

Solution to Differential Equation

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Problem Statement

For the differential equation $x(x^2 - 1)\frac{dy}{dx} = 1$, find the particular solution given $y(2) = 0$.

Theoretical Solution - 1

$$\frac{dy}{dx} = \frac{1}{x(x^2 - 1)}$$

Using partial fractions:

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x-1} + \frac{1}{x+1} \right) - \frac{1}{x}$$

Theoretical Solution - 2

Integrating both sides:

$$y = \frac{1}{2} \log |x - 1| + \frac{1}{2} \log |x + 1| - \log |x| + \log c$$

Applying Initial Condition

Substituting $y(2) = 0$:

$$0 = \log \left(c \sqrt{\frac{3}{4}} \right)$$

Solving for c :

$$c = 2\sqrt{3}$$

Particular Solution

The particular solution becomes:

$$y = \log \left(2 \sqrt{\frac{x^2 - 1}{3x}} \right)$$

Simulation - Trapezoid Rule

For a general interval, say a, b , split up the intervals into n parts such that

$$h = \frac{b - a}{n} \quad (1)$$

Consider the points:

$$x_0 = a \quad (2)$$

$$x_n = b \quad (3)$$

$$x_{i+1} = x_i + h \quad (4)$$

Difference Equation

Deriving the difference equation:

$$f(x) = \frac{1}{x(x^2 - 1)} \quad (5)$$

$$A \approx \frac{h}{2} ((f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))) \quad (6)$$

$$A \approx \frac{h}{2} \left(f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) \right) \quad (7)$$

On further simplifying, we have

$$y_{n+1} - y_n = (x_{n+1} - x_n) \left(\frac{f(x_n)}{2} + \frac{f(x_{n+1})}{2} \right) \quad (8)$$

$$y_{n+1} = y_n + (x_{n+1} - x_n) \left(\frac{f(x_n)}{2} + \frac{f(x_{n+1})}{2} \right) \quad (9)$$

$$y_{n+1} = y_n + \frac{(x_{n+1} - x_n)}{2} \left(\frac{1}{x_n(x_n^2 - 1)} + \frac{1}{x_{n+1}(x_{n+1}^2 - 1)} \right) \quad (10)$$

Laplace Transform Approach

$$g(t) = \frac{1}{t(t^2 - 1)} \quad (11)$$

$$\frac{dy}{dt} = g(t) \quad (12)$$

$$sY(s) = X(s) \quad (13)$$

$$H(s) = \frac{Y(s)}{X(s)} \quad (14)$$

$$H(s) = \frac{1}{s} \quad (15)$$

Bilinear Transform

$$s = \frac{2}{h} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) \quad (16)$$

$$H(z) = \frac{h}{2} \left(\frac{1 + z^{-1}}{1 - z^{-1}} \right) \quad (17)$$

$$Y(z) = \frac{h}{2} \left(\frac{1 + z^{-1}}{1 - z^{-1}} \right) X(z), |z| > 1 \quad (18)$$

$$(1 - z^{-1}) Y(z) = \frac{h}{2} (1 + z^{-1}) X(z) \quad (19)$$

Taking inverse z-transform on both sides, we have

$$y_n - y_{n-1} = \frac{h}{2} (g(x_n) + g(x_{n-1})) \quad (20)$$

$$y_n = y_{n-1} + \frac{h}{2} (g(x_n) + g(x_{n-1})) \quad (21)$$

Runge-Kutta Method

$$y(x+h) = y(x) + y'(x, y)h \quad (22)$$

$$y'(x, y) = \frac{1}{x(x^2 - 1)} \quad (23)$$

$$y(x_0 + h) = y(x_0) + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \quad (24)$$

where,

$$k_1 = hy'(x_0, y_0) \quad (25)$$

$$k_2 = hy'\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \quad (26)$$

$$k_3 = hy'\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \quad (27)$$

$$k_4 = hy'(x_0 + h, y_0 + k_3) \quad (28)$$

Graphical Representation

