Linear Algebra

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***Abstract*—This document provides the solution to the problem no. 7 of each section under linear algebra. The figures are constructed using python.**

This documentation can be downloaded from

```
svn co https://github.com/mohit−singh−9/Summer
    −2020/tree/master/geometry/linear_algebra.git
```

1 Triangle Exercise

1.1 Problem

Find the area of the triangle formed by joining the midpoints of the sides of the triangle whose vertices are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

1.2 Solution

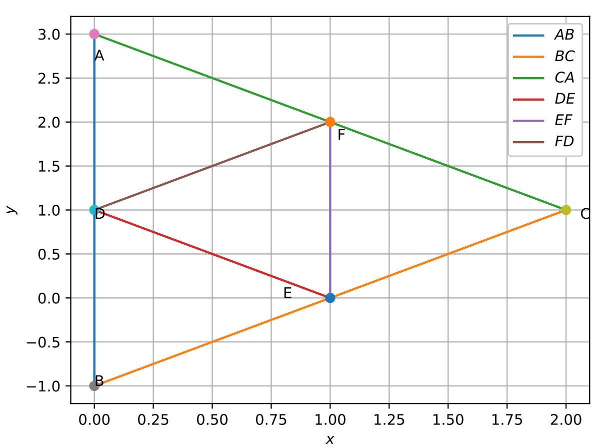


Fig. 1.0: Triangle DEF formed from midpoints of Triangle ABC

1.1. Let $\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Input values	
A	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$
B	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
C	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

TABLE 1.0: Input Table for construction

Derived values	
D	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
E	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
F	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

TABLE 1.0: Derived values

1.2. The midpoints of each side is given by

$$\mathbf{D} = \frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.1}$$

$$\mathbf{E} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.2.2}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.2.3}$$

$$\tag{1.2.4}$$

1.3. Area of a $\triangle$ ABC is given by

$$= \frac{1}{2} \left\| (\mathbf{E}-\mathbf{D}) \times (\mathbf{F}-\mathbf{D}) \right\| \tag{1.3.1}$$

$$= \frac{1}{2} \left\| \begin{pmatrix} 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| \tag{1.3.2}$$

On solving we get area of $\triangle$ DEF = 1 sq.units
.

1.4. Download the python code for finding a triangle's area from codes\triangle\area_tri_area.py

and the figure from

figs\triangle\draw_triangle.py

2 QUADRILATERAL EXERCISE

2.1 Problem

The two opposite vertices of a square are $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\2\end{pmatrix}$. Find the coordinates of other two vertices.

2.2 Solution

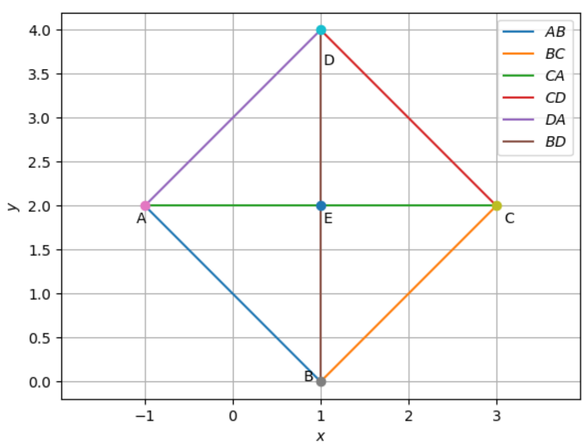


Fig. 2.0: Square ABCD

2.1. From inspection we see that the opposite vertices forms a diagonal which is parallel to x-axis. Then the diagonal formed by other two vertices is parallel to y-axis(i.e. their x coordinates are equal). Let $\mathbf{A} = \begin{pmatrix}-1\\2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\2\end{pmatrix}$.

2.2. Diagonals bisect each other at 90°. Let **B** and **D** be other two vertices.

2.3. Using the property that diagonals bisect each other at 90°, we can obtain other vertices by rotating diagonal AC by 90°about **E** in clockwise or anticlockwise direction.

2.4. The rotation matrix for a rotation of angle 90°about origin in anticlockwise direction is given by

$$\begin{pmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} \tag{2.4.1}$$

The **E** is given by

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{2.4.2}$$

$$= \begin{pmatrix}1\\2\end{pmatrix} \tag{2.4.3}$$

2.5. To make the rotation we need to shift the **E** to origin. So the change in other vectors are

$$\mathbf{A}-\mathbf{E} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{2.5.1}$$

$$\mathbf{C}-\mathbf{E} = \begin{pmatrix}2\\0\end{pmatrix} \tag{2.5.2}$$

The required matrix now is $\begin{pmatrix}-2 & 2\\0 & 0\end{pmatrix}$. Multiplying this with rotation matrix

$$= \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}-2 & 2\\0 & 0\end{pmatrix} \tag{2.5.3}$$

$$= \begin{pmatrix}0 & 0\\-2 & 2\end{pmatrix} \tag{2.5.4}$$

Now we obtained the coordinates as $\begin{pmatrix}0\\-2\end{pmatrix}$ and $\begin{pmatrix}0\\2\end{pmatrix}$. To obtain the final coordinates we will add **E** to shift to the actual position.

$$\mathbf{B} = \begin{pmatrix}0\\-2\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} \tag{2.5.5}$$

$$\mathbf{D} = \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} \tag{2.5.6}$$

Thus

$$\mathbf{B} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.5.7}$$

$$\mathbf{D} = \begin{pmatrix}1\\4\end{pmatrix} \tag{2.5.8}$$

2.6. The python code for the figure can be downloaded from

codes/quad/quad.py

3 LINE EXERCISE

3.1 Matrix

3.1.1 Problem:

3.1. Given $A = \begin{pmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{pmatrix}$. Find A+B.

3.1.2 Solution:

3.1. Since the two matrices have equal number of rows and columns, they are summable. Every element of a matrix gets added to its corresponding element in other matrix.

3.2. So

$$A + B = \begin{pmatrix} \sqrt{3}+2 & \sqrt{5}+1 & 0 \\ 0 & 6 & \frac{1}{2} \end{pmatrix} \tag{3.2.1}$$

3.3. The python code for matrix addition can be downloaded from

```
codes/line/matrix/matrix_add.py
```

3.2 Complex Numbers

3.2.1 Problem:

1. Find the modulus and argument of the complex numbers:

   a) $\dfrac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

   b) $\dfrac{1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

3.2.2 Solution:

1. A complex number $z = a + ib$ where $i = \sqrt{-1}$ is represented in vector notation as $\begin{pmatrix} a \\ b \end{pmatrix}$.
2. The multiplication of two complex numbers is not same as the multiplication of two vectors. It involves rotation of axes.
3. Suppose $z_1 = r_1 \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$ and $z_2 = r_2 \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$ be two complex numbers, then $z_1.z_2 = r_1 r_2 \begin{pmatrix} \cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) \end{pmatrix}$. Through vectors and matrices it can be realised through

$$z_1.z_2 = r_1 r_2 \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} \tag{3.2.2.3.1}$$

where $\mathbf{R} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}$ is the rotation matrix.

4. Similarly division of two complex numbers is given by $z_1.z_2^{-1} = \frac{r_1}{r_2} \begin{pmatrix} \cos(\theta_1-\theta_2) \\ \sin(\theta_1-\theta_2) \end{pmatrix}$ and through matrices multiplication as

$$z_1.z_2^{-1} = \frac{r_1}{r_2} \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} \tag{3.2.2.4.1}$$

where $\mathbf{S} = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{pmatrix}$ is the rotation matrix.

5. First converting the given vectors in polar form

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \frac{\sqrt{2}\begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix}}{\sqrt{2}\begin{pmatrix} \cos(-45^\circ) \\ \sin(-45^\circ) \end{pmatrix}} \tag{3.2.2.5.1}$$

Since this is the division of two complex numbers

$$= \frac{\sqrt{2}}{\sqrt{2}} \begin{pmatrix} \cos 45^\circ & \sin 45^\circ \\ \sin 45^\circ & -\cos 45^\circ \end{pmatrix} \begin{pmatrix} \cos(-45^\circ) \\ \sin(-45^\circ) \end{pmatrix} \tag{3.2.2.5.2}$$

$$= 1.\begin{pmatrix} \cos 90^\circ \\ \sin 90^\circ \end{pmatrix} \tag{3.2.2.5.3}$$

The magnitude is 1 and argument is 90°.

6. Here the numerator can be made a vector by taking y coordinate as 0. Also converting the vectors in polar form

$$\frac{1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \tag{3.2.2.6.1}$$

$$= \frac{1\begin{pmatrix} \cos 0^\circ \\ \sin 0^\circ \end{pmatrix}}{\sqrt{2}\begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix}} \tag{3.2.2.6.2}$$

Since its a division of two complex numbers, it can solved by

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 0^\circ & \sin 0^\circ \\ \sin 0^\circ & -\cos 0^\circ \end{pmatrix} \begin{pmatrix} \cos(45^\circ) \\ \sin(45^\circ) \end{pmatrix} \tag{3.2.2.6.3}$$

$$= 1.\begin{pmatrix} \cos(-45^\circ) \\ \sin(-45^\circ) \end{pmatrix} \tag{3.2.2.6.4}$$

The magnitude is $\frac{1}{\sqrt{2}}$ and argument is -45°.

3.3 Points and Vectors

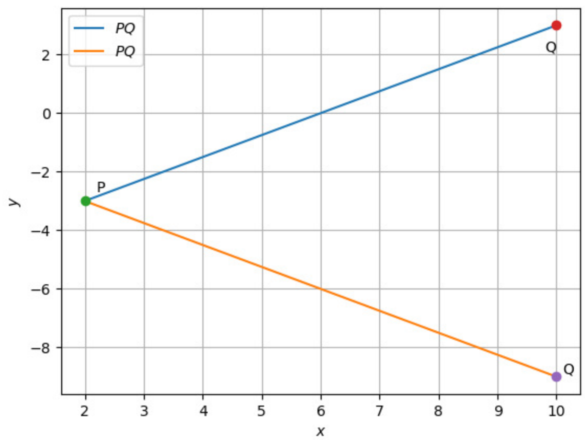
3.3.1 Problem:

1. Find the values of y for which distance between points

$$P = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, Q = \begin{pmatrix} 10 \\ y \end{pmatrix} \tag{3.3.1.1.1}$$

is 10 units.

3.3.2 Solution:



1. The distance between two points is given by equation

$$(\mathbf{P}-\mathbf{Q})^T(\mathbf{P}-\mathbf{Q}) = 10^2 \tag{3.3.2.1.1}$$
$$\left(\mathbf{P}^T-\mathbf{Q}^T\right)(\mathbf{P}-\mathbf{Q}) = 100 \tag{3.3.2.1.2}$$
$$\|P\|^2 - \mathbf{P}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{P} + \|Q\|^2 = 100 \tag{3.3.2.1.3}$$

On substituting

$$y^2 + 6y - 27 = 0 \tag{3.3.2.1.4}$$
$$(y+9)(y-3) = 0 \tag{3.3.2.1.5}$$

The roots of the above quadratic equation are -9 and 3. Therefore the values of y = -9,3

2. Both the lines in the graph have length equal to 10 units. **Q** can have two values $\begin{pmatrix} 10 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ -9 \end{pmatrix}$.
3. The python code to find the roots of the quadratic equation can be downloaded from

```
codes/line/point_vec/roots.py
```

4. The python code for the figure can be downloaded from

```
codes/line/point_vec/point_vec.py
```

3.4 Points on a Line

3.4.1 Problem:

1. Find the coordinates of points which divide the line segment joining A=$\begin{pmatrix} -2 \\ 2 \end{pmatrix}$, B=$\begin{pmatrix} 2 \\ 8 \end{pmatrix}$ into four equal parts.

3.4.2 Solution:

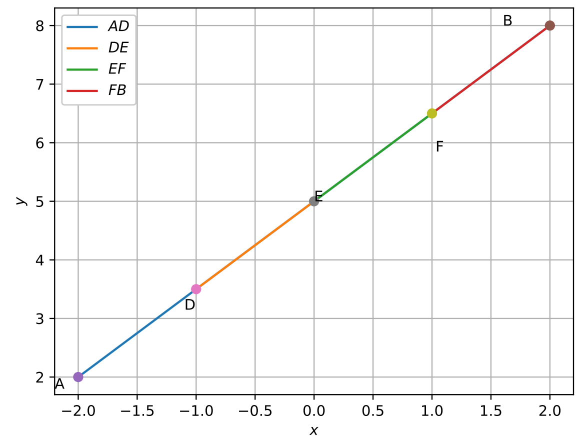


Fig. 3.4.2.0: Line segment AB

Input values	
A	$\begin{pmatrix} -2 \\ 2 \end{pmatrix}$
B	$\begin{pmatrix} 2 \\ 8 \end{pmatrix}$

TABLE 3.4.2.0: Input Table for construction

Derived values	
D	$\begin{pmatrix} -1 \\ 7/2 \end{pmatrix}$
E	$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$
F	$\begin{pmatrix} 1 \\ 13/2 \end{pmatrix}$

TABLE 3.4.2.0: Derived values

1. Let $\mathbf{D}, \mathbf{E}, \mathbf{F}$ be the points that divide the line segment into four equal parts.

2. If a point $\mathbf{X}$ divides a line segment(here AB) in the ratio of m:n then its coordinates are given by

$$\mathbf{X} = \frac{n\mathbf{B} + m\mathbf{A}}{m+n} \tag{3.4.2.2.1}$$

3. From figure, points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ divides AB in the ratio of 1:3, 2:2, 3:1 repectively. Thus there coordinates are given by

$$\mathbf{D} = \frac{1\mathbf{B} + 3\mathbf{A}}{4} = \begin{pmatrix} -1 \\ 7/2 \end{pmatrix} \tag{3.4.2.3.1}$$

$$\mathbf{E} = \frac{2\mathbf{B} + 2\mathbf{A}}{4} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{3.4.2.3.2}$$

$$\mathbf{F} = \frac{3\mathbf{B} + 1\mathbf{A}}{4} = \begin{pmatrix} 1 \\ 13/2 \end{pmatrix} \tag{3.4.2.3.3}$$

4. Download the python code for figure from

```
codes/line/point_line/line_division.py
```

3.5 Lines and Plane

3.5.1 Problem:

1. Check which of the following are solutions of the equation

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 4 \tag{3.5.1.1}$$

a) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

b) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

c) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

d) $\begin{pmatrix} \sqrt{2} \\ 4\sqrt{2} \end{pmatrix}$

e) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

3.5.2 Solution:

1. Substitute given vectors from options in the above line equation and check which will satisfy it.
2. Answer $= \begin{pmatrix} 4 \\ 0 \end{pmatrix}$
3. Also from Fig.3.5.0, the line passes through the point $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.
4. The python code for the figure

```
codes/line/lines_planes/lines_planes.py
```

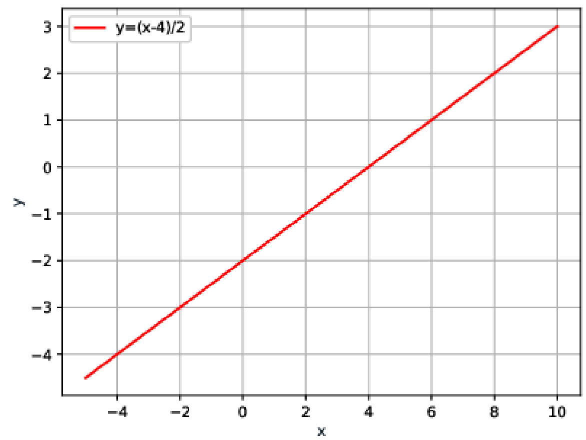


Fig. 3.5.0: Line equation: y=(x-4)/2

3.6 Motion in a Plane

3.6.1 Problem:

1. In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km/h to the north, what is the direction of the flag on the mast of the boat ?

3.6.2 Solution:

1. Let +x axis be east and +y be north direction. Also let $\mathbf{v_b}$ and $\mathbf{v_w}$ represent the velocity of boat and wind respectively along.
2. Then

$$\mathbf{v_w} = \begin{pmatrix} 72\cos 45^\circ \\ 72\sin 45^\circ \end{pmatrix} \tag{3.6.2.1}$$

$$\mathbf{v_b} = \begin{pmatrix} 0 \\ 51 \end{pmatrix} \tag{3.6.2.2}$$

3. The direction of the flag on the boat will be the relative velocity of wind w.r.t boat. So let $\mathbf{v_{wb}}$ represent the direction of flag. Then

$$\mathbf{v_{wb}} = \mathbf{v_w} - \mathbf{v_b} \tag{3.6.3.1}$$

$$= \begin{pmatrix} 36\sqrt{2} \\ 36\sqrt{2} - 51 \end{pmatrix} = \begin{pmatrix} 50.91 \\ -0.09 \end{pmatrix} \tag{3.6.3.2}$$

4. Let the angle made by $\mathbf{v_{wb}}$ w.r.t x-axis(east) be α. Then

$$\alpha = \tan\left(\frac{-0.09}{50.91}\right) \tag{3.6.4.1}$$

$$= -0.1^\circ \tag{3.6.4.2}$$

5. The direction of flag on the boat is 0.1°w.r.t east.

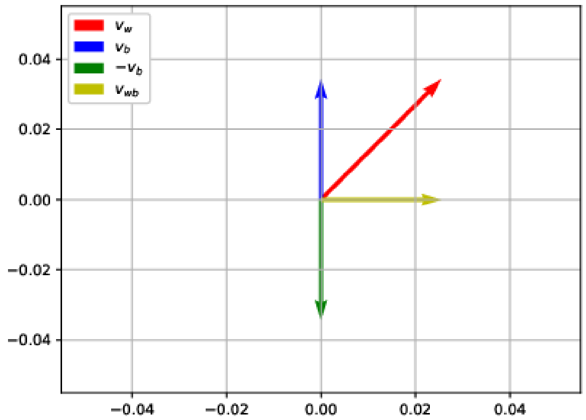


Fig. 3.6.5: Vectors representing different velocities

6. The python code for the figure can be downloaded from

```
codes/line/motion/motion.py
```

3.7 Determinants

3.7.1 Problem:

1. Find values of x, if

a) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ b) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

3.7.2 Solution:

1. If A = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$, then applying the same to abovequestion and solve the equation

$$-18 = 2x^2 - 24 \tag{3.7.1.1}$$
$$x = \pm\sqrt{3} \tag{3.7.1.2}$$

2. Following the same steps as above we get,

$$-2 = 5x - 6x \tag{3.7.2.1}$$
$$x = 2 \tag{3.7.2.2}$$

3.8 Linear Inequalities

3.8.1 Problem:

3.1. Solve $3x - 6 \geq 0$ graphically in a two dimensional plane.

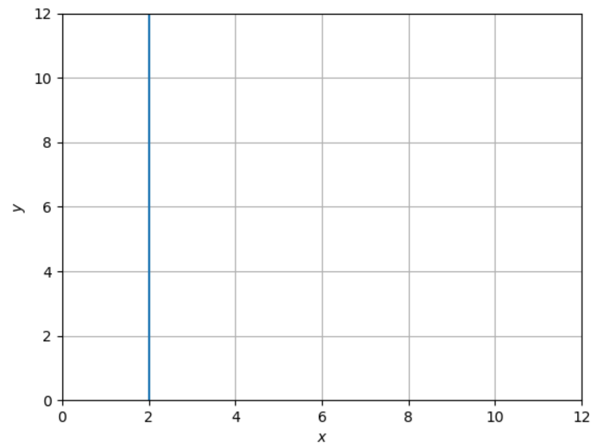


Fig. 3.0: Area satisfying x≥2

3.8.2 Solution:

3.1. If $\mathbf{x}$ is a vector then the given inequality can be represented as

$$\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} - 6 \geq 0 \tag{3.1.1}$$

On solving we get x≥ 2. No such constraint is on y. Graphically the solution is the whole region with lies to the right of line $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2$ in a 2D plane.

3.2. The python code can be downloaded from

```
codes/line/lin_ineq/lin_ineq1.py
```

3.9 Miscellaneous

3.9.1 Problem:

1. Find the distance between $\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ when
   a) PQ is parallel to the y-axis.
   b) PQ is parallel to the x-axis.

3.9.2 Solution:

1. If PQ is parallel to y axis then x coordiantes doesn't change. Therefore $x_1 = x_2 = x$. Hence, $\mathbf{P} = \begin{pmatrix} x \\ y_1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} x \\ y_2 \end{pmatrix}$. Distance between $\mathbf{P}$

and $\mathbf{Q}$ is given by

$$\sqrt{(\mathbf{P}-\mathbf{Q})^T(\mathbf{P}-\mathbf{Q})} \tag{3.9.1.1}$$

$$= \sqrt{\begin{pmatrix}0\\y_1-y_2\end{pmatrix}^T\begin{pmatrix}0\\y_1-y_2\end{pmatrix}} \tag{3.9.1.2}$$

$$= y_1 - y_2 \tag{3.9.1.3}$$

Distance between the points is $y_1 - y_2$

2. If PQ is parallel to x axis then y coordiantes doesn't change. Therefore $y_1 = y_2 = y$. Hence, $\mathbf{P} = \begin{pmatrix}x_1\\y\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}x_2\\y\end{pmatrix}$. Distance between $\mathbf{P}$ and $\mathbf{Q}$ is given by

$$\sqrt{(\mathbf{P}-\mathbf{Q})^T(\mathbf{P}-\mathbf{Q})} \tag{3.9.2.1}$$

$$= \sqrt{\begin{pmatrix}x_1-x_2\\0\end{pmatrix}^T\begin{pmatrix}x-1-x_2\\0\end{pmatrix}} \tag{3.9.2.2}$$

$$= x_1 - x_2 \tag{3.9.2.3}$$

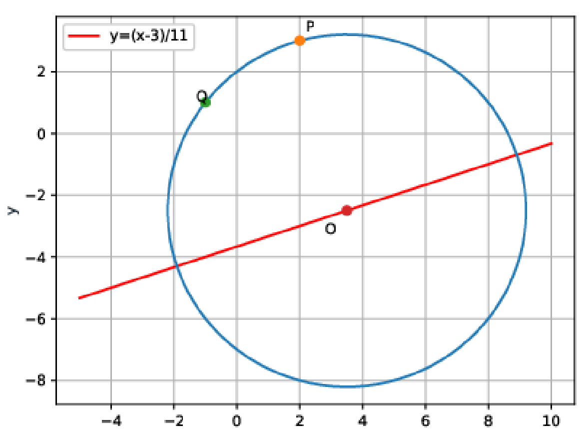
Distance between the points is $x_1 - x_2$

4 CIRCLE EXERCISE

4.1 Problem

Find the equation of the circle passing through the points $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ and and whose centre is on the line $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 11$.

4.2 Solution



Input values	
P	$\begin{pmatrix}2\\3\end{pmatrix}$
Q	$\begin{pmatrix}1\\-1\end{pmatrix}$
O	$\begin{pmatrix}7/2\\-5/2\end{pmatrix}$
Line eqn.	$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 11$

TABLE 4.0: Input Table for construction

Derived value	
r	5.7

TABLE 4.0: Derived values while construction

4.1. Let $\mathbf{O}$ be the centre of the circle and r be the radius of the circle. Since centre lies on the line, it satisfies the line equation

$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{O} = 11 \tag{4.1.1}$$

4.2. Also the circle passes through $\begin{pmatrix}2\\3\end{pmatrix}$ *and* $\begin{pmatrix}-1\\1\end{pmatrix}$.Let these points be $\mathbf{P}$ and $\mathbf{Q}$ repectively. So the distance between centre and these points will be equal to the radius.

$$\|\mathbf{P}-\mathbf{O}\| = \|\mathbf{Q}-\mathbf{O}\| = r \tag{4.2.1}$$

On solving we get the equation

$$\begin{pmatrix}6 & 4\end{pmatrix}\mathbf{O} = 11 \tag{4.2.2}$$

4.3. The equations from (4.1.1) and (4.2.2), can be solved to get $\mathbf{O}$.

$$\begin{pmatrix}1 & -3\\6 & 4\end{pmatrix}\mathbf{O} = \begin{pmatrix}11\\11\end{pmatrix} \tag{4.3.1}$$

$$\mathbf{O} = \begin{pmatrix}1 & -3\\6 & 4\end{pmatrix}^{-1}\begin{pmatrix}11\\11\end{pmatrix} \tag{4.3.2}$$

$$\mathbf{O} = \frac{1}{22}\begin{pmatrix}77\\-55\end{pmatrix} \tag{4.3.3}$$

Hence $\mathbf{O} = \begin{pmatrix}\frac{7}{2}\\\frac{-5}{2}\end{pmatrix}$

4.4. Sustituting $\mathbf{O}$ we get r= 5.7

4.5. Equation of circle is

$$\|\mathbf{x}-\mathbf{O}\| = 5.7 \tag{4.5.1}$$

4.6. The python code for the figure

codes/circle/circle.py

5 CONICS EXERCISE

5.1 Problem

Find the roots of the following quadratic equations:

1) $2x^2 - 7x + 3 = 0$
2) $2x^2 + x - 4 = 0$
3) $4x62 + 4\sqrt{3}x + 3 = 0.$
4) $2x^2 + x + 4 = 0.$

5.2 Solution

5.1. A conic section has the following equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (5.1.1)$$

The equation is expressed in vector form is as follows

$$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \quad (5.1.2)$$

a) $2x^2 - 7x + 3 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -7 & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \quad (5.1.3)$$

If $\begin{pmatrix} k \\ 0 \end{pmatrix}$ satisfies (5.1.3) then k is the root of the equation (5.1.3).
From graph, the roots are the points where the quadratic equation cuts the x-axis. A quadratic equation can have a maximum of two distinct roots.

$$2k^2 - 7k + 3 = 0 \quad (5.1.4)$$
$$(k-3)(2k-1) = 0 \quad (5.1.5)$$

From the graph in 5.1, the roots are 3 and $\frac{1}{2}$. The python code can be downloaded from

codes/conics/parabola1.py

b) $2x^2 + x - 4 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 4 = 0 \quad (5.1.6)$$

From the 5.1, the roots are 1.186 and 1.686. The python code can be downloaded from

codes/conics/parabola2.py

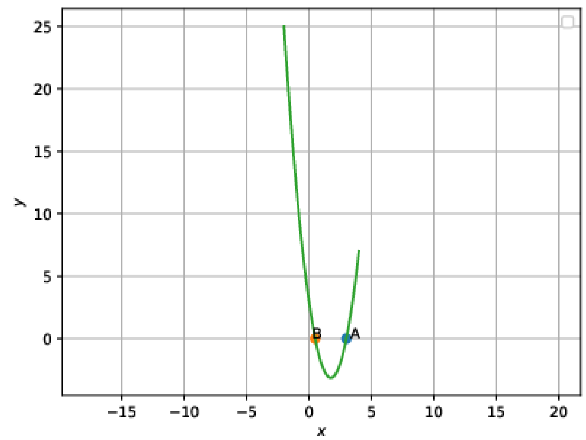


Fig. 5.1: Roots of $2x^2 - 7x + 3 = 0$

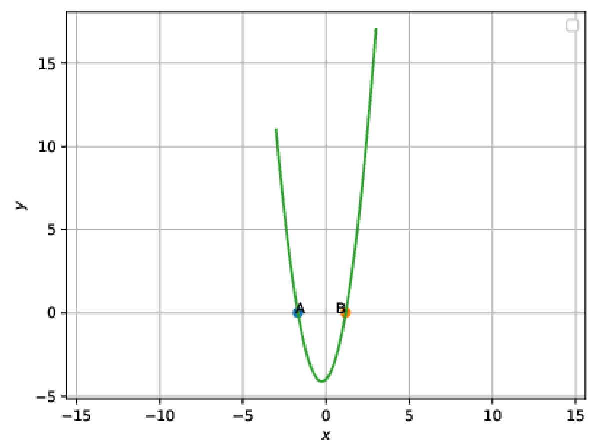


Fig. 5.1: Roots of $2x^2 + x - 4 = 0$

c) $4x^2 + 4\sqrt{3}x + 3 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4\sqrt{3} & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \quad (5.1.7)$$

From the graph in 5.1, the roots are real and equal. The root is $\frac{-\sqrt{3}}{2}$. The python code can be downloaded from

codes/conics/parabola3.py

d) $2x^2 + x + 4 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 4 = 0 \quad (5.1.8)$$

From the graph 5.1, the quadratic equation doesn't intersect x-axis. Thus it doesn't have real roots. It has ccomplex and conjugate

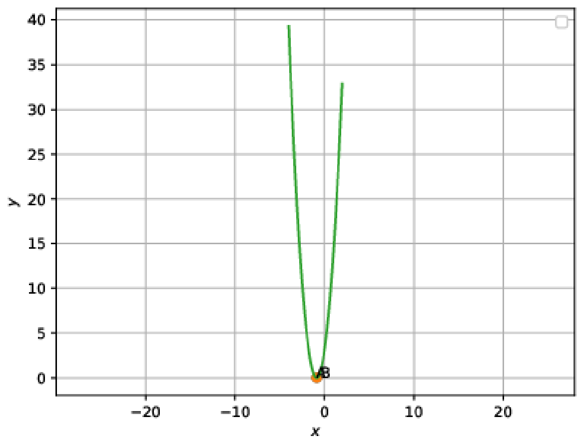


Fig. 5.1: Roots of $4x^2 + 4\sqrt{3}x + 3 = 0$

roots. The python code can be downloaded

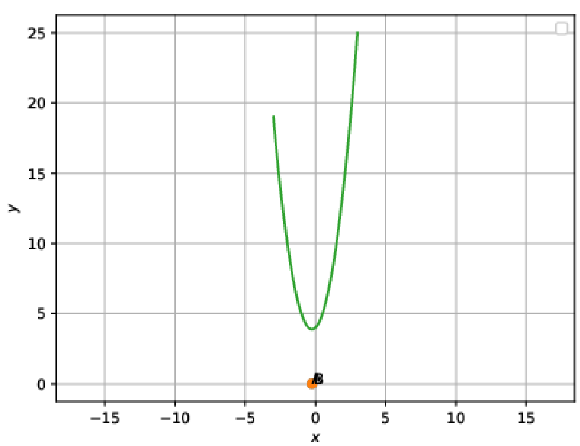


Fig. 5.1: Roots of $2x^2 + x + 4 = 0$

from

codes/conics/parabola4.py