9.4.12

EE24BTECH11012 - Bhavanisankar G S

QUESTION :
For the differential equation, $x(x^2-1)\frac{dy}{dx} = 1$, find the particular solution given that $y(2) = 0$.
SOLUTION :
Theoritical :
Consider the given equation,

$$x\left(x^2-1\right)\frac{dy}{dx} = 1 \tag{0.1}$$

$$\frac{dy}{dx} = \frac{1}{x\left(x^2-1\right)} \tag{0.2}$$

$$y(2) = 0 \tag{0.3}$$

By the method of **Partial fractions**, we have

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right) - \frac{1}{x} \tag{0.4}$$

Integrating (0.4) on both the sides, we have

$$\int dy = \int\left(\frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right) - \frac{1}{x}\right)dx \tag{0.5}$$

$$y = \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| - \log|x| + \log c \tag{0.6}$$

$$y = \log\left(\frac{c\sqrt{x^2-1}}{x}\right) \tag{0.7}$$

Substituting the initial conditions in (0.3), we have

$$0 = \log\left(\frac{c\sqrt{2^2-1}}{2}\right) \tag{0.8}$$

$$c = \frac{2}{\sqrt{3}} \tag{0.9}$$

Hence, the required particular solution becomes

$$y = \log\left(\frac{2\sqrt{x^2-1}}{\sqrt{3}x}\right) \tag{0.10}$$

Simulation :

1) For a general interval, say $[a, b]$, split up the intervals into n parts such that

$$h = \frac{b-a}{n} \tag{1.1}$$

2) Consider the points

$$x_0 = a \tag{2.1}$$
$$x_n = b \tag{2.2}$$
$$x_{i+1} = x_i + h \tag{2.3}$$

3) **Trapezoid rule** :
Summing the areas of the trapezoids formed, we have

$$f(x) = \frac{1}{x\left(x^2-1\right)} \tag{3.1}$$
$$A \approx \frac{h}{2}\left(\left(f(x_0)+f(x_1)\right)+\left(f(x_1)+f(x_2)\right)+\cdots+\left(f(x_{n-1})+f(x_n)\right)\right) \tag{3.2}$$
$$A \approx \frac{h}{2}\left(f(x_0)+2\sum_{i=1}^{n-1} f(x_i)+f(x_n)\right) \tag{3.3}$$

4) To set up the difference equation, we integrate the equation (0.2) from $n-1$ to n.

$$a = n-1 \tag{4.1}$$
$$b = n \tag{4.2}$$

On further simplifying the equation (3.3), we have

$$y_{n+1} - y_n = (x_{n+1} - x_n)\left(\frac{f(x_n)}{2} + \frac{f(x_{n+1})}{2}\right) \tag{4.3}$$
$$y_{n+1} = y_n + (x_{n+1} - x_n)\left(\frac{f(x_n)}{2} + \frac{f(x_{n+1})}{2}\right) \tag{4.4}$$
$$y_{n+1} = y_n + \frac{(x_{n+1} - x_n)}{2}\left(\frac{1}{x_n(x_n^2-1)} + \frac{1}{x_{n+1}(x_{n+1}^2-1)}\right) \tag{4.5}$$

which is th required difference equation.

5) Taking $x_0 = 2$ and $y_0 = 0$ and iterating (4.5), we can obtain the other points. The theoritical and simulation graph are shown below.

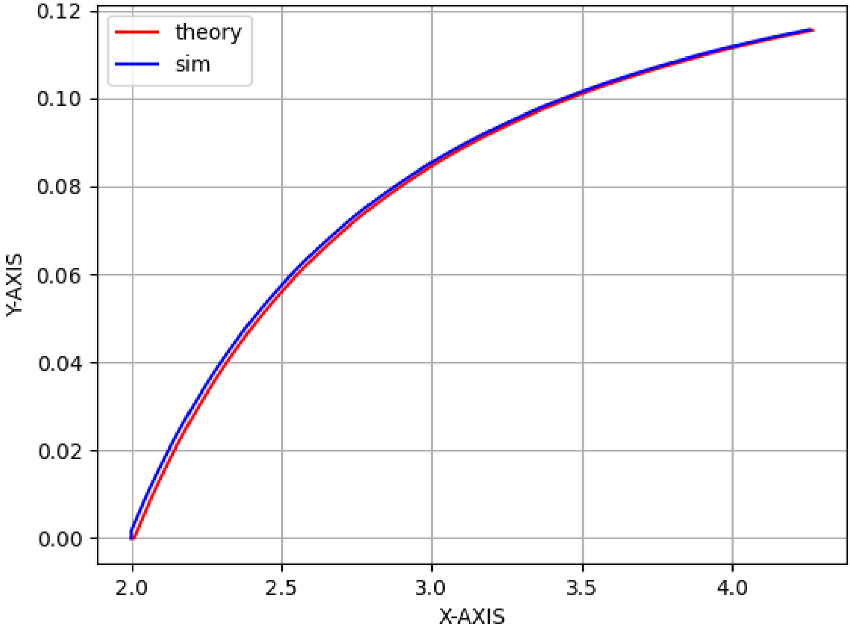

Fig. 5.1: Plot of the given question.