2102432 Linear Control Systems II — Semester 2022/2

Homework #2 — Assign: Thu January 26, 2023. — Due: Thu February 2, 2023.

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Collaborator: .... ID: .... Sec: ....

Instruction: Instruction: The goal of homework is to learn the course material and practise the problem solving. You are encouraged to collaborate on the homework, however, you must write up your homework yourself. You should not copy somebody else's homework or lend your homework to others. If you choose to collaborate, you should be able to recreate all of the steps involved in solving a problem yourself, and should do so in your writeup. Please list the names of your collaborators on the first page of homework. If any student breaks the homework regulation in the first time, his/her score for that homework will be zero. When breaking the regulation more than once, all homeworks and project will be zero.
Submit your work in one pdf file with title CUEE432-YourIntital-HW1.pdf via CourseVille by 4 pm. of the due date. Your initial is the first letter of your first name and the first letter of your last name. For example, Mr. Somchai Jaidee has initial as SJ.

1. Consider

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$$

   (a) Find the orthonormal vectors q_1 and q_2 which span is equal to the range of A.

   (b) Determine the least squares solution of $Ax = y$ using QR factorization where $y = \begin{bmatrix} 1 & 2 & 7 \end{bmatrix}^T$

   (c) Compute the norm of the error: $\|Ax_{\text{ls}} - y\|$.

2. Suppose A is skinny and full-rank. Let x_{ls} be the least-squares approximate solution of $Ax = y$, and let $y_{\text{ls}} = Ax_{\text{ls}}$. Show that the residual vector $r = y - y_{\text{ls}}$ satisfies $\|r\|^2 = \|y\|^2 - \|y_{\text{ls}}\|^2$. Also, give a brief geometric interpretation of this equality (just a few sentences and a conceptual drawing).

3. Define $p(t)$ and $\dot{p}(t)$ be the position and the velocity of the mass. Let the initial condition $p(0) = 0, \dot{p}(0) = 0$. $f(t)$ is the external force applied to the unit mass where $f(t) = x_i$ for $i - 1 < t \leq i$ and $i = 1, \ldots, 12$ Determine x which minimizes

$$\int_{t=0}^{12} f(t)^2 dt$$

   and meet the specifications: $p(12) = 1$, $\dot{p}(12) = 0$, and $p(6) = 0$. In this problem, we suggest to use MATLAB to find the solution. Plot the optimal force f and the response p and $\dot{p}$.

4. Define the function $f(x, \mu)$ as follows.

$$f(x,\mu) = \left\| \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\|^2 + \mu\|x\|^2$$

   Find x which minimizes $f(x, \mu)$ when $\mu = 1$.

1. Consider

$$A = \begin{array}{c} \begin{matrix} a_1 & a_2 \end{matrix} \\ \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \end{array}$$

(a) Find the orthonormal vectors q_1 and q_2 which span is equal to the range of A.

(b) Determine the least squares solution of $Ax = y$ using QR factorization where $y = \begin{bmatrix} 1 & 2 & 7 \end{bmatrix}^T$

(c) Compute the norm of the error: $\|Ax_{\text{ls}} - y\|$.

(a) หา q_1 จาก Gram - Schmidt

$$\tilde{q}_1 = a_1 = [1 \ 2 \ -2]^T \quad , \|q_1\| = \sqrt{1^2+2^2+(-2)^2} = \sqrt{9} = 3$$

ได้ $q_1 = \dfrac{\tilde{q}_1}{\|q_1\|} = \dfrac{1}{3}[1 \ 2 \ -2]^T = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$ //

$$\tilde{q}_2 = a_2 - (q_1^T a_2) q_1$$

$$\tilde{q}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} // \quad , \|q_2\| = \sqrt{2^2+1^2+2^2} = 3$$

$$q_2 = \frac{\tilde{q}_2}{\|q_2\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} //$$

1-2-8

(b) ได้ $Q = [q_1 \ q_2] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix}$ จาก $A = QR$ โดยที่ Q มีคุณสมบัติ $Q^T Q = I_n$

1 4 4

$\therefore \quad Q^T A = R$

ได้ R คือ $\dfrac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 9 & -9 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$ //

$$R^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$$

ได้ว่า $x_{ls} = R^{-1} Q y$

$$= \frac{1}{3} \cdot \frac{1}{9} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$x_{ls} = \frac{1}{27}\begin{bmatrix} 9 & 9 & 0 \\ 6 & 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 27 \\ 54 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} // \text{Ans}$$

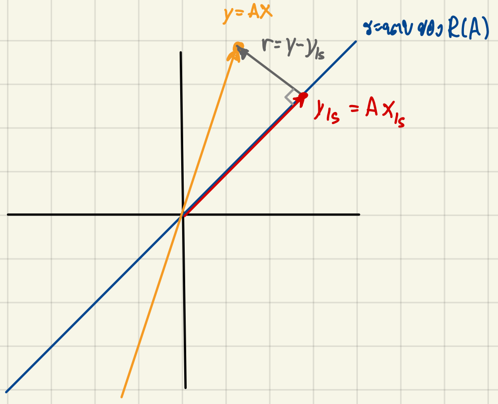
(c) จาก (B) และ ข้อกำหนดที่ให้ทราบค่า A, x_{ls}

$Ax_{ls} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$ ได้ $A x_{ls} - y = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

ได้ norm of error คือ $\|Ax_{ls} - y\| = \left\| \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \right\| = \sqrt{2^2+1^2+(-2)^2} = 3$ //

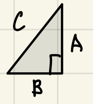
2. Suppose A is skinny and full-rank. Let x_{ls} be the least-squares approximate solution of $Ax = y$, and let $y_{ls} = Ax_{ls}$. Show that the residual vector $r = y - y_{ls}$ satisfies $\|r\|^2 = \|y\|^2 - \|y_{ls}\|^2$. Also, give a brief geometric interpretation of this equality (just a few sentences and a conceptual drawing).

วาดระนาบออกมาได้ว่า



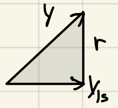
จากทฤษฎี เราจะทราบว่า ค่า y_{ls} คือ ค่าของการประมาณ y จาก x_{ls} และห่างจาก y น้อยที่สุด และต้องอยู่ในระนาบของ $R(A)$

นอกจากนี้ ค่า r คือระยะห่างของ y กับ y_s และต้องตั้งฉากกับ y_s ไปหา y
ซึ่งจะเห็นได้จากรูปว่า เราสามารถนำเอาทฤษฎีบทของพีทาโกรัส มาใช้ได้ว่า



$c^2 = A^2 + B^2$ (เมื่อ c, A, B เป็น scalar)

ได้เป็น



$$\|y\|^2 = \|r\|^2 + \|y_{ls}\|^2$$

จะได้ว่า $\|r\|^2 = \|y\|^2 - \|y_{ls}\|^2$

Ans

3. Define $p(t)$ and $\dot{p}(t)$ be the position and the velocity of the mass. Let the initial condition $p(0) = 0$, $\dot{p}(0) = 0$. $f(t)$ is the external force applied to the unit mass where $f(t) = x_i$ for $i-1 < t \le i$ and $i = 1, \ldots, 12$ Determine x which minimizes

$m=1$

$$\int_{t=0}^{12} f(t)^2 dt$$

and meet the specifications: $p(12) = 1$, $\dot{p}(12) = 0$, and $p(6) = 0$. In this problem, we suggest to use MATLAB to find the solution. Plot the optimal force f and the response p and $\dot{p}$.

จากโจทย์ ได้ condition ว่า $p(t)$ = position , $f(t) = x_i$

$\dot{p}(t)$ = velocity , $m = 1$ kg

$\ddot{p}(t)$ = accelerate

จากสูตร $F = ma$ ได้ว่า $f(t) = ma$ เมื่อ $i = 1, 2, 3, \ldots, 12$

ที่ $0 \le t \le 1$ $(i=1)$

$$f(t) = m\ddot{p}(t)$$

$$x_1 = (1)\frac{d^2p(t)}{dt^2}$$

$$\dot{p}(t) = \int_0^t x_1\, dt = \int_0^t x_1 dt = x_1 t$$ จาก initial condition ได้ว่า $\dot{p}(0) = 0$

$$\boxed{\therefore \dot{p}(1) = x_1}$$

$$p(t) = \int_0^t \dot{p}(t)dt = \int_0^t x_1(t)dt = \frac{x_1 t^2}{2}$$ จาก initial condition ได้ว่า $p(0) = 0$

ได้ $\boxed{p(1) = \frac{x_1}{2}}$

ที่ $i = 2$ ได้ $1 \le t \le 2$ $f(t) = ma$ $f(t) = ma = m x_2$

$$f(t) = ma$$

$$\ddot{p}(t)\ x_2 = \ddot{p}(t)$$

$$\therefore \int_{\dot{p}(0)} \dot{p}(t) = \int_1^t x_2(t)\, dt$$

$$\dot{p}(t) - \dot{p}(1) = x_2(t-1)$$

ได้ $\dot{p}(2) = x_2 + x_1$

$$p(t) = \int_1^t \dot{p}(t)dt = \int_1^t x_2(t-1)dt + p(1) = \left.\frac{x_2(t-1)^2}{2}\right|_1^t + \frac{x_1}{2} = \frac{x_2(t-1)^2}{2} + \frac{x_1}{2}$$

ได้ $p(2) = \frac{3}{2}x_2 + \frac{x_1}{2}$

จะเห็นความสัมพันธ์ของ p กับ x คือ

$$\begin{bmatrix} p(1) \\ \dot{p}(1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} p(2) \\ \dot{p}(2) \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} p(3) \\ \dot{p}(3) \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\vdots$

$$\begin{bmatrix} p(12) \\ \dot{p}(12) \end{bmatrix} = \begin{bmatrix} \frac{23}{2} & \frac{21}{2} & \cdots & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

หา minimize จาก $\int_0^{12} f(t)^2 dt = x_1^2 + x_2^2 + \ldots + x_{12}^2$

ได้ว่า least norm solution คือ $x_{ln} = A^T(AA^T)^{-1}y$

เมื่อ $A = \begin{bmatrix} \frac{23}{2} & \frac{21}{2} & \cdots & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & \cdots & 1 & 1 \\ \frac{11}{2} & \frac{9}{2} & \cdots & \frac{1}{2} & 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} p(12) \\ \dot{p}(12) \\ p(6) \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

ใช้ matlab ได้คำตอบคือ

```
format rational
A=[23/2 21/2 19/2 17/2 15/2 13/2 11/2 9/2 7/2 5/2 3/2 1/2
   1   1      1   1   1   1    1    1    1  1   1   1
   11/2 9/2 7/2   5/2 3/2 1/2   0    0    0   0  0   0]
y=[1
   0
   0]
xln= (A')*inv(A*A')*y
```

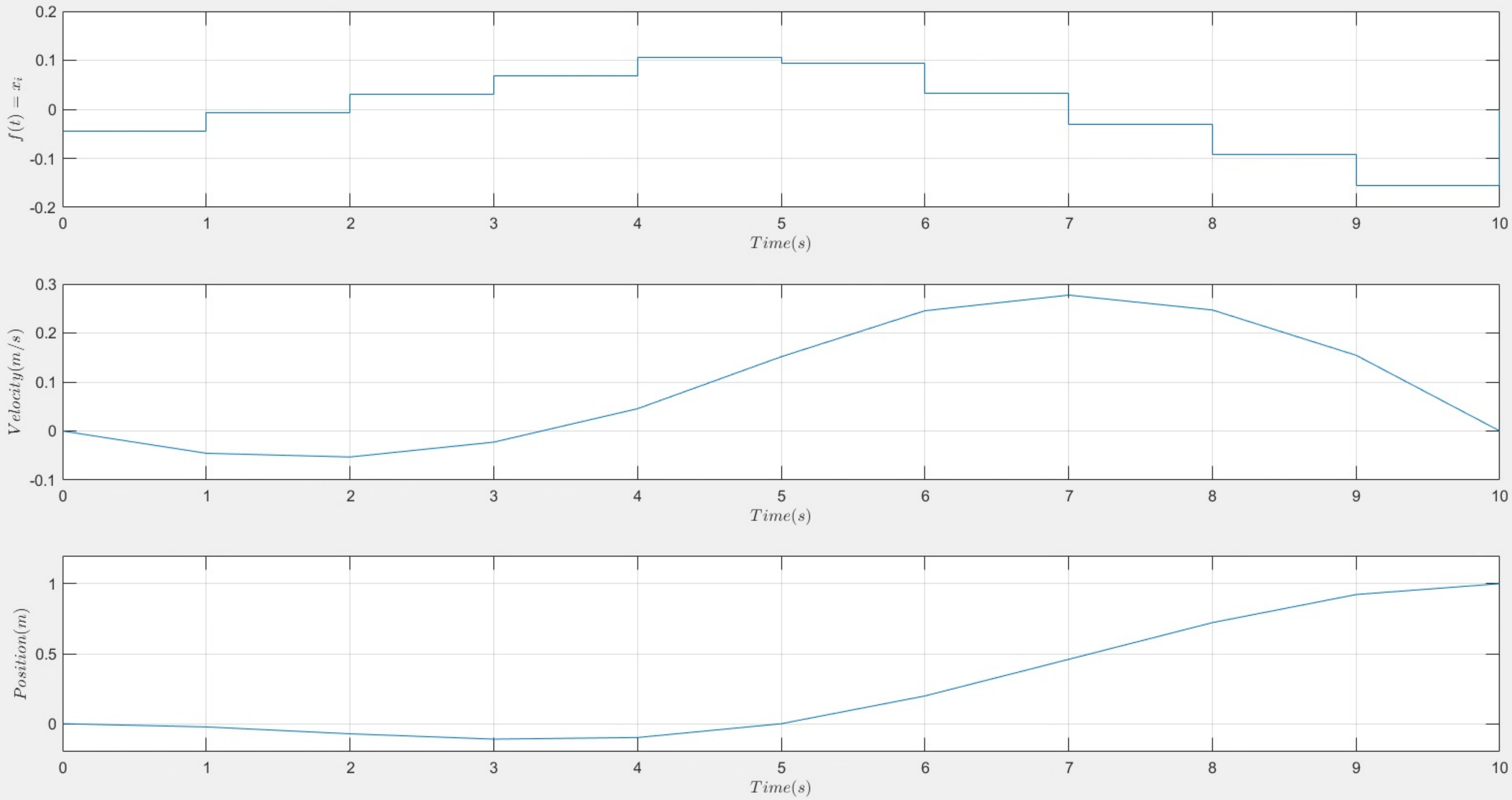
```
A =
     23/2          21/2          19/2
     1             1             1
     11/2          9/2           7/2

y =
     1
     0
     0

xln =
     -3/91
    -57/5005
     51/5005
    159/5005
    267/5005
     75/1001
     68/1001
    162/5005
    -16/5005
   -194/5005
   -372/5005
    -10/91
```

ได้กราฟ ต่างๆคือ

1) Force

2) Velocity

3) Position



4. Define the function $f(x,\mu)$ as follows.

$$f(x,\mu)=\left\|\underbrace{\begin{bmatrix}1&0\\-1&1\\1&0\end{bmatrix}}_{A}x-\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{y}\right\|^2+\mu\|x\|^2$$

Find x which minimizes $f(x,\mu)$ when $\mu=1$.

ได้ว่า $A=\begin{bmatrix}1&0\\-1&1\\1&0\end{bmatrix}$, $y=\begin{bmatrix}0\\0\\1\end{bmatrix}$

โดย norm x^2 มีสมบัติ คือ $||x||^2=x^Tx$

ได้ว่า $f(x,\mu)=||Ax-y||^2+\mu||x||^2$

$$=(Ax-y)^T(Ax-y)+\mu x^Tx$$

$$=(x^TA^T-y^T)(Ax-y)+\mu x^Tx$$

$$=x^TA^TAx\underbrace{-x^T}_{\alpha}\underbrace{A^Ty}_{\beta}-\underbrace{y^T}_{\beta}\underbrace{Ax}_{\alpha}+y^Ty+\mu x^Tx$$

$\boxed{\text{โดยที่ } \alpha\beta=\beta^T\alpha^T}$

$$=x^TA^TAx-2x^TA^Ty+y^Ty+\mu x^Tx$$

$$f(x,\mu)=x^T(A^TAx+\mu x)-2x^TA^Ty+y^Ty$$

หา x ที่ค่าน้อยที่สุดจาก $\frac{\partial f(x,\mu)}{\partial x}$ ได้ว่า

$$\frac{\partial f(x,\mu)}{\partial x}=2x^T(A^TA+\mu I)-2A^Ty=0$$

ได้ $A^Ty=x^T(A^TA+\mu I)$

$$x^T=A^Ty(A^TA+\mu I)^{-1}$$

$\boxed{\text{ต่อด้านข้าง}}$

จาก $x^T=A^Ty(A^TA+\mu I)^{-1}$ และ $\mu=1$ ได้

$$=\begin{bmatrix}1&-1&1\\0&1&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\left(\begin{bmatrix}1&-1&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0\\-1&1\\1&0\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)^{-1}$$

$$=\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}3&-1\\-1&1\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)^{-1}$$

$$=\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}4&-1\\-1&2\end{bmatrix}\right)^{-1}$$

$$=\frac{1}{7}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}2&1\\1&4\end{bmatrix}$$

$$x^T=\frac{1}{7}\begin{bmatrix}2&1\end{bmatrix}$$

$$\therefore\ x=\begin{bmatrix}\frac{2}{7}\\\frac{1}{7}\end{bmatrix}$$

// Ans