Aalto University
School of Electrical Engineering

Statistical linearization and statistical linear regression

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Learning Outcomes

1. Statistical linearization
2. Statistical linear regression
3. Discussion and connections
4. Summary

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Linear approximation of a non-linearity

- We wish to approximate the transformation

$$\mathbf{y} = \mathbf{g}(\mathbf{x}).$$

- We choose the approximation to be a linear function, say

$$\mathbf{y} = \mathbf{A}\,(\mathbf{x} - \mathbf{m}) + \mathbf{b}.$$

- One option is to use Taylor series centered at $\mathbf{x} = \mathbf{m}$, which leads to

$$\mathbf{A} = \mathbf{G}_x(\mathbf{m}),$$
$$\mathbf{b} = \mathbf{g}(\mathbf{m}).$$

- Leads to the extended Kalman filter (and smoother).
- But this is not the only option to choose $\mathbf{A}$ and $\mathbf{b}$.

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Statistical linearization

- Instead, consider minimization of the error

$$\mathrm{MSE}(\mathbf{A}, \mathbf{b}) = \mathrm{E}\left[\|\mathbf{g}(\mathbf{x}) - \mathbf{A}\,\mathbf{x} - \mathbf{b}\|^2\right]$$

  w.r.t. some $\mathbf{x} \sim p(\mathbf{x})$ with $\mathrm{E}[\mathbf{x}] = \mathbf{m}$ and $\mathrm{Cov}[\mathbf{x}] = \mathbf{P}$.

- We get

$$\begin{aligned}
&\|\mathbf{g}(\mathbf{x}) - \mathbf{A}\,(\mathbf{x}-\mathbf{m}) - \mathbf{b}\|^2 \\
&= \mathbf{g}^\mathsf{T}(\mathbf{x})\,\mathbf{g}(\mathbf{x}) - 2\,\mathbf{g}^\mathsf{T}(\mathbf{x})\,\mathbf{A}\,(\mathbf{x}-\mathbf{m}) + \mathrm{tr}\left\{\mathbf{A}\,(\mathbf{x}-\mathbf{m})\,(\mathbf{x}-\mathbf{m})^\mathsf{T}\,\mathbf{A}^\mathsf{T}\right\} \\
&\qquad + 2\,(\mathbf{x}-\mathbf{m})^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{b} - 2\mathbf{g}^\mathsf{T}(\mathbf{x})\,\mathbf{b} + \mathbf{b}^\mathsf{T}\mathbf{b}.
\end{aligned}$$

- Taking expectation gives

$$\begin{aligned}
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Statistical linearization (cont.)

- Let us set the derivatives w.r.t. $\mathbf{b}$ and $\mathbf{A}$ to zero:

$$\frac{\partial \text{MSE}}{\partial \mathbf{b}} = -2\,\mathrm{E}\left[\mathbf{g}(\mathbf{x})\right] + 2\,\mathbf{b} = 0,$$
$$\frac{\partial \text{MSE}}{\partial \mathbf{A}} = -2\,\mathrm{E}\left[\mathbf{g}(\mathbf{x})\,(\mathbf{x}-\mathbf{m})^{\mathsf{T}}\right] + 2\,\mathbf{A}\,\mathbf{P} = 0.$$

- This leads to the optimal linearization parameters

$$\mathbf{b} = \mathrm{E}\left[\mathbf{g}(\mathbf{x})\right],$$
$$\mathbf{A} = \mathrm{E}\left[\mathbf{g}(\mathbf{x})\,(\mathbf{x}-\mathbf{m})^{\mathsf{T}}\right]\,\mathbf{P}^{-1}$$
$$= \mathrm{E}\left[(\mathbf{g}(\mathbf{x})-\mathbf{b})\,(\mathbf{x}-\mathbf{m})^{\mathsf{T}}\right]\,\mathbf{P}^{-1}.$$

- Thus the linearization is (with the above $\mathbf{b}$ and $\mathbf{A}$)

$$\mathbf{y} = \mathbf{A}\,(\mathbf{x}-\mathbf{m}) + \mathbf{b}.$$

- This is called classical statistical linearization (SL).

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Statistical linearization: properties

- The expectation of SL is indeed exact:

$$\mathrm{E}\left[\mathbf{A}\,(\mathbf{x}-\mathbf{m})+\mathbf{b}\right]=\mathrm{E}\left[\mathbf{g}(\mathbf{x})\right].$$

- The covariance is not exact:

$$\begin{aligned}&\mathrm{E}\left[(\mathbf{A}\,(\mathbf{x}-\mathbf{m})+\mathbf{b}-\mathbf{b})\,(\mathbf{A}\,(\mathbf{x}-\mathbf{m})+\mathbf{b}-\mathbf{b})^{\mathsf{T}}\right]\\&=\mathrm{E}\left[\mathbf{A}\,(\mathbf{x}-\mathbf{m})\,(\mathbf{x}-\mathbf{m})^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\right]\\&=\mathbf{A}\,\mathbf{P}\,\mathbf{A}^{\mathsf{T}}\neq\mathrm{Cov}\left[\mathbf{g}(\mathbf{x})\right]\qquad\{\text{ in general }\}.\end{aligned}$$

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- But we can fix this!

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Statistical linear regression

- We can now replace the deterministic approximation on $\mathbf{g}(\mathbf{x})$ with stochastic approximation

$$\mathbf{y} = \mathbf{A}\,\mathbf{x} + \mathbf{b} + \mathbf{e},$$

where $\mathbf{e} \sim \mathrm{N}(0, \Sigma)$ is a pseudo-noise.

- The covariance is now

$$\begin{aligned}
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&= \mathbf{A}\,\mathbf{P}\,\mathbf{A}^{\mathsf{T}} + \Sigma.
\end{aligned}$$

- Thus, we can force the correct covariance by putting

$$\Sigma = \mathrm{Cov}\,[\mathbf{g}(\mathbf{x})] - \mathbf{A}\,\mathbf{P}\,\mathbf{A}^{\mathsf{T}}.$$

- This is called statistical linear regression (SLR).

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Statistical linear regression: prediction step

- Let us use the SLR approximation on the dynamic model:

$$\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1},$$

where $\mathbf{q}_{k-1} \sim \mathrm{N}(0, \mathbf{Q})$.

- When in SLR we select $\mathbf{m} = \mathbf{m}_{k-1}$ and $\mathbf{P} = \mathbf{P}_{k-1}$ we get

$$\mathbf{x}_k = \mathbf{F}\,(\mathbf{x}_{k-1} - \mathbf{m}_{k-1}) + \mathbf{b} + \mathbf{e} + \mathbf{q}_{k-1},$$

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$$\begin{aligned}
\mathbf{b} &= \mathrm{E}_{k-1}\left[\mathbf{f}(\mathbf{x}_{k-1})\right], \\
\mathbf{F} &= \mathrm{E}_{k-1}\left[(\mathbf{f}(\mathbf{x}_{k-1}) - \mathbf{b})\,(\mathbf{x}_{k-1} - \mathbf{m}_{k-1})^{\mathsf{T}}\right]\mathbf{P}_{k-1}^{-1}, \\
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Statistical linear regression: prediction step (cont.)

- This now gives the prediction step

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\begin{aligned}
\mathbf{m}_k^- &= \mathrm{E}_{k-1}[\mathbf{f}(\mathbf{x}_{k-1})], \\
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- The mean and covariance are exact by construction.
- This is indeed the Gaussian filter prediction – leading to UKFs and other sigma-point filters.

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$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{r}_k,$$

  where $\mathbf{r}_k \sim \mathrm{N}(0, \mathbf{R})$.

- Using SLR with $\mathbf{m} = \mathbf{m}_k^-$ and $\mathbf{P} = \mathbf{P}_k^-$ gives

$$\mathbf{y}_k = \mathbf{H}\,(\mathbf{x}_k - \mathbf{m}_k^-) + \mu_k + \mathbf{e}_k' + \mathbf{r}_k,$$

  with

$$\begin{aligned}
\mu_k &= \mathrm{E}_k^-[\mathbf{h}(\mathbf{x}_k)], \\
\mathbf{H} &= \mathrm{E}_k^-\left[(\mathbf{h}(\mathbf{x}_k) - \mu_k)\,(\mathbf{x}_k - \mathbf{m}_k^-)^\mathsf{T}\right]\,\left[\mathbf{P}_k^-\right]^{-1}, \\
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\boldsymbol{\mu}_k &= \mathrm{E}_k^-[\mathbf{h}(\mathbf{x}_k)], \\
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\mathrm{Cov}_k^-[\mathbf{e}'] &= \mathrm{Cov}_k^-[\mathbf{h}(\mathbf{x}_k)] - \mathbf{H}\,\mathbf{P}_k^-\,\mathbf{H}^\mathsf{T}.
\end{aligned}$$

Statistical linear regression: update step (cont.)

- The joint mean and covariance of $\mathbf{x}_k$ and $\mathbf{y}_k$ are given as

$$\mathrm{E}_k^- \left[\begin{pmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{pmatrix} \right] = \begin{pmatrix} \mathbf{m}_k^- \\ \boldsymbol{\mu}_k \end{pmatrix}$$

$$\mathrm{Cov}_k^- \left[\begin{pmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{pmatrix} \right] = \begin{pmatrix} \mathbf{P}_k^- & \mathrm{E}_k^- \left[(\mathbf{x}_k - \mathbf{m}_k^-) \, (\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)^\mathsf{T} \right] \\ (\bullet)^\mathsf{T} & \mathrm{Cov}_k^- [\mathbf{h}(\mathbf{x}_k)] + \mathbf{R} \end{pmatrix}$$

- Gaussian conditioning leads to Gaussian filter update step

$$\begin{aligned}
\boldsymbol{\mu}_k &= \mathrm{E}_k^- [\mathbf{h}(\mathbf{x}_k)] \\
\mathbf{S}_k &= \mathrm{Cov}_k^- [\mathbf{h}(\mathbf{x}_k)] + \mathbf{R} \\
\mathbf{C}_k &= \mathrm{E}_k^- \left[(\mathbf{x}_k - \mathbf{m}_k^-)(\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)^\mathsf{T} \right] \\
\mathbf{K}_k &= \mathbf{C}_k \, \mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k \, (\mathbf{y}_k - \boldsymbol{\mu}_k) \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k \, \mathbf{S}_k \, \mathbf{K}_k^\mathsf{T},
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Learning Outcomes

1. Statistical linearization
2. Statistical linear regression
3. Discussion and connections
4. Summary

Discussion and connections

- Statistical linear regression leads to the Gaussian filter – and to UKF, CKF, GHKF, etc.
- The corresponding smoother can be derived similarly.
- We have arbitrarily chosen to do the SLR w.r.t. priors $\mathrm{N}(\mathbf{m}_{k-1}, \mathbf{P}_{k-1})$ and $\mathrm{N}(\mathbf{m}_k^-, \mathbf{P}_k^-)$ –
  - We don't need to do that, we can use any $\mathbf{x} \sim p(\mathbf{x})$.
  - Leads to e.g. posterior-linearization filters and smoothers that Ángel will talk about next.
- Statistical linearization corresponds to putting $\mathbf{e} = 0$ in

$$\mathbf{y} = \mathbf{A}\,\mathbf{x} + \mathbf{b} + \mathbf{e}.$$

- We recover the statistically linearized filter.
- If we do the statistical linearization with prior means and take $\mathbf{P}_k \to 0$, $\mathbf{P}_k^- \to 0$ we obtain the EKF.

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Summary

- In statistical linearization (SL) we approximate $\mathbf{y} = \mathbf{g}(\mathbf{x})$ as $\mathbf{y} = \mathbf{A}\,(\mathbf{x} - \mathbf{m}) + \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are selected to minimize

$$\mathrm{MSE}(\mathbf{A}, \mathbf{b}) = \mathrm{E}\left[\|\mathbf{g}(|vx) - \mathbf{A}\,\mathbf{x} - \mathbf{b}\|^2 \right].$$

- In statistical linear regression (SLR) we use $\mathbf{y} = \mathbf{A}\,\mathbf{x} + \mathbf{b} + \mathbf{e}$, where $\mathbf{e}$ is a Gaussian pseudo-noise that makes the covariance exact.
- By forming a filter with SLR we get Gaussian filter, UKFs, and other related filters.
- By using SL we get the classical statistically linearized filter and also EKF as a special case.
- We can also do SL and SLR w.r.t. other than the prior – leading to posterior linearization filters (and smoothers).

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