TFApy - Time frequency analysis tools for biological time series

only working title so far..

June 7, 2019

1 Introduction

comes later...

2 Basic Wavelet Theory

Wavelet analysis itself can be a subject of pure mathematics, readers seeking a more formal introduction may find it here: Daubechies [1990], Mallat [1999]

The aim of this section is to lay down the basic principles of Wavelet analysis, with a clear focus on application.

The historically oldest way of doing frequency analysis of signals is the well known and ubiquitously used Fourier analysis. It's working principle is the decomposition of a signal into its harmonic components. A harmonic component is a Sine or Cosine with constant frequency. Mathematically the Fourier transform can be expressed as:

$$\mathcal{F}[s](f) = \int_{-\infty}^{\infty} s(t)\; e^{-2\pi i f t} dt \tag{1}$$

$$= \int_{-\infty}^{\infty} s(t)\;\; [cos(\omega t) + i\; sin(\omega t)]\, dt \tag{2}$$

Here we used the Euler identity to express the complex Exponential as the sum of Cosine and Sine and $\omega = 2\pi f$. The result is the Fourier transformed signal $\mathcal{F}[s] = \hat{s}(f)$ which is a function of the frequency f alone. This complex valued function $\hat{s}(f)$ has no direct physical meaning. It's absolute square however gives a real valued function, often denoted by the *power spectral density* or just (Fourier-)spectrum of the signal s:

$$P_{\mathcal{F}}(f) = |\hat{s}(f)|^2 \tag{3}$$

It describes the contribution of each harmonic component with fequency f to the signal.

The Fourier transform translates the signal from the *time domain* into the *frequency domain*: $\mathcal{F} : s(t) \rightarrow \hat{s}(f)$. As a corollary, all time-dependent information of the signal is lost in the frequency domain (see Figure **??**a). Therefore Fourier analysis is best suited for *stationary* signals, meaning here no varying frequencies over time. This is a situation often at least approximately found in fields like engineering (Smith et al. [1997]) or spectroscopy, but is rather rare in Biology. Fourier analysis nowadays comes in many forms an flavours, the following methods are all based on its concepts: Periodogramm, Welsh's method and Lomb-Scargle method.

Mathematically, decompositions of the form of equation 1 work by choosing a specific set of *basis functions*. For the Fourier transform, these basis functions are the Sines and Cosines which have no *localization in time* but are sharply *localized in frequency*. Each harmonic component carries exactly one frequency effective everywhere in time. The idea to reach an optimal compromise between time and frequency localization goes back to Gabor (Gabor [1946]). He introduced Gaussian modulated harmonic components:

$$\Psi(t) = \pi^{-1/4}\ e^{-t^2/2}\ e^{i\omega_0 t} \tag{4}$$

$$= \pi^{-1/4}\ e^{-t^2/2}\ [cos(\omega_0 t) + i\ sin(\omega_0 t)] \tag{5}$$

This function is also known as *Morlet Wavelet*. The basis function for time-frequency analysis are then generated by *scaling* and *translation*:

$$\Psi_{s,\tau}(t) = s^{-1/2}\ \Psi\left(\frac{t-\tau}{s}\right) \tag{6}$$

Varying the time localization τ slides the Wavelet left and right on the time axis. The scale s changes the *center frequency* of the wavelet according to $\omega_c = \omega_0/s$. Higher scales therefore generate Wavelets with lower center frequency. The Gaussian envelope suppresses the harmonic component with frequency ω_c farer away from τ. This frequency ω_c is conventionally taken as the Fourier equivalent (or pseudo-) frequency of a Morlet Wavelet with scale s. It's noteworthy to state that Wavelets in general are not as sharply localized in frequency as their harmonic counterparts, it's a tradeoff necessary for the gain in time localization.

Torrence and Compo [1998]

3 Optimal Filtering - Do's and Dont's

4 Readout - Along the Ridge

5 Application UI and API

References

Daubechies, I. (1990). The wavelet transform, time-frequency localization and signal analysis. *IEEE transactions on information theory*, 36(5):961–1005.

Gabor, D. (1946). Theory of communication. part 1: The analysis of information. *Journal of the Institution of Electrical Engineers-Part III: Radio and Communication Engineering*, 93(26):429–441.

Mallat, S. (1999). *A wavelet tour of signal processing*. Elsevier.

Smith, S. W. et al. (1997). The scientist and engineer's guide to digital signal processing.

Torrence, C. and Compo, G. P. (1998). A practical guide to wavelet analysis. *Bulletin of the American Meteorological society*, 79(1):61–78.