Math Document Template

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***Abstract*—This is a document explaining for a question on the concept of linear algebra.**

Download all python codes from

```
svn co https://github.com/Ashuwin/summer_20/
    trunk/linear_algebra/codes
```

and latex-tikz codes from

```
svn co https://github.com/Ashuwin/summer_20/
    trunk/linear_algebra/figs
```

1 Triangle

1.1 Problem

1. Find the area of triangle whose vertices are
   a) $\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}2\\-4\end{pmatrix}$
   b) $\begin{pmatrix}-5\\-1\end{pmatrix},\begin{pmatrix}3\\-5\end{pmatrix},\begin{pmatrix}5\\2\end{pmatrix}$

1.2 Solution

1. The area of triangle ABC:
   Solution: The area of triangle ABC using cross product is obtained as:

$$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\| \tag{1.2.1.1}$$

$$\frac{1}{2}\left\|(\begin{pmatrix}-1\\0\end{pmatrix}-\begin{pmatrix}2\\3\end{pmatrix})\times(\begin{pmatrix}2\\-4\end{pmatrix}-\begin{pmatrix}2\\3\end{pmatrix})\right\| \tag{1.2.1.2}$$

$$\frac{1}{2}\left\|\begin{pmatrix}-3\\-3\end{pmatrix}\times\begin{pmatrix}0\\-7\end{pmatrix}\right\| = \frac{21}{2} \tag{1.2.1.3}$$

Area of $\triangle ABC = 10.5 units^2$ and it is found in the following python code:

```
codes/triangle/tri_area_ABC.py
```

$\triangle ABC$ in Fig.1.2.1 is generated using the following python code

```
codes/triangle/triangle1.py
```

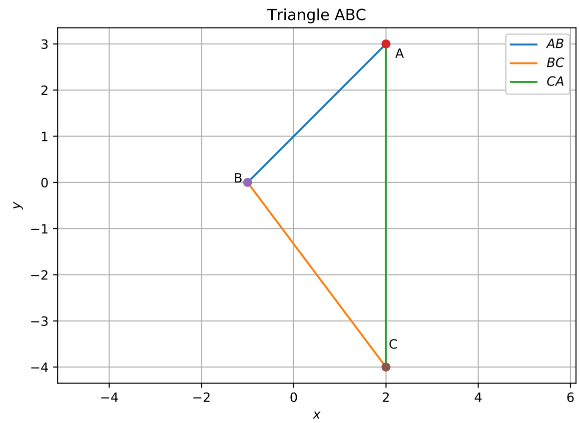

Fig. 1.2.1: Triangle ABC using python

2. The area of triangle PQR:
   Solution: The area of triangle PQR using Heron's formula is obtained as:

$$\frac{1}{2}\left\|(\mathbf{Q}-\mathbf{P})\times(\mathbf{R}-\mathbf{P})\right\| \tag{1.2.2.1}$$

$$\frac{1}{2}\left\|(\begin{pmatrix}3\\-5\end{pmatrix}-\begin{pmatrix}-5\\-1\end{pmatrix})\times(\begin{pmatrix}5\\2\end{pmatrix}-\begin{pmatrix}-5\\-1\end{pmatrix})\right\| \tag{1.2.2.2}$$

$$\frac{1}{2}\left\|\begin{pmatrix}8\\-4\end{pmatrix}\times\begin{pmatrix}10\\3\end{pmatrix}\right\| = \frac{64}{2} \tag{1.2.2.3}$$

Area of $\triangle PQR = 32 units^2$ and it is found in the following python code:

```
codes/triangle/tri_area_PQR.py
```

$\triangle PQR$ in Fig.1.2.2 is generated using the following python code

```
codes/triangle/triangle2.py
```

2 Quadrilateral

2.1 Problem

1. Find the area of the quadrilateral whose vertices are, taken in order, are $\begin{pmatrix}-4\\2\end{pmatrix},\begin{pmatrix}-3\\-5\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}$

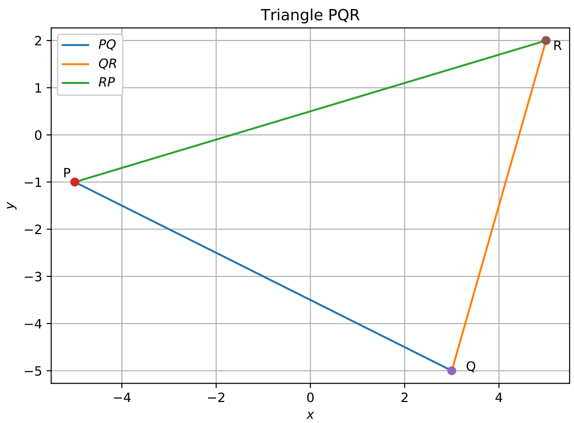

Fig. 1.2.2: Triangle PQR using python

2.2 Solution

1. The area of triangle ABC:
   Solution: The area of triangle ABC using cross product is obtained as:
$$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\| \tag{2.2.1.1}$$
$$\frac{1}{2}\left\|(\begin{pmatrix}-3\\-5\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix})\times(\begin{pmatrix}3\\-2\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix})\right\| \tag{2.2.1.2}$$
$$\frac{1}{2}\left\|\begin{pmatrix}1\\-7\end{pmatrix}\times\begin{pmatrix}7\\-4\end{pmatrix}\right\|=\frac{45}{2} \tag{2.2.1.3}$$
   Area of $\triangle ABC = 22.5units^2$ and it is found in the following python code:

```
codes/tri_area_ABC.py
```

2. The area of triangle ACD:
   Solution: The area of triangle ACD using Heron's formula is obtained as:
$$\frac{1}{2}\left\|(\mathbf{C}-\mathbf{A})\times(\mathbf{D}-\mathbf{A})\right\| \tag{2.2.2.1}$$
$$\frac{1}{2}\left\|(\begin{pmatrix}3\\-2\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix})\times(\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix})\right\| \tag{2.2.2.2}$$
$$\frac{1}{2}\left\|\begin{pmatrix}7\\-4\end{pmatrix}\times\begin{pmatrix}6\\1\end{pmatrix}\right\|=\frac{31}{2} \tag{2.2.2.3}$$
   Area of $\triangle ACD = 15.5units^2$ and it is found in the following python code:

```
codes/tri_area_ACD.py
```

3. The area of quadrilateral $ABCD$:
   Solution: Area of Quadrilateral $ABCD$ = Area of $\triangle ABC$ + Area of $\triangle ACD$ = $38units^2$
4. Quadrilateral $ABCD$ in Fig.2.2.4 is generated using the following python code

```
codes/quadrilateral/quad.py
```

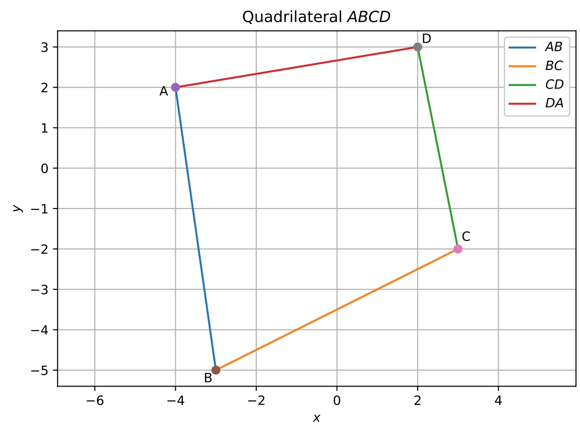

Fig. 2.2.4: Quadrilateral $ABCD$ using python

3 Line exercises

3.1 Complex numbers

3.1.1 Problem:

1. Find the conjugate of $\dfrac{\begin{pmatrix}3\\-2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}}{\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}}$

3.1.2 Solution:

1. A complex number $\begin{pmatrix}a\\b\end{pmatrix}$ can be represented as 2 x 2 matrix:
$$\begin{pmatrix}a & -b\\ b & a\end{pmatrix}$$
   Multiplying the given complex numbers after converting them to a 2 x 2 matrix,
$$\frac{\begin{pmatrix}3 & 2\\-2 & 3\end{pmatrix}\begin{pmatrix}2 & -3\\3 & 2\end{pmatrix}}{\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}} \tag{3.1.2.1.1}$$
$$\frac{\begin{pmatrix}12 & -5\\5 & 12\end{pmatrix}}{\begin{pmatrix}4 & -3\\3 & 4\end{pmatrix}} \tag{3.1.2.1.2}$$

Converting the matrices back to complex numbers,

$$\frac{\begin{pmatrix}12\\5\end{pmatrix}}{\begin{pmatrix}4\\3\end{pmatrix}} \tag{3.1.2.1.3}$$

Multiplying the conjugate of denominator to both numerator and denominator,

$$\frac{\begin{pmatrix}12\\5\end{pmatrix}\begin{pmatrix}4\\-3\end{pmatrix}}{\begin{pmatrix}4\\3\end{pmatrix}\begin{pmatrix}4\\-3\end{pmatrix}} \tag{3.1.2.1.4}$$

Multiplying the complex numbers after converting them to a 2 x 2 matrix,

$$\frac{\begin{pmatrix}12 & -5\\5 & 12\end{pmatrix}\begin{pmatrix}4 & 3\\-3 & 4\end{pmatrix}}{\begin{pmatrix}4 & -3\\3 & 4\end{pmatrix}\begin{pmatrix}4 & 3\\-3 & 4\end{pmatrix}} \tag{3.1.2.1.5}$$

$$\frac{\begin{pmatrix}63 & 16\\-16 & 63\end{pmatrix}}{\begin{pmatrix}25 & 0\\0 & 25\end{pmatrix}} \tag{3.1.2.1.6}$$

$$\frac{1}{25}\begin{pmatrix}63\\-16\end{pmatrix} \tag{3.1.2.1.7}$$

Conjugate of the complex number $= \frac{1}{25}\begin{pmatrix}63\\16\end{pmatrix}$

3.2 *Points and Vectors*

3.2.1 Problem: Find the point on the x-axis which is equidistant from

$$\begin{pmatrix}2\\-5\end{pmatrix}, \begin{pmatrix}-2\\9\end{pmatrix}$$

3.2.2 Solution:

1. From the given information,

$$\left\|\mathbf{x} - \begin{pmatrix}2\\-5\end{pmatrix}\right\|^2 = \left\|\mathbf{x} - \begin{pmatrix}-2\\9\end{pmatrix}\right\|^2 \tag{3.2.2.1.1}$$

$\|\mathbf{x}\|^2 + \left\|\begin{pmatrix}2\\-5\end{pmatrix}\right\|^2 - 2\begin{pmatrix}2 & -5\end{pmatrix}\mathbf{x} = \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}-2\\9\end{pmatrix}\right\|^2 - 2\begin{pmatrix}-2 & 9\end{pmatrix}\mathbf{x}$

which can be simplified to obtain

$$\begin{pmatrix}8 & -28\end{pmatrix}\mathbf{x} = -56 \tag{3.2.2.1.2}$$

Choose $\mathbf{x} = \begin{pmatrix}x\\0\end{pmatrix}$ as the point lies on the x-axis

$$\begin{pmatrix}8 & -28\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = -56 \tag{3.2.2.1.3}$$

$$\implies x = -7 \tag{3.2.2.1.4}$$

The point is $\begin{pmatrix}-7\\0\end{pmatrix}$

The Lines in Fig.3.2.2.1 is generated using the following python code

```
codes/line/point_vector/point_vector.py
```

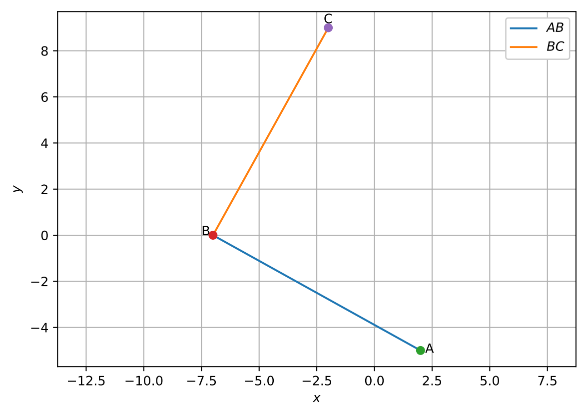

Fig. 3.2.2.1: Lines generated using python

3.3 *Points on a line*

3.3.1 Problem: If $\mathbf{A} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\-4\end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB

3.3.2 Solution:

1. $\mathbf{A} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\-4\end{pmatrix}$

   Then $\mathbf{P}$ that divides $\mathbf{A}, \mathbf{B}$ in the ratio k:1 is

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{3.3.2.1.1}$$

   For the given problem,$k = \frac{3}{4}$

   Using the equation 3.3.2.1.1, the desired point is

$$\mathbf{P} = \frac{\frac{3}{4}\begin{pmatrix}2\\-4\end{pmatrix} + 1\begin{pmatrix}-2\\-2\end{pmatrix}}{\frac{3}{4}+1} \tag{3.3.2.1.2}$$

$$\mathbf{P} = \begin{pmatrix}-2/7\\-20/7\end{pmatrix} \tag{3.3.2.1.3}$$

The following python code plots the Fig.**??**

```
codes/point_line/int_sec.py
```

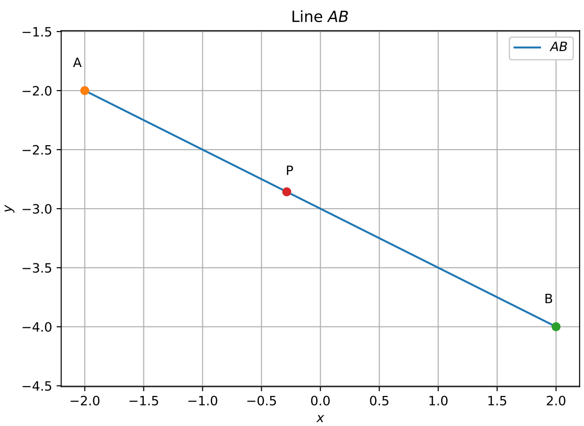

Fig. 3.3.2.1: Line AB using python

3.4 *Lines and Planes*

3.4.1 Problem: Write four solutions for each of the following equations

a)$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 7$

b)$\begin{pmatrix}\pi & 1\end{pmatrix}\mathbf{x} = 9$

c)$\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} = 0$

3.4.2 Solution:

1. $\mathbf{x}$ are randomly chosen and substituted in the equation and solutions are found.

a)

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 7 \tag{3.4.2.1.1}$$

Solution: Let $\mathbf{x} = \begin{pmatrix}a\\0\end{pmatrix}$ Substituting in equation 3.4.2.1.1, $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = 7$
$\implies a = \frac{7}{2}$
Let $\mathbf{x} = \begin{pmatrix}0\\b\end{pmatrix}$ Substituting in equation 3.4.2.1.1, $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}0\\b\end{pmatrix} = 7$
$\implies b = 7$
Let $\mathbf{x} = \begin{pmatrix}c\\1\end{pmatrix}$ Substituting in equation 3.4.2.1.1, $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}c\\1\end{pmatrix} = 7$
$\implies c = 3$
Let $\mathbf{x} = \begin{pmatrix}1\\d\end{pmatrix}$ Substituting in equation 3.4.2.1.1, $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1\\d\end{pmatrix} = 7$
$\implies d = 5$

b)

$$\begin{pmatrix}\pi & 1\end{pmatrix}\mathbf{x} = 9 \tag{3.4.2.1.2}$$

Solution: Let $\mathbf{x} = \begin{pmatrix}a\\0\end{pmatrix}$ Substituting in equation 3.4.2.1.2, $\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = 9$
$\implies a = \frac{9}{\pi}$
Let $\mathbf{x} = \begin{pmatrix}0\\b\end{pmatrix}$ Substituting in equation 3.4.2.1.2, $\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}0\\b\end{pmatrix} = 9$
$\implies b = 9$
Let $\mathbf{x} = \begin{pmatrix}c\\1\end{pmatrix}$ Substituting in equation 3.4.2.1.2, $\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}c\\1\end{pmatrix} = 9$
$\implies c = \frac{8}{\pi}$
Let $\mathbf{x} = \begin{pmatrix}1\\d\end{pmatrix}$ Substituting in equation 3.4.2.1.2, $\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}1\\d\end{pmatrix} = 9$
$\implies d = 9 - \pi$

c)

$$\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} = 0 \tag{3.4.2.1.3}$$

Solution: Let $\mathbf{x} = \begin{pmatrix}a\\0\end{pmatrix}$ Substituting in equation 3.4.2.1.3, $\begin{pmatrix}1 & -4\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = 0$
$\implies a = 0$
Let $\mathbf{x} = \begin{pmatrix}0\\b\end{pmatrix}$ Substituting in equation 3.4.2.1.3, $\begin{pmatrix}1 & -4\end{pmatrix}\begin{pmatrix}0\\b\end{pmatrix} = 0$
$\implies b = 0$

Let $\mathbf{x} = \begin{pmatrix} c \\ 1 \end{pmatrix}$ Substituting in equation 3.4.2.1.3, $\begin{pmatrix} 1 & -4 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = 0$
$\implies c = 4$
Let $\mathbf{x} = \begin{pmatrix} 1 \\ d \end{pmatrix}$ Substituting in equation 3.4.2.1.3, $\begin{pmatrix} 1 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ d \end{pmatrix} = 0$
$\implies d = \frac{1}{4}$

3.5 Motion in a plane

3.5.1 Problem:

1. A man can swim with a speed of $4.0km/h$ in still water. How long does he take to cross a river $1.0km$ wide if the river flows steadily at $3.0km/h$ and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank?

3.5.2 Solution:

1. Let the man be at point $\mathbf{M} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
   The speed of the man is $\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$
   The speed of the river is $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$
   Since the swimmer dive the river normal to the flow of river, therefore time taken by swimmer to cross the river,
$$t = \frac{d}{\|\mathbf{u}\|} = \frac{1km}{4km/h} = 15mins$$
   Distance covered down the river $= t \times \|\mathbf{v}\|$
$$x = \frac{1}{4}hr \times 3km/h = 750m$$
   The code for diagrammatic representation(3.5.2.1) of the solution is

```
codes/line/motion_plane/man_river.py
```

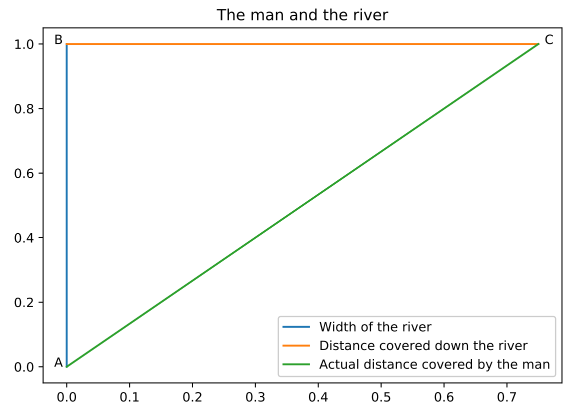

Fig. 3.5.2.1

3.6 Matrix

3.6.1 Problem:

1. Find the values of x,y and z from the following equations:
   a) $\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix}$
   b) $\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$
   c) $\begin{pmatrix} x+y+z \\ x+y \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

3.6.2 Solution:

1. This problem is solved by comparing the respective elements in both the matrices
   a)
$$\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix} \tag{3.6.2.1.1}$$
$$x = 1, y = 4, z = 3 \tag{3.6.2.1.2}$$
   b)
$$\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix} \tag{3.6.2.1.3}$$
$$5 + z = 5 \tag{3.6.2.1.4}$$
$$\implies z = 0 \tag{3.6.2.1.5}$$
$$x + y = 6 \tag{3.6.2.1.6}$$
$$xy = 8 \tag{3.6.2.1.7}$$
$$x = 4, y = 2 \tag{3.6.2.1.8}$$
$$x = 2, y = 4 \tag{3.6.2.1.9}$$
   $x = 4, y = 2, z = 0$ or $x = 2, y = 4, z = 0$
   c) $\begin{pmatrix} x+y+z \\ x+y \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

Expressing it as $Ax = b$ and $x = A^{-1}b$,

$$\begin{pmatrix}1 & 1 & 1\\1 & 1 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}9\\5\\7\end{pmatrix} \tag{3.6.2.1.10}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\1 & 1 & 0\\0 & 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}9\\5\\7\end{pmatrix} \tag{3.6.2.1.11}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix} \tag{3.6.2.1.12}$$

$$x = 2, y = 3, z = 4 \tag{3.6.2.1.13}$$

3.7 Determinants

3.7.1 Problem:

1. If $A = \begin{vmatrix}1 & 1 & -2\\2 & 1 & -3\\5 & 4 & -9\end{vmatrix}$, find $|A|$.

3.7.2 Solution:

1. To find the value of the determinant,

$$\begin{pmatrix}1 & 1 & -2\\2 & 1 & -3\\5 & 4 & -9\end{pmatrix} \tag{3.7.2.1.1}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix}1 & 1 & -2\\0 & -1 & 1\\5 & 4 & -9\end{pmatrix} \tag{3.7.2.1.2}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 5R_1} \begin{pmatrix}1 & 1 & -2\\0 & -1 & 1\\0 & -1 & 1\end{pmatrix} \tag{3.7.2.1.3}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix}1 & 1 & -2\\0 & -1 & 1\\0 & 0 & 0\end{pmatrix} \tag{3.7.2.1.4}$$

Det $|A| = 1 \times -1 \times 0 = 0$

The value of the determinant is found in the following python code

```
codes/line/determinants/det.py
```

3.8 Linear Inequalities

3.8.1 Problem:

1. Solve $3x + 2y > 6$ graphically

3.8.2 Solution:

1. Let $3x+2y = 6$ intersects the x-axis and y-axis at **A** and **B** respectively.

   a) Let $\mathbf{A} = \begin{pmatrix}x\\0\end{pmatrix}$

$$3x = 6 \tag{3.8.2.1.1}$$

$$\implies x = 2 \tag{3.8.2.1.2}$$

$$\mathbf{A} = \begin{pmatrix}2\\0\end{pmatrix} \tag{3.8.2.1.3}$$

   b) Let $\mathbf{B} = \begin{pmatrix}0\\y\end{pmatrix}$

$$2y = 6 \tag{3.8.2.1.4}$$

$$\implies y = 3 \tag{3.8.2.1.5}$$

$$\mathbf{B} = \begin{pmatrix}0\\3\end{pmatrix} \tag{3.8.2.1.6}$$

   c) Origin $= \begin{pmatrix}0\\0\end{pmatrix}$ does not satisfy the equation $3x + 2y < 6$.
   $\implies$ The solution is the right side of the line $3x + 2y = 6$

2. The following python code is the diagrammatic representation of the solution in Fig.3.8.2.2

```
codes/linear_inequalities/linear_inequalities.
    py
```

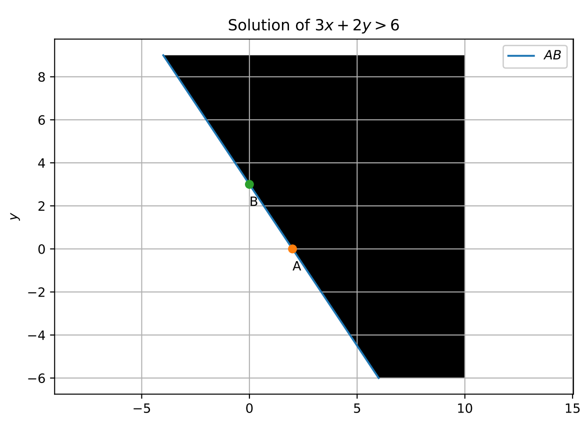

Fig. 3.8.2.2

3.9 Miscellaneous

3.9.1 Problem:

1. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid- point of the base is at the origin. Find vertices of the triangle.

3.9.2 Solution:

1. Let $\triangle ABC$ be an equilateral triangle. Let AB be the base and $\mathbf{M}$ be the midpoint.
   a) $\mathbf{A} = \begin{pmatrix} 0 \\ m \end{pmatrix}$ (as point $\mathbf{A}$ lies on the y-axis)
   $\mathbf{B} = \begin{pmatrix} 0 \\ n \end{pmatrix}$ (as point $\mathbf{B}$ lies on the y-axis)
   $\mathbf{M} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (as point $\mathbf{M}$ lies on the origin)
   $\mathbf{C} = \begin{pmatrix} p \\ 0 \end{pmatrix}$ (as point $\mathbf{A}$ lies on the x-axis)

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{C}-\mathbf{A}\| = 2a \tag{3.9.2.1.1}$$

   b) $\mathbf{M}$ is the mid-point of $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{3.9.2.1.2}$$
$$\implies m = -n \tag{3.9.2.1.3}$$
$$\|\mathbf{A}-\mathbf{B}\| = 2a \tag{3.9.2.1.4}$$
$$\implies m = -n = a \tag{3.9.2.1.5}$$

   c)

$$\|\mathbf{C}-\mathbf{A}\| = 2a \tag{3.9.2.1.6}$$
$$\sqrt{p^2+a^2} = 2a \tag{3.9.2.1.7}$$
$$\implies p = \pm\sqrt{3}a \tag{3.9.2.1.8}$$

   d) The vertices are,
   $\mathbf{A} = \begin{pmatrix} 0 \\ a \end{pmatrix}$
   $\mathbf{B} = \begin{pmatrix} 0 \\ -a \end{pmatrix}$
   $\mathbf{C} = \begin{pmatrix} \pm\sqrt{3}a \\ 0 \end{pmatrix}$
   $\triangle ABC$ in Fig.3.9.2.1 is generated using the following python code

```
codes/line/miscellaneous/tri_equi.py
```

4 Circle

4.1 Circle Example

4.1.1 Problem:

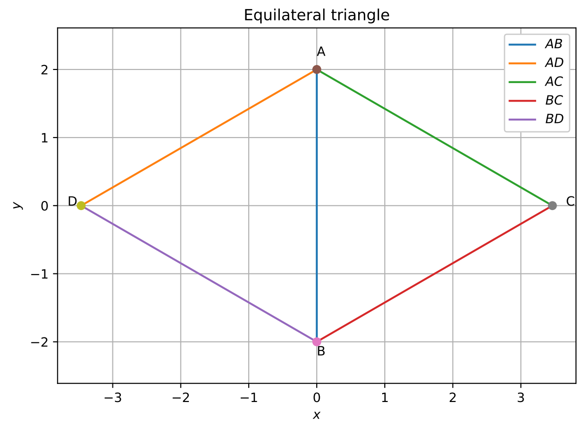

Fig. 3.9.2.1: Triangles ABC and ABD using python

1. Find the area of the region in the first quadrant enclosedd by the x-axis, the line $\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0$, and the circle $\|\mathbf{x}\| = 1$.

4.1.2 Solution:

1. Let $\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0$ intersects the circle $\|\mathbf{x}\| = 1$ at point $\mathbf{A} = \begin{pmatrix} a \\ b \end{pmatrix}$ in the first quadrant. The circle $\|\mathbf{x}\| = 1$ intersects the x-axis at point $\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the first quadrant.

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0 \tag{4.1.2.1.1}$$
$$\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \tag{4.1.2.1.2}$$
$$a = b \tag{4.1.2.1.3}$$
$$\|\mathbf{x}\| = 1 \tag{4.1.2.1.4}$$
$$\sqrt{a^2+b^2} = 1 \tag{4.1.2.1.5}$$
$$From(4.1.2.1.3), 2a^2 = 1 \tag{4.1.2.1.6}$$
$$a = \frac{1}{\sqrt{2}} = b \tag{4.1.2.1.7}$$

Finding scalar products,

$$\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{4.1.2.1.8}$$

$$(\mathbf{A}-\mathbf{O})^T(\mathbf{B}-\mathbf{O}) = \|\mathbf{A}-\mathbf{O}\|\,\|\mathbf{B}-\mathbf{O}\|\cos\angle AOB \tag{4.1.2.1.9}$$

$$\left(\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\right)^T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \left\|\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\right\|\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|\cos\angle AOB \tag{4.1.2.1.10}$$

$$\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\end{pmatrix} = 1\times 1\times\cos\angle AOB \tag{4.1.2.1.11}$$

$$\cos\angle AOB = \frac{1}{\sqrt{2}} \tag{4.1.2.1.12}$$

$$\angle AOB = 45^\circ \tag{4.1.2.1.13}$$

Area of the circle $= \pi\times r^2$ Area of the sector AOB:

$$\frac{\angle AOB}{360}\pi\times r^2 \tag{4.1.2.1.14}$$

$$\frac{45}{360}\pi\times 1^2 \tag{4.1.2.1.15}$$

$$\frac{\pi}{8} \tag{4.1.2.1.16}$$

$$0.3927 units^2 \tag{4.1.2.1.17}$$

Required area = Area of the sector AOB = 0.3927 $units^2$
The circle in Fig.4.1.2.1 is generated using the following python code

```
codes/circle/example/circle.py
```

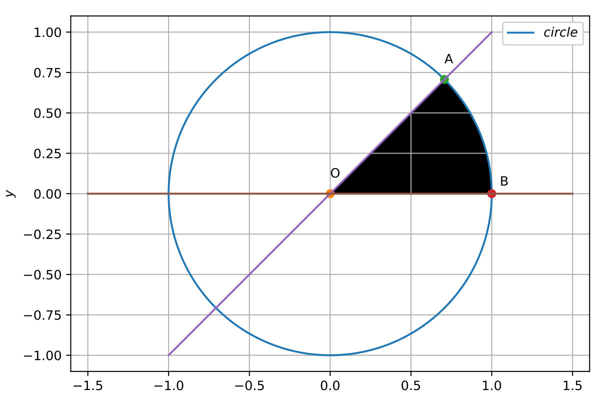

Fig. 4.1.2.1: Circle generated using python

4.2 Circle Exercise

4.2.1 Problem:

1. Find the equation of the circle passing through the points $\begin{pmatrix}4\\1\end{pmatrix}$ and $\begin{pmatrix}6\\5\end{pmatrix}$ and whose centre is on the line $\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = 16$.

4.2.2 Solution:

1. The vector form of general equation of circle,
$$\mathbf{x}^T\mathbf{x} - 2\mathbf{O}^T\mathbf{x} + F = 0 \tag{4.2.2.1.1}$$
whose centre is $\mathbf{O}$.
2. Point $\mathbf{A} = \begin{pmatrix}4\\1\end{pmatrix}$ lies on the circle. So, point A satisfies the equation 4.2.2.1.1
$$\begin{pmatrix}4\\1\end{pmatrix}^T\begin{pmatrix}4\\1\end{pmatrix} - 2\mathbf{O}^T\begin{pmatrix}4\\1\end{pmatrix} + F = 0$$
$$2\mathbf{O}^T\begin{pmatrix}4\\1\end{pmatrix} - F = 17 \tag{4.2.2.2.1}$$
3. Point $\mathbf{B} = \begin{pmatrix}6\\5\end{pmatrix}$ lies on the circle. So, point B satisfies the equation 4.2.2.1.1
$$\begin{pmatrix}6\\5\end{pmatrix}^T\begin{pmatrix}6\\5\end{pmatrix} - 2\mathbf{O}^T\begin{pmatrix}6\\5\end{pmatrix} + F = 0 \tag{4.2.2.3.1}$$
$$2\mathbf{O}^T\begin{pmatrix}6\\5\end{pmatrix} - F = 61 \tag{4.2.2.3.2}$$
4. Centre $\mathbf{O}$ lies on the line $\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = 16$
$$\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{O} = 16 \tag{4.2.2.4.1}$$
5. Solving equations 4.2.2.2.1, 4.2.2.3.2 and 4.2.2.4.1
$$\mathbf{O} = \begin{pmatrix}3\\4\end{pmatrix}, F = 15$$
$\implies$ Equation of the circle is $x^2 + y^2 - 6x - 8y + 15 = 0$
The vector form is
$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} + 15 = 0$$
6. The circle in Fig.4.2.2.6 is generated using the following python code

```
codes/circle/circle.py
```

5 Conics

5.1 Conics Example

5.1.1 Problem:

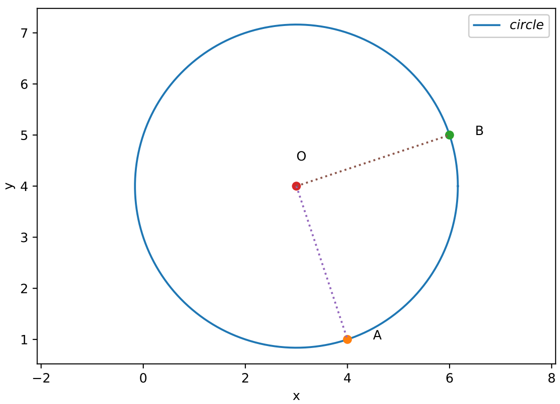

Fig. 4.2.2.6: Circle generated using python

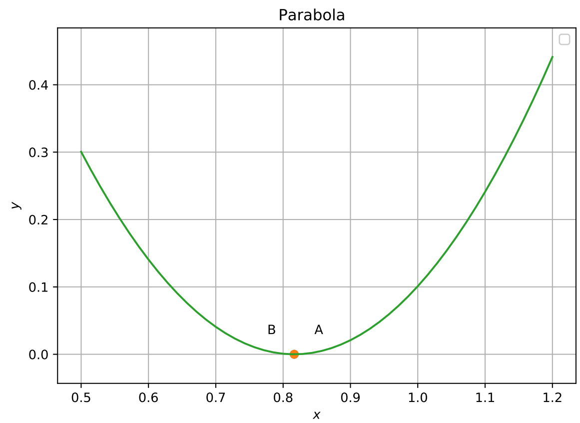

Fig. 5.1.2.1: Parabola

1. Find the roots of the quadratic equations $3x^2 - 2\sqrt{6}x + 2 = 0$

5.1.2 Solution:

1. For a general polynomial equation of degree 2 ,
   $p(x,y) \implies Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
   The vector form is
   $$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \tag{5.1.2.1.1}$$
   $3x^2 - 2\sqrt{6}x + 2 = 0$
   The vector form from the equation is 5.2.2.1.1 is
   $$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2\sqrt{6} & 0 \end{pmatrix} \mathbf{x} + 2 = 0 \tag{5.1.2.1.2}$$
   The values of $\mathbf{x}$ are found in the following python code

   ```
   codes/conics/example/conics.py
   ```

   $\mathbf{x} = \begin{pmatrix} 0.81649658 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.81649658 \\ 0 \end{pmatrix}$
   which can be verified from the Fig.5.1.2.1.
   The following python code generates the fig.5.1.2.1

   ```
   codes/conics/example/conics.py
   ```

5.2 Conics Exercise

5.2.1 Problem:

1. Find the roots of the quadratic equations:
   a) $x^2 - 3x - 10 = 0$
   b) $2x^2 + x - 6 = 0$
   c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
   d) $2x^2 - x + \frac{1}{8} = 0$
   e) $100x^2 - 20x + 1 = 0$

5.2.2 Solution:

1. For a general polynomial equation of degree 2 ,
   $p(x,y) \implies Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
   The vector form is
   $$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \tag{5.2.2.1.1}$$
   a) $x^2 - 3x - 10 = 0$
   The vector form from the equation is 5.2.2.1.1 is
   $$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -3 & 0 \end{pmatrix} \mathbf{x} - 10 = 0 \tag{5.2.2.1.2}$$
   The values of $\mathbf{x}$ are found in the following python code

   ```
   codes/conics/exercise/conics_1.py
   ```

   $\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
   which can be verified from the Fig.5.2.2.1.
   The following python code generates the fig.5.2.2.1

   ```
   codes/conics/exercise/conics_1.py
   ```

   b) $2x^2 + x - 6 = 0$
   The vector form from the equation is

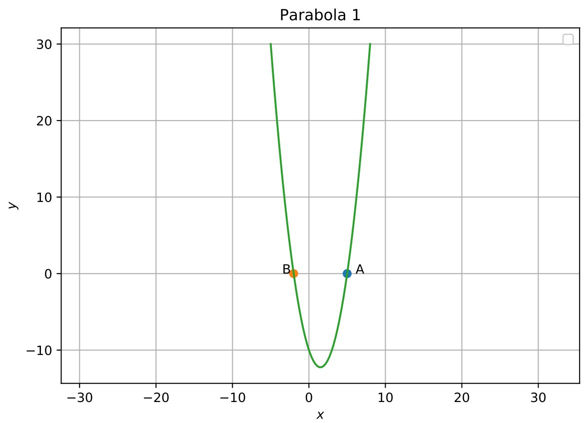

Fig. 5.2.2.1: Parabola 1

5.2.2.1.1 is

$$\mathbf{x}^T\begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x}-6=0 \quad (5.2.2.1.3)$$

The values of $\mathbf{x}$ are found in the following python code

```
codes/conics/exercise/conics_2.py
```

$\mathbf{x}=\begin{pmatrix}1.5\\0\end{pmatrix},\begin{pmatrix}-2\\0\end{pmatrix}$
which can be verified from the Fig.5.2.2.1. The following python code generates the fig.5.2.2.1

```
codes/conics/exercise/conics_2.py
```

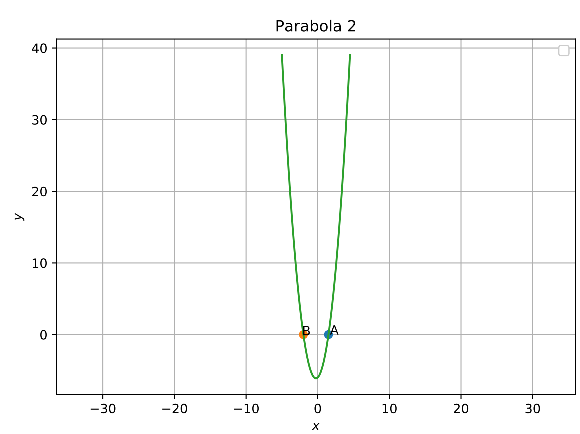

Fig. 5.2.2.1: Parabola 2

c) $\sqrt{2}x^2+7x+5\sqrt{2}=0$
The vector form from the equation is 5.2.2.1.1 is

$$\mathbf{x}^T\begin{pmatrix}\sqrt{2} & 0\\0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}7 & 0\end{pmatrix}\mathbf{x}+5\sqrt{2}=0 \quad (5.2.2.1.4)$$

The values of $\mathbf{x}$ are found in the following python code

```
codes/conics/exercise/conics_3.py
```

$\mathbf{x}=\begin{pmatrix}-1.414\\0\end{pmatrix},\begin{pmatrix}-3.535\\0\end{pmatrix}$ which can be verified from the Fig.5.2.2.1. The following python code generates the fig.5.2.2.1

```
codes/conics/exercise/conics_3.py
```

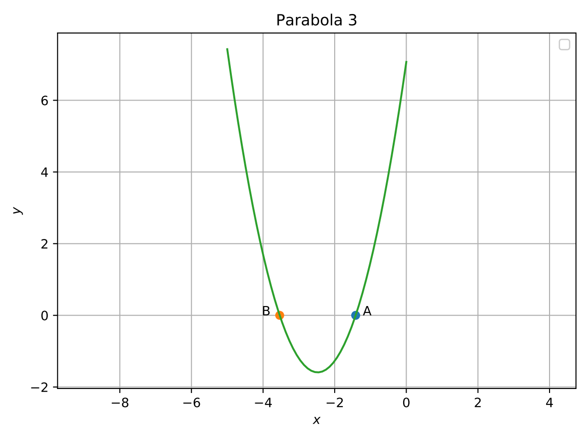

Fig. 5.2.2.1: Parabola 3

d) $2x^2-x+\frac{1}{8}=0$
The vector form from the equation is 5.2.2.1.1 is

$$\mathbf{x}^T\begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}-1 & 0\end{pmatrix}\mathbf{x}+\frac{1}{8}=0 \quad (5.2.2.1.5)$$

The values of $\mathbf{x}$ are found in the following python code

```
codes/conics/exercise/conics_4.py
```

$\mathbf{x}=\begin{pmatrix}0.25\\0\end{pmatrix},\begin{pmatrix}0.25\\0\end{pmatrix}$
which can be verified from the Fig.5.2.2.1. The following python code generates the fig.5.2.2.1

```
codes/conics/exercise/conics_4.py
```

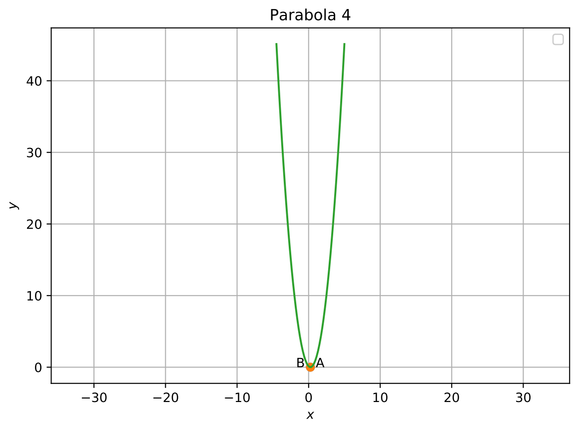

Fig. 5.2.2.1: Parabola 4

e) $100x^2 - 20x + 1 = 0$
The vector form from the equation is 5.2.2.1.1 is

$$\mathbf{x}^T \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -20 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{5.2.2.1.6}$$

The values of $\mathbf{x}$ are found in the following python code

```
codes/conics/exercise/conics_5.py
```

$\mathbf{x} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$ which can be verified from the Fig.5.2.2.1. The following python code generates the fig.5.2.2.1

```
codes/conics/exercise/conics_5.py
```

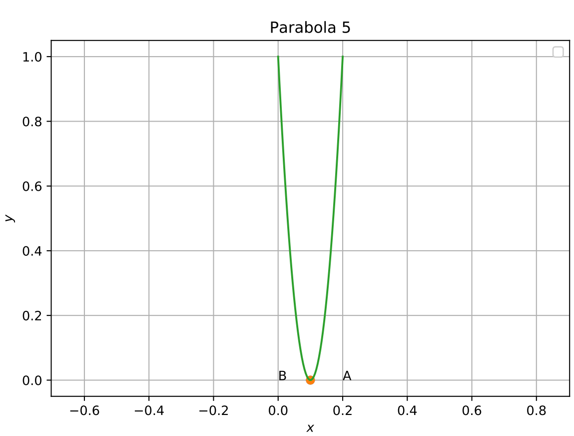

Fig. 5.2.2.1: Parabola 5