11.16.3.8.5

EE24BTECH11012 - Bhavanisankar G S

QUESTION :
Three coins are tossed once. Find the probability of getting no head.
SOLUTION :

Variable name	Description
$\mathbf{S}$	Sample space
$\mathbf{X}$	Random variable corresponding to the number of heads
p	Toss corresponding to head
$F_{\mathbf{X}}(x)$	Cumulative distribution function (CDF)
$p_{\mathbf{X}}(x)$	Probability Mass function (PMF)

Let us assume the random variable to be the sum of three Bernoulli Random Variables.

$$\mathbf{X} = \mathbf{X_1} + \mathbf{X_2} + \mathbf{X_3} \tag{0.1}$$

$$\mathbf{X_i} = \begin{cases} 1 & , \text{Outcome - head} \\ 0 & , \text{Outcome - tail} \end{cases} \tag{0.2}$$

$$\implies p_{X_i}(k) = \begin{cases} 1 - p & , k = 0 \\ p & , k = 1 \end{cases} \tag{0.3}$$

Considering all the outcomes as equally likely, we have

$$p = \frac{1}{2} \tag{0.4}$$

For the given question, let $\mathbf{X}$ denote the number of heads. The sample space corresponding to the given scenario is tabulated below.

Event	Sample space
$p_{\mathbf{X}}(0)$	$\{TTT\}$
$p_{\mathbf{X}}(1)$	$\{TTH, THT, HTT\}$
$p_{\mathbf{X}}(2)$	$\{HHT, HTH, THH\}$
$p_{\mathbf{X}}(3)$	$\{HHH\}$

By the properties of Z-transform of **Probability Mass Function**, we have

$$M_{\mathbf{X}}(z) = M_{\mathbf{X_1}}(z)M_{\mathbf{X_2}}(z)M_{\mathbf{X_3}}(z) \tag{0.5}$$

$$M_{\mathbf{X_1}} = \sum_{n=-\infty}^{\infty} p_{\mathbf{X_1}}(n)z^{-n} = (1-p) + (p)\, z^{-1} \tag{0.6}$$

$$M_{\mathbf{X_2}} = \sum_{n=-\infty}^{\infty} p_{\mathbf{X_2}}(n)z^{-n} = (1-p) + (p)\, z^{-1} \tag{0.7}$$

$$M_{\mathbf{X_3}} = \sum_{n=-\infty}^{\infty} p_{\mathbf{X_3}}(n)z^{-n} = (1-p) + (p)\, z^{-1} \tag{0.8}$$

$$M_{\mathbf{X}}(z) = \left((1-p) + (p)\, z^{-1}\right)^3 \tag{0.9}$$

$$= \sum_{k=-\infty}^{\infty} {}^3C_k \,(1-p)^{3-k}\, p^k z^{-k} \tag{0.10}$$

$$p_{\mathbf{X}}(k) = {}^3C_k \,(1-p)^{3-k}\, p^k \tag{0.11}$$

Substituting (0.4) in (0.11), we have

$$p_{\mathbf{X}}(k) = \frac{{}^3C_k}{8} \tag{0.12}$$

The PMF is then given by -

$$p_{\mathbf{X}}(k) = \begin{cases} \frac{{}^3C_k}{8} & 0 \le k \le 3, k \in \mathbb{W} \\ 0 & otherwise \end{cases} \tag{0.13}$$

$$\implies p_{\mathbf{X}}(k) = \begin{cases} \frac{1}{8} & k = 0 \\ \frac{3}{8} & k = 1 \\ \frac{3}{8} & k = 2 \\ \frac{1}{8} & k = 3 \\ 0 & \text{otherwise} \end{cases} \tag{0.14}$$

The corresponding **Cumulative Distribution Function** can then be written as -

$$F_X(k) = \sum_{i=-\infty}^{k} {}^3C_i \left(\frac{1}{2}\right)^3 = \begin{cases} 0 & k < 0 \\ {}^3C_0\left(\frac{1}{2}\right)^3 = \frac{1}{8} & 0 \le k < 1 \\ {}^3C_1\left(\frac{1}{2}\right)^3 + {}^3C_0\left(\frac{1}{2}\right)^3 = \frac{4}{8} & 1 \le k < 2 \\ {}^3C_2\left(\frac{1}{2}\right)^3 + {}^3C_1\left(\frac{1}{2}\right)^3 + {}^3C_0\left(\frac{1}{2}\right)^3 = \frac{7}{8} & 2 \le k < 3 \\ {}^3C_3\left(\frac{1}{2}\right)^3 + {}^3C_2\left(\frac{1}{2}\right)^3 + {}^3C_1\left(\frac{1}{2}\right)^3 + {}^3C_0\left(\frac{1}{2}\right)^3 = 1 & k \ge 3 \end{cases} \tag{0.15}$$

$$\implies F_{\mathbf{X}}(k) = Pr(\mathbf{X} \le k) = \begin{cases} 0 & k < 0 \\ \frac{1}{8} & 0 \le k < 1 \\ \frac{1}{2} & 1 \le k < 2 \\ \frac{7}{8} & 2 \le k < 3 \\ 1 & k \ge 3 \end{cases} \tag{0.16}$$

$$F_{\mathbf{X}}(0) = P(\mathbf{X} <= 0) \tag{0.17}$$

$$= \frac{1}{8} \tag{0.18}$$

Simulation :

1) Generate a random number using the **rand()** function.
2) Restrict the random number to either 0 or 1 by using rand() % 2 operator, and assign it to head (H) and tail (T).
3) Count the number of favourable outcomes by iterating for a large number of trials.
4) Divide it by the total number of trials to get the desired PMF.
5) CDF can then be simulated by summing the required PMFs.

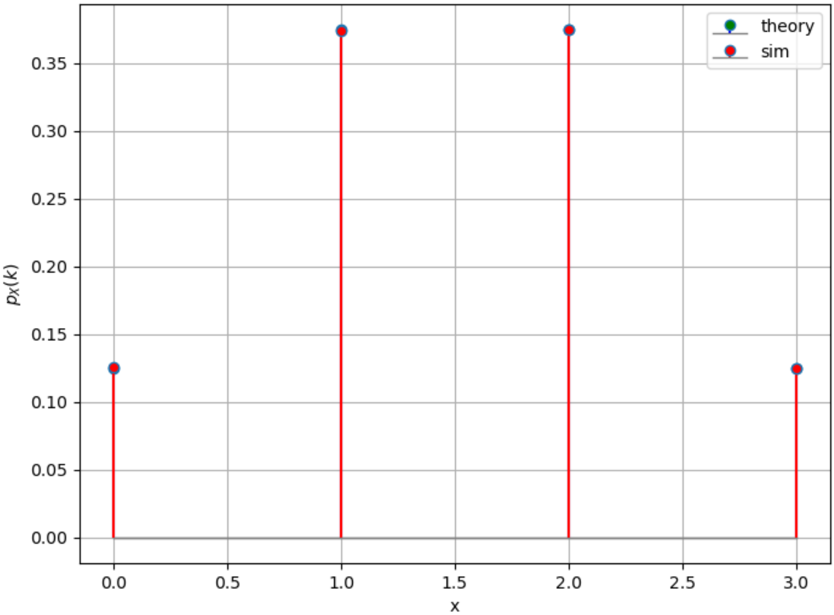

Fig. 5.1: Probability Mass Funtion

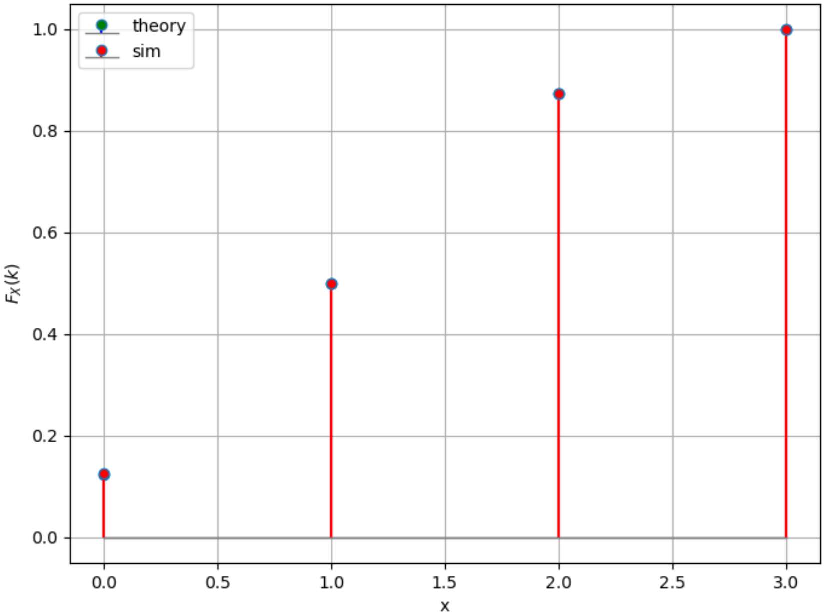

Fig. 5.2: Cumulative Distribution Function