

9.2.1

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LAPLACE TRANSFORMS

- **Notation** : Laplace transform of a function $f(x)$ is denoted as $\mathcal{L}(f(x))$, i.e.,

$$\mathcal{L}(f(x)) = F(s) = \int_0^{\infty} f(x)e^{-sx}dx \quad (0.1)$$

- It is a linear transformation, since integration is a linear operation.
- **Laplace transform of some functions :**

$$f(x) = 0 \implies F(s) = 0 \quad (0.2)$$

$$f(x) = 1 \implies F(s) = \frac{1}{s} \text{ for } \operatorname{Re}(s) > 0 \quad (0.3)$$

$$f(x) = x^n \implies F(s) = \frac{n!}{s^{n+1}} \text{ for } \operatorname{Re}(s) > 0 \quad (0.4)$$

$$f(x) = e^{ax} \implies F(s) = \frac{1}{s-a} \text{ for } \operatorname{Re}(s) > a \quad (0.5)$$

$$f(x) = \sin ax \implies F(s) = \frac{a}{s^2 + a^2} \text{ for } \operatorname{Re}(s) > 0 \quad (0.6)$$

$$f(x) = \cos ax \implies F(s) = \frac{s}{s^2 + a^2} \text{ for } \operatorname{Re}(s) > 0 \quad (0.7)$$

- **Some other useful results include :**

$$\mathcal{L}(f'(x)) = sF(s) - f(0^-) \quad (0.8)$$

$$\mathcal{L}(f''(x)) = s^2F(s) - sf(0^-) - f'(0^-) \quad (0.9)$$

- **Laplace transform of unit step function $u(t)$:**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (0.10)$$

From (0.1)

$$\mathcal{L}(u(t)) = \int_0^{\infty} u(t)e^{-st}dt \quad (0.11)$$

For all non-negative values, $u(t) = 1$. Hence, the integral becomes,

$$F(s) = \int_0^{\infty} (1)e^{-st}dt \quad (0.12)$$

$$F(s) = \left[\frac{e^{-st}}{-s} \right]_0^{\infty} = \frac{1}{s}, \text{ for } \operatorname{Re}(s) > 0 \quad (0.13)$$

- **Laplace transform of $e^{at}u(t)$:**

From (0.1)

$$\mathcal{L}(e^{at}u(t)) = \int_0^{\infty} e^{at}u(t)e^{-st}dt \quad (0.14)$$

$$F(s) = \int_0^{\infty} e^{(a-s)t}dt \quad (0.15)$$

$$F(s) = \left[\frac{e^{(a-s)t}}{a-s} \right]_0^{\infty} = \frac{1}{s-a}, \text{ for } \operatorname{Re}(s) > a \quad (0.16)$$

When $a = 1$,

$$F(s) = \mathcal{L}(e^t u(t)) = \frac{1}{s-1} \text{ for } \operatorname{Re}(s) > 1 \quad (0.17)$$

Z-TRANSFORMS

- **Notation :**

$$Y(z) = \sum_{n \rightarrow -\infty}^{\infty} y_n z^{-n} \quad (0.18)$$

- **Z-transform of $u(t)$:**

From (0.18)

$$Y(z) = \sum_{t \rightarrow -\infty}^{\infty} u(t)z^{-t} \quad (0.19)$$

From (0.10), this can be simplified to

$$Y(z) = \sum_{t=0}^{\infty} (1)z^{-t} \quad (0.20)$$

$$Y(z) = \frac{1}{1-z^{-1}}, \text{ for } |z| > 1 \quad (0.21)$$

- **Z-transform of $a^t u(t)$:**

From (0.18)

$$Y(z) = \sum_{t \rightarrow -\infty}^{\infty} a^t u(t)z^{-t} \quad (0.22)$$

From (0.10), this can be simplified to

$$Y(z) = \sum_{t=0}^{\infty} a^t z^{-t} \quad (0.23)$$

$$Y(z) = \sum_{t=0}^{\infty} (az^{-1})^t \quad (0.24)$$

$$Y(z) = \frac{1}{1-az^{-1}}, \text{ for } |z| > |a| \quad (0.25)$$

- **Some other useful results :**

$$Y(u_{n-1}) = z^{-1}Y(u_n) \quad (0.26)$$

$$Y(u_{n+1}) = z(Y(u_n) - u_0) \quad (0.27)$$

QUESTION:

Consider the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$. Verify that $y = e^x + 1$ is a solution for it, given the initial conditions $y(0) = 2$ and $y'(0) = 1$.

SOLUTION:

Consider the differential equation,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0 \quad (0.28)$$

Integrating (0.28) on both the sides, we have,

$$\int \left(\frac{d^2y}{dx^2} - \frac{dy}{dx} \right) dx = \int (0) dx \quad (0.29)$$

$$\frac{dy}{dx} - y = k \quad (0.30)$$

where, k is a constant. Applying the initial conditions, we have

$$k = 1 - 2 = -1 \quad (0.31)$$

$$\frac{dy}{dx} = y - 1 \quad (0.32)$$

Applying Laplace transform to (0.28) on both sides, we have

$$\mathcal{L}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) = \mathcal{L}(0) \quad (0.33)$$

$$\mathcal{L}\left(\frac{d^2y}{dx^2}\right) - \mathcal{L}\left(\frac{dy}{dx}\right) = \mathcal{L}(0) \quad (0.34)$$

$$(s^2F(s) - sf(0^-) - f'(0^-)) - (sF(s) - f(0^-)) = 0 \quad (0.35)$$

$$F(s)(s^2 - s) - f(0^-)(s - 1) - f'(0^-) = 0 \quad (0.36)$$

$$F(s) = \frac{f(0^-)(s - 1) + f'(0^-)}{s^2 - s} \quad (0.37)$$

$$\mathcal{L}(f(x)) = \frac{f(0^-) - f'(0^-)}{s} + \frac{f'(0^-)}{s - 1} \quad (0.38)$$

Substituting the initial conditions, $y(0^-) = 2$ and $y'(0^-) = 1$, we have

$$f(x) = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{1}{s - 1}\right) \quad (0.39)$$

$$f(x) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s - 1}\right) \quad (0.40)$$

From, (0.13) and (0.17), it can be deduced that

$$f(x) = u(x) + e^x u(x) \quad (0.41)$$

$$= u(x) (1 + e^x) \quad (0.42)$$

Hence, verified.

ALGORITHM :

$$x_0 = 0 \quad (0.43)$$

$$y_0 = 2 \quad (0.44)$$

$$h = 0.01 \quad (0.45)$$

$$x_{n+1} = x_n + h \quad (0.46)$$

$$y_{n+1} = y_n + h(y') \quad (0.47)$$

From (0.32),

$$y_{n+1} = y_n + h(y_n - 1) \quad (0.48)$$

which is the required difference equation.

Applying unilateral (one-sided) z-transform on both sides of (0.48), we have

$$Y(y_{n+1}) = Y(y_n (1 + h) - h) \quad (0.49)$$

$$zY(z) - zy_0 = (1 + h)Y(z) - h \frac{1}{1 - z^{-1}} \quad (0.50)$$

$$Y(z)(z - (1 + h)) = zy_0 - \frac{h}{1 - z^{-1}} \quad (0.51)$$

Substituting $y_0 = 2$ and simplifying, we have

$$Y(z) = \frac{1}{1 - (1 + h)z^{-1}} + \frac{1}{1 - z^{-1}} \quad (0.52)$$

From (0.21) and (0.25), applying Inverse-Z-Transform on both sides, we have

$$y_n = (1 + h)^n u(n) + (1)u(n) \quad (0.53)$$

$$= u(n) (1 + (1 + h)^n) \quad (0.54)$$

The plot corresponding to (0.54) is given below.

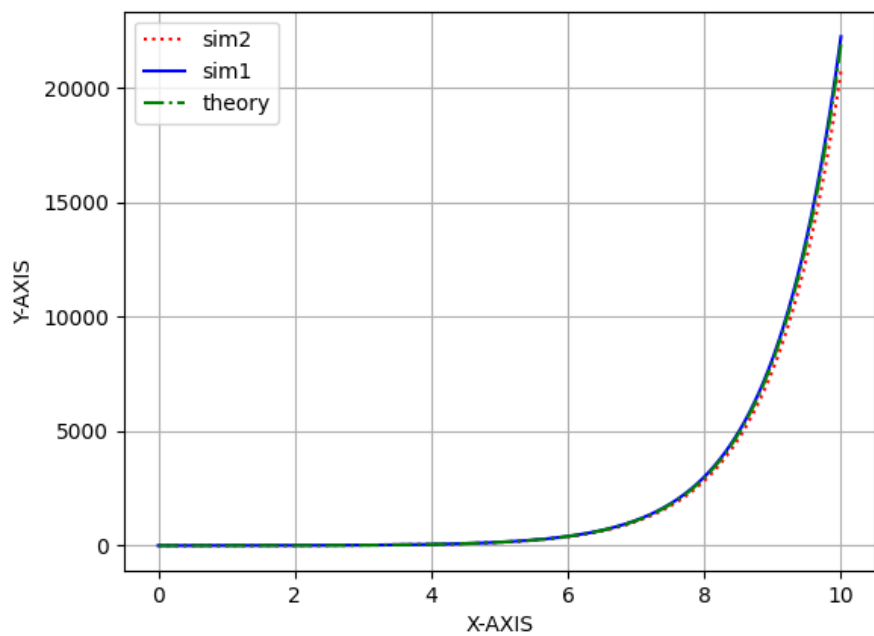


Fig. 0.1: A plot of the given question.