2.8)$$

Let line 1.2.1 and line 1.2.2 meet at point C. Then,

$$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.2.9)$$

$$\mathbf{C} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.2.10)$$

$$\mathbf{C} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \quad (1.2.11)$$

So, $\triangle ABC$ is formed by intersection of 1.1.1, 1.2.1 and 1.2.2. The following Python code generates Fig. 1.2 The lines 1.1.1 and 1.2.1 and the triangle ABC formed by the two lines and y-axis are plotted in the figure below

codes/triangle/linesandtri.py

2 QUADRILATERAL

2.1 Problem

Draw Quadrilateral in cartesian plane whose vertices are $\begin{pmatrix} -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \end{pmatrix}$.

Also Find its area.

2.2 Solution

2.1. Vertices of the quadrilateral are,

$$\mathbf{A} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \mathbf{D} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \quad (2.1.1)$$

Quadrilateral ABCD is drawn by joining its vertices A and B, B and C, C and D, D and

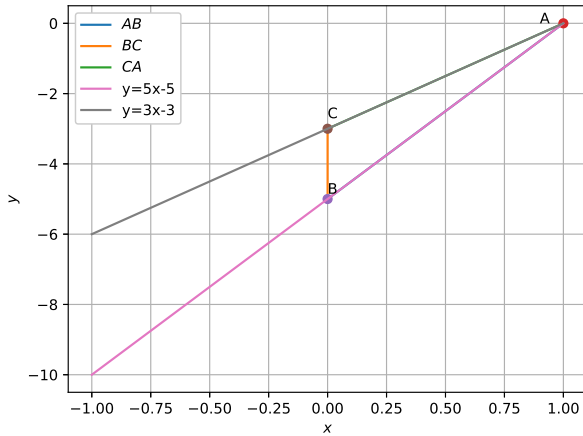


Fig. 1.2: Plot of lines and the Triangle ABC

A. The following Python code generates Fig. 2.1

```
codes/quad/quad.py
```

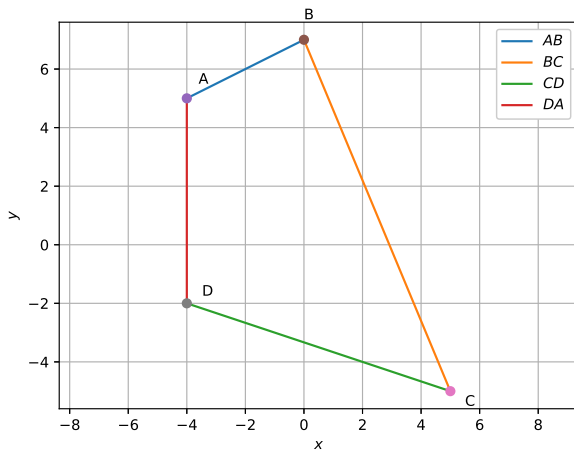


Fig. 2.1: Quadrilateral ABCD

2.2. From Figure 2.1 Area of the Quadrilateral ABCD can be given as

$$Ar(\triangle ABC) + Ar(\triangle BCD) \quad (2.2.1)$$

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| + \frac{1}{2} \|(\mathbf{C} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D})\| \quad (2.2.2)$$

For two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\|\mathbf{a} \times \mathbf{b}\| = |a_1 b_2 - a_2 b_1| \quad (2.2.3)$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \quad (2.2.4)$$

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \quad (2.2.5)$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 5 \\ -12 \end{pmatrix} \quad (2.2.6)$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} 9 \\ -3 \end{pmatrix} \quad (2.2.7)$$

Using 2.2.3

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| = \frac{1}{2} |(-28)| = 14 \quad (2.2.8)$$

$$\frac{1}{2} \|(\mathbf{C} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D})\| = \frac{1}{2} |(-15 + 108)| = 46.5 \quad (2.2.9)$$

Substituting the above values in equation 2.2.2, We get

$$Area = 14 + 46.5 = 60.5 \text{ sq. units} \quad (2.2.10)$$

3 LINE

3.1 Complex numbers

3.1.1 Problem: Find multiplicative inverse of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

3.1.2 Solution:

3.1. Complex number $\begin{pmatrix} a \\ b \end{pmatrix}$ is expressed as a matrix as

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad (3.1.1)$$

If complex number $\begin{pmatrix} x \\ y \end{pmatrix}$ is the multiplicative inverse of $\begin{pmatrix} a \\ b \end{pmatrix}$, Then their product must be

complex number $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.1.2)$$

$$\Rightarrow \begin{pmatrix} ax - by & -(bx + ay) \\ bx + ay & ax - by \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.1.3)$$

$$\Rightarrow ax - by = 1 \quad (3.1.4)$$

$$bx + ay = 0 \quad (3.1.5)$$

From equations 3.1.4 and 3.1.5

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.6)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} \quad (3.1.7)$$

Using 3.1.7 The multiplicative inverse of complex number $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is $\begin{pmatrix} 0.15384615 \\ 0.23076923 \end{pmatrix}$

The python code for above problem is

```
codes/line/comp/comp.py
```

3.2 Points and vectors

3.2.1 Problem: A town B is located 36km east and 15 km north of town A. How would you find the distance from town A to town B without actually measuring it.

3.2.2 Solution:

3.1. We represent town A by **A** and town B by **B**. If **A** is taken to be the origin and East and North directions are considered to be +ve x-axis and +ve y-axis respectively, Then **B** is given as

$$\mathbf{B} = \begin{pmatrix} 36 \\ 15 \end{pmatrix} \quad (3.1.1)$$

The distance d between A and B is given by

$$d = \|\mathbf{A} - \mathbf{B}\| \quad (3.1.2)$$

$$d = 39\text{km} \quad (3.1.3)$$

The following Python code generates Fig. 3.1

```
codes/line/towns/towns.py
```

So the distance between Town A and Town B is 39km.

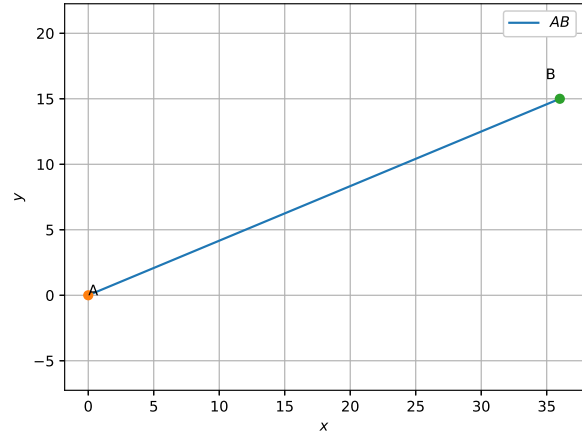


Fig. 3.1: Position of Towns A and B

3.3 Points on a Line

3.4 Problem

Find the ratio in which line segment joining points $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$ $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ is divide by $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$

3.4.1 Solution:

3.1.

$$\mathbf{A} = \begin{pmatrix} -3 \\ 10 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 6 \\ -8 \end{pmatrix} \mathbf{C} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} \quad (3.1.1)$$

Let **C** divide AB in ratio k:1. Then by section formulae,

$$\mathbf{C} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (3.1.2)$$

$$\begin{pmatrix} -1 \\ 6 \end{pmatrix} = \frac{1}{k + 1} \begin{pmatrix} 6k - 3 \\ -8k + 10 \end{pmatrix} \quad (3.1.3)$$

$$k = \frac{2}{7} \quad (3.1.4)$$

So **C** divides AB in ratio 2:7

The following Python code generates Fig. 3.1

```
codes/line/section/section.py
```

3.5 Lines and planes

3.5.1 Problem: Find two solutions for each of the following equations.

$$(a) (4 \ 3) \mathbf{x} = 12 \quad (1.1)$$

$$(b) (2 \ 5) \mathbf{x} = 0 \quad (1.2)$$

$$(c) (0 \ 3) \mathbf{x} = 4 \quad (1.3)$$

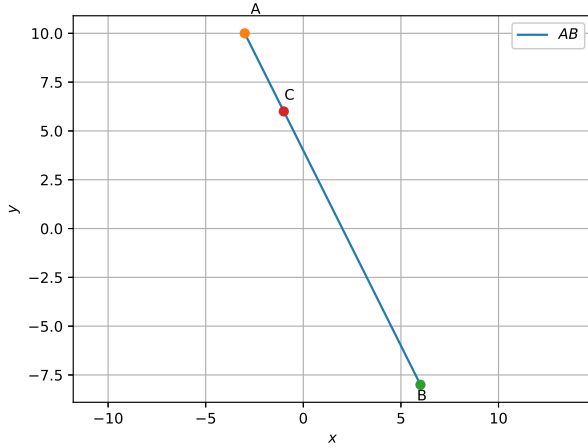


Fig. 3.1: C divides AB in ratio k:1

3.5.2 Solution:

3.1. A point \mathbf{c} lying on the line

$$\begin{pmatrix} a & b \end{pmatrix} \mathbf{x} = d \quad (3.1.1)$$

at a distance λ from point \mathbf{x} lying on the same line is given as

$$\mathbf{c} = \mathbf{x} + \frac{\lambda}{\sqrt{a^2 + b^2}} \begin{pmatrix} b \\ -a \end{pmatrix} \quad (3.1.2)$$

$$\lambda = \sqrt{a^2 + b^2} \implies \mathbf{c} = \mathbf{x} + \begin{pmatrix} b \\ -a \end{pmatrix} \quad (3.1.3)$$

3.2. Equation of y axis is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (3.2.1)$$

3.3.

$$(a) \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \quad (3.3.1)$$

The line meets y-axis at point \mathbf{y}_1 given using 3.2.1 as,

3.4.

$$\begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix} \mathbf{y}_1 = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (3.4.1)$$

$$\mathbf{y}_1 = \begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (3.4.2)$$

$$\mathbf{y}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad (3.4.3)$$

Another point \mathbf{c}_1 on the line is found using

equation 3.1.3

$$\mathbf{c}_1 = \mathbf{y}_1 + \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (3.4.4)$$

$$\implies \mathbf{c}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.4.5)$$

$$(b) \begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.6)$$

The line meets y-axis at point \mathbf{y}_2 given using 3.2.1 as,

3.5.

$$\begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix} \mathbf{y}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.5.1)$$

$$\mathbf{y}_1 = \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.5.2)$$

$$\mathbf{y}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.5.3)$$

Another point \mathbf{c}_2 on the line is found using equation 3.1.3

$$\mathbf{c}_2 = \mathbf{y}_2 + \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad (3.5.4)$$

$$\implies \mathbf{c}_2 = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad (3.5.5)$$

$$(c) \begin{pmatrix} 0 & 3 \end{pmatrix} \mathbf{x} = 4 \quad (3.5.6)$$

The line meets y-axis at point \mathbf{y}_2 given using 3.2.1 as,

3.6.

$$\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} \mathbf{y}_1 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (3.6.1)$$

$$\mathbf{y}_1 = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (3.6.2)$$

$$\mathbf{y}_1 = \begin{pmatrix} 0 \\ \frac{4}{3} \end{pmatrix} \quad (3.6.3)$$

Another point \mathbf{c}_2 on the line is found using equation 3.1.3

$$\mathbf{c}_2 = \mathbf{y}_2 + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.6.4)$$

$$\implies \mathbf{c}_2 = \begin{pmatrix} 3 \\ \frac{4}{3} \end{pmatrix} \quad (3.6.5)$$

The python code for the above problem , plotting the figure 3.6 is available at

codes/line/pointonline2/pointonline.py

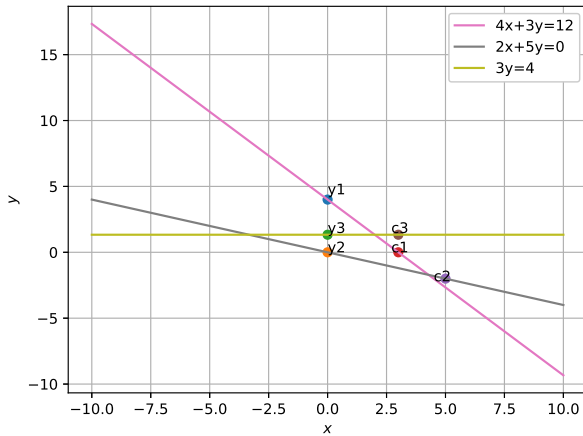


Fig. 3.6: Plot of the three lines and the points on them

3.6 Motion in a plane

3.6.1 Problem: Rain is falling vertically with a speed of 35 m/s. A woman rides a bicycle with a speed of 12 m/s in east to west direction. What is the direction in which she should hold the umbrella?

3.6.2 Solution:

3.1. Let us take vertically upward direction as +ve y-axis and west to east direction as +ve x-axis. The woman experiences rain in the direction of the relative velocity of the rain wrt her own velocity. The velocity of rain = $\mathbf{v}_r = \begin{pmatrix} 0 \\ -35 \end{pmatrix}$

The velocity of woman = $\mathbf{v}_w = \begin{pmatrix} -12 \\ 0 \end{pmatrix}$
relative velocity of rain wrt woman is given as

$$\mathbf{v}_{rw} = \mathbf{v}_r - \mathbf{v}_w \Rightarrow \begin{pmatrix} 12 \\ -35 \end{pmatrix} \quad (3.1.1)$$

She must hold the umbrella opposite to the direction of rain she experiences. So the woman must hold the umbrella along the direction of $-\mathbf{v}_{rw}$. So the woman must hold the umbrella forward at an angle θ to the vertical where

$$\theta = \tan^{-1} \left(\frac{12}{35} \right) \quad (3.1.2)$$

The following python code generates figure 3.1 which illustrates the velocities \mathbf{v}_r , \mathbf{v}_w , \mathbf{v}_{rw}

codes/line/rain/rain.py

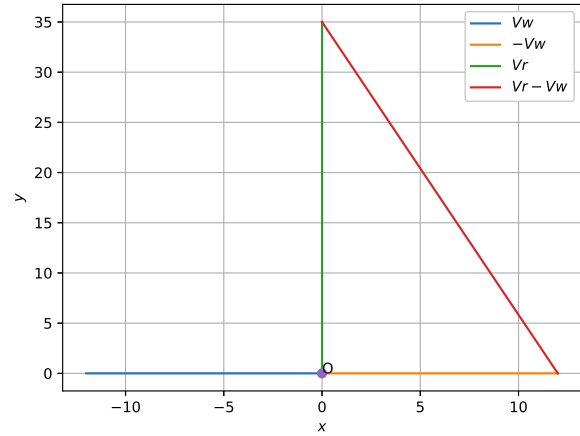


Fig. 3.1: Direction of umbrella

3.7 Matrix

3.7.1 Problem: If a matrix has 18 elements, what are the possible orders it can have. What if it has 5 elements?

3.7.2 Solution:

3.1. A matrix with n elements can be represented as a matrix of order $(r \times c)$ if and only if n, r and c are all natural numbers (Here r is the number of rows and c is the number of columns in the matrix.). This is possible only if r is a divisor of n .

3.2. So the total possible orders a matrix with n elements can have is equal to the total number of divisors of n .

The following python code finds the total possible orders (d) for a matrix of n elements.

codes/line/matrix/matrix.py

So a matrix of 18 elements has 6 possible orders and a matrix of 5 elements can have 2 possible orders.

3.8 Determinants

3.8.1 Problem: If $\mathbf{A} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$ Show that:

$$|2\mathbf{A}| = 4|\mathbf{A}| \quad (2.1)$$

3.8.2 Solution:

3.1. Determinant of a (2×2) matrix is calculated as follows

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (3.1.1)$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} \quad (3.1.2)$$

$$2\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 8 & 4 \end{pmatrix} \quad (3.1.3)$$

Using 3.1.1 ,3.1.2, 3.1.3

$$|A| = 2 - 8 = -6 \implies 4|A| = -24 \quad (3.1.4)$$

$$|2A| = 8 - 32 = -24 \quad (3.1.5)$$

$$\implies |2A| = 4|A| \quad (3.1.6)$$

3.9 Linear Inequalities

3.9.1 Problem: Solve

$$\frac{3x-4}{2} \geq \frac{x+1}{4} \quad (1.1)$$

Show graph of solutions on numberline.

3.9.2 Solution:

3.1.

$$\frac{3x-4}{2} \geq \frac{x+1}{4} - 1 \quad (3.1.1)$$

$$\frac{3x-4}{2} - \frac{x+1}{4} \geq -1 \quad (3.1.2)$$

3.2. Make RHS positive by multiplying with -1 on both sides , Inequality changes.

$$-\frac{3x-4}{2} + \frac{x+1}{4} \leq 1 \quad (3.2.1)$$

3.3. Convert \leq sign to $=$ sign by adding slack variable s on the LHS such that s is non-negative , That is

$$s \geq 0 \implies -\frac{3x-4}{2} + \frac{x+1}{4} + s = 1 \quad (3.3.1)$$

$$\implies -5x + 9 + 4s = 4 \quad (3.3.2)$$

$$\implies x = 1 + \frac{4s}{5} \quad (3.3.3)$$

$$\implies x \geq 1 \quad (3.3.4)$$

The following code marks the solution of inequality on numberline as shown in figure 3.3

```
codes/line/ineq/ineq.py
```

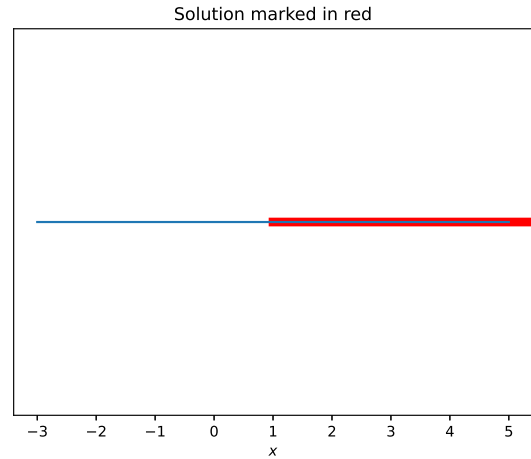


Fig. 3.3: Solution of the inequality

3.10 Miscellaneous

3.10.1 Problem: In $\triangle ABC$ Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (3.1)$$

3.10.2 Solution:

3.1. The vertices of $\triangle ABC$ are taken as follows:

Solution: See Table. 3.1

vertex	coordinates
A	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
B	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$
C	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

TABLE 3.1: To construct triangle ABC

3.2. The midpoints of sides are given as

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \quad (3.2.1)$$

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \quad (3.2.2)$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \quad (3.2.3)$$

3.3. Also Centroid divides median in ratio 2:1. So by section formulae \mathbf{O} is given as

$$\mathbf{O} = \frac{2\mathbf{D} + \mathbf{A}}{3} \quad (3.3.1)$$

The derived coordinates are listed in 3.3.

Vector	Coordinates
D	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$
E	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
F	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
O	$\begin{pmatrix} 3 \\ \frac{2}{3} \end{pmatrix}$

TABLE 3.3: Derived coordinates of triangle ABC

3.10.3 Proof:

3.1. Substituting 3.2.1 in 3.3.1 we get

$$\Rightarrow \mathbf{O} = \frac{\mathbf{B} + \mathbf{C} + \mathbf{A}}{3} \quad (3.1.1)$$

The following code plots figure 3.1

codes/line/median/median.py

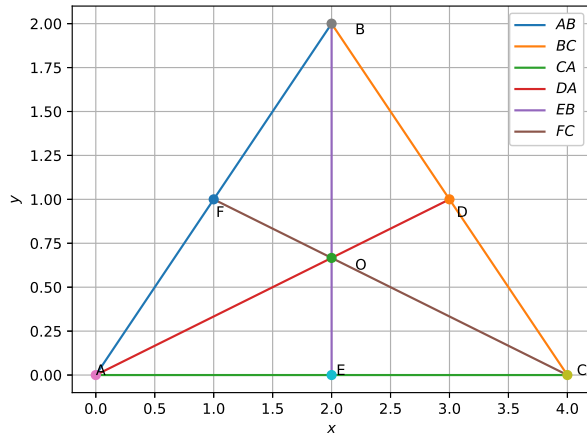


Fig. 3.1: Triangle ABC with centroid O

4 CIRCLE

4.1 Example

4.1.1 Problem: Find the equation of circle which passes through point $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and whose centre lies on the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X} = 2 \quad (1.1)$$

4.1.2 Solution:

4.1. Let \mathbf{O} be the centre, r be the radius and $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ be the points lying on the circle.

Any point on the circle is equidistant from its centre

$$\Rightarrow \|\mathbf{A} - \mathbf{O}\| = \|\mathbf{B} - \mathbf{O}\| = r \quad (4.1.1)$$

$$\Rightarrow \|\mathbf{A} - \mathbf{O}\|^2 - \|\mathbf{B} - \mathbf{O}\|^2 = 0 \quad (4.1.2)$$

$$\Rightarrow (\mathbf{A} - \mathbf{O})^T (\mathbf{A} - \mathbf{O}) \quad (4.1.3)$$

$$- (\mathbf{B} - \mathbf{O})^T (\mathbf{B} - \mathbf{O}) = 0 \quad (4.1.4)$$

$$\Rightarrow (\mathbf{A} - \mathbf{B})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (4.1.5)$$

Also centre \mathbf{O} lies on the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X} = 2 \quad (4.1.6)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{O} = 2 \quad (4.1.7)$$

(4.1.5) and (4.1.7), can be combined to form the matrix equation

$$\mathbf{NO} = \mathbf{c} \quad (4.1.8)$$

$$\Rightarrow \mathbf{O} = \mathbf{N}^{-1} \mathbf{c} \quad (4.1.9)$$

where

$$\mathbf{N} = \begin{pmatrix} (\mathbf{A} - \mathbf{B})^T \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \end{pmatrix} \quad (4.1.10)$$

$$\mathbf{c} = \begin{pmatrix} \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \\ 2 \end{pmatrix} \quad (4.1.11)$$

radius r of the circle is given as

$$r = \|\mathbf{A} - \mathbf{O}\| \quad (4.1.12)$$

Therefore the equation of the circle is

$$\|\mathbf{X} - \mathbf{O}\| = r \quad (4.1.13)$$

Where \mathbf{O} and r are derived using equations 4.1.9 and 4.1.12

The following code calculates centre and radius and plots figure 4.1

codes/circle1/circle1.py

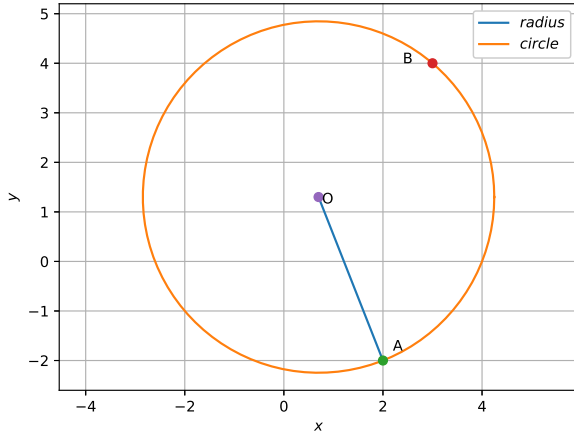


Fig. 4.1: Circle with centre at \mathbf{O} and radius r

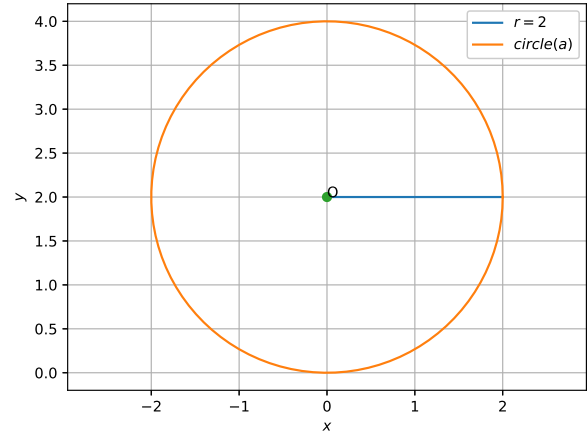


Fig. 4.2: Circle with centre at $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and radius 2

4.2 Exercise

4.2.1 Problem: Sketch the circles with

$$(a) \text{ centre } \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \text{ radius } 2 \quad (1.1)$$

$$(b) \text{ centre } \begin{pmatrix} -2 \\ 32 \end{pmatrix}, \text{ radius } 4 \quad (1.2)$$

$$(c) \text{ centre } \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, \text{ radius } \frac{1}{12} \quad (1.3)$$

$$(d) \text{ centre } \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ radius } \sqrt{2} \quad (1.4)$$

$$(e) \text{ centre } \begin{pmatrix} -a \\ -b \end{pmatrix}, \text{ radius } \sqrt{a^2 - b^2} \quad (1.5)$$

4.2.2 Solution:

4.1. Let \mathbf{O} be the centre, r be the radius of the circle. Any point \mathbf{X} lying on the circle is at a distance r from \mathbf{O} .

Therefore the equation of the circle is

$$\|\mathbf{X} - \mathbf{O}\| = r \quad (4.1.1)$$

4.2.

$$(a) \mathbf{O} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, r = 2 \quad (4.2.1)$$

The following code sketches the circle 4.2.1 in figure 4.2 using the equation 4.1.1

```
codes/circle2/circle2a.py
```

4.3.

$$(b) \mathbf{O} = \begin{pmatrix} -2 \\ 32 \end{pmatrix}, r = 4 \quad (4.3.1)$$

The following code sketches the circle 4.3.1 in figure 4.3 using the equation 4.1.1

```
codes/circle2/circle2b.py
```

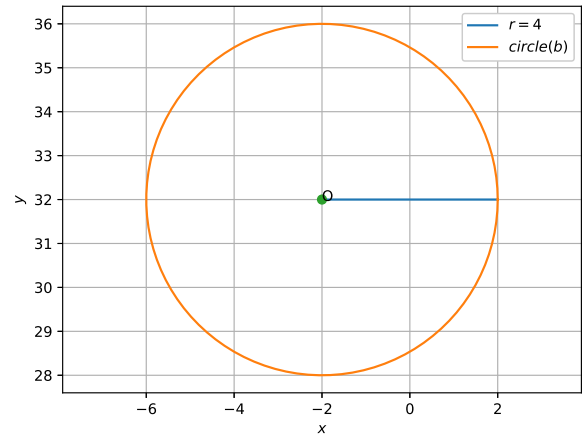


Fig. 4.3: Circle with centre at $\begin{pmatrix} -2 \\ 32 \end{pmatrix}$ and radius 4

4.4.

$$(c) \mathbf{O} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, r = \frac{1}{12} \quad (4.4.1)$$

The following code sketches the circle 4.4.1 in figure 4.4 using the equation 4.1.1

```
codes/circle2/circle2c.py
```

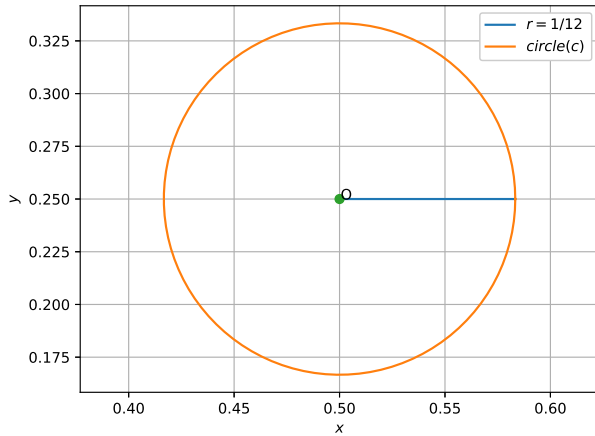



Fig. 4.4: Circle with centre at $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$

4.5.

$$(d) \mathbf{O} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, r = \sqrt{2} \quad (4.5.1)$$

The following code sketches the circle 4.5.1 in figure 4.5 using the equation 4.1.1

```
codes/circle2/circle2d.py
```

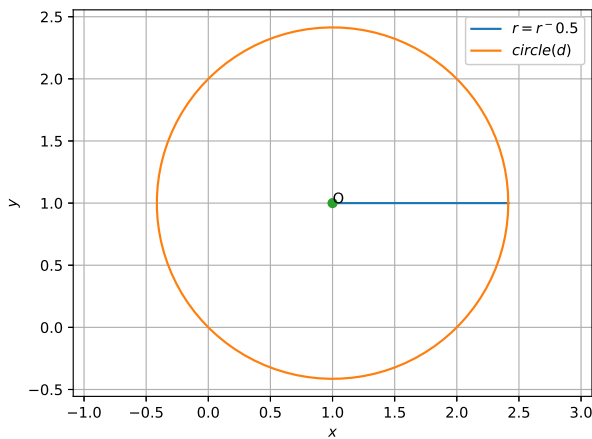


Fig. 4.5: Circle with centre at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and radius $\sqrt{2}$

4.6.

$$(e) \mathbf{O} = \begin{pmatrix} -a \\ -b \end{pmatrix}, r = \sqrt{a^2 - b^2} \quad (4.6.1)$$

The parameters used to sketch the circle are

taken as

$$a = 5, b = 4 \implies \mathbf{O} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \quad (4.6.2)$$

$$r = \sqrt{5^2 - 4^2} = 3 \quad (4.6.3)$$

The following code sketches the circle 4.6.3 in figure 4.6 using the equation 4.1.1

```
codes/circle2/circle2e.py
```

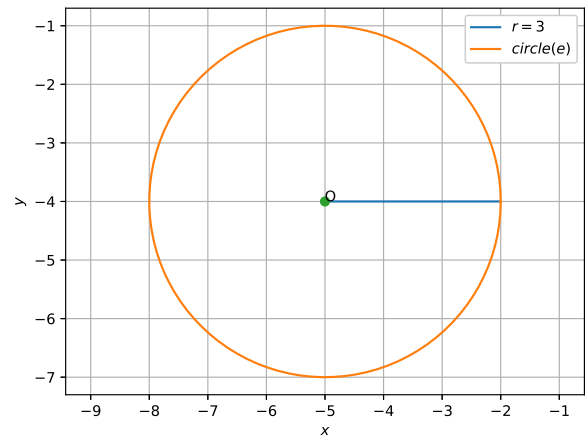


Fig. 4.6: Circle with centre at $\begin{pmatrix} -5 \\ -4 \end{pmatrix}$ and radius 3

5 CONICS

5.1 Example

5.1.1 Problem: Find $p(0)$, $p(1)$, $p(2)$ for each of the following polynomials

$$(a) p(y) = y^2 \quad (b) p(x) = (x - 1)(x + 1) \quad (6.1)$$

5.1.2 Solution:

5.1. (a) To find $p(0)$ we substitute 0 in place of the variable y in $p(y)$. Similarly we find $p(1)$ and $p(2)$

$$p(y) = y^2 \quad (5.1.1)$$

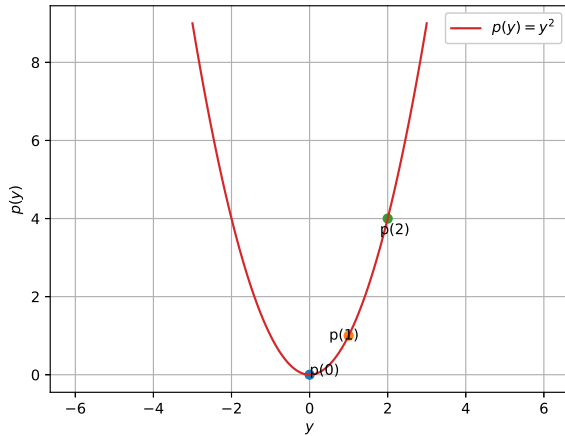
$$\implies p(0) = 0 \quad (5.1.2)$$

$$p(1) = 1 \quad (5.1.3)$$

$$p(2) = 4 \quad (5.1.4)$$

The following code sketches the graph of 5.1.1 in figure 5.1

```
codes/conic1/conic1a.py
```

Fig. 5.1: Graph of $p(y)$

5.2. (b) Similarly we find $p(0)$, $p(1)$ and $p(2)$ of $p(x)$ by replacing x

$$p(x) = (x - 1)(x + 1) \quad (5.2.1) \quad 5.2.$$

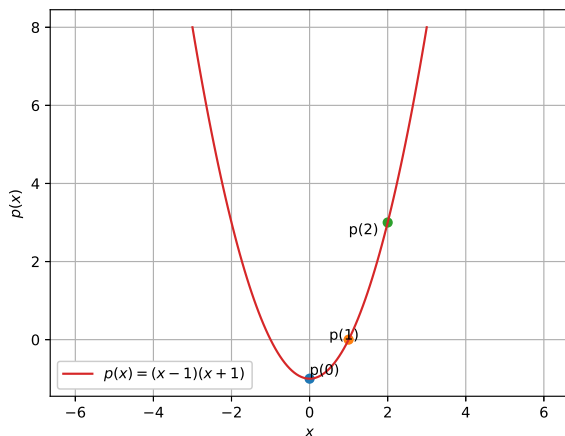
$$\Rightarrow p(0) = -1 \quad (5.2.2)$$

$$p(1) = 0 \quad (5.2.3)$$

$$p(2) = 3 \quad (5.2.4)$$

The following code sketches the graph of 5.2.1 in figure 5.2

```
codes/conic1/conic1b.py
```

Fig. 5.2: Graph of $p(x)$

5.2 Exercise

5.2.1 Problem: Factorise:

$$(a) 12x^2 - 7x + 1 \quad (2.1)$$

$$(b) 6x^2 + 5x - 6 \quad (2.2)$$

$$(c) 2x^2 + 7x + 3 \quad (2.3)$$

$$(d) 3x^2 - x - 4 \quad (2.4)$$

5.2.2 Solution:

5.1. A conic section has the following equation 5.1.1

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (5.1.1)$$

The equation is expressed in vector form as follows The vector form is

$$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \quad (5.1.2)$$

If x_1 is root of an equation, Then $(x - x_1)$ is a factor

$$(a) 12x^2 - 7x + 1 \quad (5.2.1)$$

can be expressed according to 5.1.2 as

$$\mathbf{x}^T \begin{pmatrix} 12 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -7 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (5.2.2)$$

To find roots using 5.2.2, substitute

$$y = 0 \quad (5.2.3)$$

$$\Rightarrow 12x^2 - 7x + 1 = 0 \quad (5.2.4)$$

$$x = \frac{1}{3}, \frac{1}{4} \quad (5.2.5)$$

Hence $(x - \frac{1}{3})$ and $(x - \frac{1}{4})$ are the factors

$$\Rightarrow (3x - 1)(4x - 1) = 12x^2 - 7x + 1 \quad (5.2.6)$$

The following code sketches the graph of 5.2.1 in figure 5.2

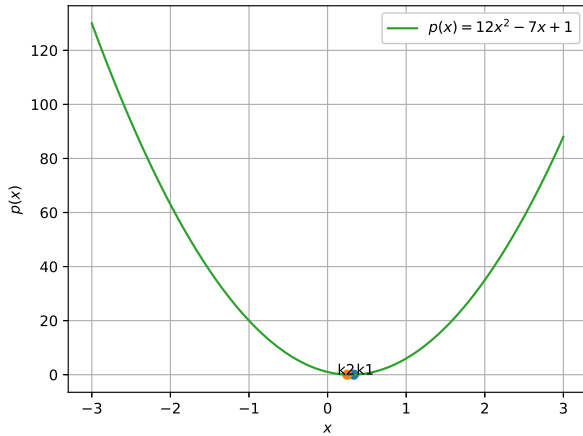
```
codes/conic2/conic2a.py
```

5.3.

$$(b) 6x^2 + 5x - 6 \quad (5.3.1)$$

can be expressed according to 5.1.2 as

$$\mathbf{x}^T \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5 & 0 \end{pmatrix} \mathbf{x} - 6 = 0 \quad (5.3.2)$$

Fig. 5.2: Graph of $12x^2 - 7x + 1$

Substituting $y = 0$ in equation 5.3.2 to find roots,

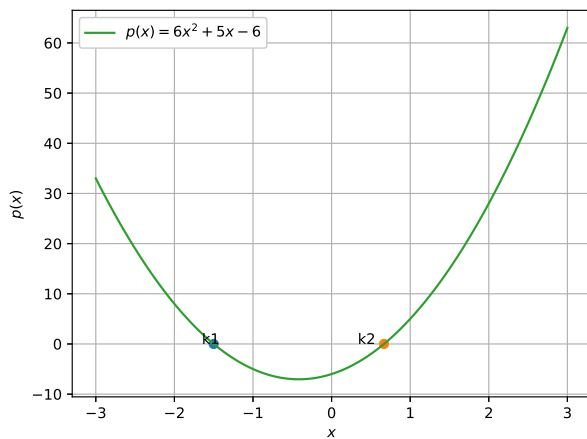
$$\Rightarrow 6x^2 + 5x - 6 = 0 \quad (5.3.3)$$

$$x = \frac{-3}{2}, \frac{2}{3} \quad (5.3.4)$$

$$(2x + 3)(3x - 2) = 6x^2 + 5x - 6 \quad (5.3.5)$$

The following code sketches the graph of 5.3.1 in figure 5.3

```
codes/conic2/conic2b.py
```

Fig. 5.3: Graph of $6x^2 + 5x - 6$

5.4.

$$(c) 2x^2 + 7x + 3 \quad (5.4.1)$$

can be expressed according to 5.1.2 as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \quad (5.4.2)$$

Substituting $y = 0$ in equation 5.4.2,

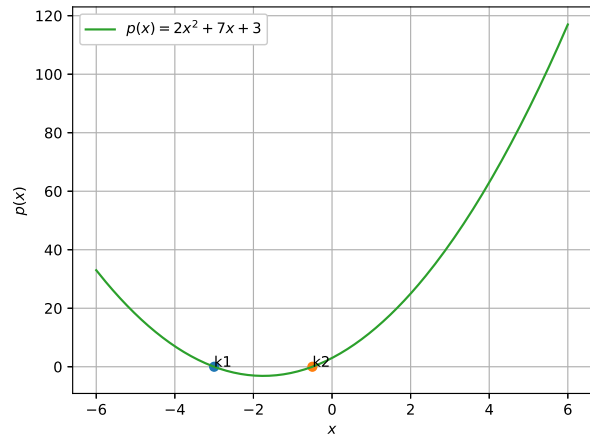
$$\Rightarrow 2x^2 + 7x + 3 = 0 \quad (5.4.3)$$

$$x = \frac{-1}{2}, -3 \quad (5.4.4)$$

$$(2x + 1)(x + 3) = 2x^2 + 7x + 3 \quad (5.4.5)$$

The following code sketches the graph of 5.4.1 in figure 5.4

```
codes/conic2/conic2c.py
```

Fig. 5.4: Graph of $2x^2 + 7x + 3$

5.5.

$$(d) 3x^2 - x - 4 \quad (5.5.1)$$

can be expressed according to 5.1.2 as

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} - 4 = 0 \quad (5.5.2)$$

Substituting $y = 0$ in equation 5.4.2,

$$\Rightarrow 3x^2 - x - 4 = 0 \quad (5.5.3)$$

$$x = \frac{4}{3}, -1 \quad (5.5.4)$$

$$(3x - 4)(x + 1) = 3x^2 - x - 4 \quad (5.5.5)$$

The following code sketches the graph of 5.5.1 in figure 5.5

```
codes/conic2/conic2d.py
```

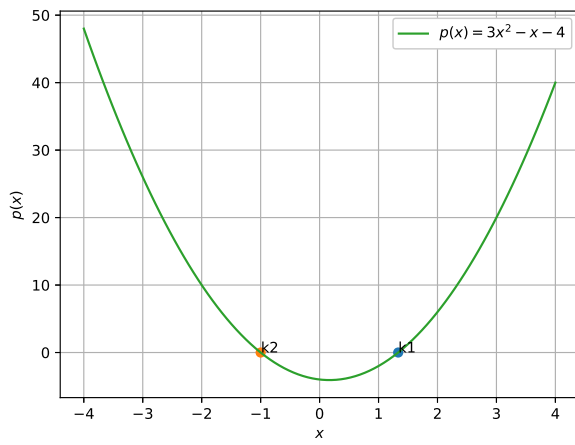


Fig. 5.5: Graph of $3x^2 - x - 4$