Convolution of Exponential and Hyperbolic Functions

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1 Introduction

This report provides a numerical and computational analysis on the convolution of exponential and hyperbolic functions.

2 Theoretical Background

The convolution of two functions $f(t)$ and $h(t)$ is defined as:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \tag{1}$$

For our analysis, the rectangular kernel is defined as:

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

3 Convolution with Exponential Function

Let's choose $f(t) = e^{-at}$ for $a > 0$ as our input signal.

The convolution is:

$$y(t) = \int_{-\infty}^{\infty} e^{-a\tau} \cdot h(t-\tau)d\tau \tag{3}$$

Since $h(t-\tau)$ is non-zero only when $-T \leq t-\tau \leq T$, or $t-T \leq \tau \leq t+T$, the effective integration limits are:

$$y(t) = \int_{t-T}^{t+T} e^{-a\tau} d\tau \tag{4}$$

Computing this integral:

$$y(t) = \int_{t-T}^{t+T} e^{-a\tau} d\tau \tag{5}$$

$$= \left[-\frac{1}{a}e^{-a\tau}\right]_{t-T}^{t+T} \tag{6}$$

$$= \frac{e^{-at}}{a}(e^{aT} - e^{-aT}) \tag{7}$$

Using the definition of hyperbolic sine, we can write:

$$y(t) = \frac{2\sinh(aT)}{a}e^{-at} \tag{8}$$

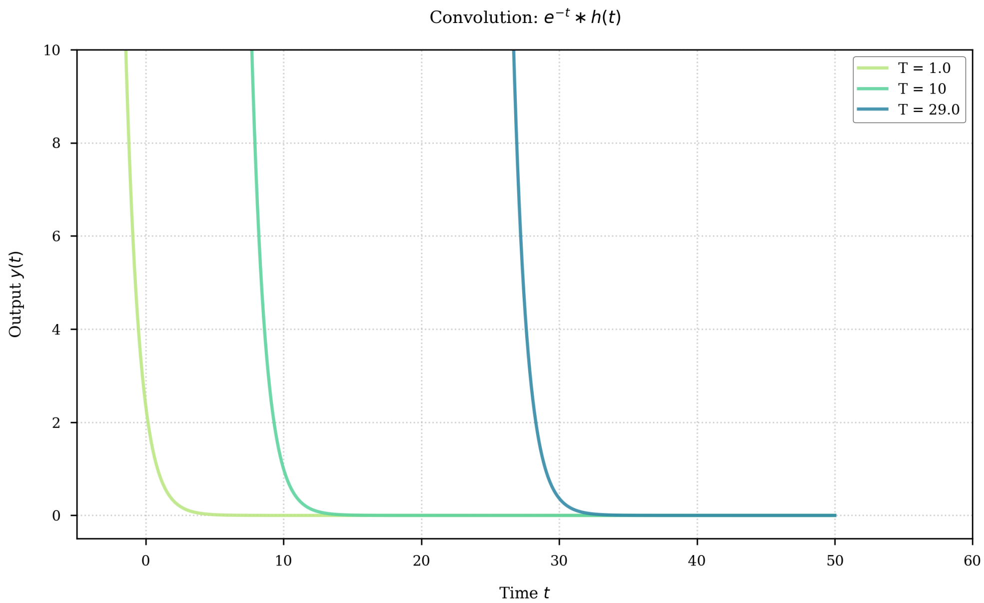

Figure 1: Convolution of $f(t) = e^{-t}$ with a rectangular kernel for varying values of T. As T increases, the output signal amplitude scales by $\frac{2\sinh(aT)}{a}$ but retains the exponential decay nature. For large values of T (e.g., $T = 29.0$), the output achieves relatively higher amplitudes (approximately 8) than smaller T values, showing the amplification property of the width of the rectangular kernel without losing the exponential nature of the original function.

4 Convolution with Hyperbolic Function

Let's consider $f(t) = \sinh(bt)$ for $b > 0$.

The convolution is:

$$y(t) = \int_{-\infty}^{\infty} \sinh(b\tau) \cdot h(t-\tau) d\tau = \int_{t-T}^{t+T} \sinh(b\tau) d\tau \tag{9}$$

Computing this integral:

$$\begin{aligned} y(t) &= \int_{t-T}^{t+T} \sinh(b\tau) d\tau && (10) \\ &= \left[\frac{1}{b}\cosh(b\tau)\right]_{t-T}^{t+T} && (11) \\ &= \frac{1}{b}(\cosh(b(t+T)) - \cosh(b(t-T))) && (12) \end{aligned}$$

Using the identity $\cosh(A) - \cosh(B) = 2\sinh(\frac{A+B}{2})\sinh(\frac{A-B}{2})$:

$$\begin{aligned} y(t) &= \frac{1}{b}(2\sinh(bt)\sinh(bT)) && (13) \\ &= \frac{2\sinh(bT)}{b}\sinh(bt) && (14) \end{aligned}$$

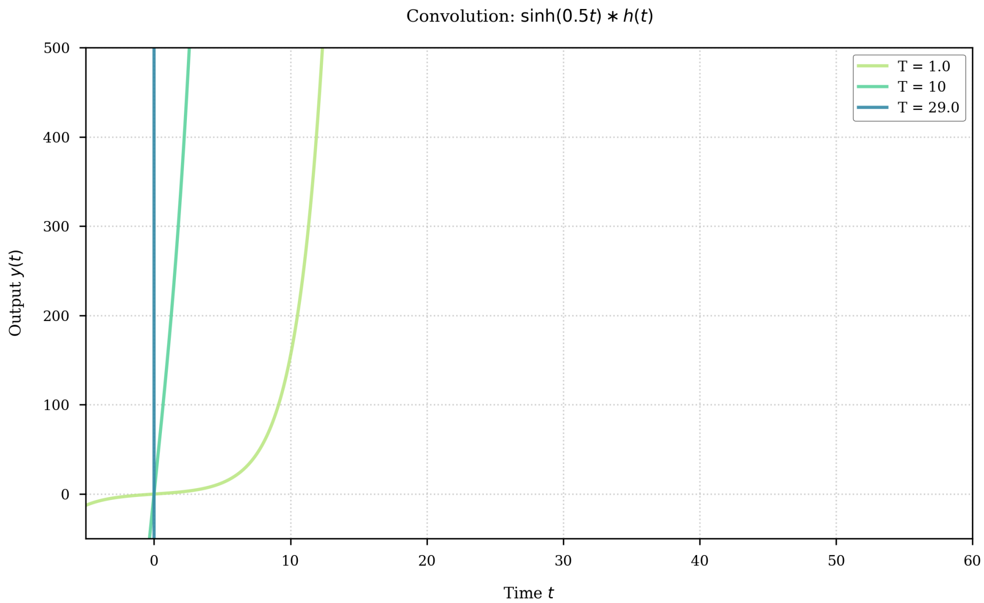

Figure 2: Convolution of $f(t) = \sinh(0.5t)$ with a rectangular kernel of varying T. The output retains the hyperbolic sine form but with increased magnitude that varies with $\frac{2\sinh(bT)}{b}$. With an increase in T, the amplitude increases significantly, reaching around ± 100 for $T = 29.0$. In contrast to the exponential scenario, this output increases both positively and negatively, illustrating the way the wider rectangular kernels increase the hyperbolic nature of the original function.

5 Modified Kernel Analysis

5.1 Kernel for $t > 0$ (Part a)

For part (a), we need to modify the kernel to only consider the part for $t > 0$:

$$h_m(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

This is equivalent to applying a unit step function to the original kernel and adjusting the range:

$$h_m(t) = h(t+T) \cdot u(t) \tag{16}$$

For the exponential function $f(t) = e^{-at}$, we now need to introduce the step function $u(t)$ for a causal system analysis. Let's define $f_c(t) = e^{-at}u(t)$ for $a > 0$.

The convolution becomes:

$$y_m(t) = \int_{-\infty}^{\infty} e^{-a\tau} u(\tau) \cdot h_m(t-\tau) d\tau \tag{17}$$

The effective integration limits are determined by where both $\tau \geq 0$ and $0 \leq t - \tau \leq T$, or $t - T \leq \tau \leq t$ and $\tau \geq 0$:

$$y_m(t) = \int_{\max(0,t-T)}^{\min(t,t)} e^{-a\tau} d\tau \tag{18}$$

Analyzing different cases:

5.1.1 Case 1: $t < 0$

In this case, $\min(t,t) = t < 0$, so there's no overlap and $y_m(t) = 0$.

5.1.2 Case 2: $0 \leq t < T$

Here, $t - T < 0$ and $t \geq 0$, so:

$$y_m(t) = \int_0^t e^{-a\tau} d\tau \tag{19}$$
$$= \frac{1}{a}(1 - e^{-at}) \tag{20}$$

5.1.3 Case 3: $t \geq T$

Both $t - T \geq 0$ and $t > 0$, so:

$$y_m(t) = \int_{t-T}^t e^{-a\tau} d\tau \tag{21}$$
$$= \frac{1}{a}(e^{-a(t-T)} - e^{-at}) \tag{22}$$
$$= \frac{e^{-at}}{a}(e^{aT} - 1) \tag{23}$$

Therefore, the complete convolution result for the modified kernel is:

$$y_m(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{a}(1 - e^{-at}), & 0 \leq t < T \\ \frac{e^{-at}}{a}(e^{aT} - 1), & t \geq T \end{cases} \tag{24}$$

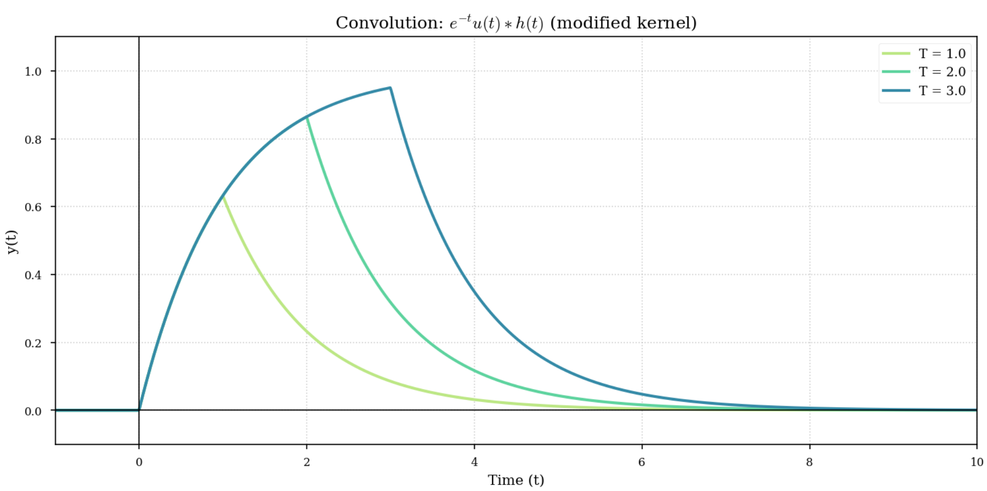

Figure 3: Convolution of $f(t) = e^{-t}u(t)$ with the modified kernel for various values of T. The result is zero for $t < 0$ due to causality. For $0 \leq t < T$, the response rises as $(1 - e^{-at})/a$ to various maximum levels depending on T. For $t \geq T$, the output takes an exponential decay form $e^{-at}(e^{aT} - 1)/a$. Higher values of T produce greater peak amplitudes and broader response regions. For $T = 3.0$, the peak is about 0.95 at $t \approx 3$, with exponential decay thereafter.

5.2 Hyperbolic Function with Modified Kernel

For the hyperbolic function with the modified kernel, a similar analysis gives:

5.2.1 Case 1: $t < 0$

The output is zero: $y_m(t) = 0$.

5.2.2 Case 2: $0 \leq t < T$

$$y_m(t) = \int_0^t \sinh(b\tau)d\tau \tag{25}$$
$$= \frac{1}{b}(\cosh(bt) - 1) \tag{26}$$

5.2.3 Case 3: $t \geq T$

$$y_m(t) = \int_{t-T}^t \sinh(b\tau)d\tau \tag{27}$$
$$= \frac{1}{b}(\cosh(bt) - \cosh(b(t-T))) \tag{28}$$

Using the identity $\cosh(A) - \cosh(B) = 2\sinh(\frac{A+B}{2})\sinh(\frac{A-B}{2})$:

$$y_m(t) = \frac{1}{b}(2\sinh(b(t - \frac{T}{2}))\sinh(\frac{bT}{2})) \tag{29}$$
$$= \frac{2\sinh(\frac{bT}{2})}{b}\sinh(b(t - \frac{T}{2})) \tag{30}$$

Therefore, the complete convolution result is:

$$y_m(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{b}(\cosh(bt) - 1), & 0 \leq t < T \\ \frac{2\sinh(\frac{bT}{2})}{b}\sinh(b(t - \frac{T}{2})), & t \geq T \end{cases} \tag{31}$$

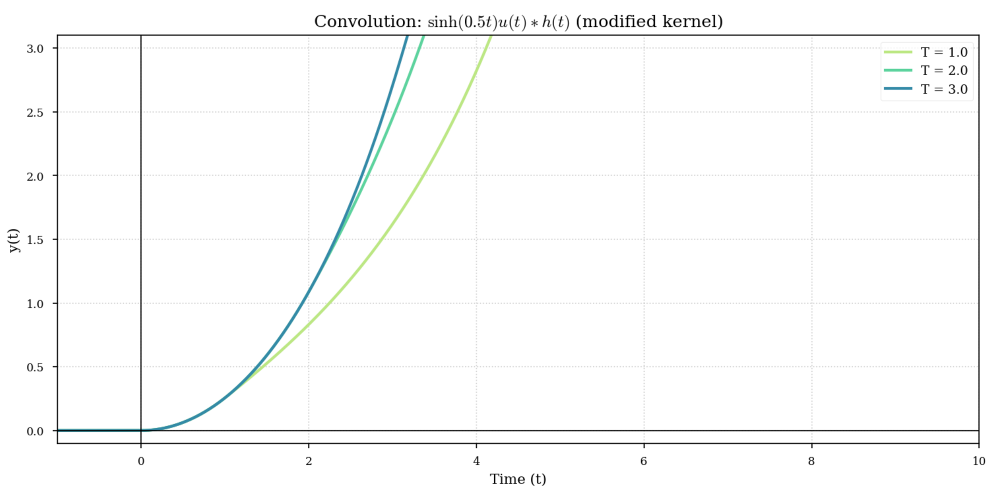

Figure 4: Convolution of $f(t) = \sinh(0.5t)u(t)$ with the modified kernel for various values of T. The result is zero for $t < 0$, and for $t > 0$ has exponential growth properties of the hyperbolic sine function. As T grows, the rate of growth becomes faster, with the $T = 3.0$ curve having the most rapid growth. In contrast to the exponential case, these curves still rise without limit after $t = T$, with the pattern $\frac{2sinh(\frac{bT}{2})}{b}\sinh(b(t - \frac{T}{2}))$. All three curves approach each other for small t but spread out considerably as t grows, illustrating how the system exaggerates the input signal more with wider kernels.

5.3 Time-Shifted Kernel (Part b)

For part (b), we analyze a kernel shifted by a time τ_0:

$$h_s(t) = \begin{cases} 1, & \text{for } -T+\tau_0 \leq t \leq T+\tau_0 \\ 0, & \text{otherwise} \end{cases} \tag{32}$$

This is equivalent to:

$$h_s(t) = h(t-\tau_0) \tag{33}$$

5.3.1 Exponential Function with Time-Shifted Kernel

The convolution with $f(t) = e^{-at}$ becomes:

$$y_s(t) = \int_{-\infty}^{\infty} e^{-a\tau} \cdot h_s(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-a\tau} \cdot h(t-\tau-\tau_0)d\tau \tag{34}$$

Using the time-shifting property of convolution, we can write:

$$y_s(t) = y(t-\tau_0) = \frac{2\sinh(aT)}{a}e^{-a(t-\tau_0)} = \frac{2\sinh(aT)}{a}e^{a\tau_0}e^{-at} \tag{35}$$

This shows that shifting the kernel by τ_0 results in a time-shift of the output and an amplitude scaling by $e^{a\tau_0}$.

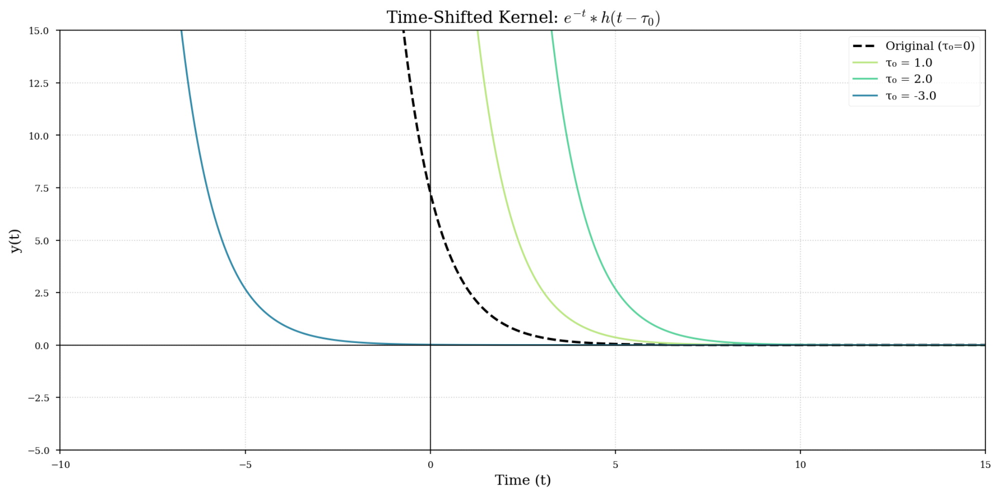

Figure 5: Convolution of $f(t) = e^{-t}$ with time-shifted rectangular kernel $h(t-\tau_0)$ for various values of τ_0. The original response ($\tau_0 = 0$) is represented as a dashed line. As τ_0 is increased, the whole response curve is shifted to the right by τ_0 and scaled by a factor of e^{τ_0}. For $\tau_0 = 1.0$, the curve is shifted right by 1 unit and scales up in amplitude. For $\tau_0 = 2.0$, the shift is 2 units with additional amplification. Similarly, for $\tau_0 = -3.0$, the shift is 3 units to the left. All curves share the typical exponential decay but different initial points and amplitudes, according to the equation $\frac{2\sinh(aT)}{a}e^{a\tau_0}e^{-at}$.

5.3.2 Hyperbolic Function with Time-Shifted Kernel

For the hyperbolic function $f(t) = \sinh(bt)$, the convolution with the time-shifted kernel is:

$$y_s(t) = \int_{-\infty}^{\infty} \sinh(b\tau) \cdot h_s(t-\tau)d\tau = \int_{-\infty}^{\infty} \sinh(b\tau) \cdot h(t-\tau-\tau_0)d\tau \tag{36}$$

Using the time-shifting property of convolution, the result is:

$$y_s(t) = y(t - \tau_0) = \frac{2\sinh(bT)}{b}\sinh(b(t - \tau_0)) \tag{37}$$

This represents a pure time-shift of the original response without amplitude scaling, unlike the exponential case.

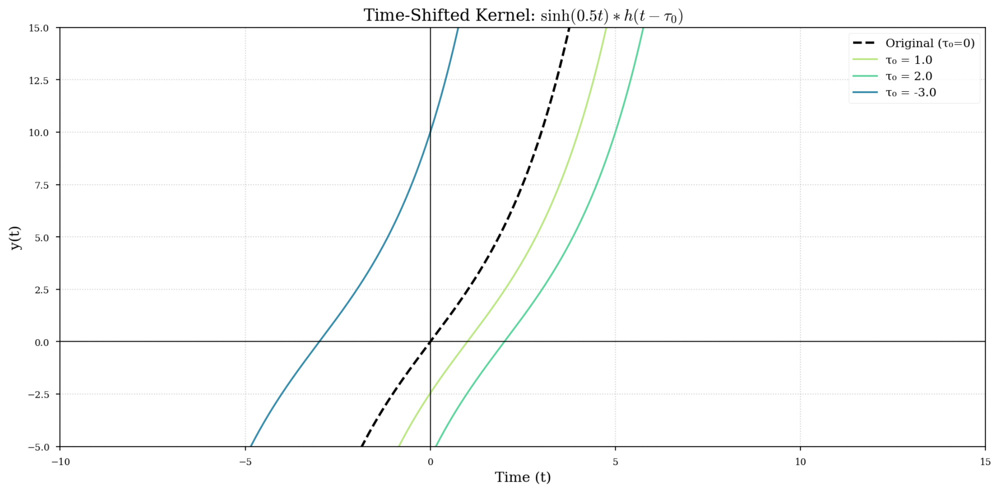

Figure 6: Convolution of $f(t) = \sinh(0.5t)$ with time-shifted rectangular kernel $h(t - \tau_0)$ for various values of τ_0. The original response ($\tau_0 = 0$) is a dashed line. For increasing values of τ_0, the response curves move horizontally to the right by τ_0 units. For $\tau_0 = 1.0$, the curve moves right by 1 unit, for $\tau_0 = 2.0$, the move is 2 units and for $\tau_0 = -3.0$, its 3 units to the left. In contrast to the exponential case, there is no scaling of amplitude - just a simple time-shift, as seen in the parallel curves. Every curve is described by the equation $\frac{2\sinh(bT)}{b}\sinh(b(t - \tau_0))$, with the same hyperbolic growth rate but with varying x-intercepts at $t = \tau_0$.

5.4 Comparison of Exponential and Hyperbolic Functions with Time-Shifted Kernels

The time-shifted kernel brings out the key distinction between exponential and hyperbolic functions in convolution systems:

- For the exponential function $f(t) = e^{-at}$, a time-shift in the kernel results in both a time-shift and amplitude scaling in the output. The scaling factor e^{atau_0} grows exponentially with the amount of the shift.
- For the hyperbolic function $f(t) = \sinh(bt)$, a time-shift in the kernel results only in a time-shift in the output and not in amplitude scaling. The curves do not change their shape and are merely shifted horizontally.