EE1060 - DETT GROUP QUIZ 2

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Contents

1 What is convolution ?

Convolution is a mathematical operation on two functions, f and g, as the integral of the product of the two functions after one is reflected about the y-axis and shifted, i.e., If we are convolving a signal with a kernel, at each time t, we are sliding the kernel over the signal and computing how much they overlap. This is useful in various fields of study like finding the system response given the impulse response, signal processing and in probability.
Given two functions $f(x)$ and $g(x)$, convolution of the two signals,

$$x(t) = f(t) * g(t) \tag{1}$$
$$= \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \tag{2}$$

Some properties of convolution :

1. **Commutativity** :
$$f(t) * g(t) = g(t) * f(t)$$
2. **Distributivity** :
$$f(t) * [g(t) + h(t)] = [f(t) * g(t)] + [f(t) * h(t)]$$
3. **Associativity** :
$$f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$$
4. **Time shifting property** : If we are given $y(t) = x_1(t) * x_2(t)$, then
$$x_1(t) * x_2(t-T) = y(t-T)$$
$$x_1(t-T) * x_2(t) = y(t-T)$$
$$x_1(t-T_1) * x_2(t-T_2) = y(t-T_1-T_2)$$
5. **Width property** : If the duration of signals $f(t)$ and $g(t)$ are T_1 and T_2 respectively, then the duration of signal convolving $f(t)$ and $g(t)$ is equal to $T_1 + T_2$

2 Convolution of sinc function

Given kernel,

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

and the function

$$f(t) = sinc(t)$$
$$= \frac{sint}{t}$$

Let $y(t) = x(t) * f(t)$, then

$$\begin{aligned} y(t) &= h(t) * f(t) \\ &= f(t) * h(t) \\ &= \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \end{aligned}$$

$$h(t-\tau) = \begin{cases} 1, & \text{for } -T \leq t-\tau \leq T \\ 0, & \text{otherwise} \end{cases}$$

$$h(t-\tau) = \begin{cases} 1, & \text{for } \tau - T \leq t \leq \tau + T \\ 0, & \text{otherwise} \end{cases}$$

Therefore, the convolution becomes,

$$y(t) = \int_{t-T}^{t+T} \frac{sin(\tau)}{\tau} * 1d\tau$$

$$y(t) = \int_{t-T}^{t+T} \frac{sin(\tau)}{\tau} d\tau$$

This integration can be numerically computed and can be seen to be the following -

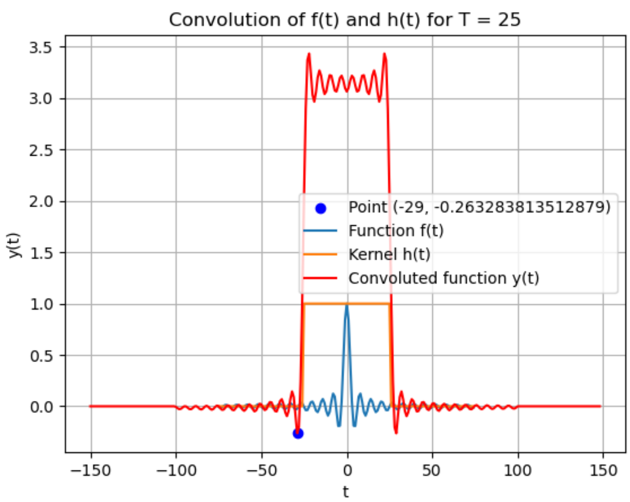

Figure 1: T = 25

For different values of T, the change in $y(t)$ is depicted in the following -

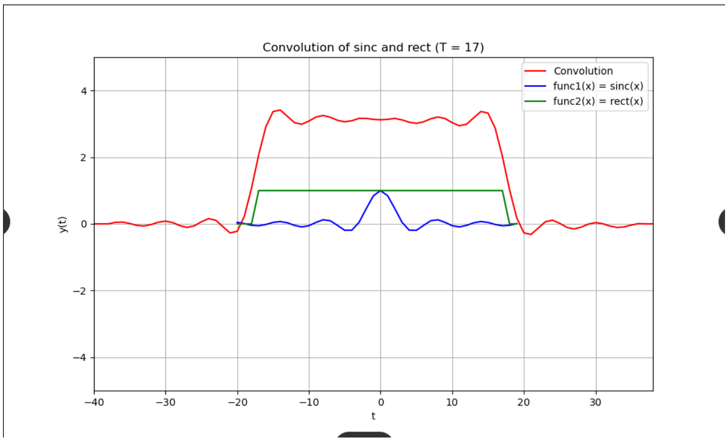

Figure 2: T = 17

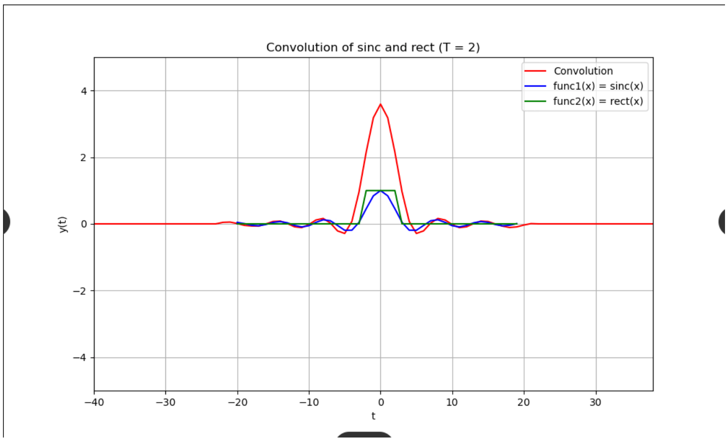

Figure 3: T = 2

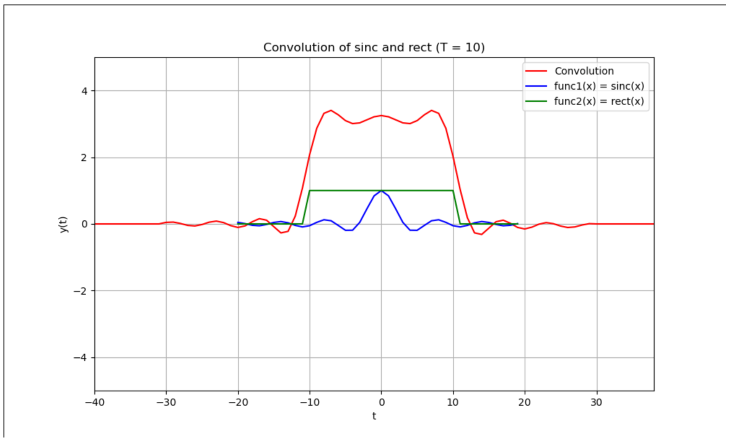

Figure 4: T = 10

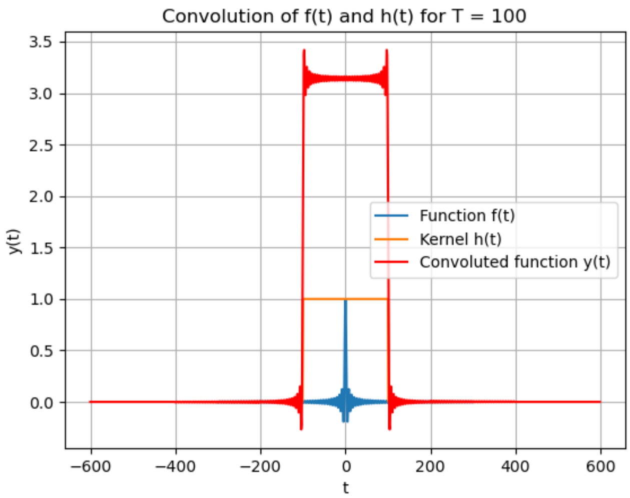

Figure 5: T = 100

It can be seen that as T increases, $y(t)$ becomes more square.

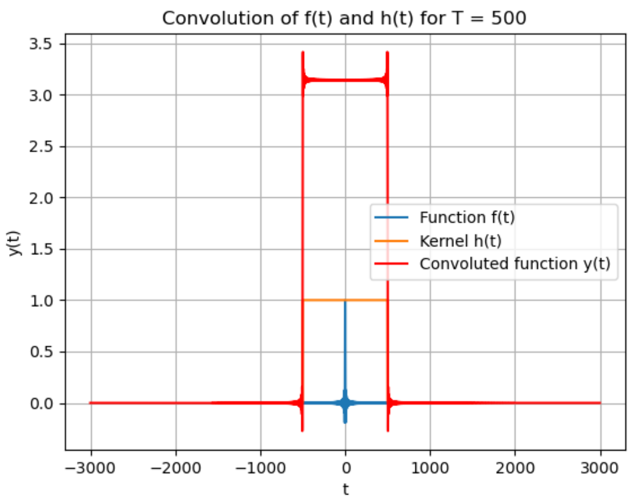

Figure 6: T = 500

2.1 Considering the kernel for $t > 0$

The response of the system for different values of T are as follows -

It can be seen that the width of the convolution becomes half and it becomes shifted

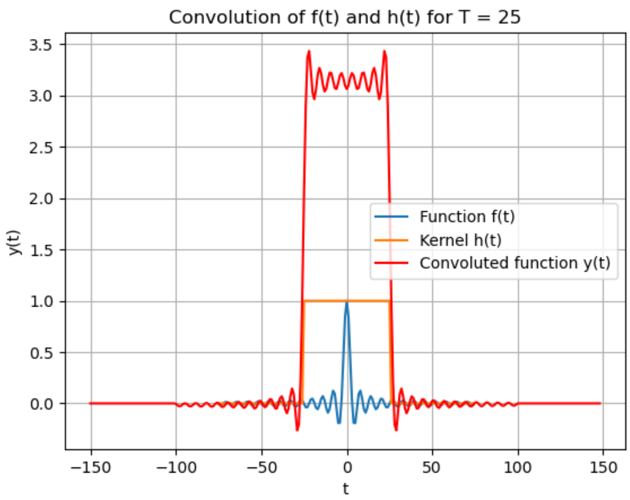

Figure 7: T = 25

to the right.

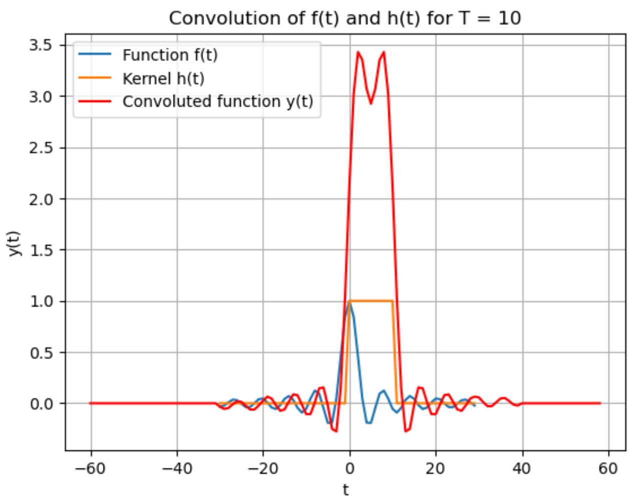

Figure 8: T = 10

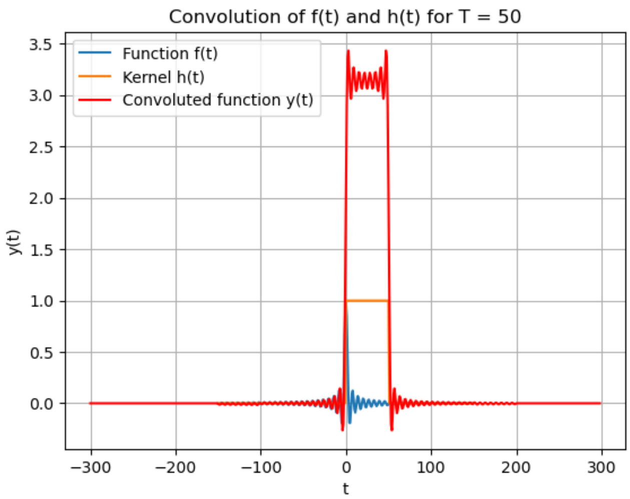

Figure 9: T = 50

2.2 Shifting the kernel by t_0

Shifting of kernel means moving the kernel to the left or right by an amount, i.e., if the given kernel is shifted by an amount t_0, then it becomes,

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t - t_0 \leq T \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} 1, & \text{for } -T + t_0 \leq t \leq T + t_0 \\ 0, & \text{otherwise} \end{cases}$$

If we are convolving a signal with the kernel, at each time t, we are sliding the kernel over the signal and computing how much they overlap. So, when we shift the kernel by t_0, we are changing when the kernel has its maximum influence. This can physically mean applying a response t_0 time units early (or) t_0 time units late depending upon the sign

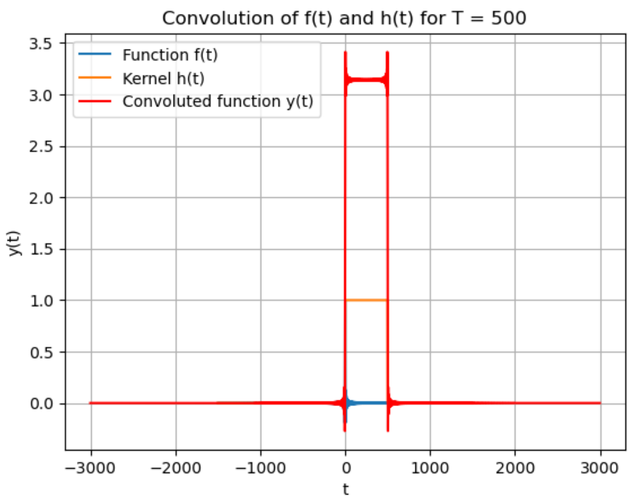

Figure 10: T = 500

of t_0, i.e., if $t_0 > 0$, the response is delayed and is advanced for $t_0 < 0$. This results in the convoluted signal also to be shifted by the same amount, as in figures 11 and 12 -

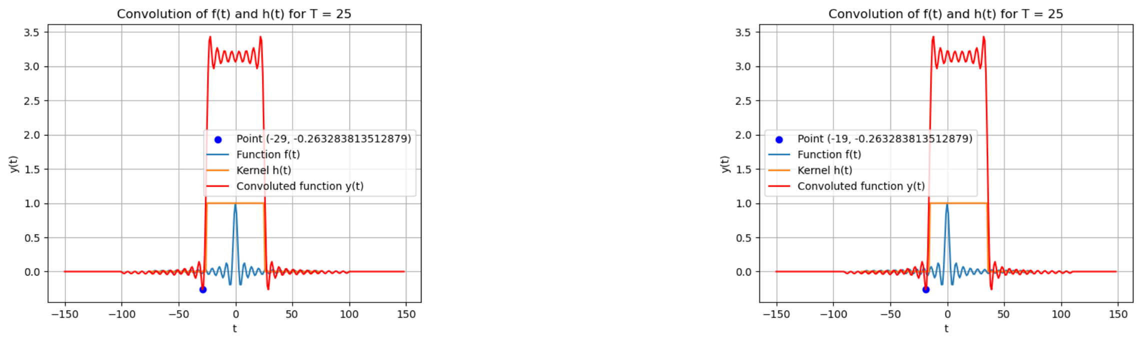

Figure 11: Shifting toward right

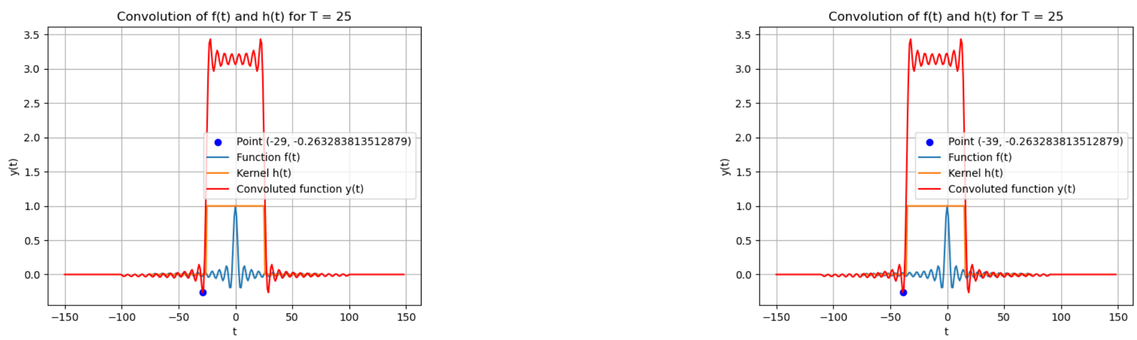

Figure 12: Shifting toward left

Shifting of the kernel by 10 units results in shifting of convoluted function by 10 units.

3 Significance of time-shift

1. In time-delayed systems, a positive shift (t_0) means that the system responds later to changes in the input. This can cause the system to react too slowly, which may lead to oscillations or instabilities in feedback systems, especially if the system relies on feedback loops (like in control systems or biological systems).

2. A negative shift ($t < 0$) might result in the system anticipating the input and responding before it occurs. This could cause overreaction or instability if the system is not designed to handle anticipatory behavior.

3. In devices like thermostat, the system may respond to temperature changes with a delay because of the time it takes to heat/cool the room. By shifting of kernel, if the delay becomes too large, then the system might not be able to track the input accurately, leading to instability (or) oscillations.

4. In the context of signal processing, shifting the kernel corresponds to a phase shift in the signal, e.g., in delay filters, the impulse response can be delayed.

5. In biological systems, time delays are common in processes like hormonal responses, neural signals, or metabolic pathways. Shifting the kernel models how these systems react over time. For example, the delayed release of insulin after glucose intake could be modeled using a time-shifted kernel.

4 Convolution of step function

For the kernel in (2), we shall find the convolution with $f(t) = u(t)$

4.1 Analytical Convolution

Since $u(\tau) = 0$ for $\tau < 0$, the integral becomes:

$$y(t) = \int_0^\infty h(t-\tau)\, d\tau$$

We determine the overlap of $h(t-\tau)$ within its support $[-T, T]$, resulting in:

$$t - T \leq \tau \leq t + T$$

Intersecting this with $\tau \geq 0$, the integration limits become:

$$\max(0, t-T) \leq \tau \leq t + T$$

Thus:

$$y(t) = \int_{\max(0,t-T)}^{t+T} 1\, d\tau = t + T - \max(0, t-T)$$

Breaking into cases:

$$y(t) = \begin{cases} 0, & t < -T \\ t + T, & -T \leq t < T \\ 2T, & t \geq T \end{cases}$$

4.2 Scenario Analysis

4.2.1 Causal Kernel

Modified kernel:

$$h(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

Now:

$$y(t) = \int_0^\infty h(t-\tau)\, d\tau = \int_{\max(0,t-T)}^{t} 1\, d\tau = \min(t, T)$$

Thus:

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t < T \\ T, & t \ge T \end{cases}$$

4.2.2 Shifted Kernel

Let $h(t) \to h(t - \tau_0)$. Then:

$$y(t) = \int_0^\infty u(\tau) h(t - \tau - \tau_0)\, d\tau = y_{\text{original}}(t - \tau_0)$$

The output is simply delayed by τ_0.
The corresponding graphs are shown below -

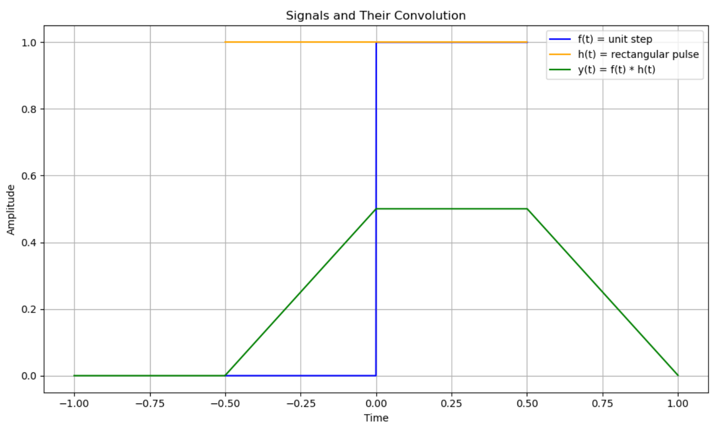

Figure 13: T = 0.5

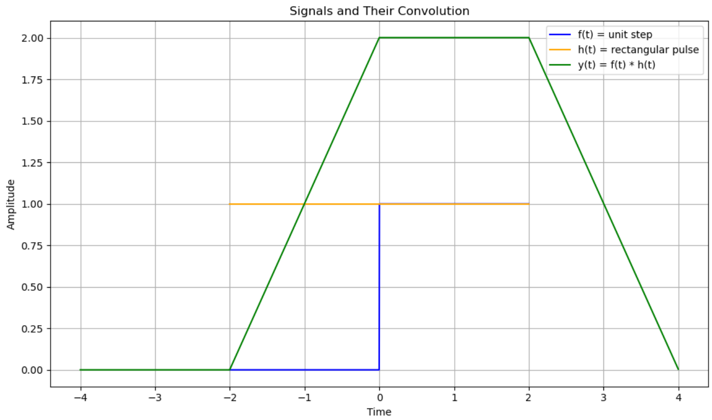

Figure 14: T = 2

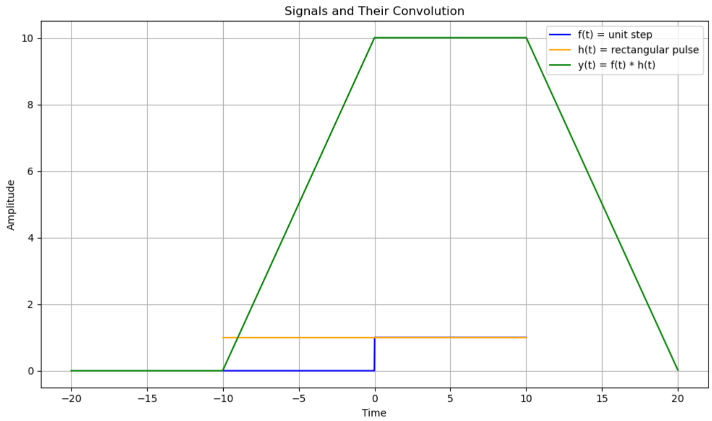

Figure 15: T = 10

5 Applications of convolution - Filters (Image processing)

5.1 Gaussian filter

1. A Gaussian filter is a type of digital filter that utilizes a Gaussian distribution function to smooth and reduce noise in signals or images. It achieves this by averaging neighboring values with weights based on a Gaussian curve, giving more weight to the central value and less to the outer values.

2. One dimensional Gaussian -

$$G_1(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \tag{3}$$

Two-dimensional Gaussian -

$$G_2(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{4}$$

3. **How it works ?**

   - A Gaussian filter, is a low-pass filter, that uses a kernel that represents the weighting of neighboring pixels.
   - The kernel is then convolved with the image (i.e., it is moved across the image), and the values of the kernel are multiplied by the corresponding pixel values in the image.
   - The products are then summed, and the result is divided by the sum of the kernel values, producing a weighted average of the pixel's neighborhood.
   - The weighted average becomes the new value of the pixel at the center of the kernel's position, smoothing out the image.

4. The corresponding figure is shown below -

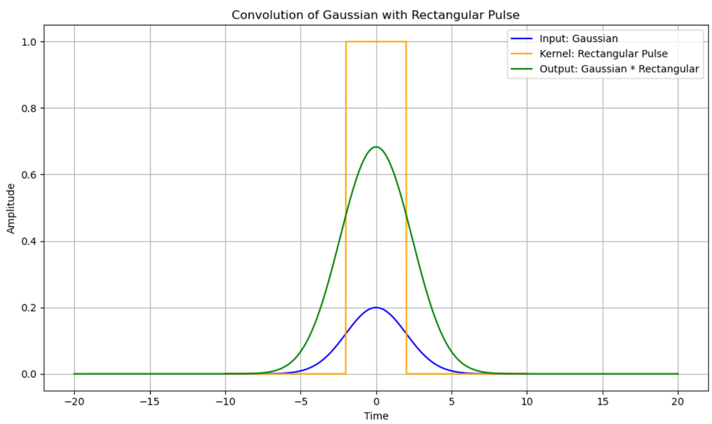

Figure 16: T = 2

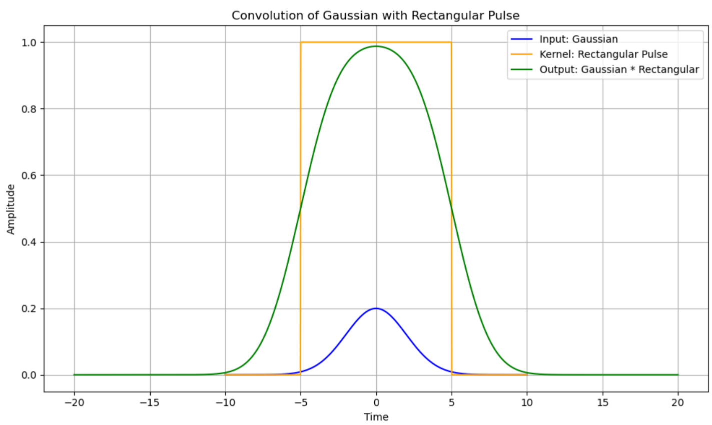

Figure 17: T = 5

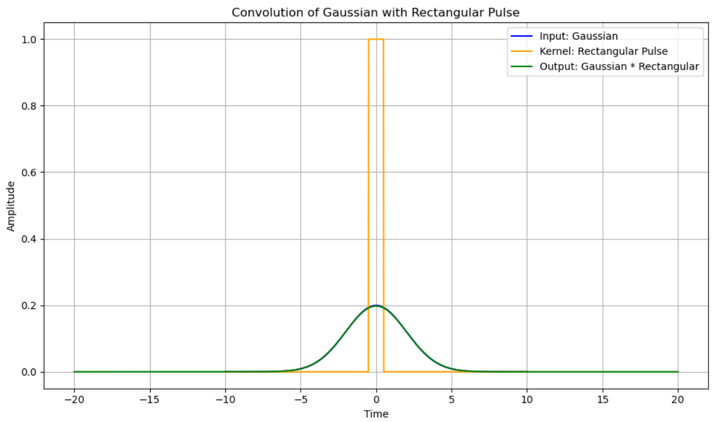

Figure 18: T = 0.5

5. In the case of Figure 18, since the kernel itself is bell-shaped, convolving a narrow Gaussian will produce an output being very similar to the original signal, just slightly smoothed or blurred.

6. Gaussian filters are used because -

    - Smooth
    - Decay to zero rapidly
    - Simple analytical formula
    - **Central Limit Theorem :** Limit of applying filters many times is some Gaussian.
    - Separable

$$G_2(x, y) = G_1(x)G_1(y)$$

5.2 Laplacian filter

1. A Laplacian filter is a second-order derivative operator that is commonly used in image processing for edge detection and enhancement.

2. It is often implemented using a convolution kernel, which is a small matrix that is slid over the image and multiplied with the surrounding pixels to produce a new pixel value.

3. The Laplacian filter, when combined with a Gaussian blur (Laplacian of Gaussian), can be used to detect edges in an image.

4. Image corresponding to this is given below -

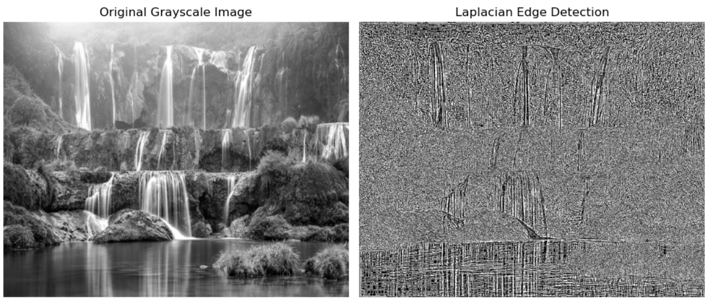

Figure 19: Edge detection using convolution