EECS151 : Introduction to Digital Design and ICs

Lecture 6 – Combinational Logic

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September 14, 2021. Apple announces A15 Bionic processor
- 6 cores
- New GPU
- New neural engine
- …

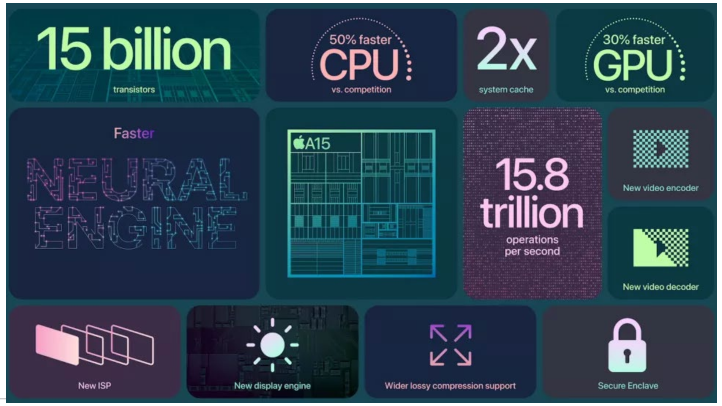


Image source: Apple

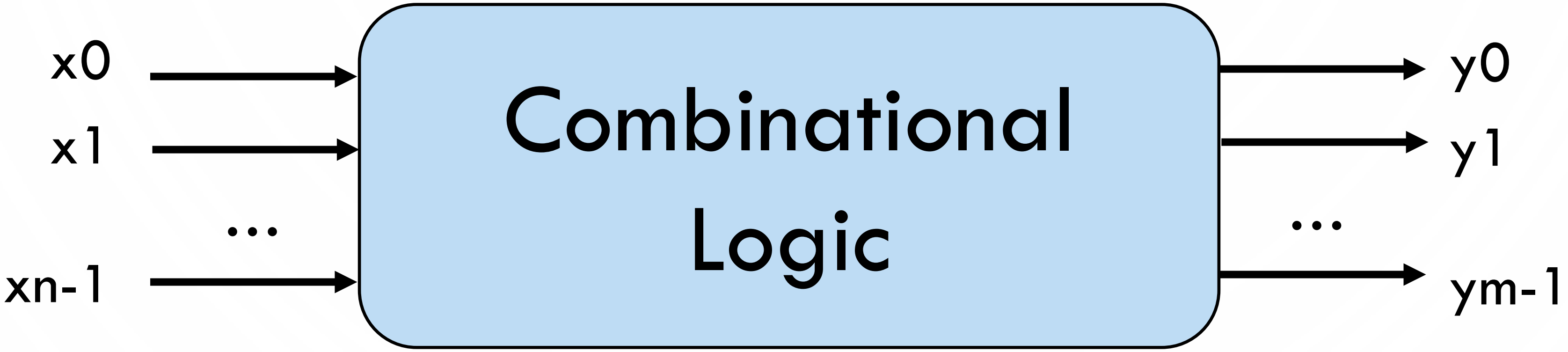
Review

- Sequential logic uses flip-flops and (sometimes) latches
- Flip-flops and latches are inferred in Verilog
  - Always blocks
- Practice is the best way to learn a new language…
- Blocking and non-blocking assignments

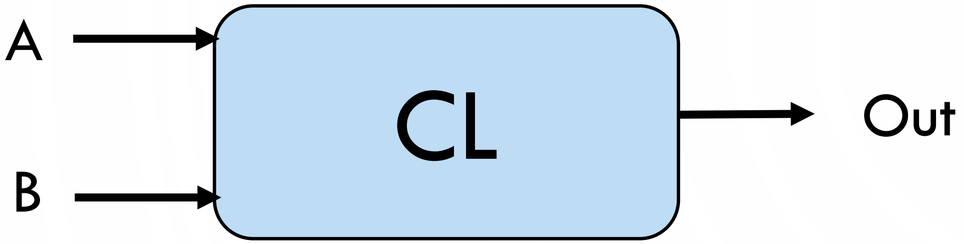
Combinational Logic

Combinational Logic

- The outputs depend *only* on the current values of the inputs.
  - Memoryless: compute the output values using the current inputs.



Combinational Logic Example



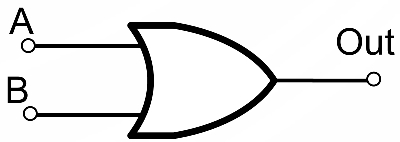
Boolean Equations:

```
Out = A OR B
    = A + B
```

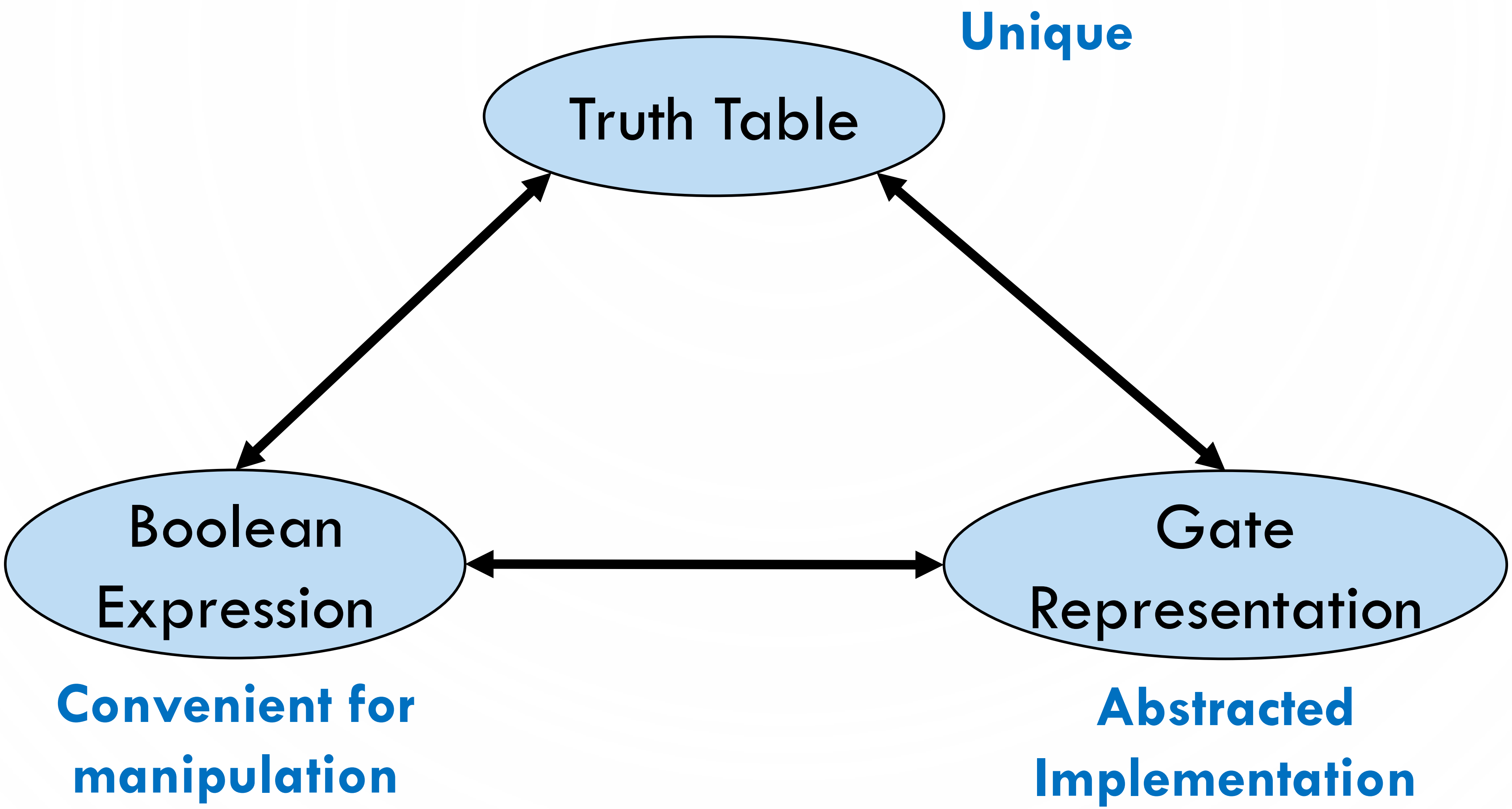
Truth Table Description:

A	B	Out
0	0	0
0	1	1
1	0	1
1	1	1

Gate Representations:



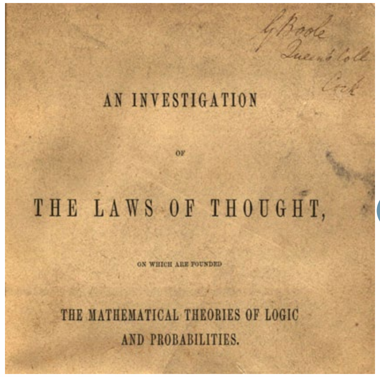
Relationship Among Representations



Boolean Algebra

Boolean Algebra Background

- Logic: The study of the principles of reasoning.
- The 19th Century Mathematician, George Boole, developed a math. system (algebra) involving logic, Boolean Algebra.
  - His variables took on TRUE, FALSE.
- Later Claude Shannon (father of information theory) showed (in his Master's thesis!) how to map Boolean Algebra to digital circuits.


AN INVESTIGATION
OF
THE LAWS OF THOUGHT,
ON WHICH ARE FOUNDED
THE MATHEMATICAL THEORIES OF LOGIC
AND PROBABILITIES.

georgeboole.com
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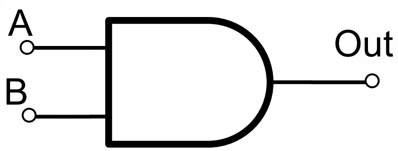
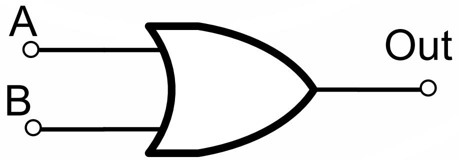
http://hdl.handle.net/1721.1/11173

Boolean Algebra Fundamentals

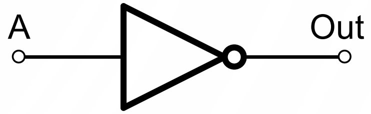
- Two elements {0, 1}
- Two binary operators: AND (·) OR (+)
- One unary operator: NOT (‾ , ′)



A	B	Out
0	0	0
0	1	0
1	0	0
1	1	1



A	B	Out
0	0	0
0	1	1
1	0	1
1	1	1



A	Out
0	1
1	0

Axioms of Boolean Algebra

Axiom	Dual	Name
$B = 0$ if $B \neq 1$	$B = 1$ if $B \neq 0$	Binary field
$\overline{0} = 1$	$\overline{1} = 0$	NOT
$0 \cdot 0 = 0$	$1 + 1 = 1$	AND/OR
$1 \cdot 1 = 1$	$0 + 0 = 0$	AND/OR
$0 \cdot 1 = 0 \cdot 1 = 0$	$1 + 0 = 0 + 1 = 1$	AND/OR

In mathematical logic, axioms are given

- Anything else can be derived from these axioms

Each axiom has a dual

Boolean Operations

- Given two variables (A, B), 16 logic functions

A	B	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_A	F_B	F_C	F_D	F_E	F_F
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Boolean Algebra Theorems

- Null elements, identities:
  - A+0=A, A•1=A
  - A+1=1, A•0=0
- Idempotency:
  - A+A=A, A•A=A
- Complements:
  - A+A´=1, A•A´=0
- Involution:
  - (A')' = A
- Commutativity:
  - A+B=B+A, A•B=B•A
- Associativity:
  - (A + B) + C= A + (B + C)= A + B + C
  - (A • B) • C = A • (B • C) = A • B • C
- Distributivity:
  - A • (B+C) = (A•B) + (A•C)
  - A + (B•C) = (A+B) • (A+C)
- Covering:
  - A • (A+B) = A, A + (A•B) = B
- Consensus
  - (A • B) + (A' • C) + (B • C) = (A • B) + (A' • C)

Literals are variables or their complements

Proving Distributive Law

- $A \cdot (B+C) = (A \cdot B) + (A \cdot C)$

A	B	C	(B+C)	A • (B+C)	(A•B)	(A•C)	(A•B) + (A•C)
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

Proving Distributive Law

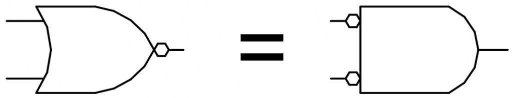
- A • (B+C) = (A•B) + (A•C)

A	B	C	(B+C)	A • (B+C)	(A•B)	(A•C)	(A•B) + (A•C)
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

DeMorgan's Law

- Theorem for complementing a complex function.

(A + B)' = A' B'



A	B	A'	B'	(A + B)'	A' B'
0	0				
0	1				
1	0				
1	1				

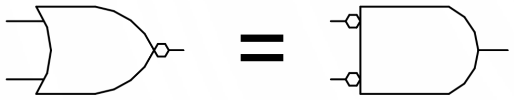
(A B)' = A' + B'

A	B	A'	A'	(A B)'	A' + B'
0	0				
0	1				
1	0				
1	1				

DeMorgan's Law

- Procedure for complementing a complex function.

$(A + B)' = A' B'$



A	B	A'	B'	(A + B)'	A' B'
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

$(A B)' = A' + B'$

A	B	A'	B'	(A B)'	A' + B'
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

Canonical Forms

- Two types:
  - Sum of Products (SOP)
  - Product of Sums (POS)
- Sum of Products
  - a.k.a Disjunctive normal form, minterm expansion
  - Minterm: a product (AND) involving all inputs
  - SOP: Summing minterms for which the output is True

Minterms	a	b	c	f	f'
a'b'c'	0	0	0	0	1
a'b'c'	0	0	1	0	1
a'b c'	0	1	0	0	1
a'b c	0	1	1	1	0
a b'c'	1	0	0	1	0
a b'c	1	0	1	1	0
a b c'	1	1	0	1	0
a b c	1	1	1	1	0

One product (and) term for each 1 in f:

f = a'bc + ab'c' + ab'c + abc' + abc

f' = a'b'c' + a'b'c + a'bc'

Sum of Products (cont.)

- Canonical Forms are usually not minimal:
- Example:

```
f   = a'bc + ab'c' + ab'c + abc' + abc   (xy' + xy = x)
    = a'bc + ab' + ab
    = a'bc + a
    = a + bc

f'  = a'b'c' + a'b'c + a'bc'
    = a'b' + a'bc'
    = a' ( b' + bc' )
    = a' ( b' + c' )
    = a'b' + a'c'
```

$$a + a'b = a + b$$

- Recall distributive theorem
  $a+bc = (a+b)(a+c)$

Canonical Forms

- Two types:
  - Sum of Products (SOP)
  - Product of Sums (POS)
- Product of Sums:
  - a.k.a. conjunctive normal form, maxterm expansion
  - Maxterm: a sum (OR) involving all inputs
  - POS: Product (AND) maxterms for which the output is FALSE
  - Can obtain POSs from applying DeMorgan's law to the SOPs of F (and vice versa)

Maxterms	a	b	c	f	f'
a+b+c	0	0	0	0	1
a+b+c'	0	0	1	0	1
a+b'+c	0	1	0	0	1
a+b'+c'	0	1	1	1	0
a'+b+c	1	0	0	1	0
a'+b+c'	1	0	1	1	0
a'+b'+c	1	1	0	1	0
a'+b'+c'	1	1	1	1	0

One sum (**or**) term for each **0** in **f**:

```
f  = (a+b+c)(a+b+c')(a+b'+c)
f' = (a+b'+c')(a'+b+c)(a'+b+c')
     (a'+b'+c)(a+b+c')
```

Quiz

- Derive the product-of-sums form of $\bar{Y}$ based on the truth table.

a) $\bar{Y} = (A + B)(A + \bar{B})$

b) $\bar{Y} = A\bar{B} + AB$

c) $\bar{Y} = \bar{A}\bar{B} + \bar{A}B$

A	B	Y	$\bar{Y}$
0	0	0	1
0	1	0	1
1	0	1	0
1	1	1	0

Boolean Simplification

Example: Full Adder (FA) Carry out

$$
\begin{aligned}
co &= a'bc + ab'c + abc' + abc \\
&= a'bc + ab'c + abc' + abc + abc \\
&= a'bc + abc + ab'c + abc' + abc \\
&= (a' + a)bc + ab'c + abc' + abc \\
&= (1)bc + ab'c + abc' + abc \\
&= bc + ab'c + abc' + abc + abc \\
&= bc + ab'c + abc + abc' + abc \\
&= bc + a(b' + b)c + abc' + abc \\
&= bc + a(1)c + abc' + abc \\
&= bc + ac + ab(c' + c) \\
&= bc + ac + ab(1) \\
&= bc + ac + ab
\end{aligned}
$$

c	b	c	s	co
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Why do Boolean simplification?

- Minimize number of gates in circuit
  - Gates take area
- Minimize amount of wiring in circuit
  - Wiring takes space and is difficult to route
  - Physical gates have limited number of inputs
- Minimize number of gate levels
  - Faster is better
- How to systematically simplify Boolean logics?
  - Use tools!

Practical methods for Boolean simplification

- Still based on Boolean algebra, but more systematic
- 2-level simplification -> multilevel
- Key tool: The Uniting Theorem

$$ab' + ab = a\ (b' + b) = a\ (1) = a$$

ab	f
00	0
01	0
10	1
11	1

$$f = ab' + ab = a(b'+b) = a$$

b values change within rows

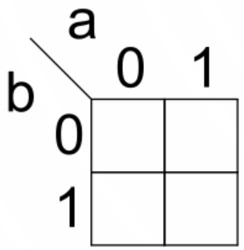
a values don't change

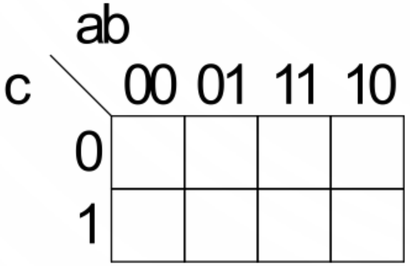
b is eliminated, **a** remains

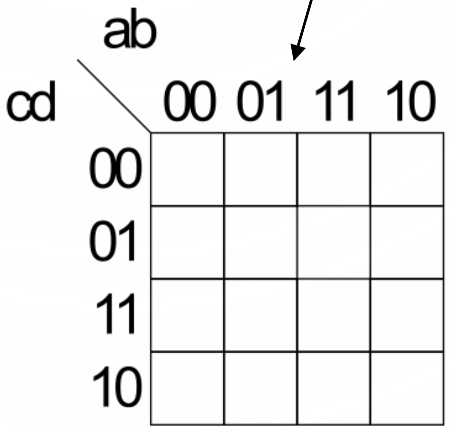
Karnaugh Map Method

- K-map is an alternative method of representing the truth table and to help visual the adjacencies.

Note: “Gray code” labeling.







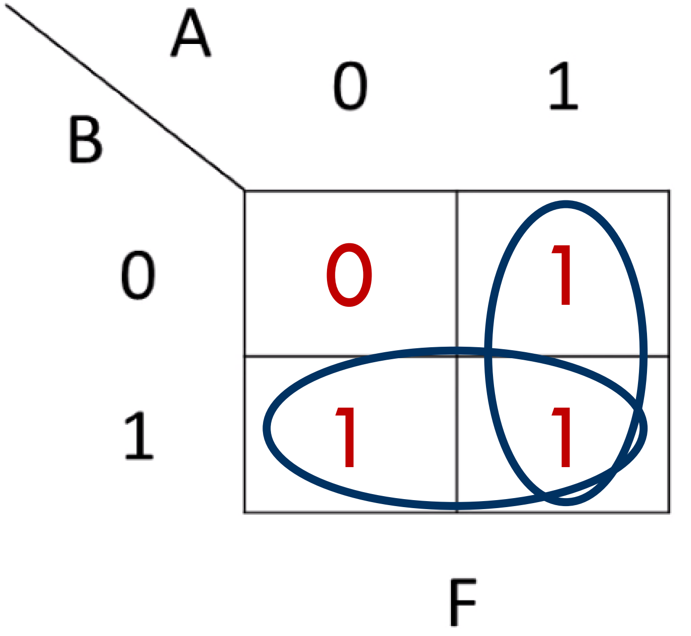
Karnaugh Map Method

- Adjacent groups of 1's represent product terms

OR

A	B	F
0	0	0
0	1	1
1	0	1
1	1	1

Karnaugh Map



$F = A + B$

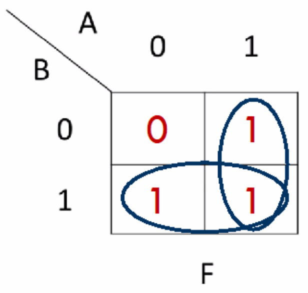
Karnaugh Map Method

1. Draw K-map of the appropriate number of variables.
2. Fill in map with function values from truth table.
3. Form groups of 1's.
   - ✓ Dimensions of groups must be even powers of two (1x1, 1x2, 1x4, …, 2x2, 2x4, …)
   - ✓ <u>Form as large as possible groups and as few groups as possible.</u>
   - ✓ Groups can overlap (this helps make larger groups)
   - ✓ Remember K-map is periodical in all dimensions (groups can cross over edges of map and continue on other side)
4. For each group write a product term.
   - The term includes the "constant" variables (use the uncomplemented variable for a constant 1 and complemented variable for constant 0)
5. Form Boolean expression as sum-of-products.

OR

A	B	F
0	0	0
0	1	1
1	0	1
1	1	1

Karnaugh Map



F = A + B

Karnaugh Map Method

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$F = A + B$

BC \ A	0	1
00	0	1
01	0	1
11	1	1
10	1	1

Product-of-Sums Version

1. Form groups of 0's instead of 1's.
2. For each group write a sum term.
   - the term includes the "constant" variables (use the uncomplemented variable for a constant 0 and complemented variable for constant 1)
3. Form Boolean expression as product-of-sums.

cd \ ab	00	01	11	10
00	1	0	0	1
01	0	1	0	0
11	1	1	1	1
10	1	1	1	1

$$f = (b' + c + d)(a' + c + d')(b + c + d')$$

Karnaugh Maps with Don’t Cares

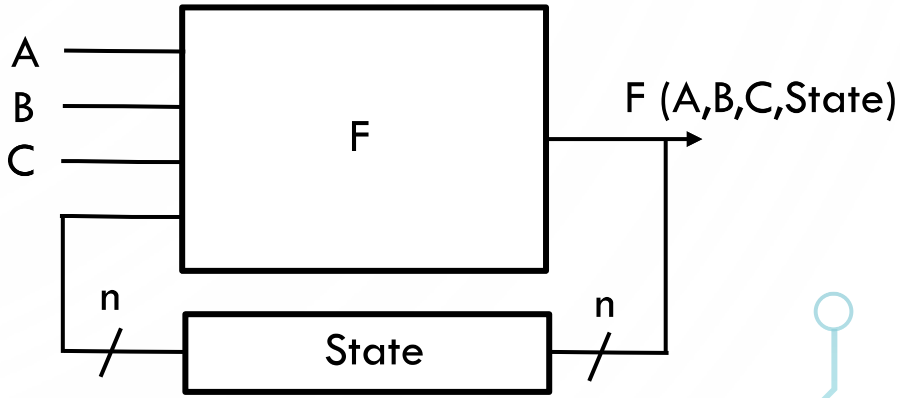
- Don’t cares (x’s) in the truth table can be either 0’s or 1’s

CD \ AB	00	01	11	10
00	1	1	X	1
01	1	0	X	1
11	1	1	X	X
10	1	0	X	X

Finite-State Machines

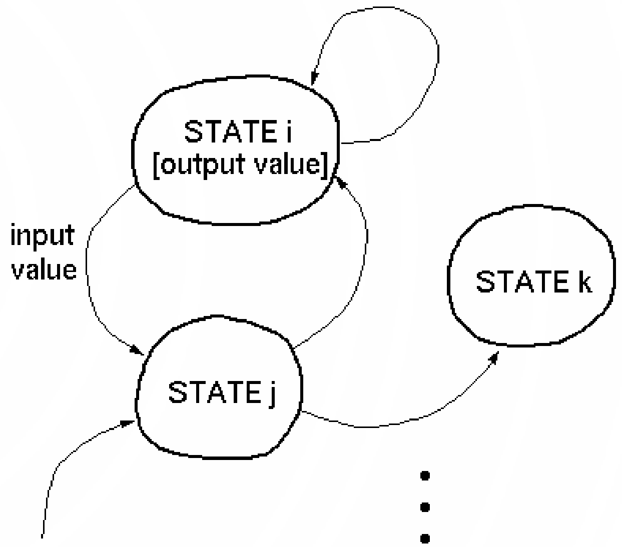
Sequential logic

- Combinational logic:
  - Memoryless: the outputs only dependent on the current inputs.
- Sequential logic:
  - Memory: the outputs depend on both current and previous values of the inputs.
    - Distill the prior inputs into a smaller amount of information, i.e., states.
  - State: the information about a circuit
    - Influences the circuit's future behavior
    - Stored in Flip-flops and Latches
  - Finite State Machines:
    - Useful representation for designing sequential circuits
    - As with all sequential circuits: output depends on present and past inputs
    - We will first learn how to design by hand then how to implement in Verilog.



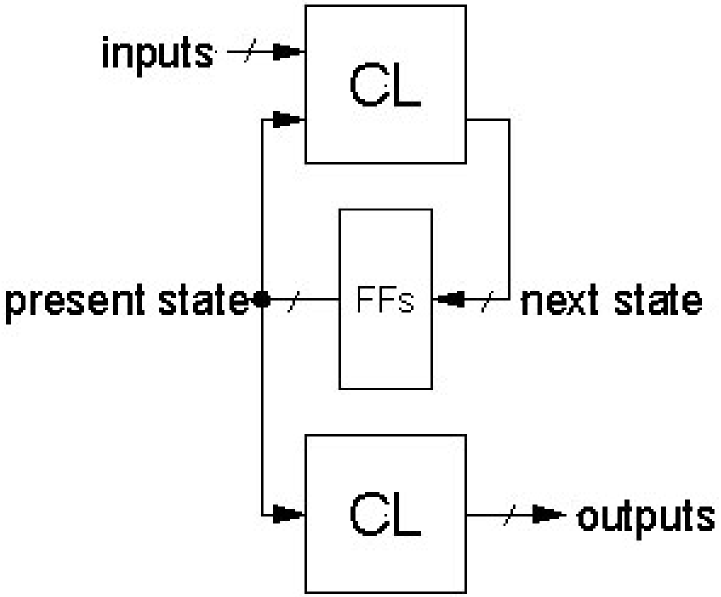
Finite State Machines

- A sequential circuit which has
  - External inputs
  - Externally visible outputs
  - Internal states



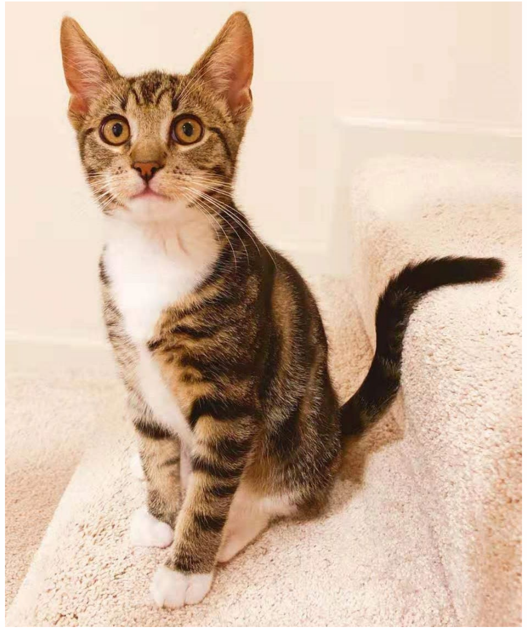
State Transition Diagram

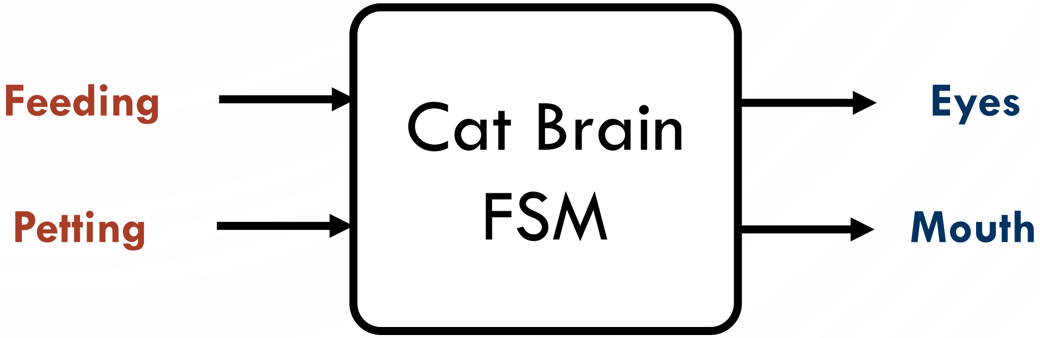
- Consists of:
  - State register
    - Stores current state
    - Loads previously calculated next state
    - # of states <= 2^(# of FFs)
  - Combinational logic
    - Computes the next state
    - Computes the outputs



FSM Example

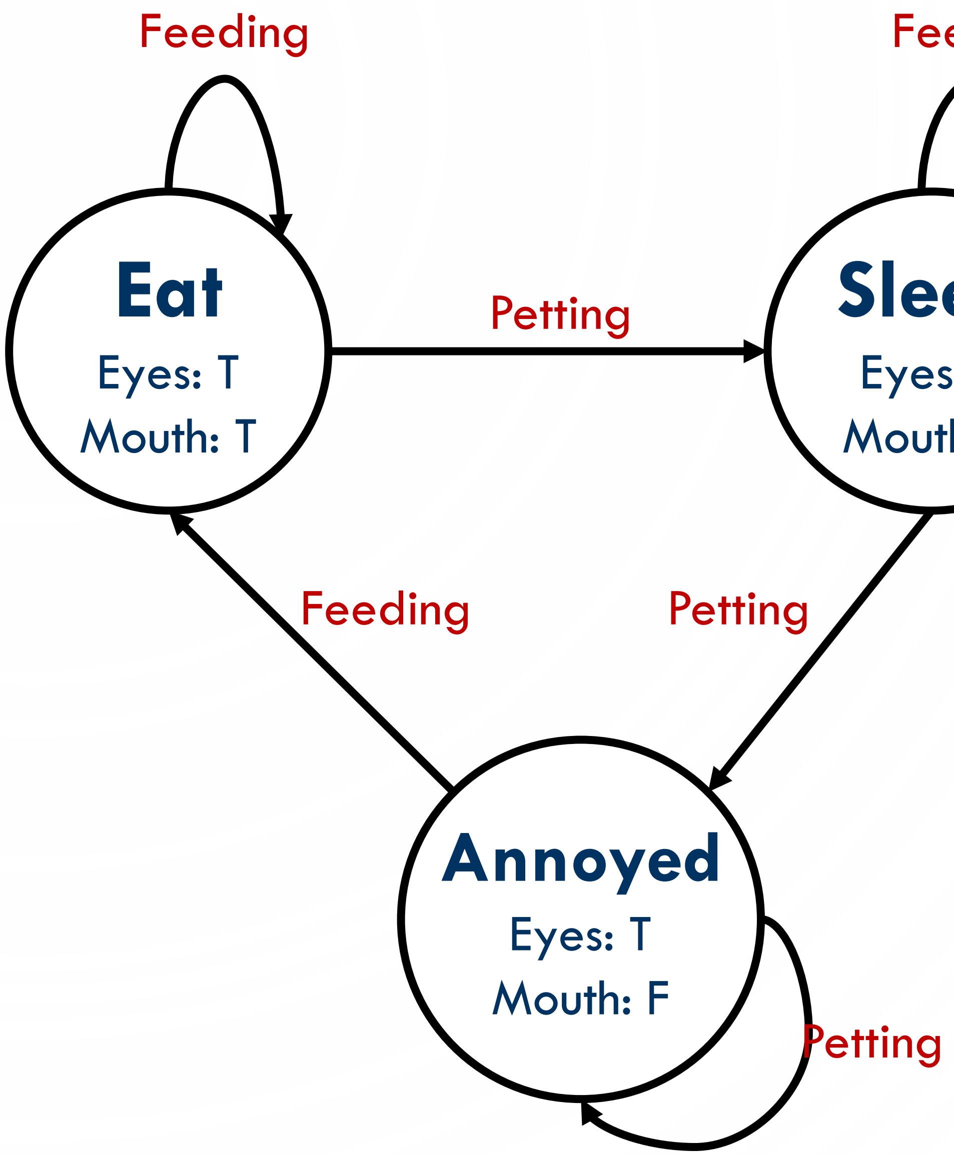
- Cat Brain (Simplified…)
  - Inputs:
    - Feeding
    - Petting
  - Outputs:
    - Eyes: open or close
    - Mouth: open or close
  - States:
    - Eating
    - Sleeping
    - Annoyed…





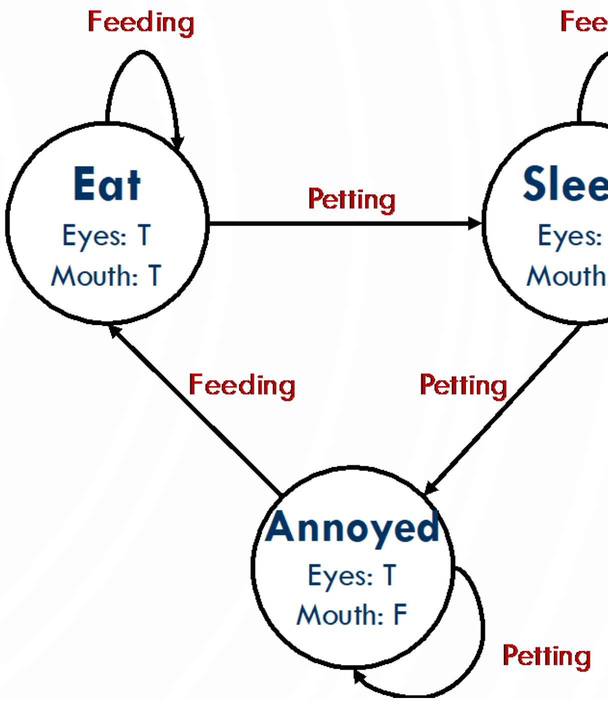
FSM State Transition Diagram

- States:
  - Circles
- Outputs:
  - Labeled in each state
  - Arcs
- Inputs:
  - Arcs



FSM Symbolic State Transition Table

Current State	Inputs	Next State
Eat	Feeding	Eat
Eat	Petting	Sleep
Sleep	Feeding	Sleep
Sleep	Petting	Annoyed
Annoyed	Feeding	Eat
Annoyed	Petting	Annoyed

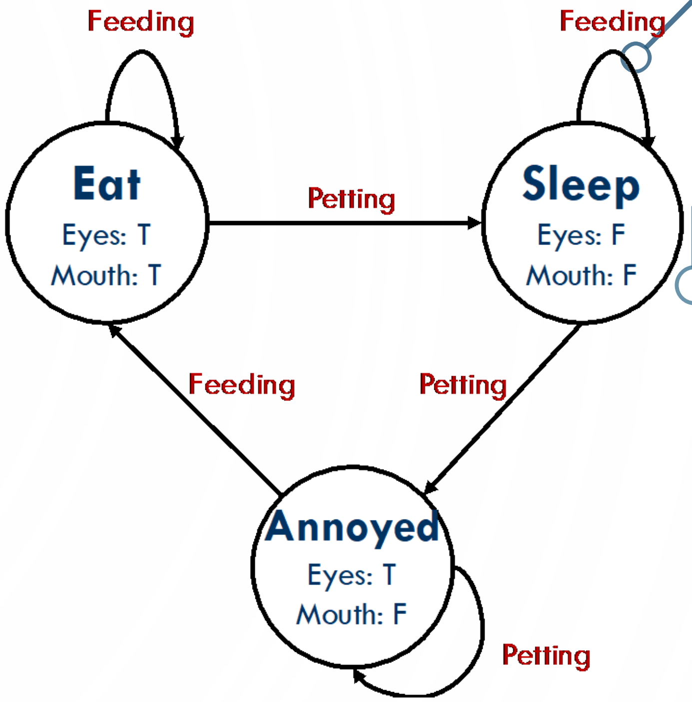


FSM Encoded State Transition Table

State	Encoding
Eat	00
Sleep	01
Annoyed	10

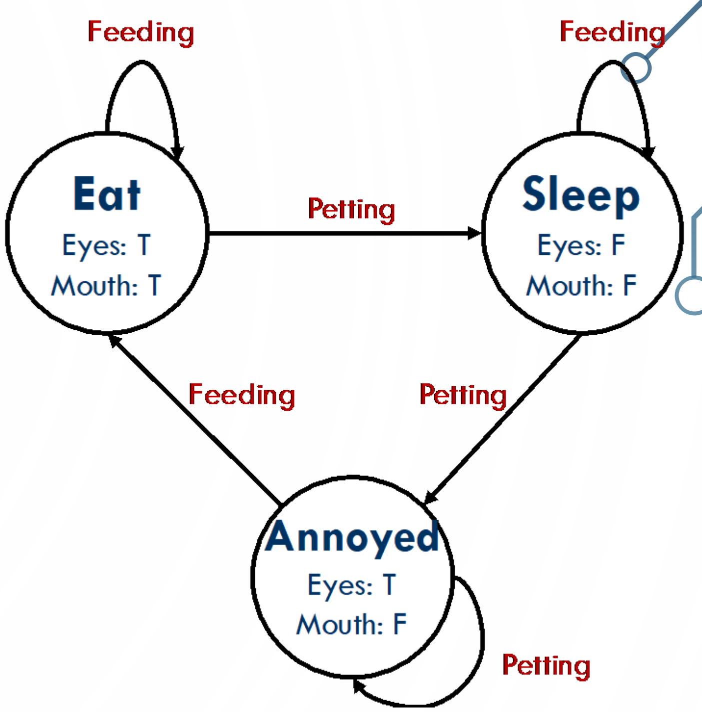
Current State		Input	Next State	
S1	S0	X	S1'	S0'
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	0
1	0	1	1	0

$$S0' = \overline{S1}\,\overline{S0}X + \overline{S1}S0\overline{X} = \overline{S1}\left(\overline{S0}X + S0\overline{X}\right) = \overline{S1}(S0 \oplus X)$$
$$S1' = \overline{S1}S0X + S1\overline{S0}X = (S1 \oplus S0)X$$



Current State	Inputs	Next State
Eat	Feeding	Eat
Eat	Petting	Sleep
Sleep	Feeding	Sleep
Sleep	Petting	Annoyed
Annoyed	Feeding	Eat
Annoyed	Petting	Annoyed

FSM Output Table



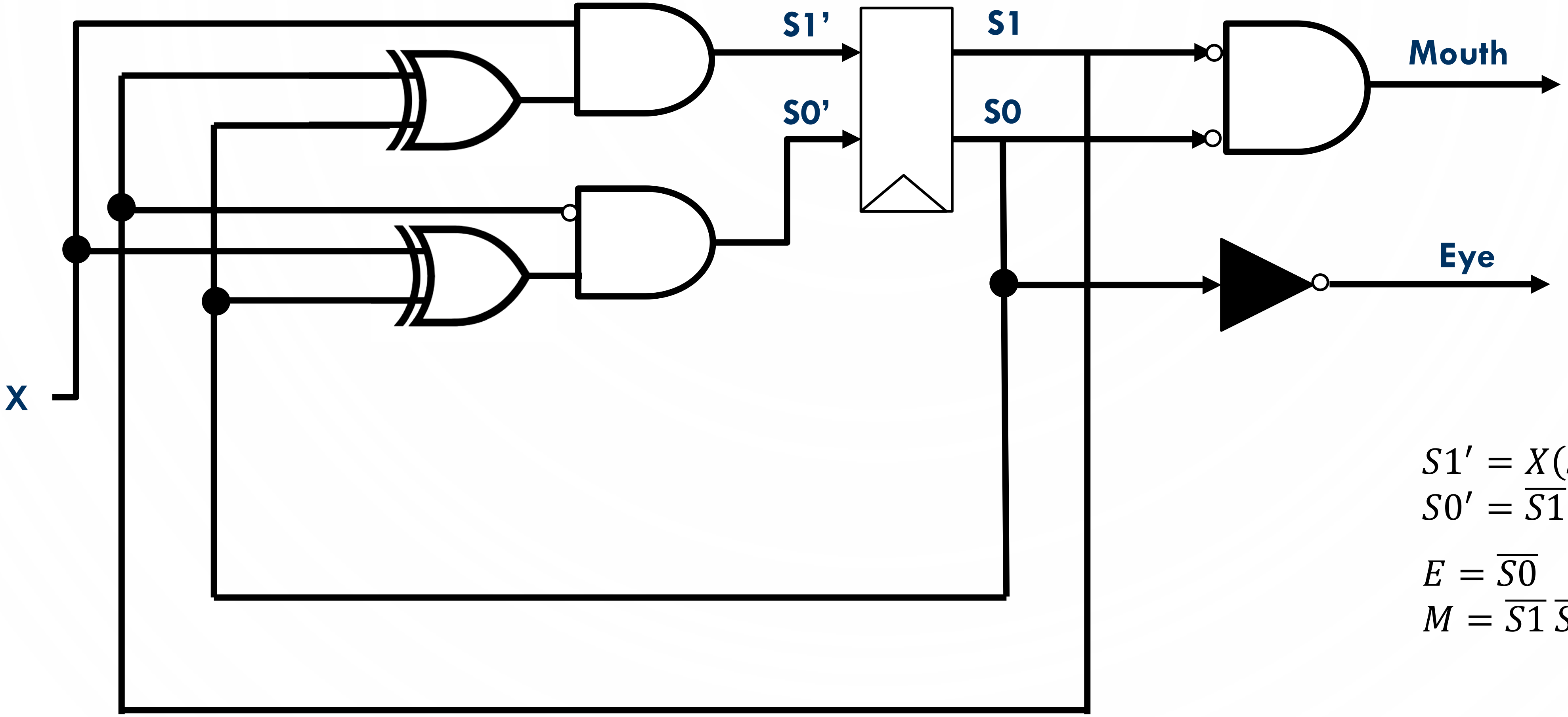
State	Encoding
Eat	00
Sleep	01
Annoyed	10

Current State		Outputs	
S1	**S0**	**E**	**M**
0	0	1	1
0	1	0	0
1	0	1	0

Outputs		Encoding
Eyes	Mouth	
Open	Open	11
Close	Close	00
Open	Close	10

$$E = \overline{S1}\,\overline{S0} + S1\overline{S0} = \overline{S0}$$
$$M = \overline{S1}\,\overline{S0}$$

FSM Gate Representation



$$S1' = X(S0 \oplus S1)$$
$$S0' = \overline{S1}(S0 \oplus X)$$

$$E = \overline{S0}$$
$$M = \overline{S1}\,\overline{S0}$$

Summary

- Combinational logic:
  - The outputs only depend on the current values of the inputs (memoryless)
  - The functional specification of a combinational circuit can be expressed as:
    - A truth table
    - A Boolean equation
- Boolean algebra
  - Deal with variables that are either True or False
  - Map naturally to hardware logic gates
  - Use theorems of Boolean algebra and Karnaugh maps to simplify equations
- Finite state machines: Common example of sequential logic
- Common job interview questions ☺