Solution to Differential Equation

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Table Of Contents

- Problem statement
- Theoritical solution
- Trapezoidal rule
- Laplace transform
- Bilinear transform
- RK method
- Figure

Problem Statement

For the differential equation $x(x^2 - 1)\frac{dy}{dx} = 1$, find the particular solution given $y(2) = 0$.

Theoretical Solution

$$\frac{dy}{dx} = \frac{1}{x(x^2 - 1)}$$

Using partial fractions:

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x+1}\right) - \frac{1}{x}$$

Integrating both sides:

$$y = \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| - \log|x| + \log c$$

Continued

Substituting $y(2) = 0$:

$$0 = \log\left(c\sqrt{\frac{3}{4}}\right)$$

Solving for c:

$$c = 2\sqrt{3}$$

The particular solution becomes:

$$y = \log\left(2\sqrt{\frac{x^2 - 1}{3x}}\right)$$

Simulation - Trapezoid Rule

For a general interval, say $[a, b]$, split up the intervals into n parts such that

$$h = \frac{b - a}{n} \tag{1}$$

Consider the points:

$$x_0 = a \tag{2}$$

$$x_n = b \tag{3}$$

$$x_{i+1} = x_i + h \tag{4}$$

Difference Equation

Deriving the difference equation:

$$f(x) = \frac{1}{x\left(x^2 - 1\right)} \tag{5}$$

$$A \approx \frac{h}{2}\left(\left(f(x_0) + f(x_1)\right) + \left(f(x_1) + f(x_2)\right) + \cdots + \left(f(x_{n-1}) + f(x_n)\right)\right) \tag{6}$$

$$A \approx \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right) \tag{7}$$

By integrating the differential equation from $n-1$ to n, we have

$$y_{n+1} - y_n = (x_{n+1} - x_n)\left(\frac{f(x_n)}{2} + \frac{f(x_{n+1})}{2}\right) \tag{8}$$

$$y_{n+1} = y_n + (x_{n+1} - x_n)\left(\frac{f(x_n)}{2} + \frac{f(x_{n+1})}{2}\right) \tag{9}$$

$$y_{n+1} = y_n + \frac{(x_{n+1} - x_n)}{2}\left(\frac{1}{x_n(x_n^2 - 1)} + \frac{1}{x_{n+1}(x_{n+1}^2 - 1)}\right) \tag{10}$$

Laplace Transform Approach

$$g(t) = \frac{1}{t(t^2 - 1)} \tag{11}$$

$$\frac{dy}{dt} = g(t) \tag{12}$$

$$sY(s) = X(s) \tag{13}$$

$$H(s) = \frac{Y(s)}{X(s)} \tag{14}$$

$$H(s) = \frac{1}{s} \tag{15}$$

Bilinear Transform

$$s = \frac{2}{h}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{16}$$

$$H(z) = \frac{h}{2}\left(\frac{1 + z^{-1}}{1 - z^{-1}}\right) \tag{17}$$

$$Y(z) = \frac{h}{2}\left(\frac{1 + z^{-1}}{1 - z^{-1}}\right) X(z), |z| > 1 \tag{18}$$

$$\left(1 - z^{-1}\right) Y(z) = \frac{h}{2}\left(1 + z^{-1}\right) X(z) \tag{19}$$

Taking inverse z-transform on both sides, we have

$$y_n - y_{n-1} = \frac{h}{2}\left(g(x_n) + g(x_{n-1})\right) \tag{20}$$

$$y_n = y_{n-1} + \frac{h}{2}\left(g(x_n) + g(x_{n-1})\right) \tag{21}$$

Runge-Kutta Method

$$y(x+h) = y(x) + y'(x,y)h \tag{22}$$

$$y'(x,y) = \frac{1}{x\left(x^2-1\right)} \tag{23}$$

$$y(x_0+h) = y(x_0) + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \tag{24}$$

where,

$$k_1 = hy'(x_0, y_0) \tag{25}$$

$$k_2 = hy'(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) \tag{26}$$

$$k_3 = hy'\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \tag{27}$$

$$k_4 = hy'\left(x_0 + h, y_0 + k_3\right) \tag{28}$$

Graphical Representation