

# TFAPy - Time frequency analysis tools for biological time series

only working title so far..

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## 1 Introduction

comes later...

## 2 Basic Wavelet Theory

In this section we aim to lay down the basic principles of wavelet analysis as employed in our signal analysis tool. We mainly follow the authors of ?, albeit the more mathematical subtleties are moved to the Appendix to allow for a wider audience. Readers seeking a more formal introduction may find it here: ??

### Fourier Analysis

The historically earliest way of performing frequency analysis of periodic signals is the well known and ubiquitously used Fourier analysis. It's working principle is the decomposition of a signal  $f(t)$  into its harmonic components. A harmonic component is a Sine or Cosine with constant frequency. Mathematically the Fourier transform can be expressed as:

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad (1)$$

$$= \int_{-\infty}^{\infty} f(t) [\cos(\omega t) + i \sin(\omega t)] dt \quad (2)$$

Here we used the Euler identity to express the complex Exponential as the sum of Cosine and Sine. The result is the Fourier transformed signal  $\mathcal{F}[f] = \hat{f}(\omega)$  which is a function of the frequency  $\omega$  alone. This complex valued function  $\hat{f}(\omega)$  has no direct physical meaning. It's absolute square however gives a real valued function, often denoted by the *power spectral density* or just (Fourier-)spectrum of the signal  $f$ :

$$P_{\mathcal{F}}(\omega) = |\hat{f}(\omega)|^2 \quad (3)$$

It describes the contribution of each harmonic component with frequency  $\omega$  to the signal (Figure 1a and b).

The Fourier transform translates the signal from the *time domain* into the *frequency domain*:  $\mathcal{F} : f(t) \rightarrow \hat{f}(\omega)$ . As a corollary, all time-dependent information of the signal is lost in the frequency domain (see Figure 1d). Therefore Fourier analysis is best suited for stationary or time-invariant signals, meaning no varying frequencies over time. This is a situation often at least approximately found in fields like engineering (?) or spectroscopy, but is rather rare in Biology. Fourier analysis nowadays comes in many forms and flavours, the following methods (among others) are all based on its concepts: Periodogram, Welch's method and Lomb-Scargle method.

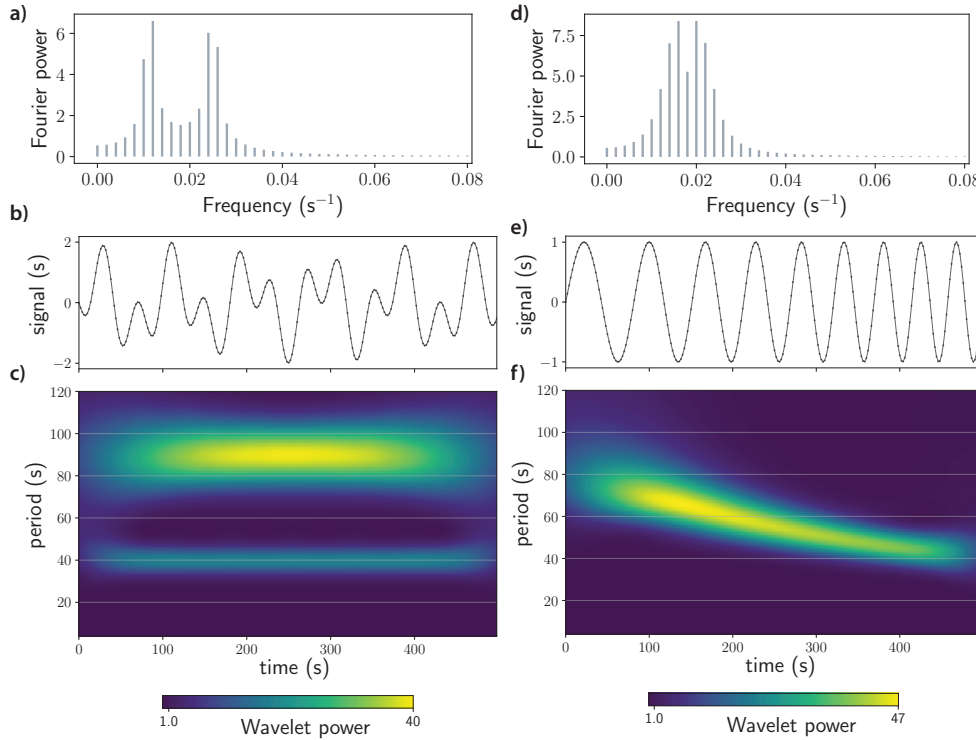


Figure 1: a) Fourier power spectrum showing two peaks corresponding to the two harmonic components present in the signal. b) Synthetic signal is harmonic composition with two periods  $T_1 = 40\text{s}$  ( $f_1 = 0.025\text{Hz}$ ) and  $T_2 = 90\text{s}$  ( $f_2 = 0.011\text{Hz}$ ). c) Wavelet power spectrum, both harmonic components are prominently featured. d) Fourier power spectrum shows a range of frequencies but no information about their time dependence. e) Synthetic signal sweeps linearly from  $T_2$  to  $T_1$ . f) Wavelet power spectrum shows time-resolved (instantaneous) periods.

## Wavelets and Wavelet transform

Mathematically, decompositions of the form of equation 1 work by choosing a specific set of *basis functions*. For the Fourier transform, these basis functions

are the Sines and Cosines which have no *localization in time* but are sharply *localized in frequency*. Each harmonic component carries exactly one frequency effective everywhere in time. The idea to reach an optimal compromise between time and frequency localization goes back to Gabor (?). He introduced Gaussian modulated harmonic components:

$$\Psi(t) = \pi^{-1/4} e^{-t^2/2} e^{i\omega_0 t} \quad (4)$$

$$= \pi^{-1/4} e^{-t^2/2} [\cos(\omega_0 t) + i \sin(\omega_0 t)] \quad (5)$$

This function is also known as *Morlet wavelet*. The basis function for time-frequency analysis are then generated from the *mother wavelet* by *scaling* and *translation*:

$$\Psi_{s,\tau}(t) = s^{-1/2} \Psi\left(\frac{t-\tau}{s}\right) \quad (6)$$

Varying the time localization  $\tau$  slides the wavelet left and right on the time axis. The scale  $s$  changes the *center frequency* of the Morlet wavelet according to  $\omega_c(s) \approx \omega_0/s$  (see also Appendix equation ??). Higher scales therefore generate wavelets with lower center frequency. The Gaussian envelope suppresses the harmonic component with frequency  $\omega_c$  farer away from  $\tau$ , therewith localizes the wavelet in time. This frequency  $\omega_c(s)$  is conventionally taken as the Fourier equivalent (or pseudo-) frequency of a Morlet wavelet with scale  $s$ . It's noteworthy to state that wavelets in general are not as sharply localized in frequency as their harmonic counterparts. It's a tradeoff imposed by the uncertainty principle to gain localization in time (?).

The wavelet transform of a signal  $f(t)$  is given by the following integral expression:

$$\mathcal{W}[f](\tau, s) = \int_{-\infty}^{\infty} f(t) s^{-1/2} \Psi^*\left(\frac{t-\tau}{s}\right) dt \quad (7)$$

For a fixed scale, this equation has the form of a complex convolution. One useful interpretation is to treat it as a cross-correlation between the signal and the wavelet of scale  $s$  (or center frequency  $\omega_c(s)$ ): the translation variable  $\tau$  slides the wavelet along the signal, and as the wavelet decays fastly away from  $\tau$  only the instantaneous correlation of the wavelet with center frequency  $\omega_c$  and the signal around  $\tau$  contributes to the integral. By using an array of wavelets with different frequencies, this allows to scan for multiple frequencies in the signal in a time-resolved manner (see Figure 1f).

The result of the transform:  $\mathcal{W} : f(t) \rightarrow \tilde{f}(\tau, \omega)$  is a complex valued function of two variables, frequency  $\omega$  and time localization  $\tau$ . Here and in the following we implicitly convert scales to frequency via the corresponding central frequencies of the Morlet wavelets. (in our tool the user never has to deal with scales directly). To obtain a physically meaningful quantity, one now defines the *wavelet power spectrum*:

$$P_{\mathcal{W}}(\tau, \omega) = \frac{|\tilde{f}(\tau, \omega)|^2}{\sigma^2} \quad (8)$$

By stacking the transformations frequency-ordered one constructs a two-dimensional

$(\tau - \omega)$  representation of the wavelet power spectrum, where the power itself is usually color coded (Figure 1c and f). Using a dense set of frequencies to scan for approximates the *continuous Wavelet transform*.

## Wavelet Power Interpretation

We adopted the normalization with the variance of the signal  $\sigma^2$  from ref. ? as it allows for a natural and statistical interpretation of the wavelet power. To see this, it's important to note that the *time averaged* wavelet power spectrum is an unbiased estimator for the true Fourier power spectrum of a signal (?):

$$\int_{-\infty}^{+\infty} P_{\mathcal{W}}(\tau, \omega) d\tau \rightarrow P_{\mathcal{F}}(\omega) \quad (9)$$

This allows to directly compare theoretical Fourier power spectra of stationary stochastic processes to the empirical wavelet power. White noise normalized by it's variance has a mean Fourier power of one, hence the normalized Wavelet power of one corresponds to the expected power for white noise (Figure ??). Extending these arguments to the random fluctuations

## 3 Optimal Filtering - Do's and Dont's

## 4 Readout - Along the Ridge

## 5 Application UI and API

## 6 Appendix

### 6.1 General Wavelet Properties

Here we want to briefly discuss a few mathematical properties of Wavelets in general, formal introductions may for example be found in ? and ?. An important property of Wavelets is the vanishing average:

$$\int_{-\infty}^{\infty} \Psi_{s,\tau}(t) dt = 0 \quad (10)$$

this is often referred to as “admissability criterion”. Grossmann and Morlet have shown, that for a faithful decomposition and re-synthesis this condition has to be fulfilled (?). In Fourier space this is equivalent to  $\hat{\Psi}_{s,\tau}(0) = 0$ , meaning a vanishing zero-frequency component.

To ensure equal decomposition on all scales, a normalization to unit energy is required:

$$\int_{-\infty}^{\infty} |\Psi_{s,\tau}(t)|^2 dt = 1 \quad (11)$$

These two conditions imply that a Wavelet itself is oscillatory and decaying in time.

### 6.1.1 Center frequency

Wavelets, as opposed to the Fourier basis functions, don't only have one frequency yet they are localized in frequency space. A way to still associate a single frequency to a specific Wavelet with scale  $s$  is the so called center frequency defined by:

$$\omega_c(s) = \int_{-\infty}^{\infty} \omega |\widehat{\Psi}_{s,\tau}(\omega)|^2 d\omega \quad (12)$$

From equation ?? and Plancherel's theorem it follows that the Fourier power spectrum of a Wavelet is also normalized:  $\int |\widehat{\Psi}_{s,\tau}(\omega)|^2 d\omega = 1$ , and therefore  $\omega_c$  simply is its mean frequency.

## 6.2 Morlet Wavelet Properties

Taking the usual definition (?) of the Morlet mother wavelet:

$$\Psi(t) = \pi^{-1/4} e^{-t^2/2} e^{i\omega_0 t}. \quad (13)$$

Its Fourier transform reads:

$$\widehat{\Psi}(\omega) = \sqrt{2\pi} e^{-\frac{1}{2}(\omega - \omega_0)^2}, \quad (14)$$

which is a Gaussian with mean  $\omega_0$ . If we check for admissibility:

$$\widehat{\Psi}(0) = \sqrt{2\pi} e^{-\frac{1}{2}\omega_0^2}, \quad (15)$$

we see that the condition is not fulfilled. In the literature this form of the Morlet wavelet is still the most prominent due to its simplicity. To at least approximately comply with the admissibility criterion, an  $\omega_0 \geq 2\pi$  is often chosen. With that we get:

$$\widehat{\Psi}(0) = \sqrt{2\pi} e^{-\frac{1}{2}4\pi^2} \approx 3.78 \cdot 10^{-9}, \quad (16)$$

which is very close to zero. From that we find for the center frequency of the Morlet wavelet with scale  $s$  using equations ?? and ??:  $\omega_c(s) = \frac{\omega_0}{s}$

However, the authors in ? give the Morlet center frequency as

$$\omega_c = \frac{s}{2}(\omega_0 + \sqrt{2 + \omega_0^2}) \quad (17)$$

which as shown is inconsistent with the definition of equation ?. This formula actually comes from the exactly admissible form of the Morlet wavelet ?:

$$\Psi_{adm}(t) \sim e^{-\frac{t^2}{2}} \left( e^{i\omega_0 t} - e^{-\frac{1}{2}\omega_0^2} \right) \quad (18)$$

Its Fourier transform is given by:

$$\widehat{\Psi}_{adm}(\omega) \sim e^{-\frac{1}{2}(\omega - \omega_0)^2} - e^{-\frac{1}{2}(\omega_0^2 + \omega^2)}, \quad (19)$$

and readily fulfils the admissibility criterion  $\widehat{\Psi}_{adm}(0) = 0$ . For  $\omega_0 = 2\pi$  the

deviation between the two center frequency definitions is around 1.25%, with the latter one being more exact.

## References

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