9.2.1

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LAPLACE TRANSFORMS

- **Notation** : Laplace transform of a function $f(x)$ is denoted as $\mathcal{L}(f(x))$, i.e.,

$$\mathcal{L}(f(x)) = F(s) = \int_0^\infty f(x)e^{-sx}dx \tag{0.1}$$

- It is a linear transformation, since integration is a linear operation.
- **Laplace transform of some functions :**

$$f(x) = 0 \implies F(s) = 0 \tag{0.2}$$

$$f(x) = 1 \implies F(s) = \frac{1}{s} \text{ for } s > 0 \tag{0.3}$$

$$f(x) = x^n \implies F(s) = \frac{n!}{s^{n+1}} \text{ for } s > 0 \tag{0.4}$$

$$f(x) = e^{at} \implies F(s) = \frac{1}{s-a} \text{ for } s > a \tag{0.5}$$

$$f(x) = \sin ax \implies F(s) = \frac{a}{s^2 + a^2} \text{ for } s > 0 \tag{0.6}$$

$$f(x) = \cos ax \implies F(s) = \frac{s}{s^2 + a^2} \text{ for } s > 0 \tag{0.7}$$

- **Some other useful results include :**

$$\mathcal{L}(f'(x)) = sF(s) - f(0^-) \tag{0.8}$$

$$\mathcal{L}(f''(x)) = s^2 F(s) - sf(0^-) - f'(0^-) \tag{0.9}$$

- **Laplace transform of unit step function $u(t)$:**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{0.10}$$

From (0.1)

$$\mathcal{L}(u(t)) = \int_0^\infty u(t)e^{-st}dt \tag{0.11}$$

For all non-negative values, $u(t) = 1$. Hence, the integral becomes,

$$F(s) = \int_0^\infty (1)e^{-st}dt \tag{0.12}$$

$$F(s) = \left[\frac{e^{-st}}{-s} \right]_0^\infty = \frac{1}{s}, \text{ for } s > 0 \tag{0.13}$$

- **Laplace transform of $e^{at}u(t)$:**
  From (0.1)

$$\mathcal{L}\left(e^{at}u(t)\right) = \int_0^\infty e^{at}u(t)e^{-st}dt \qquad (0.14)$$

$$F(s) = \int_0^\infty e^{(a-s)t}dt \qquad (0.15)$$

$$F(s) = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty = \frac{1}{s-a}, \text{ for } s > a \qquad (0.16)$$

When $a = 1$,

$$F(s) = \mathcal{L}\left(e^t u(t)\right) = \frac{1}{s-1} \text{ for } s > 1 \qquad (0.17)$$

Z-TRANSFORMS

- **Notation** :

$$Y(z) = \sum_{n \to -\infty}^\infty y_n z^{-n} \qquad (0.18)$$

- **Z-transform of $u(t)$:**
  From (0.18)

$$Y(z) = \sum_{t \to -\infty}^\infty u(t)z^{-t} \qquad (0.19)$$

From (0.10), this can be simplified to

$$Y(z) = \sum_{t=0}^\infty (1)z^{-t} \qquad (0.20)$$

$$Y(z) = \frac{1}{1-z^{-1}}, \text{ for } |z| > 1 \qquad (0.21)$$

- **Z-transform of $a^t u(t)$:**
  From (0.18)

$$Y(z) = \sum_{t \to -\infty}^\infty a^t u(t)z^{-t} \qquad (0.22)$$

From (0.10), this can be simplified to

$$Y(z) = \sum_{t=0}^\infty a^t z^{-t} \qquad (0.23)$$

$$Y(z) = \sum_{t=0}^\infty \left(az^{-1}\right)^t \qquad (0.24)$$

$$Y(z) = \frac{1}{1-az^{-1}}, \text{ for } |z| > |a| \qquad (0.25)$$

- **Some other useful results** :

$$Y(u_{n-1}) = z^{-1} Y(u_n) \tag{0.26}$$

$$Y(u_{n+1}) = z \left(Y(u_n) - u_0 \right) \tag{0.27}$$

QUESTION:

Consider the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$. Verify that $y = e^x + 1$ is a solution for it, given the initial condtitions $y(0) = 2$ and $y'(0) = 1$.

SOLUTION:

Consider the differential equation,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0 \tag{0.28}$$

Integrating (0.28) on both the sides, we have,

$$\int \left(\frac{d^2y}{dx^2} - \frac{dy}{dx} \right) dx = \int (0) \, dx \tag{0.29}$$

$$\frac{dy}{dx} - y = k \tag{0.30}$$

where, k is a constant. Applying the initial conditions, we have

$$k = 1 - 2 = -1 \tag{0.31}$$

$$\frac{dy}{dx} = y - 1 \tag{0.32}$$

Applying Laplace transform to (0.28) on both sides, we have

$$\mathcal{L}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx} \right) = \mathcal{L}(0) \tag{0.33}$$

$$\mathcal{L}\left(\frac{d^2y}{dx^2} \right) - \mathcal{L}\left(\frac{dy}{dx} \right) = \mathcal{L}(0) \tag{0.34}$$

$$\left(s^2 F(s) - s f(0^-) - f'(0^-) \right) - \left(s F(s) - f(0^-) \right) = 0 \tag{0.35}$$

$$F(s)\left(s^2 - s \right) - f(0^-)(s - 1) - f'(0^-) = 0 \tag{0.36}$$

$$F(s) = \frac{f(0^-)(s - 1) + f'(0^-)}{s^2 - s} \tag{0.37}$$

$$\mathcal{L}(f(x)) = \frac{f(0^-) - f'(0^-)}{s} + \frac{f'(0^-)}{s - 1} \tag{0.38}$$

Substituting the initial conditions, $y(0^-) = 2$ and $y'(0^-) = 1$, we have

$$f(x) = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{1}{s - 1} \right) \tag{0.39}$$

$$f(x) = \mathcal{L}^{-1}\left(\frac{1}{s} \right) + \mathcal{L}^{-1}\left(\frac{1}{s - 1} \right) \tag{0.40}$$

From, (0.13) and (0.17), it can be deduced that

$$f(x) = u(x) + e^x u(x) \tag{0.41}$$

$$= u(x)(1 + e^x) \tag{0.42}$$

Hence, verified.

ALGORITHM :

$$x_0 = 0 \tag{0.43}$$

$$y_0 = 2 \tag{0.44}$$

$$h = 0.01 \tag{0.45}$$

$$x_{n+1} = x_n + h \tag{0.46}$$

$$y_{n+1} = y_n + h(y') \tag{0.47}$$

From (0.32),

$$y_{n+1} = y_n + h(y_n - 1) \tag{0.48}$$

which is the required difference equation.

Applying unilateral (one-sided) z-transform on both sides of (0.48), we have

$$Y(y_{n+1}) = Y(y_n(1 + h) - h) \tag{0.49}$$

$$zY(z) - zy_0 = (1 + h)Y(z) - h\frac{1}{1 - z^{-1}} \tag{0.50}$$

$$Y(z)(z - (1 + h)) = zy_0 - \frac{h}{1 - z^{-1}} \tag{0.51}$$

Substituting $y_0 = 2$ and simplifying, we have

$$Y(z) = \frac{1}{1 - (1 + h)z^{-1}} + \frac{1}{1 - z^{-1}} \tag{0.52}$$

From (0.21) and (0.25), applying Inverse-Z-Transform on both sides, we have

$$y_n = (1 + h)^n u(n) + (1)u(n) \tag{0.53}$$

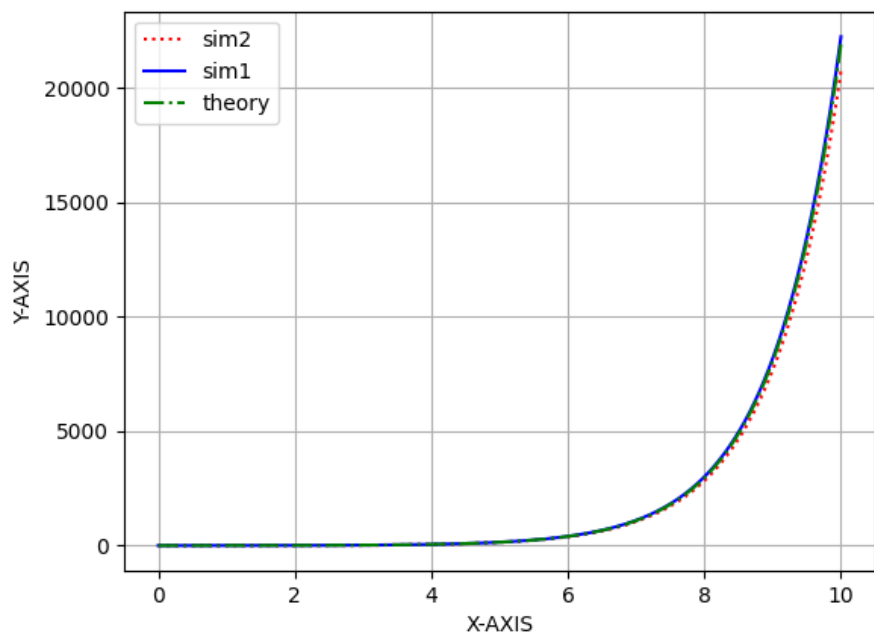$$= u(n)(1 + (1 + h)^n) \tag{0.54}$$

The plot corresponding to (0.55) is given below.

Fig. 0.1: A plot of the given question.