10.3.2.4.1

EE24BTECH11012
BHAVANISANKAR G S

January 21, 2025

Question

Determine if the system of equations

$$x + y = 5 \tag{1}$$

$$2x + 2y = 10 \tag{2}$$

is consistent or not. If consistent, obtain the solution graphically.

Solution Outline

1. Formulate the given system of equations in matrix form.
2. Factorize the matrix as

$$\mathbf{A} = \mathbf{LU} \tag{3}$$
$$\mathbf{L} = \mathit{Lowertriangular} \tag{4}$$
$$\mathbf{U} = \mathit{Uppertriangular} \tag{5}$$

3. Solve the equation

$$\mathbf{LUx} = \mathbf{B} \tag{6}$$
$$\implies \mathbf{Ly} = \mathbf{B} \tag{7}$$
$$\mathbf{Ux} = \mathbf{y} \tag{8}$$

Algorithm

Let $\mathbf{A}$ be an $n \times n$ matrix. Initialize $\mathbf{L}$ to an $n \times n$ Identity matrix. Initialize $\mathbf{U}$ to a zero matrix.

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{9}$$

$$U = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{10}$$

For each row i from 0 to $n-1$:

1. For each column j from i to $n-1$:

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \tag{11}$$

2. For each row j from $i+1$ to $n-1$:

$$L_{ji} = \frac{1}{U_{ii}} \left(A_{ij} - \sum_{k=0}^{i-1} L_{jk} U_{ki} \right) \tag{12}$$

Repeat the above step for all $i = 0, 1, \ldots, n-1$

After all the iterations

$$\mathbf{A} = \mathbf{LU} \tag{13}$$

Solution

For the given question,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \tag{14}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \tag{15}$$

$$\mathbf{U} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{16}$$

Using (7), we have

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \tag{17}$$

By the method of **Forward substitution**, we have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{18}$$

Similarly,

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{19}$$

It can be seen that

$$x_1 + x_2 = 5 \tag{20}$$
$$0 = 0 \tag{21}$$

which is the same as the equation given in question, indicating that the given system of equations has infinite number of solutions, and hence, the system is consistent.

Cramer's Rule

This question can also be solved by **Cramer's Rule**. According to this rule, the solution of the equation

$$\mathbf{Ax} = \mathbf{b} \tag{22}$$

is given by

$$\mathbf{x_i} = \frac{D_i}{|A|} \tag{23}$$

where, D_i is the discriminant of the matrix in which the i^{th} column of **D** replaced by the vector **b**.
If $|A| = 0$, then (in case of a 2×2 matrix)

1. If $D_1 = 0$ and $D_2 = 0$, then the system has infinitely many solutions.
2. If $D_i \neq 0$ for some $i < 2$, then the system has no solution.

Solution - Cramer's Rule

For the given question,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \tag{24}$$

$$\implies |A| = 0 \tag{25}$$

$$\mathbf{D_1} = \begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix} \tag{26}$$

$$\implies |D_1| = 0 \tag{27}$$

$$\mathbf{D_2} = \begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix} \tag{28}$$

$$\implies |D_1| = 0 \tag{29}$$

Hence, the system has infinitely many solutions, and is consistent.

QR method

Consider

$$\mathbf{A} = \mathbf{QR} \tag{30}$$

$$\mathbf{Q} - \mathit{Orthonormal} \tag{31}$$

$$\mathbf{R} - \mathit{Uppertriangular} \tag{32}$$

The equation (22) can be simplified to

$$\mathbf{QRx} = \mathbf{B} \tag{33}$$

$$\mathbf{Rx} = \mathbf{Q}^T\mathbf{B} \tag{34}$$

This can then be solved using **Back substitution**.

Solution - QR method

Using the **Gram-Schmidt orthogonalisation**, we have

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \tag{35}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & 0 \end{pmatrix} \tag{36}$$

$$\implies \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} \tag{37}$$

which again simplifies to (19) indicating that the given system is consistent with infinite number of solutions.

Plot