9.2.1

EE24BTECH11012 - Bhavanisankar G S

LAPLACE TRANSFORMS

- **Notation** : Laplace transform of a function $f(x)$ is denoted as $\mathcal{L}(f(x))$, i.e.,

$$\mathcal{L}(f(x)) = F(s) = \int_0^\infty f(x)e^{-sx}dx \tag{0.1}$$

- It is a linear transformation, since integration is a linear operation.
- **Laplace transform of some functions :**

$$f(x) = 0 \implies F(s) = 0 \tag{0.2}$$
$$f(x) = 1 \implies F(s) = \frac{1}{s} \tag{0.3}$$
$$f(x) = x^n \implies F(s) = \frac{n!}{s^{n+1}} \tag{0.4}$$
$$f(x) = e^{at} \implies F(s) = \frac{1}{s-a} \tag{0.5}$$
$$f(x) = \sin ax \implies F(s) = \frac{a}{s^2+a^2} \tag{0.6}$$
$$f(x) = \cos ax \implies F(s) = \frac{s}{s^2+a^2} \tag{0.7}$$

- **Some other useful results include :**

$$\mathcal{L}(f'(x)) = sF(s) - f(0^-) \tag{0.8}$$
$$\mathcal{L}(f''(x)) = s^2F(s) - sf(0^-) - f'(0^-) \tag{0.9}$$

- **Laplace transform of unit step function $u(t)$:**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{0.10}$$

From (0.1)

$$\mathcal{L}(u(t)) = \int_0^\infty u(t)e^{-st}dt \tag{0.11}$$

For all non-negative values, $u(t) = 1$. Hence, the integral becomes,

$$F(s) = \int_0^\infty (1)e^{-st}dt \tag{0.12}$$
$$F(s) = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s} \tag{0.13}$$

- **Laplace transform of $e^{at}u(t)$:**
  From (0.1)

$$\mathcal{L}\left(e^{at}u(t)\right) = \int_0^\infty e^{at}u(t)e^{-st}dt \tag{0.14}$$

$$F(s) = \int_0^\infty e^{(a-s)t}dt \tag{0.15}$$

$$F(s) = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty = \frac{1}{s-a} \tag{0.16}$$

  When $a = 1$,

$$F(s) = \mathcal{L}\left(e^{t}u(t)\right) = \frac{1}{s-1} \tag{0.17}$$

QUESTION:
Consider the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$. Verify that $y = e^x + 1$ is a solution for it, given the initial condtitions $y(0) = 2$ and $y'(0) = 1$.
SOLUTION:
Consider the differential equation,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0 \tag{0.18}$$

Integrating (0.18) on both the sides, we have,

$$\int\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)dx = \int (0)\,dx \tag{0.19}$$

$$\frac{dy}{dx} - y = k \tag{0.20}$$

where, k is a constant. Applying the initial conditions, we have

$$k = 1 - 2 = -1 \tag{0.21}$$

$$\frac{dy}{dx} = y - 1 \tag{0.22}$$

Applying Laplace transform to (0.18) on both sides, we have

$$\mathcal{L}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) = \mathcal{L}(0) \tag{0.23}$$

$$\mathcal{L}\left(\frac{d^2y}{dx^2}\right) - \mathcal{L}\left(\frac{dy}{dx}\right) = \mathcal{L}(0) \tag{0.24}$$

$$\left(s^2F(s) - sf(0^-) - f'(0^-)\right) - (sF(s) - f(0^-)) = 0 \tag{0.25}$$

$$F(s)\left(s^2 - s\right) - f(0^-)(s-1) - f'(0^-) = 0 \tag{0.26}$$

$$F(s) = \frac{f(0^-)(s-1) + f'(0^-)}{s^2 - s} \tag{0.27}$$

$$\mathcal{L}(f(x)) = \frac{f(0^-) - f'(0^-)}{s} + \frac{f'(0^-)}{s-1} \tag{0.28}$$

Substituting the initial conditions, $y(0^-) = 2$ and $y'(0^-) = 1$, we have

$$f(x) = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{1}{s-1}\right) \tag{0.29}$$

$$f(x) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) \tag{0.30}$$

From, (0.13) and (0.17), it can be deduced that

$$f(x) = 1 + e^x \tag{0.31}$$

Hence, verified.
ALGORITHM :

$$x_0 = 0 \tag{0.32}$$

$$y_0 = 2 \tag{0.33}$$

$$h = 0.01 \tag{0.34}$$

$$x_{n+1} = x_n + h \tag{0.35}$$

$$y_{n+1} = y_n + h(y') \tag{0.36}$$

From (0.22),

$$y_{n+1} = y_n + h(y_n - 1) \tag{0.37}$$

which is the required difference equation.

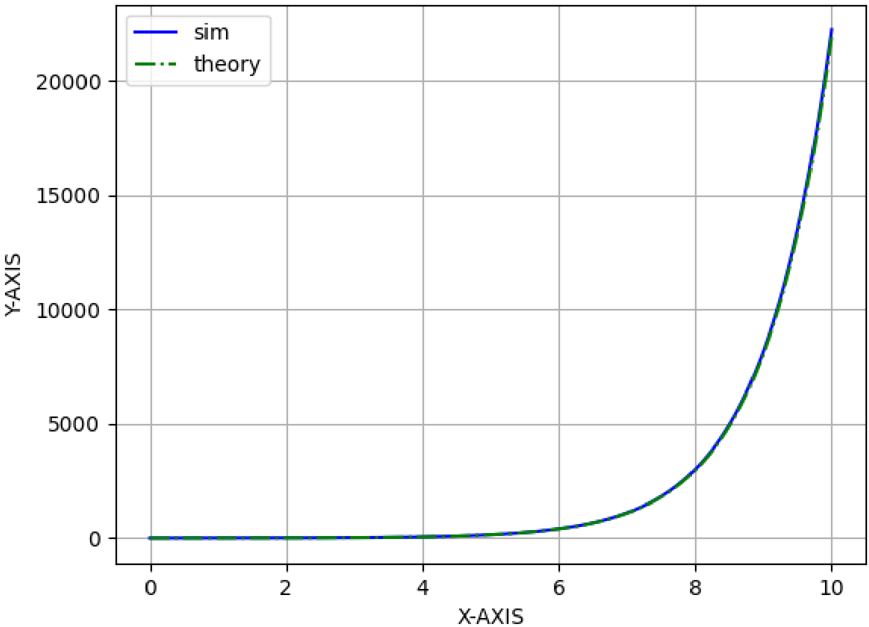

Fig. 0.1: A plot of the given question.