11.16.3.8.5

EE24BTECH11012
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Question

Three coins are tossed once. Find the probability of getting no head.

Solution Outline

1. Define a random variable.
2. Devise the PMF and CDF of the random variable.
3. Deduce the required probability from the CDF expression.

Variables Used :

Variable name	Description
$\mathbf{S}$	Sample space
$\mathbf{X}$	Random variable corresponding to the number of heads
p	Toss corresponding to head
$F_{\mathbf{X}}(x)$	Cumulative distribution function (CDF)
$p_{\mathbf{X}}(x)$	Probability Mass function (PMF)

Let us assume the random variable to be the sum of three Bernoulli Random Variables.

$$\mathbf{X} = \mathbf{X_1} + \mathbf{X_2} + \mathbf{X_3} \tag{1}$$

$$\mathbf{X_i} = \begin{cases} 1 & , \text{ Outcome - head} \\ 0 & , \text{ Outcome - tail} \end{cases} \tag{2}$$

$$\implies p_{X_i}(k) = \begin{cases} 1 - p & , k = 0 \\ p & , k = 1 \end{cases} \tag{3}$$

Considering all the outcomes as equally likely, we have

$$p = \frac{1}{2} \tag{4}$$

Solution

For the given question, let **X** denote the number of heads. The sample space corresponding to the given scenario is tabulated below.

Event	Sample space
$p_{\mathbf{X}}(0)$	$\{TTT\}$
$p_{\mathbf{X}}(1)$	$\{TTH, THT, HTT\}$
$p_{\mathbf{X}}(2)$	$\{HHT, HTH, THH\}$
$p_{\mathbf{X}}(3)$	$\{HHH\}$

By the properties of Z-transform of **Probability Mass Function**, we have

$$M_{\mathbf{X}}(z) = M_{\mathbf{X}_1}(z) M_{\mathbf{X}_2}(z) M_{\mathbf{X}_3}(z) \tag{5}$$

$$M_{\mathbf{X}_1} = \sum_{n=-\infty}^{\infty} p_{\mathbf{X}_1}(n) z^{-n} = (1-p) + (p)\, z^{-1} \tag{6}$$

$$M_{\mathbf{X}_2} = \sum_{n=-\infty}^{\infty} p_{\mathbf{X}_2}(n) z^{-n} = (1-p) + (p)\, z^{-1} \tag{7}$$

$$M_{\mathbf{X}_3} = \sum_{n=-\infty}^{\infty} p_{\mathbf{X}_3}(n) z^{-n} = (1-p) + (p)\, z^{-1} \tag{8}$$

$$M_{\mathbf{X}}(z) = \left((1-p) + (p)\, z^{-1}\right)^3 \tag{9}$$

$$= \sum_{k=-\infty}^{\infty} {}^{3}C_k \,(1-p)^{3-k}\, p^k z^{-k} \tag{10}$$

$$p_{\mathbf{X}}(k) = {}^{3}C_k \,(1-p)^{3-k}\, p^k \tag{11}$$

Substituting (4) in (11), we have

$$p_{\mathbf{X}}(k) = \frac{{}^3C_k}{8} \tag{12}$$

The PMF is then given by -

$$p_{\mathbf{X}}(k) = \begin{cases} \frac{{}^3C_k}{8} & 0 \leq k \leq 3, k \in \mathbb{W} \\ 0 & otherwise \end{cases} \tag{13}$$

$$\implies p_{\mathbf{X}}(k) = \begin{cases} \frac{1}{8} & k = 0 \text{ or } k = 3 \\ \frac{3}{8} & k = 1 \text{ or } k = 2 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

CDF

The corresponding **Cumulative Distribution Function** can then be written as -

$$F_X(k) = \sum_{i=-\infty}^{k} {}^{3}C_i \left(\frac{1}{2}\right)^3 \tag{15}$$

$$= \begin{cases} 0 & k < 0 \\ {}^{3}C_0 \left(\frac{1}{2}\right)^3 & 0 \leq k < 1 \\ {}^{3}C_1 \left(\frac{1}{2}\right)^3 + {}^{3}C_0 \left(\frac{1}{2}\right)^3 & 1 \leq k < 2 \\ {}^{3}C_2 \left(\frac{1}{2}\right)^3 + {}^{3}C_1 \left(\frac{1}{2}\right)^3 + {}^{3}C_0 \left(\frac{1}{2}\right)^3 & 2 \leq k < 3 \\ {}^{3}C_3 \left(\frac{1}{2}\right)^3 + {}^{3}C_2 \left(\frac{1}{2}\right)^3 + {}^{3}C_1 \left(\frac{1}{2}\right)^3 + {}^{3}C_0 \left(\frac{1}{2}\right)^3 & k \geq 3 \end{cases} \tag{16}$$

$$\implies F_{\mathbf{X}}(k) = Pr(\mathbf{X} \leq k) = \begin{cases} 0 & k < 0 \\ \frac{1}{8} & 0 \leq k < 1 \\ \frac{1}{2} & 1 \leq k < 2 \\ \frac{7}{8} & 2 \leq k < 3 \\ 1 & k \geq 3 \end{cases} \tag{17}$$

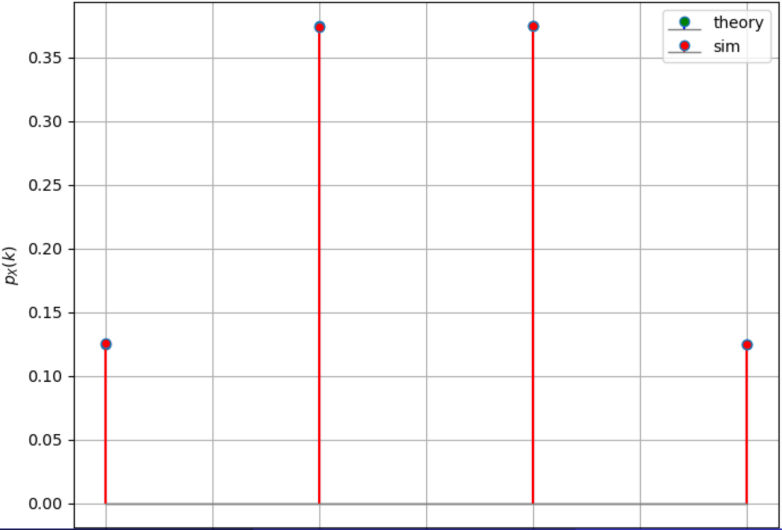
$$F_{\mathbf{X}}\left(0\right) = P\left(\mathbf{X} <= 0\right) \tag{18}$$

$$= \frac{1}{8} \tag{19}$$

Simulation

1. Generate a random number using the **rand()** function.
2. Restrict the random number to either 0 or 1 by using rand() % 2 operator, and assign it to head (H) and tail (T).
3. Count the number of favourable outcomes by iterating for a large number of trials.
4. Divide it by the total number of trials to get the desired PMF.
5. CDF can then be simulated by summing the required PMFs.

PMF - Plot



CDF - Plot