8.3.12

EE24BTECH11012 - Bhavanisankar G S

QUESTION :
Find the area bounded by the curves $\left\{(x, y) : y \geq x^2 \text{ and } y = |x|\right\}$
SOLUTION :
Theoritical :

1) **FINDING THE POINT OF INTERSECTION** :

$$y = x^2 \tag{1.1}$$
$$y = |x| \tag{1.2}$$

By the symmetry of the equations, the required area is double the area of

$$y = x^2 \text{ for } x \geq 0 \tag{1.3}$$
$$y = x \text{ for } x \geq 0 \tag{1.4}$$

Clearly, the points of intersection are

$$x = 0 \tag{1.5}$$
$$x = 1 \tag{1.6}$$

2) **EVALUATING THE INTEGRAL** :
From the graph, it can be seen that

$$x \geq x^2 \text{ for } 0 \leq x \leq 1 \tag{2.1}$$

Hence, the integral becomes

$$A = 2\left(\int_0^1 \left(x - x^2\right) dx\right) \tag{2.2}$$
$$A = 2\left(\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1\right) \tag{2.3}$$
$$A = 2\left(\frac{1}{2} - \frac{1}{3}\right) \tag{2.4}$$
$$A = 2\left(\frac{1}{6}\right) \tag{2.5}$$
$$A = \frac{1}{3} \tag{2.6}$$

Hence, the area bounded by the given curves is $\frac{1}{3}$.

Simulation :

1) For a general interval, say $[a, b]$, split up the intervals into n parts such that
$$h = \frac{b-a}{n} \tag{1.1}$$
2) Consider the points
$$x_0 = a \tag{2.1}$$
$$x_n = b \tag{2.2}$$
$$x_{i+1} = x_i + h \tag{2.3}$$
3) **Trapezoid rule** :
Summing the areas of the trapezoids formed, we have
$$f(x) = x - x^2 \tag{3.1}$$
$$A \approx \frac{h}{2}\left((f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))\right) \tag{3.2}$$
$$A \approx \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right) \tag{3.3}$$
In the given question,
$$a = 0 \tag{3.4}$$
$$b = 1 \tag{3.5}$$
Clearly,
$$f(a) = f(b) = 0 \tag{3.6}$$
since both the curves have $(0, 0)$ and $(1, 1)$ as their common points. Simplifying from (1.1) and (3.3), we have
$$A \approx \frac{1}{n}\left(\sum_{i=1}^{n-1} x_i - x_i^2\right) \tag{3.7}$$
$$A \approx \frac{1}{n}\left(\sum_{i=1}^{n-1} \frac{i}{n} - (\frac{i}{n})^2\right) \tag{3.8}$$
$$A \approx \frac{1}{n^2}\left(\sum_{i=1}^{n-1}\left(i - \frac{i^2}{n}\right)\right) \tag{3.9}$$

Consider

$$A_{n+1} = A_n + \frac{h}{2}(y_n + y_{n+1}) \tag{3.10}$$
$$A_{n+1} = A_n + \frac{h}{2}(y_n + (y_n + hy'_n)) \tag{3.11}$$
$$A_{n+1} = A_n + \frac{h}{2}(y_n + (y_n + h(1 - 2x_n))) \tag{3.12}$$
$$A_{n+1} = A_n + \frac{h}{2}(2y_n + h(1 - 2x_n)) \tag{3.13}$$

which is the required difference equation.

4) Equation (3.9) can be coded to obtain the area bounded by the two curves.
5) It can be seen that the approximate solution turns out to be 0.3333333299999998.

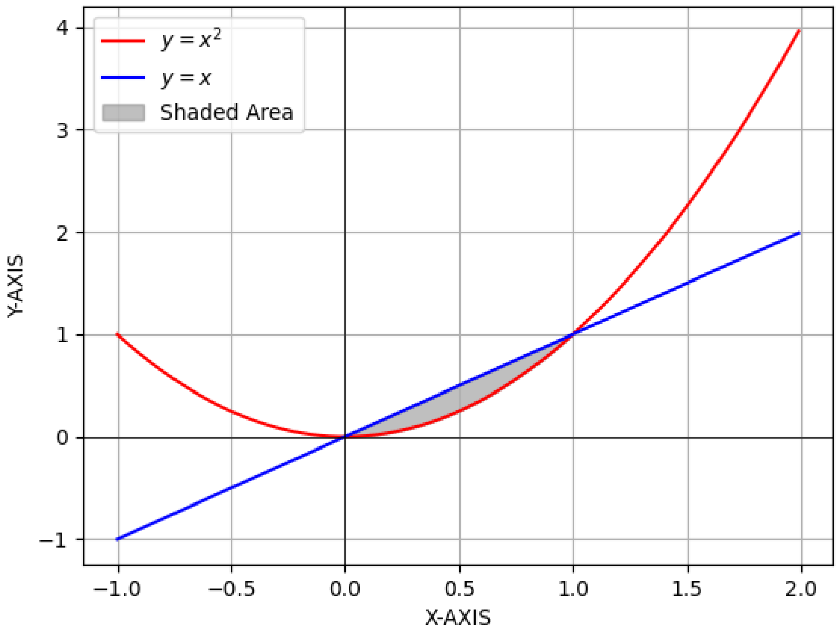

Fig. 5.1: Plot of the given question.