9.2.1

EE24BTECH11012 - Bhavanisankar G S

LAPLACE TRANSFORMS

- Transformation applied on a function results in a function of another variable, unlike an operation which when applied on a function yields another function but of the same variable.
- Laplace transform is a very useful technique used to solve complex equations using **integral transformations**.
- Any equation of the form $\int_a^b f(t)K(p,t)dt = F(p)$ is called an integral transformation, and the function $K(p,t)$ is called the **Kernel function**.
  When $a = 0$ and $b = \infty$, then the integral transformation is called the Laplace transformation.
- Notation : Laplace transform of a function $f(x)$ is denoted as $\mathcal{L}(f(x))$, i.e.,

$$\mathcal{L}(f(x)) = F(s) = \int_0^\infty f(x)e^{-sx}dx$$

  .
- It is a linear transformation, since integration is a linear operation.
- Laplace transform of some functions :

$$f(x) = 0 \implies F(s) = 0 \tag{0.1}$$

$$f(x) = 1 \implies F(s) = \frac{1}{s} \tag{0.2}$$

$$f(x) = x^n \implies F(s) = \frac{n!}{s^{n+1}} \tag{0.3}$$

$$f(x) = e^{at} \implies F(s) = \frac{1}{s-a} \tag{0.4}$$

$$f(x) = \sin ax \implies F(s) = \frac{a}{s^2+a^2} \tag{0.5}$$

$$f(x) = \cos ax \implies F(s) = \frac{s}{s^2+a^2} \tag{0.6}$$

- Some other useful results include :

$$\mathcal{L}(f'(x)) = sF(s) - f(0^-) \tag{0.7}$$

$$\mathcal{L}(f'(x)) = s^2F(s) - sf(0^-) - f'(0^-) \tag{0.8}$$

QUESTION:
Consider the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$. Verify that $y = e^x + 1$ is a solution for it.
SOLUTION:
Consider the differential equation,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$$

Applying Laplace transform on both sides, we have

$$\mathcal{L}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) = \mathcal{L}(0) \tag{0.9}$$

$$\mathcal{L}\left(\frac{d^2y}{dx^2}\right) - \mathcal{L}\left(\frac{dy}{dx}\right) = \mathcal{L}(0) \tag{0.10}$$

$$\left(s^2F(s) - sf(0^-) - f'(0^-)\right) - (sF(s) - f(0^-)) = 0 \tag{0.11}$$

$$F(s)\left(s^2 - s\right) - f(0^-)(s-1) - f'(0^-) = 0 \tag{0.12}$$

$$F(s) = \frac{f(0^-)(s-1) + f'(0^-)}{s^2 - s} \tag{0.13}$$

$$\mathcal{L}(f(x)) = \frac{f(0^-) - f'(0^-)}{s} + \frac{f'(0^-)}{s-1} \tag{0.14}$$

$$(f(x)) = \mathcal{L}^{-1}\left(\frac{f(0^-) - f'(0^-)}{s} + \frac{f'(0^-)}{s-1}\right) \tag{0.15}$$

$$\implies (f(x)) = (f(0^-) - f'(0^-))\,\mathcal{L}^{-1}\left(\frac{1}{s}\right) + f'(0^-)\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) \tag{0.16}$$

$$f(x) = (f(0^-) - f'(0^-)) + f'(0^-)e^x \tag{0.17}$$

When $f'(0^-) = 1$ and $f(0^-) = 2$, we have $y = e^x + 1$, which is the required solution. Hence verified.

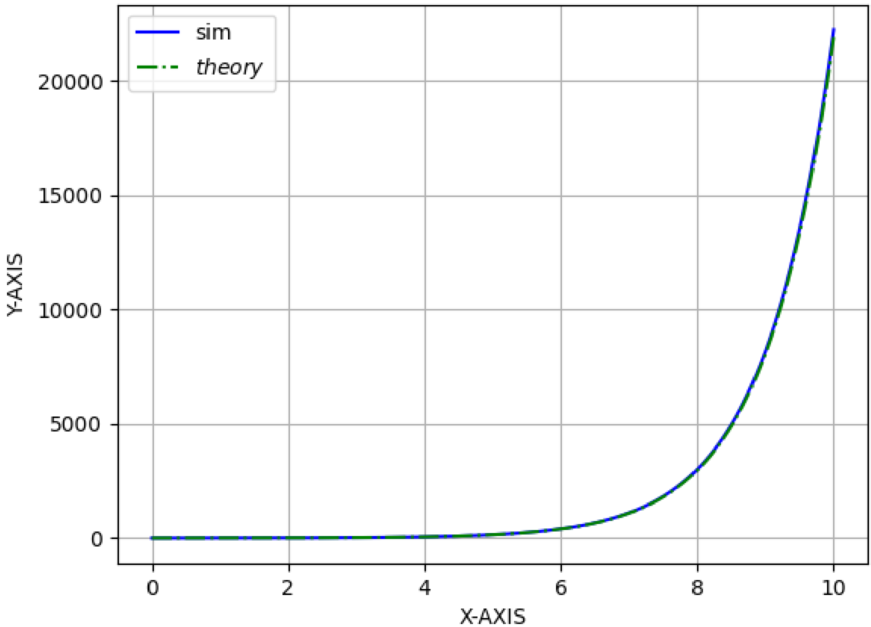

Fig. 0.1: A plot of the given question.