9.4.12

EE24BTECH11012 - Bhavanisankar G S

QUESTION :
For the differential equation, $x(x^2-1)\frac{dy}{dx} = 1$, find the particular solution given that $y(2) = 0$.
SOLUTION :
Theoritical :
Consider the given equation,

$$x\left(x^2-1\right)\frac{dy}{dx} = 1 \tag{0.1}$$

$$\frac{dy}{dx} = \frac{1}{x\left(x^2-1\right)} \tag{0.2}$$

$$y(2) = 0 \tag{0.3}$$

By the method of **Partial fractions**, we have

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right) - \frac{1}{x} \tag{0.4}$$

Integrating (0.4) on both the sides, we have

$$\int dy = \int\left(\frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right) - \frac{1}{x}\right)dx \tag{0.5}$$

$$y = \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| - \log|x| + \log c \tag{0.6}$$

$$y = \log\left(\frac{c\sqrt{x^2-1}}{x}\right) \tag{0.7}$$

Substituting the initial conditions in (0.3), we have

$$0 = \log\left(\frac{c\sqrt{2^2-1}}{2}\right) \tag{0.8}$$

$$c = \frac{2}{\sqrt{3}} \tag{0.9}$$

Hence, the required particular solution becomes

$$y = \log\left(\frac{2\sqrt{x^2-1}}{\sqrt{3}x}\right) \tag{0.10}$$

Simulation :

1) For a general interval, say $[a, b]$, split up the intervals into n parts such that

$$h = \frac{b-a}{n} \tag{1.1}$$

2) Consider the points

$$x_0 = a \tag{2.1}$$
$$x_n = b \tag{2.2}$$
$$x_{i+1} = x_i + h \tag{2.3}$$

3) **Trapezoid rule** :
Summing the areas of the trapezoids formed, we have

$$f(x) = \frac{1}{x\left(x^2-1\right)} \tag{3.1}$$
$$A \approx \frac{h}{2}\left(\left(f(x_0)+f(x_1)\right)+\left(f(x_1)+f(x_2)\right)+\cdots+\left(f(x_{n-1})+f(x_n)\right)\right) \tag{3.2}$$
$$A \approx \frac{h}{2}\left(f(x_0)+2\sum_{i=1}^{n-1} f(x_i)+f(x_n)\right) \tag{3.3}$$

4) To set up the difference equation, we integrate the equation (0.2) from $n-1$ to n.

$$a = n-1 \tag{4.1}$$
$$b = n \tag{4.2}$$

On further simplifying the equation (3.3), we have

$$y_{n+1} - y_n = (x_{n+1} - x_n)\left(\frac{f(x_n)}{2} + \frac{f(x_{n+1})}{2}\right) \tag{4.3}$$
$$y_{n+1} = y_n + (x_{n+1} - x_n)\left(\frac{f(x_n)}{2} + \frac{f(x_{n+1})}{2}\right) \tag{4.4}$$
$$y_{n+1} = y_n + \frac{(x_{n+1} - x_n)}{2}\left(\frac{1}{x_n(x_n^2-1)} + \frac{1}{x_{n+1}(x_{n+1}^2-1)}\right) \tag{4.5}$$

which is the required difference equation.

5) Taking $x_0 = 2$ and $y_0 = 0$ and iterating (4.5), we can obtain the other points.

Another approach :
Consider (0.2). Let the Laplace transform of RHS be $X(s)$. Then,

$$g(t) = \frac{1}{t(t^2-1)} \tag{5.1}$$
$$\frac{dy}{dt} = g(t) \tag{5.2}$$
$$\tag{5.3}$$

Applying Laplace transform on both the sides of (5.2) , we have

$$sY(s) = X(s) \tag{5.4}$$

$$\tag{5.5}$$

The transfer function, $H(s)$ can then be defined as

$$H(s) = \frac{Y(s)}{X(s)} \tag{5.6}$$

$$H(s) = \frac{1}{s} \tag{5.7}$$

Applying **Bi-linear transform** on both sides of (5.7), i.e., converting s-domain into z-domain, we have

$$s = \frac{2}{h}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{5.8}$$

$$H(z) = \frac{h}{2}\left(\frac{1 + z^{-1}}{1 - z^{-1}}\right) \tag{5.9}$$

$$Y(z) = \frac{h}{2}\left(\frac{1 + z^{-1}}{1 - z^{-1}}\right)X(z) \tag{5.10}$$

$$\left(1 - z^{-1}\right)Y(z) = \frac{h}{2}\left(1 + z^{-1}\right)X(z) \tag{5.11}$$

$$\tag{5.12}$$

Taking **Inverse z-transform** on both the sides of (5.11) , we have

$$y_n - y_{n-1} = \frac{h}{2}\left(g(x_n) + g(x_{n-1})\right) \tag{5.13}$$

$$y_n = y_{n-1} + \frac{h}{2}\left(g(x_n) + g(x_{n-1})\right) \tag{5.14}$$

which is the required difference equation.
It can be seen that (4.5) and (5.14) are essentially the same.
Using RK method :
By the method of **Finite differences**, we have

$$y(x + h) = y(x) + y'(x, y)h \tag{5.15}$$

According to the RK method, (5.15) can be written as

$$y(x_0 + h) = y(x_0) + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \tag{5.16}$$

where,

$$k_1 = hy'(x_0, y_0) \tag{5.17}$$

$$k_2 = hy'\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \tag{5.18}$$

$$k_3 = hy'\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \tag{5.19}$$

$$k_4 = hy'(x_0 + h, y_0 + k_3) \tag{5.20}$$

Iterating (5.16), a graph can be plotted.
Theoritical and simulation graphs of the methods described above are plotted below.

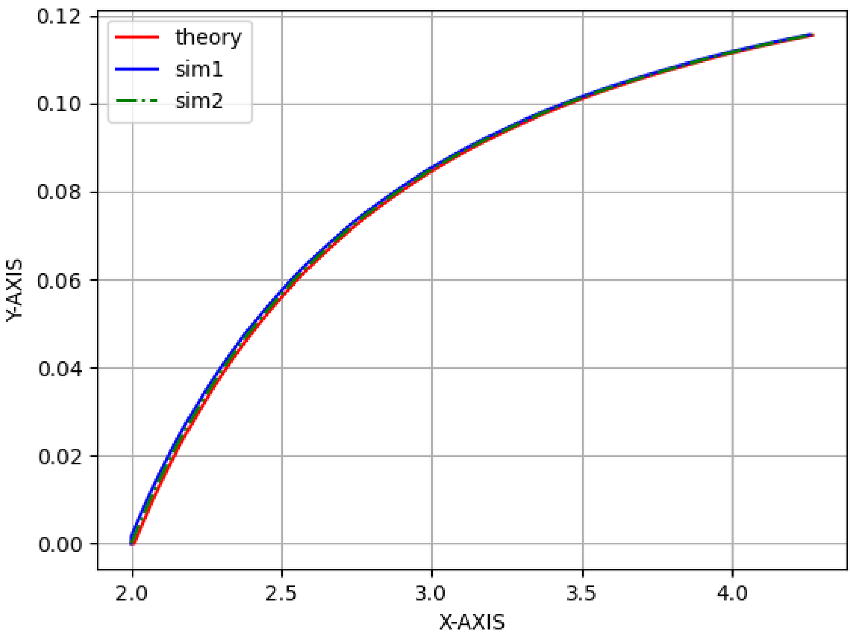

Fig. 5.1: Plot of the given question.