Parallel line through trapezium

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***Abstract*—This document proves that a line passing through a trapezium which is parallel to the parallel sides of a trapezium cuts the non-parallel sides in equal ratios.**

Download python codes from

```
svn co https://github.com/krishnajakodali/
    Summer2020/trunk/geometry/codes
```

and latex-tikz codes from

```
svn co https://github.com/krishnajakodali/
    Summer2020/trunk/geometry/figs
```

1 Problem

ABCD is a trapezium with $AB \parallel DC$. E and F are points on the non-parallel sides AD and BC respectively such that EF is parallel to AB.
Show that:

$$\frac{AE}{ED} = \frac{BF}{FC}$$

2 Construction

2.1. The trapezium ABCD looks like the Fig. 2.1. with $AB \parallel DC \parallel EF$

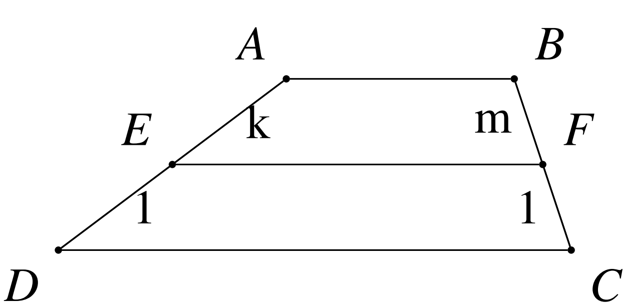

Fig. 2.1: Trapezium ABCD by Latex-Tikz

2.2. List the design parameters for construction
Solution: See Table. 2.2

Parameter	Description	Value
Length of AB	a	4
Length of CD	c	9
x-coordinate of A	x_a	4
Height of trapezium	h	3
AE / ED ratio	k	1

TABLE 2.2: To construct trapezium ABCD

2.3. Find the coordinates of the various points in Fig. 2.1
Solution: From the given information,

$$\mathbf{D} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.3.1}$$

$$\mathbf{C} = \begin{pmatrix} c \\ 0 \end{pmatrix} \tag{2.3.2}$$

$\because$ **A** x coordinate is given , y coordinate is same as the height of the trapezium.

$$\implies \mathbf{A} = \begin{pmatrix} x_a \\ h \end{pmatrix} \tag{2.3.3}$$

Also, **B** is at a distance from **A** and AB is parallel to x axis.
From (2.3.3),$\mathbf{A} = \begin{pmatrix} x_a \\ h \end{pmatrix}$ Let $\mathbf{B} = \begin{pmatrix} x_b \\ h \end{pmatrix}$

$$a^2 = \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A} - \mathbf{B}\|^T \|\mathbf{A} - \mathbf{B}\| \tag{2.3.4}$$

$$= \mathbf{A}^T\mathbf{A} + \mathbf{B}^T\mathbf{B} - \mathbf{A}^T\mathbf{B} - \mathbf{B}^T\mathbf{A} \tag{2.3.5}$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T\mathbf{B} \quad \left(\because \mathbf{A}^T\mathbf{B} = \mathbf{B}^T\mathbf{A}\right) \tag{2.3.6}$$

$$= x_a^2 + h^2 + x_b^2 + h^2 - 2(x_a x_b + h^2) \tag{2.3.7}$$

$$= (x_a - x_b)^2 \tag{2.3.8}$$

$$x_b = x_a + a \tag{2.3.9}$$

$$\tag{2.3.10}$$

E divides AD in the ratio k: 1.

$$\frac{AD}{ED} = k + 1 \tag{2.3.11}$$

$$\mathbf{E} = \frac{k\mathbf{D} + \mathbf{A}}{k + 1} \tag{2.3.12}$$

$$\tag{2.3.13}$$

Let **F** divide BC in ratio m:1.

$$\mathbf{F} = \frac{m\mathbf{C} + \mathbf{B}}{m + 1} \tag{2.3.14}$$

$$\tag{2.3.15}$$

EF is parallel to DC and hence the x axis. So we equate y coordinates of E and F to find m. from 2.3.13 and 2.3.15

$$\frac{h}{k+1} = \frac{h}{m+1} \tag{2.3.16}$$

$$m = k \tag{2.3.17}$$

$$\mathbf{F} = \frac{k\mathbf{C} + \mathbf{B}}{k + 1} \tag{2.3.18}$$

$$\tag{2.3.19}$$

The values are listed in 2.3.

D	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
C	$\begin{pmatrix} 9 \\ 0 \end{pmatrix}$
A	$\begin{pmatrix} 4 \\ 3 \end{pmatrix}$
B	$\begin{pmatrix} 8 \\ 3 \end{pmatrix}$
E	$\begin{pmatrix} 2 \\ 1.5 \end{pmatrix}$
F	$\begin{pmatrix} 8.5 \\ 1.5 \end{pmatrix}$

TABLE 2.3: Derived coordinates trapezium ABCD

2.4. Draw Fig. 2.1 using python
Solution: The following Python code generates Fig. 2.1

```
codes/prob.py
```

and the equivalent latex-tikz code generating Fig. 2.1 is

```
figs/prob.tex
```

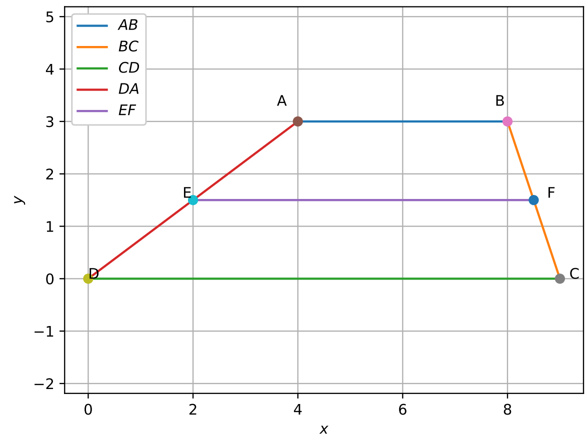

Fig. 2.4: Trapezium ABCD generated using python

The above latex code can be compiled as a standalone document as

```
figs/prob_alone.tex
```

3 Solution

3.1. Let F divide CD in the ratio m:1.Then using section formulae,

$$\mathbf{F} = \frac{m\mathbf{C} + \mathbf{B}}{m + 1} \tag{3.1.1}$$

Also using 2.3.13

$$\mathbf{E} = \frac{k\mathbf{D} + \mathbf{A}}{k + 1} \tag{3.1.2}$$

$$\implies \mathbf{E} - \mathbf{F} = \frac{k\mathbf{D} + \mathbf{A}}{k + 1} - \frac{m\mathbf{C} + \mathbf{B}}{m + 1} \tag{3.1.3}$$

If D is taken as the origin then $\mathbf{D} = \mathbf{0}$. So 3.1.3 becomes

$$\implies \mathbf{E} - \mathbf{F} = \frac{\mathbf{A}}{k + 1} - \frac{m\mathbf{C} + \mathbf{B}}{m + 1} \tag{3.1.4}$$

3.2. AB is parallel to DC. So for some constant l

$$\implies \mathbf{C} - \mathbf{D} = l(\mathbf{B} - \mathbf{A}) \tag{3.2.1}$$

$$\implies \mathbf{C} = l(\mathbf{B} - \mathbf{A}) \tag{3.2.2}$$

3.3. Substituting 3.2.2 in 3.1.4 We get

$$\implies \mathbf{E} - \mathbf{F} = \frac{\mathbf{A}}{k + 1} - \frac{ml(\mathbf{B} - \mathbf{A}) + \mathbf{B}}{m + 1} \tag{3.3.1}$$

$$\implies \mathbf{E} - \mathbf{F} = \mathbf{A}\left(\frac{1}{k + 1} + \frac{lm}{m + 1}\right) - \mathbf{B}\left(\frac{ml + 1}{m + 1}\right) \tag{3.3.2}$$

But $EF \parallel AB$.So,For some w

$$\implies \mathbf{E} - \mathbf{F} = w(\mathbf{A} - \mathbf{B}) \tag{3.3.3}$$

Comparing 3.3.3 and 3.3.2. We have

$$\frac{1}{k+1} + \frac{lm}{m+1} = \frac{ml+1}{m+1} = w \tag{3.3.4}$$

$$\implies m + 1 + (k+1)\,lm = (lm+1)(k+1) \tag{3.3.5}$$

3.3.5 must hold true for all values of l,k and m.Thus m = k

Hence F divides BC in ratio k:1, same as the ratio in which E divides AD.