9.7.4

EE24BTECH11012 - Bhavanisankar G S

QUESTION :
Prove that $x^2 - y^2 = c\left(x^2 + y^2\right)^2$ is the general solution of differential equation, $\left(x^3 - 3xy^2\right)dx = \left(y^3 - 3x^2y\right)dy$, where c is a parameter. Considering $y(1) = 1$, find the particular solution of the given differential equation.
SOLUTION :

$$\left(x^3 - 3xy^2\right)dx = \left(y^3 - 3x^2y\right)dy \tag{0.1}$$
$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \tag{0.2}$$

Putting

$$y = vx \tag{0.3}$$

in (0.2), we have

$$v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} \tag{0.4}$$
$$x\frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v} \tag{0.5}$$

Separating the variables, we have

$$\frac{v^3 - 3v}{1 - v^4}dv = \frac{1}{x}dx \tag{0.6}$$

By the method of **partial fractions**, we have

$$\left(\frac{-2v}{1 + v^2} - \frac{v}{1 - v^2}\right)dv = \frac{1}{x}dx \tag{0.7}$$

Integrating on both the sides of (0.7), we have

$$\int\left(\frac{-2v}{1 + v^2} - \frac{v}{1 - v^2}\right)dv = \int\frac{1}{x}dx \tag{0.8}$$
$$\log\left(\frac{\sqrt{1 - v^2}}{1 + v^2}\right) = \log x + \log c, c - \text{ constant} \tag{0.9}$$
$$\left(\frac{\sqrt{1 - v^2}}{1 + v^2}\right) = cx \tag{0.10}$$

Substituting (0.3) in (0.10) and rearranging, we have

$$\frac{\sqrt{x^2 - y^2}}{x^2 + y^2} = c \tag{0.11}$$

$$x^2 - y^2 = c\left(x^2 + y^2\right)^2 \tag{0.12}$$

Hence proved.
By the **Forward-Euler method**, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{0.13}$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \text{ for small } h \tag{0.14}$$

$$y_{n+1} \approx y_n + h\left(y'\right) \tag{0.15}$$

Substituting (0.2) in (0.15), we have

$$y_{n+1} = y_n + h\left(\frac{x_n^3 - 3x_n y_n^2}{y_n^3 - 3x_n^2 y_n}\right) \tag{0.16}$$

which is the required difference equation.
Algorithm :
Take

$$x_0 = 1 \tag{0.17}$$

$$y_0 = 1 \tag{0.18}$$

$$h = 0.01 \tag{0.19}$$

The simulation plot can be plotted by iterating y_n ((0.16)) for different x_n as

$$x_{n+1} = x_n + h \tag{0.20}$$

The plot for computational and theoritical solution are given below.

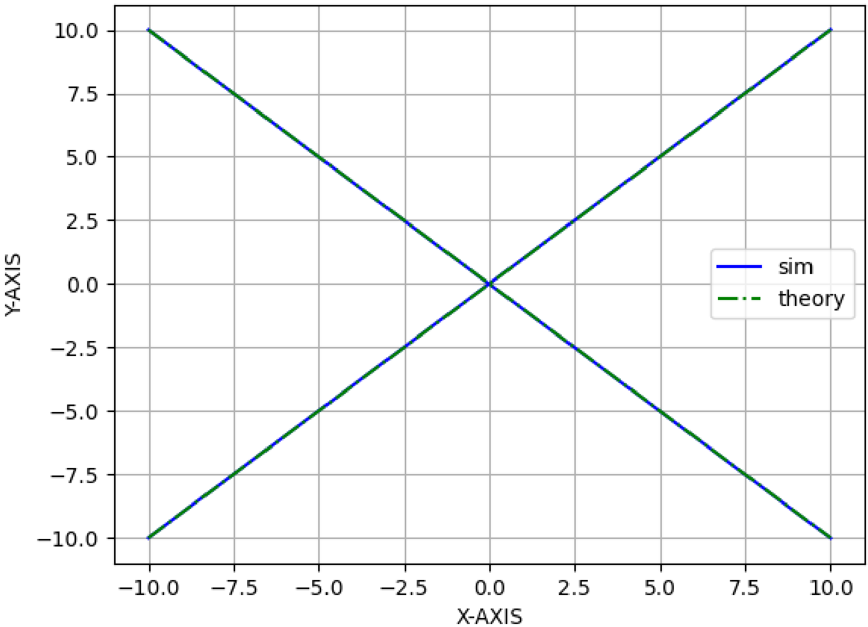

Fig. 0.1: Computational and theoritical solution.