Convolution

Gunda Srihaas

1 Question

Compute the convolution of a given signal f(t) with a rectangular kernel h(t), analytically. The rectangular kernel is defined as:

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Derive the convolution expression $y(t) = (f * h)(t)$ in terms of known functions, and analyze the system's behavior for various values of the kernel duration T and the input signal $f(t)$. Additionally, investigate the following scenarios:

a. Modify the kernel to only consider the part of the kernel for $t > 0$. How does this affect the convolution result?

b. Shift the kernel by a time τ_0. Analyze how the shift impacts the convolution output and discuss the significance of this shift in the context of time-delayed systems

2 Solution

2.1 Taking $f(t) = sin(\omega t)$

The convolution is defined as:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\, d\tau$$

Given,

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Which implies in convolution,

$$-T \leq t - \tau \leq T$$
$$t - T \leq \tau \leq t + T$$

$\implies h(t-\tau) = 1$ in $(t-T, t+T)$
Thus, the convolution integral becomes:

$$y(t) = \int_{t-T}^{t+T} f(\tau)\, d\tau$$

For $f(t) = \sin(\omega t)$, the convolution becomes:

$$y(t) = \int_{t-T}^{t+T} \sin(\omega\tau)\, d\tau$$

$$\begin{aligned} y(t) &= \int_{t-T}^{t+T} \sin(\omega\tau)\, d\tau \\ &= \left[-\frac{1}{\omega} \cos(\omega\tau) \right]_{t-T}^{t+T} \\ &= -\frac{1}{\omega} \left[\cos(\omega(t+T)) - \cos(\omega(t-T)) \right] \end{aligned}$$

Using the trigonometric identity:

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

Let:

$$\begin{aligned} A &= \omega(t+T) \\ B &= \omega(t-T) \end{aligned}$$

Then:

$$\begin{aligned} \frac{A+B}{2} &= \omega t \\ \frac{A-B}{2} &= \omega T \end{aligned}$$

Substituting:

$$\begin{aligned} y(t) &= -\frac{1}{\omega} \left[-2 \sin(\omega t) \sin(\omega T) \right] \\ &= \frac{2}{\omega} \sin(\omega T) \sin(\omega t) \end{aligned}$$

Final Result

$$y(t) = \frac{2}{\omega} \sin(\omega T) \sin(\omega t)$$

2.2 Taking any periodic $f(t)$

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L}\right)$$

By substitution of $f(t)$ and and reducing the integral we get

$$y(t) = \int_{t-T/2}^{t+T/2} \left[A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi\tau}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi\tau}{L}\right) \right] d\tau$$

DC Term

$$\int_{t-T/2}^{t+T/2} A_0 \, d\tau = A_0 T$$

Sine part

$$B_n \int \sin\left(\frac{n\pi\tau}{L}\right) d\tau = -\frac{L}{n\pi} \cos\left(\frac{n\pi\tau}{L}\right)$$

Evaluating from $t - T/2$ to $t + T/2$:

$$B_n \cdot \frac{2L}{n\pi} \cos\left(\frac{n\pi t}{L}\right) \sin\left(\frac{n\pi T}{2L}\right)$$

Cosine part

$$\begin{aligned}
y_{\cos}(t) &= \int_{t-T}^{t+T} A_n \cos\left(\frac{n\pi\tau}{L}\right) d\tau \\
&= A_n \left[\frac{L}{n\pi} \sin\left(\frac{n\pi\tau}{L}\right) \right]_{t-T}^{t+T} \\
&= A_n \cdot \frac{L}{n\pi} \left[\sin\left(\frac{n\pi(t+T)}{L}\right) - \sin\left(\frac{n\pi(t-T)}{L}\right) \right]
\end{aligned}$$

Using the trigonometric identity:

$$\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

Let:

$$\begin{aligned}
C &= \frac{n\pi(t+T)}{L} = \frac{n\pi t}{L} + \frac{n\pi T}{L} \\
D &= \frac{n\pi(t-T)}{L} = \frac{n\pi t}{L} - \frac{n\pi T}{L}
\end{aligned}$$

Then:

$$\frac{C+D}{2} = \frac{n\pi t}{L}$$
$$\frac{C-D}{2} = \frac{n\pi T}{L}$$

Substituting:

$$\begin{aligned} y_{\cos}(t) &= A_n \cdot \frac{L}{n\pi}\left[2\cos\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi T}{L}\right)\right] \\ &= A_n \cdot \frac{2L}{n\pi}\sin\left(\frac{n\pi T}{L}\right)\cos\left(\frac{n\pi t}{L}\right) \end{aligned}$$

Final Result

$$y(t) = 2A_0T + \sum_{n=1}^{\infty} \frac{2L}{n\pi}\sin\left(\frac{n\pi T}{L}\right)\left[A_n \sin\left(\frac{n\pi t}{L}\right) + B_n \cos\left(\frac{n\pi t}{L}\right)\right]$$

Convolution of a Signal with a Rectangular Kernel

Given a signal $f(t)$ and a rectangular kernel $h(t)$ defined as:

$$h(t) = \begin{cases} 1, & \text{for } -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

The convolution $y(t) = (f * h)(t)$ is:

$$y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

Part 1: General Convolution Expression

The kernel $h(t-\tau)$ is 1 when $-T \le t-\tau \le T$, i.e., $t-T \le \tau \le t+T$. Thus:

$$y(t) = \int_{t-T}^{t+T} f(\tau)\,d\tau$$

Part 2: Analysis for Various T and $f(t)$

- **Large T:** More smoothing of $f(t)$, attenuates high frequencies.
- **Small T:** Less smoothing, $y(t) \approx f(t)$.
- **Constant $f(t) = c$:** $y(t) = 2Tc$.

- **Linear** $f(t) = at + b$**:**

$$
\begin{aligned}
y(t) &= \int_{t-T}^{t+T} (a\tau + b)\, d\tau \\
&= 2aTt + 2Tb
\end{aligned}
$$

Part a: Modified Kernel ($t > 0$)

Define:

$$
h_{\text{modified}}(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}
$$

The convolution becomes:

$$
y_{\text{modified}}(t) = \int_{t-T}^{t} f(\tau)\, d\tau
$$

Difference:

- Original: Integrates symmetrically around t ($[t - T, t + T]$).
- Modified: Only integrates past values ($[t - T, t]$), making it causal.

Part b: Shifted Kernel by τ_0

Shifted kernel:

$$
h_{\text{shifted}}(t) = h(t - \tau_0) = \begin{cases} 1, & \text{for } \tau_0 - T \leq t \leq \tau_0 + T \\ 0, & \text{otherwise} \end{cases}
$$

The convolution is:

$$
\begin{aligned}
y_{\text{shifted}}(t) &= \int_{t-\tau_0-T}^{t-\tau_0+T} f(\tau)\, d\tau \\
&= y(t - \tau_0)
\end{aligned}
$$

Interpretation: Shifting the kernel by τ_0 shifts the output by τ_0.

Specific Cases for $f(t)$

Case 1: $f(t) = \sin(\omega t)$

$$
\begin{aligned}
y(t) &= \int_{t-T}^{t+T} \sin(\omega\tau)\, d\tau \\
&= \frac{2\sin(\omega T)}{\omega} \sin(\omega t)
\end{aligned}
$$

For shifted kernel:

$$
y_{\text{shifted}}(t) = \frac{2\sin(\omega T)}{\omega} \sin(\omega(t - \tau_0))
$$

Case 2: Fourier Series Representation

Let:

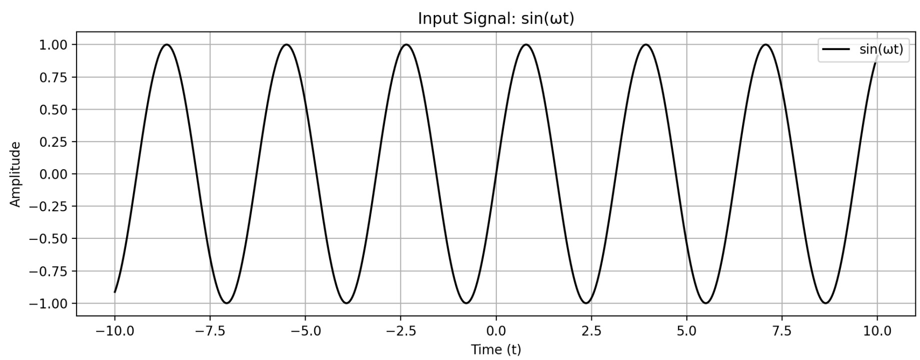
$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L}\right)$$

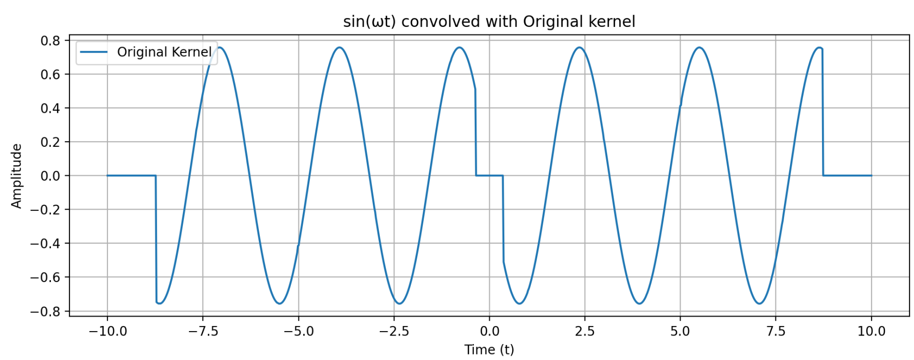
Then:

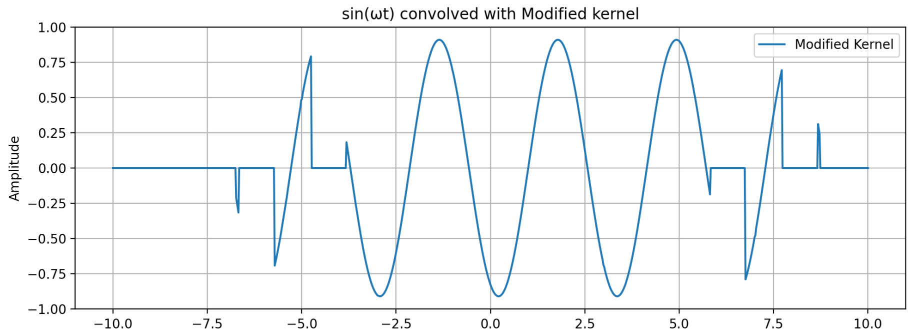
$$y(t) = 2TA_0 + \sum_{n=1}^{\infty} \frac{2L}{n\pi} \sin\left(\frac{n\pi T}{L}\right) \left[A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)\right]$$

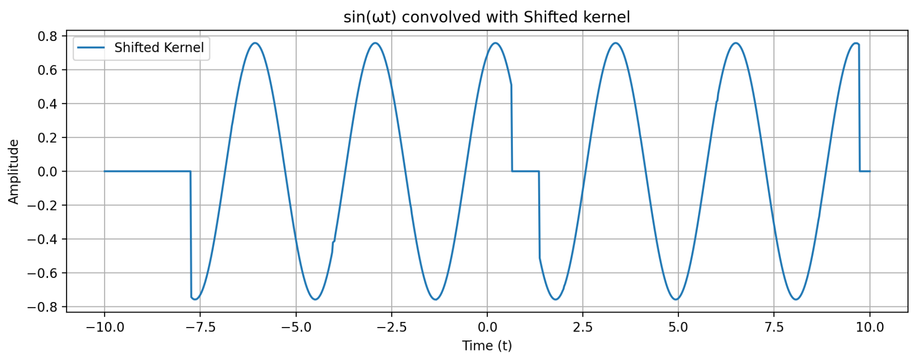
For shifted kernel:

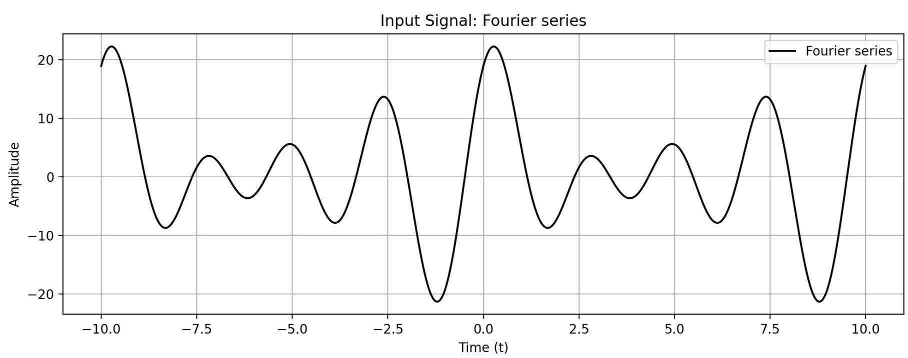
$$\begin{aligned} y_{\text{shifted}}(t) = 2TA_0 + \sum_{n=1}^{\infty} \frac{2L}{n\pi} \sin\left(\frac{n\pi T}{L}\right) \\ \times \left[A_n \cos\left(\frac{n\pi(t-\tau_0)}{L}\right) + B_n \sin\left(\frac{n\pi(t-\tau_0)}{L}\right)\right] \end{aligned}$$

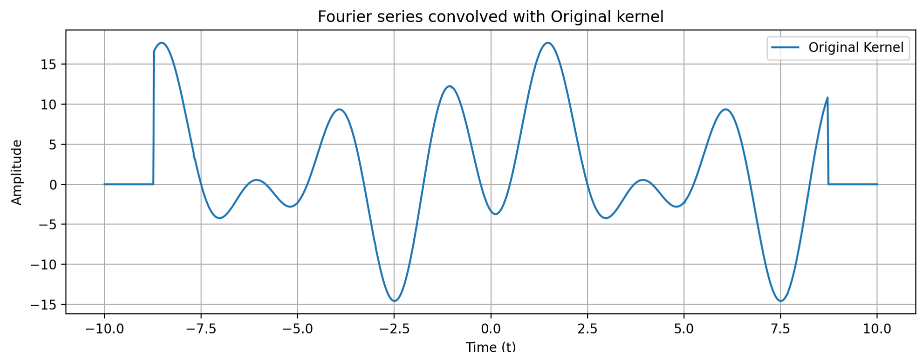


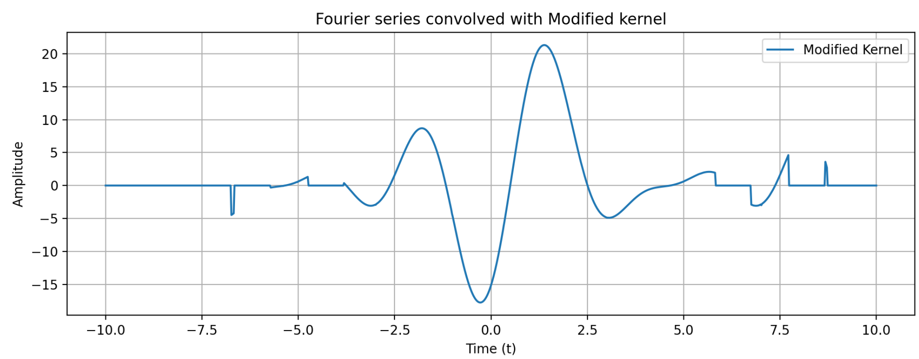


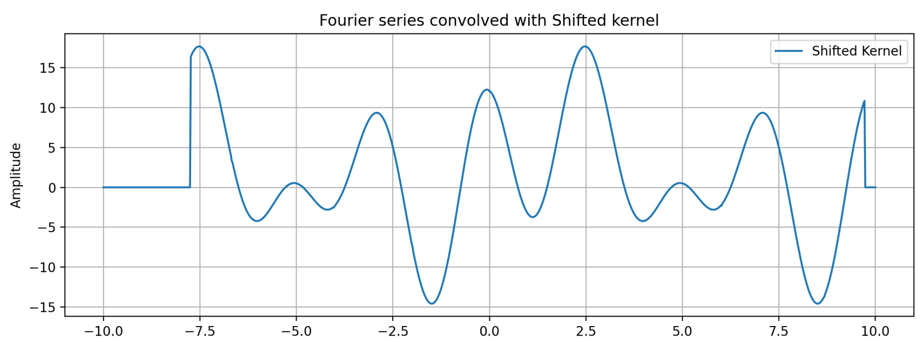

sin(ωt) convolved with Modified kernel
Modified Kernel
Amplitude
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0


sin(ωt) convolved with Shifted kernel
Shifted Kernel
Amplitude
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
Time (t)


Input Signal: Fourier series
Fourier series
Amplitude
20
10
0
−10
−20
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
Time (t)


Fourier series convolved with Original kernel
Original Kernel
Amplitude
Time (t)


Fourier series convolved with Modified kernel
Modified Kernel
Amplitude
Time (t)


Fourier series convolved with Shifted kernel
Shifted Kernel
Amplitude
Time (t)