9.2.1

EE24BTECH11012 - Bhavanisankar G S

LAPLACE TRANSFORMS

- **Notation** : Laplace transform of a function $f(x)$ is denoted as $\mathcal{L}(f(x))$, i.e.,

$$\mathcal{L}(f(x)) = F(s) = \int_0^\infty f(x)e^{-sx}dx \tag{0.1}$$

- It is a linear transformation, since integration is a linear operation.
- **Laplace transform of some functions :**

$$f(x) = 0 \implies F(s) = 0 \tag{0.2}$$

$$f(x) = 1 \implies F(s) = \frac{1}{s} \text{ for } Re(s) > 0 \tag{0.3}$$

$$f(x) = x^n \implies F(s) = \frac{n!}{s^{n+1}} \text{ for } Re(s) > 0 \tag{0.4}$$

$$f(x) = e^{at} \implies F(s) = \frac{1}{s-a} \text{ for } Re(s) > a \tag{0.5}$$

$$f(x) = \sin ax \implies F(s) = \frac{a}{s^2+a^2} \text{ for } Re(s) > 0 \tag{0.6}$$

$$f(x) = \cos ax \implies F(s) = \frac{s}{s^2+a^2} \text{ for } Re(s) > 0 \tag{0.7}$$

- **Some other useful results include :**

$$\mathcal{L}(f'(x)) = sF(s) - f(0^-) \tag{0.8}$$

$$\mathcal{L}(f''(x)) = s^2F(s) - sf(0^-) - f'(0^-) \tag{0.9}$$

- **Laplace transform of unit step function** $u(t)$ **:**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{0.10}$$

From (0.1)

$$\mathcal{L}(u(t)) = \int_0^\infty u(t)e^{-st}dt \tag{0.11}$$

For all non-negative values, $u(t) = 1$. Hence, the integral becomes,

$$F(s) = \int_0^\infty (1)e^{-st}dt \tag{0.12}$$

$$F(s) = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}, \text{ for } Re(s) > 0 \tag{0.13}$$

- **Laplace transform of** $e^{at}u(t)$ **:**
  From (0.1)

$$\mathcal{L}\left(e^{at}u(t)\right) = \int_0^\infty e^{at}u(t)e^{-st}dt \tag{0.14}$$

$$F(s) = \int_0^\infty e^{(a-s)t}dt \tag{0.15}$$

$$F(s) = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty = \frac{1}{s-a}, \text{ for } Re(s) > a \tag{0.16}$$

  When $a = 1$,

$$F(s) = \mathcal{L}\left(e^{t}u(t)\right) = \frac{1}{s-1} \text{ for } Re(s) > 1 \tag{0.17}$$

Z-TRANSFORMS

- **Notation** :

$$Y(z) = \sum_{n\to-\infty}^{\infty} y_n z^{-n} \tag{0.18}$$

- **Z-transform of** $u(t)$:
  From (0.18)

$$Y(z) = \sum_{t\to-\infty}^{\infty} u(t)z^{-t} \tag{0.19}$$

  From (0.10), this can be simplified to

$$Y(z) = \sum_{t=0}^{\infty}(1)z^{-t} \tag{0.20}$$

$$Y(z) = \frac{1}{1-z^{-1}}, \text{ for } |z| > 1 \tag{0.21}$$

- **Z-transform of** $a^t u(t)$:
  From (0.18)

$$Y(z) = \sum_{t\to-\infty}^{\infty} a^t u(t)z^{-t} \tag{0.22}$$

  From (0.10), this can be simplified to

$$Y(z) = \sum_{t=0}^{\infty} a^t z^{-t} \tag{0.23}$$

$$Y(z) = \sum_{t=0}^{\infty} \left(az^{-1}\right)^t \tag{0.24}$$

$$Y(z) = \frac{1}{1-az^{-1}}, \text{ for } |z| > |a| \tag{0.25}$$

- **Some other useful results** :

$$Y(u_{n-1}) = z^{-1}Y(u_n) \tag{0.26}$$

$$Y(u_{n+1}) = z\,(Y(u_n) - u_0) \tag{0.27}$$

QUESTION:
Consider the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$. Verify that $y = e^x + 1$ is a solution for it, given the initial condtitions $y(0) = 2$ and $y'(0) = 1$.
SOLUTION:
Consider the differential equation,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0 \tag{0.28}$$

Integrating (0.28) on both the sides, we have,

$$\int \left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) dx = \int (0)\, dx \tag{0.29}$$

$$\frac{dy}{dx} - y = k \tag{0.30}$$

where, k is a constant. Applying the initial conditions, we have

$$k = 1 - 2 = -1 \tag{0.31}$$

$$\frac{dy}{dx} = y - 1 \tag{0.32}$$

Applying Laplace transform to (0.28) on both sides, we have

$$\mathcal{L}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) = \mathcal{L}\,(0) \tag{0.33}$$

$$\mathcal{L}\left(\frac{d^2y}{dx^2}\right) - \mathcal{L}\left(\frac{dy}{dx}\right) = \mathcal{L}\,(0) \tag{0.34}$$

$$\left(s^2F(s) - sf(0^-) - f'(0^-)\right) - (sF(s) - f(0^-)) = 0 \tag{0.35}$$

$$F(s)\left(s^2 - s\right) - f(0^-)\,(s - 1) - f'(0^-) = 0 \tag{0.36}$$

$$F(s) = \frac{f(0^-)\,(s - 1) + f'(0^-)}{s^2 - s} \tag{0.37}$$

$$\mathcal{L}\,(f(x)) = \frac{f(0^-) - f'(0^-)}{s} + \frac{f'(0^-)}{s - 1} \tag{0.38}$$

Substituting the initial conditions, $y(0^-) = 2$ and $y'(0^-) = 1$, we have

$$f(x) = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{1}{s - 1}\right) \tag{0.39}$$

$$f(x) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s - 1}\right) \tag{0.40}$$

From, (0.13) and (0.17), it can be deduced that

$$f(x) = u(x) + e^x u(x) \tag{0.41}$$
$$= u(x)(1 + e^x) \tag{0.42}$$

Hence, verified.
ALGORITHM :

$$x_0 = 0 \tag{0.43}$$
$$y_0 = 2 \tag{0.44}$$
$$h = 0.01 \tag{0.45}$$
$$x_{n+1} = x_n + h \tag{0.46}$$
$$y_{n+1} = y_n + h(y') \tag{0.47}$$

From (0.32),

$$y_{n+1} = y_n + h(y_n - 1) \tag{0.48}$$

which is the required difference equation.
Applying unilateral (one-sided) z-transform on both sides of (0.48), we have

$$Y(y_{n+1}) = Y(y_n(1 + h) - h) \tag{0.49}$$
$$zY(z) - zy_0 = (1 + h)Y(z) - h\frac{1}{1 - z^{-1}} \tag{0.50}$$
$$Y(z)(z - (1 + h)) = zy_0 - \frac{h}{1 - z^{-1}} \tag{0.51}$$

Substituting $y_0 = 2$ and simplifying, we have

$$Y(z) = \frac{1}{1 - (1 + h)z^{-1}} + \frac{1}{1 - z^{-1}} \tag{0.52}$$

From (0.21) and (0.25), applying Inverse-Z-Transform on both sides, we have

$$y_n = (1 + h)^n u(n) + (1)u(n) \tag{0.53}$$
$$= u(n)(1 + (1 + h)^n) \tag{0.54}$$

The plot corresponding to (0.54) is given below.

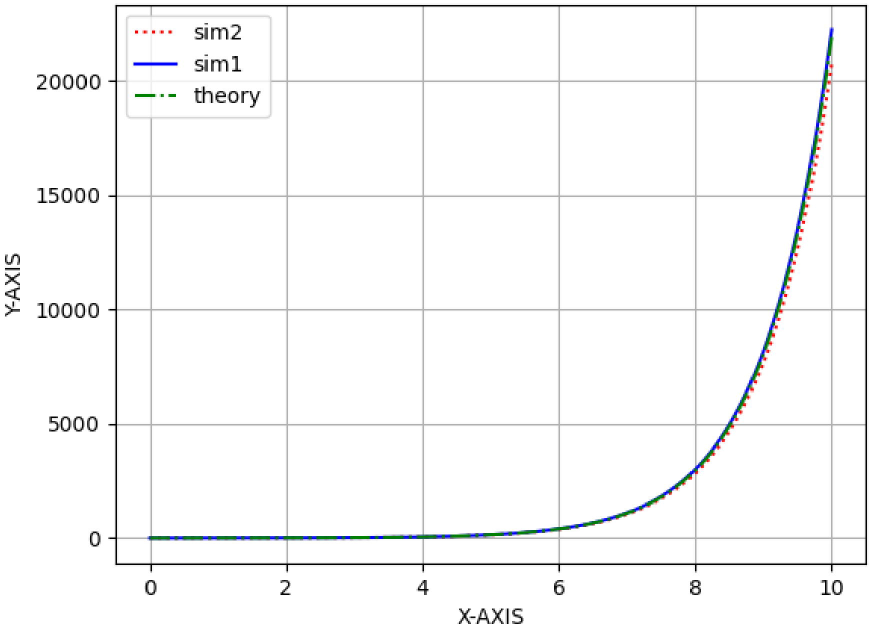

Fig. 0.1: A plot of the given question.